# Complex spaces with locally product metrics: general theory.

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**Summary** - *Locally product complex spaces are introduced with particular reference to the Calculus of Variations on complex manifolds. The goedesics of such spaces possess unusual features associated with the fact that many of the connection coefficients are tensorial in character. A restricted partial covariant derivative is introduced together with a curvature tensor. The latter is found to be invariant under a large class of gauge-like transformations. This invariance property leads naturally to the introduction of almost totally decomposable complex spaces. Necessary and sufficient conditions for a locally product complex Riemannian space to be almost totally decomposable are discussed. The counterpart of the Einstein tensor is also exhibited.*

## § 1. - Introduction.

We consider a $2n$-dimensional real manifold $X_{2n}$ (of class $C^\infty$) referred to local coordinates $(x^j, y^j)$, $(i, j, \dots = 1, \dots, n)$.

Corresponding to each point $P$ of $X_{2n}$ we introduce complex numbers $z^j$,

$$z^j = x^j + iy^j, \qquad (i^2 = -1), \tag{1.1}$$

which may be regarded as the complex coordinates of $P$ (with respect to the given coordinate system). If there exist complex coordinate neighbourhoods $U(z^j)$, $U(\bar{z}^j)$, (where $\bar{z}^j$ refer to another local coordinate system), such that in the intersection of these neighbourhoods we have

$$\begin{cases} \bar{z}^j = \bar{z}^j(z^h), \\ \det\left|\dfrac{\partial \bar{z}^j}{\partial z^h}\right| \neq 0, \end{cases} \tag{1.2}$$

where $\bar{z}^j(z^h)$ are holomorphic functions of $z^h$, the space $X_{2n}$ is said to admit a *complex structure*. Under these circumstances $X_{2n}$ is called a *complex space* of complex dimension $n$ and is usually denoted by $C_n$.

With (1.1) we may associate the conjugate complex

$$z^{j*} = x^j - iy^j,$$

so that (1.2) carries with it the corresponding conjugate complex transformation

$$\bar{z}^{j*} = \bar{z}^{j*}(z^{h*}). \tag{1.3}$$

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The $2n$ quantities $X^j$, $X^{j^*}$ defined over $C_n$ will be the components of a contravariant vector field if, under (1.2) and (1.3), they trasnform according to [1]

$$\bar{X}^j = \frac{\partial \bar{z}^j}{\partial z^h} X^h, \qquad \bar{X}^{j^*} = \frac{\partial \bar{z}^{j^*}}{\partial z^{h^*}} X^{h^*}.$$

The form of the general transformation laws for tensors should be clear from this example.

A metric may be defined over $C_n$ by introducing a tensor $g$ of covariant valency 2, such that the «distance» between two given points $z^i$, $z^{i^*}$ and $z^i + dz^i$, $z^{i^*} + dz^{i^*}$ is given by

$$ds^2 = g_{hj} dz^h dz^j + g_{hj^*} dz^h dz^{j^*} + g_{h^*j^*} dz^{h^*} dz^{j^*},$$

where it is usually assumed that $g_{hj}$, $g_{hj^*}$, $g_{h^*j^*}$ are symmetric.

Those complex spaces $C_n$ which are such that

$$g_{kj^*} = 0$$

are called *Locally Product Complex Finsler Spaces* if, in addition, the $g$ are functions of $(z^i, z^{i^*}, \dot{z}^i, \dot{z}^{i^*})$, where a dot denotes differentiation with respect to an arbitrary parameter $\tau$. We shall denote such spaces by $C_n^F$. If

$$\begin{cases} g_{hj} = g_{hj}(z^i, z^{i^*}, \dot{z}^i), \\ g_{h^*j^*} = g_{h^*j^*}(z^i, z^{i^*}, \dot{z}^{i^*}), \end{cases}$$

we say that $C_n^F$ is *tangent decomposable*, whereas if

$$\begin{cases} g_{hj} = g_{hj}(z^i, \dot{z}^i), \\ g_{h^*j^*} = g_{h^*j^*}(z^{i^*}, \dot{z}^{i^*}), \end{cases}$$

we speak of a *totally decomposable* $C_n^F$. If the $g$ are functions of $z^i$ and $z^{i^*}$ alone, $C_n^F$ is called a *Locally Product Complex Riemannian Space*, and will be denoted by $C_n^R$.

An investigation of locally product metrics on a *real* Riemannian space has been made [2], but no detailed analysis appears to exist for either $C_n^R$ or $C_n^F$. It is the purpose of this paper to consider certain problems in the Calculus of Variations on complex manifolds which give rise to $C_n^F$ spaces and to discuss the unusual features which such spaces possess, with particular

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(1) Unless otherwise stated the summation convention is used throughout this paper.

(2) See Tachibana [6] and Yano [7], Chapter X.

reference to the geodesics. Many of these features are still present in the Riemannian case. In order to gain insight into the general problem, $C_n^R$ spaces are therefore investigated and the concept of a restricted partial covariant derivative is introduced. This leads to a natural definition of a restricted curvature tensor. This curvature tensor is found to be invariant under a large class of transformations which are gauge-like in character. The significance of this transformation is discussed by introducing the notion of an almost totally decomposable space.

## § 2. - Locally Product Complex Finsler Spaces.

We shall consider real scalar Lagrange functions $L$ of the form

$$L = L(z^i, z^{i*}, \dot{z}^i, \dot{z}^{i*}), \tag{2.1}$$

where

$$z^j = x^j + iy^j,$$
$$z^{j*} = x^j - iy^j,$$

($x^j$, $y^j$ real, $i^2 = -1$) and the dot denotes differentation with respect to an arbitrary real parameter $\tau$. In order to ensure that the so-called action integral associated with (2.1), viz.

$$\int L d\tau, \tag{2.2}$$

is parameter invariant, it is necessary and sufficient that $L$ satisfies the condition (3)

$$\frac{\partial L}{\partial \dot{z}^i}\dot{z}^i + \frac{\partial L}{\partial \dot{z}^{i*}}\dot{z}^{i*} = L. \tag{2.3}$$

In the sequel *we consider only those Lagrangians which are of this form.* From (2.3) it can be shown that (4)

$$L^2 = \frac{1}{2}\frac{\partial^2 L^2}{\partial \dot{z}^j \partial \dot{z}^h}\dot{z}^j\dot{z}^h + \frac{\partial^2 L^2}{\partial \dot{z}^j \partial \dot{z}^{h*}}\dot{z}^j\dot{z}^{h*} + \frac{1}{2}\frac{\partial^2 L^2}{\partial \dot{z}^{j*} \partial \dot{z}^{h*}}\dot{z}^{h*}\dot{z}^{j*},$$

which may be rewritten as

$$L^2 = g_{jh}\dot{z}^j\dot{z}^h + g_{jh*}\dot{z}^j\dot{z}^{h*} + g_{j*h*}\dot{z}^{j*}\dot{z}^{h*}, \tag{2.4}$$

(3) For the definition of partial differentiation with respect to a complex variable, see e.g. Bochner and Martin [2], p. 37. On the necessary and sufficient condition (2.3) see Rund [4], p. 17.

(4) Rund [4], p. 48.

where we have defined

$$(2.5)\qquad \begin{cases} g_{jh} = \dfrac{1}{2}\,\partial^2 L^2/\partial\dot{z}^j\partial\dot{z}^h, \\ g_{jh^*} = \partial^2 L^2/\partial\dot{z}^j\partial\dot{z}^{h^*}, \\ g_{j^*h^*} = \dfrac{1}{2}\,\partial^2 L^2/\partial\dot{z}^{j^*}\partial\dot{z}^{h^*}. \end{cases}$$

It is evident that the quantities $g_{jh}$, $g_{jh^*}$, $g_{j^*h^*}$ are symmetric in their lower indices and in general will be functions of the $4n$ complex variables $(z^i,\ z^{i^*},\ \dot{z}^i,\ \dot{z}^{i^*})$.

We shall restrict our considerations to holomorphic transformations of the type

$$(2.6)\qquad \begin{aligned} \bar{z}^i &= \bar{z}^i(z^i), \\ \bar{z}^{i^*} &= \bar{z}^{i^*}(z^{j^*}), \end{aligned}$$

where we further assume that the Jacobian does not vanish in the region considered. Under (2.6) the quantities defined by (2.5) will transform like the components of tensors of covariant valency two (5).

It is the purpose of this paper to consider in detail *the consequences which arise from assuming that* (6)

$$(2.7)\qquad g_{ij^*} = 0.$$

From (2.5) and (2.7) it is evident that (2.4) reduces to

$$(2.8)\qquad L^2 = g_{ij}\dot{z}^i\dot{z}^j + g_{i^*j^*}\dot{z}^{i^*}\dot{z}^{j^*},$$

where

$$(2.9)\qquad \begin{cases} \partial g_{ij}/\partial\dot{z}^{h^*} = 0, \\ \partial g_{i^*j^*}/\partial\dot{z}^h = 0, \end{cases}$$

(5) See e.g. Bochner [1], p. 786 et seq.
(6) For the case $g_{ij} = 0$, $g_{i^*j^*} = 0$, see Rund [4].

from which we conclude that

$$
(2.10)\qquad \begin{aligned} g_{ij} &= g_{ij}(z^h, z^{h*}, \dot{z}^h), \\ g_{i*j*} &= g_{i*j*}(z^h, z^{h*}, \dot{z}^{h*}). \end{aligned}
$$

If we introduce a parameter $s$ characterised by

$$
(2.11)\qquad L^2 = (ds/d\tau)^2,
$$

we see that with (2.8) we may associate a *locally product complex Finsler space* with metric

$$
(2.12)\qquad ds^2 = g_{ij}dz^i dz^j + g_{i*j*}dz^{i*}dz^{j*}.
$$

Furthermore, in view of (2.10), this space is *tangent decomposable.*

Since $L^2$ is homogeneous of degree two in $\dot{z}^j$, $\dot{z}^{j*}$ it follows that

$$
(2.13)\qquad \frac{\partial g_{ij}}{\partial \dot{z}^h}\dot{z}^h = 0, \qquad \frac{\partial g_{i*j*}}{\partial \dot{z}^{h*}}\dot{z}^{h*} = 0,
$$

which, from (2.5), also implies that

$$
(2.14)\qquad \frac{\partial g_{ij}}{\partial \dot{z}^h}\dot{z}^j = 0, \qquad \frac{\partial g_{i*j*}}{\partial \dot{z}^{h*}}\dot{z}^{j*} = 0.
$$

However, since $L$ is assumed real we have

$$
(2.15)\qquad g_{ij} = (g_{i*j*})^*,
$$

so that the two equations in (2.13) are not independent but are the complex conjugates of each other. A similar remark applies to (2.14). In fact we will in general find that properties associated with $g_{i*j*}dz^{i*}dz^{j*}$ of (2.12) may be derived from the corresponding properties of $g_{ij}\, dz^i dz^j$ by taking the conjugate complex.

We adopt the following notation. Upper case Latin letters $A, B, C, \ldots$ (to which the summation convention still applies) are to run from 1 to $2n$ where

$$
\mathrm{A} = \begin{cases} i & \textit{when} \quad 1 \le A \le n, \\ i^* & \textit{when} \quad n+1 \le A \le 2n, \end{cases}
$$

so that, for example,

$$(g_{AB}) = \begin{pmatrix} g_{ij} & 0 \\ 0 & g_{i^*j^*} \end{pmatrix},$$

and (2.12) becomes

$$ds^2 = g_{AB} dz^A dz^B . \tag{2.16}$$

It is furthermore assumed that

$$\det |g_{AB}| \neq 0,$$

which is equivalent to the two conditions

$$\begin{cases} \det |g_{ij}| \neq 0, \\ \det |g_{i^*j^*}| \neq 0. \end{cases}$$

However, from (2.15), either of these two conditions is implied by the other so that in fact *we need only demand that* $\det |g_{ij}| \neq 0$. The quantities $g^{AB}$, $g^{ij}$ and $g^{i^*j^*}$ are the elements of the inverse matrices of $g_{AB}$, $g$ and $g_{i^*j^*}$ respectively. They are uniquely defined and enjoy the properties

$$g^{AB} g_{BC} = \delta^A_C , \qquad g^{ij} g_{ik} = \delta^j_k . \qquad g^{i^*j^*} g_{j^*k^*} = \delta^{i^*}_{k^*} .$$

It is not difficult to show that

$$(g^{AB}) = \begin{pmatrix} g^{ij} & 0 \\ 0 & g^{i^*j^*} \end{pmatrix}$$

and

$$g^{ij} = (g^{i^*j^*})^* .$$

In this notation (2.13) and (2.14) read

$$\begin{cases} \dfrac{\partial g_{AB}}{\partial \dot{z}^C} \dot{z}^C = 0, \\[2ex] \dfrac{\partial g_{AB}}{\partial \dot{z}^C} z^A = 0. \end{cases} \tag{2.17}$$

By means of (2.16) and (2.17) we could, if we so desired, introduce connection coefficients together with partial covariant differentation and the curvature tensors in a manner very similar to that of the real variable Finsler space (7).

However, if such a procedure is adopted it is difficult to gain any insight into the properties peculiar to the spaces under consideration. We therefore, for the time being, follow a slightly different line of thought.

We first state

LEMMA 1: - (a) $\partial g_{ij}/\partial z^{h^*}$ *and* $\partial g_{i^*j^*}/\partial z^h$ *are tensors under the transformation* (2.6).

(b) *If* $X^{i_1 \ldots i_r}_{j_1 \ldots j_s}$ *and* $X^{i_1^* \ldots i_r^*}_{j_1^* \ldots j_s^*}$ *are tensor fields then so are*

$$\partial X^{i_1 \ldots i_r}_{j_1 \ldots j_s}/\partial z^{h^*} \text{ and } \partial X^{i_1^* \ldots i_r^*}_{j_1^* \ldots j_s^*}/\partial z^h.$$

Although this result is somewhat unexpected, the proof is obvious.

We now turn to the geodesics which arise from the Lagrange function (2.8) when $L$ is substituted in the Euler-Lagrange equations corresponding to (2.2), viz.

$$\left\{\begin{aligned} &\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{z}^i}\right) - \frac{\partial L}{\partial z^i} = 0, \\ &\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{z}^{i^*}}\right) - \frac{\partial L}{\partial z^{i^*}} = 0. \end{aligned}\right. \tag{2.18}$$

It is evident that the second of these is the conjugate complex of the first so we need consider only one of them.

From (2.8), (2.9) (2.13) and (2.14) we find

$$\frac{\partial L}{\partial \dot{z}^i} = \frac{1}{L} g_{ij}\dot{z}^j,$$

together with

$$\frac{\partial L}{\partial z^i} = \frac{1}{2L}\left(\frac{\partial g_{hj}}{\partial z^i}\dot{z}^h\dot{z}^j + \frac{\partial g_{h^*j^*}}{\partial z^i}\dot{z}^{h^*}\dot{z}^{j^*}\right).$$

We substitute these relations in (2.18) and, after simplification by means of (2.13) and (2.14) we find that *the equations characterising geodesics are*

$$g_{ij}\ddot{z}^j + [hj, i]\dot{z}^h\dot{z}^j = \frac{\dot{L}}{L} g_{ij}\dot{z}^j + F_{ih^*}\dot{z}^{h^*}, \tag{2.19}$$

(7) Rund [5].

*where*

$$[hj,\ i]=\frac{1}{2}\left(\frac{\partial g_{ji}}{\partial z^h}+\frac{\partial g_{ih}}{\partial z^j}-\frac{\partial g_{hj}}{\partial z^i}\right) \tag{2.20}$$

*and*

$$F_{ih^*}=\frac{1}{2}\frac{\partial g_{h^*j^*}}{\partial z^i}\dot{z}^{j^*}-\frac{\partial g_{ij}}{\partial z^{h^*}}\dot{z}^j. \tag{2.21}$$

From Lemma 1 we see that the $F_{ih^*}$ *are the componentes of a tensor of covariant valency two.* We notice that the first term on the right hand side of (2.19) may be removed as usual by choosing $\tau=s$ and using (2.11). However, we cannot eliminate the second term which by analogy with the corresponding real variable case, will be called *the force vector.* We remark that the force vector arises by virtue of the dependence of $g_{ij}$ on $z^{k^*}$ and $g_{i^*j^*}$ on $z^k$, and may therefore be regarded as an «interaction» term.

If we define

$$F_{i^*h}=\frac{1}{2}\frac{\partial g_{hj}}{\partial z^{i^*}}\dot{z}^j-\frac{\partial g_{i^*j^*}}{\partial z^h}\dot{z}^{j^*}, \tag{2.22}$$

together with

$$[i^*j^*,\ k^*]=\frac{1}{2}\left(\frac{\partial g_{j^*k^*}}{\partial z^{i^*}}+\frac{\partial g_{k^*i^*}}{\partial z^{j^*}}-\frac{\partial g_{i^*j^*}}{\partial z^{k^*}}\right), \tag{2.23}$$

and observe that

$$F_{ih^*}=(F_{i^*h})^*,$$

$$[ij,\ k]=[i^*j^*,\ k^*]^*,$$

then the conjugate complex of (2.19) is

$$g_{i^*j^*}\ddot{z}^{j^*}+[h^*j^*,\ i^*]\dot{z}^{h^*}\dot{z}^{j^*}=\frac{\dot{L}}{L}g_{i^*j^*}\dot{z}^{j^*}+F_{i^*h}\dot{z}^h.$$

From (2.21) and (2.22) we have the identities

$$\left\{\begin{aligned}\frac{\partial F_{ih^*}}{\partial \dot{z}^j}&=\frac{\partial F_{jh^*}}{\partial \dot{z}^i}=-2\frac{\partial F_{h^*i}}{\partial \dot{z}^j}=-2\frac{\partial F_{h^*j}}{\partial \dot{z}^i}=-\frac{\partial g_{ij}}{\partial z^{h^*}},\\ \frac{\partial F_{i^*h}}{\partial \dot{z}^{j^*}}&=\frac{\partial F_{j^*h}}{\partial \dot{z}^{i^*}}=-2\frac{\partial F_{hi^*}}{\partial \dot{z}^{j^*}}=-2\frac{\partial F_{hj^*}}{\partial \dot{z}^{i^*}}=-\frac{\partial g_{i^*j^*}}{\partial z^h}.\end{aligned}\right. \tag{2.24}$$

In view of Lemma 1, *all these quantities are tensors.* When (2.24) is substituted back in (2.21) we find

$$F_{ih^*} = \frac{\partial F_{ih^*}}{\partial \dot{z}^j}\dot{z}^j + \frac{\partial F_{ih^*}}{\partial \dot{z}^{j^*}}\dot{z}^{j^*},$$

so that $F_{ih^*}$ *is homogeneous of degree one in* $\dot{z}^j$, $\dot{z}^{j^*}$.

We can thus state

THEOREM 1. - *A necessary and sufficient condition for the space* (2.12) *to be totally decomposable is that* $F_{ih^*}$ *is independent of both* $\dot{z}^j$ *and* $\dot{z}^{j^*}$, *or, equivalently, that* $F_{ih^*}=0$.

In view of the tensorial character of (2.24), these are invariant conditions.

## § 3. - Locally Product Complex Riemannian Spaces.

Although it is possible to continue with a study of locally product Finsler spaces along the lines suggested in the previous section, we notice that the interesting and unusual features of this theory (viz. Lemma 1 and the presence of $F_{ih^*}$ in the geodesic equation (2.19)) persist irrespective of the dependence of $g_{ij}$ on $\dot{z}^h$. In order to gain some insight into the subject, *we therefore restrict ourselves in the sequel to locally product complex Riemannian spaces*, i. e. to metrics of the form (8) (compare with (2.12))

$$ds^2 = g_{ij}dz^i dz^j + g_{i^*j^*}dz^{i^*}dz^{j^*}, \tag{3.1}$$

where

$$g_{ij} = g_{ij}(z^h,\ z^{h^*})$$

and

$$g_{i^*j^*} = g_{i^*j^*}(z^h,\ z^{h^*}).$$

Using the notation introduced in Section 2, we may define the Christoffel symbols of the first and second kinds by

$$[AB,\ C] = \frac{1}{2}\left(\frac{\partial g_{BC}}{\partial z^A} + \frac{\partial g_{CA}}{\partial z^B} - \frac{\partial g_{AB}}{\partial z^C}\right)$$

and

$$\left\{ {A \atop BC} \right\} = g^{AD}[BC,\ D] \tag{3.2}$$

(8) This implies that there exists a $J$ other than $I$ for which $J^2=I$.

respectively. This is consistent with (2.20) and (2.23). In general these quantities are not tensors but, in the case of $\left\{ {A \atop BC} \right\}$, transform under (2.6) according to

$$\text{(3.3)} \qquad \overline{\left\{ {R \atop ST} \right\}} \frac{\partial \bar{z}^S}{\partial z^B} \frac{\partial \bar{z}^T}{\partial z^C} \frac{\partial z^A}{\partial \bar{z}^R} = \left\{ {A \atop BC} \right\} - \frac{\partial^2 \bar{z}^R}{\partial z^B \partial z^C} \frac{\partial z^A}{\partial \bar{z}^R}.$$

At this stage we could pursue the programme outlined following (2.17) and introduce the partial covariant derivative of a tensor field $X^{A_1 \dots A_r}_{B_1 \dots B_s}$ by

$$\text{(3.4)} \qquad X^{A_1 \dots A_r}_{B_1 \dots B_s \mid C} = \partial X^{A_1 \dots A_r}_{B_1 \dots B_s} / \partial z^C + \sum_{\mu=1}^{r} \left\{ {A_\mu \atop HC} \right\} X^{A_1 \dots A_{\mu-1} H A_{\mu+1} \dots A_r}_{B_1 \dots B_s} + \\ - \sum_{\mu=1}^{s} \left\{ {H \atop B_\mu C} \right\} X^{A_1 \dots A_r}_{B_1 \dots B_{\mu-1} H B_{\mu+1} \dots B_s}.$$

The curvature tensor $H_A{}^B{}_{CD}$ would then be introduced by means of the integrability conditions

$$\text{(3.5)} \qquad X^A{}_{\mid BC} - X^A{}_{\mid CB} = H_D{}^A{}_{BC} X^D,$$

according to which

$$\text{(3.6)} \qquad H_D{}^A{}_{BC} = \frac{\partial}{\partial z^C} \left\{ {A \atop D\ B} \right\} - \frac{\partial}{\partial z^B} \left\{ {A \atop D\ C} \right\} + \left\{ {R \atop D\ B} \right\} \left\{ {A \atop C\ R} \right\} - \left\{ {R \atop D\ C} \right\} \left\{ {A \atop B\ R} \right\}.$$

However, as we have already noted, this notation and procedure tends to obscure those aspects of the theory which are of interest to us. In fact, we shall show that the above approach, (3.4) to (3.6), may be replaced by a similar one which has the advantage of exhibiting explicitly the unusual features of this space. This approach depends upon

LEMMA 2. - (a) *All Christoffel symbols involving one or two asterisks are tensors under* (2.6). *Equivalently, the only non-tensorial Christoffel symbols of the second kind are* $\left\{ {i \atop jk} \right\}$ *and* $\left\{ {i^* \atop j^*k^*} \right\}$.

(b) $\partial \left\{ {i \atop jk} \right\} / \partial z^{h^*}$ *and* $\partial \left\{ {i^* \atop j^*k^*} \right\} / \partial z^h$ *are tensors.*

The proof of this follows directly from (3.3).

The consequences of the first part of this Lemma manifest themselves immediately if the right hand side of (3.4) is expanded fully. It is obvious that a number of these terms will in general involve Christoffel symbols of the second kind with one or two asterisks and will consequently be tensors. We are thus lead, in a fairly natural manner, to the definition of *the restric-*

*ted partial covariant derivative of a tensor field* $X^{A_1\dots A_r}_{B_1\dots B_s}$, viz.

(3.7)
$$X^{A_1\dots A_r}_{B_1\dots B_s\,;\,C} = \partial X^{A_1\dots A_r}_{B_1\dots B_s}/\partial z^C + \sum_{\mu=1}^{r} \Gamma_H{}^{A_\mu}{}_C X^{A_1\dots A_{\mu-1}HA_{\mu+1}\dots A_r}_{B_1\dots B_s} +$$
$$- \sum_{\mu=1}^{s} \Gamma_{B_\mu}{}^H{}_C X^{A_1\dots A_r}_{B_1\dots B_{\mu-1}HB_{\mu+1}\dots B_s},$$

where

$$\Gamma_B{}^A{}_C = \begin{cases} \left\{ {A \atop B\ C} \right\} & \textit{if } \left\{ {A \atop B\ C} \right\} \textit{ is not a tensor,} \\ 0 & \textit{if } \left\{ {A \atop B\ C} \right\} \textit{ is a tensor.} \end{cases}$$

A few remarks on the properties of the restricted partial covariant derivative are perhaps in order. It should be clear that (3.7) may be derived from (3.4) by omitting in the latter all (additive) terms which are tensorial. As a result of this the restricted partial covariant derivative of a tensor is once more a tensor, while, if $\varphi$ is a scalar

$$\varphi\,;\,_A = \partial\varphi/\partial z^A.$$

The usual product rule will also hold. However, we do have to sacrifice the strict counterpart of Ricci's Lemma since in general

$$g_{AB\,;\,C} \neq 0,$$

as may be seen from

LEMMA 3.

$$g_{ij\,;\,k} = 0, \qquad g^{ij}{}_{;\,k} = 0, \qquad g_{i^*j^*\,;\,k^*} = 0, \qquad g^{i^*j^*}{}_{;\,k^*} = 0,$$
$$g_{ij\,;\,k^*} = \partial g_{ij}/\partial z^{k^*}, \qquad g^{ij}{}_{;\,k^*} = \partial g^{ij}/\partial z^{k^*},$$
$$g_{i^*j^*\,;\,k} = \partial g_{i^*j^*}/\partial z^k, \qquad g^{*j^*}{}_{;\,k} = \partial g^{i^*j^*}/\partial z^k,$$
$$\delta^A_{B\,;\,C} = 0.$$

Consequently, *a necessary and sufficient condition for Ricci's Lemma to hold strictly* [9], viz. $g_{AB\,;\,C} = 0$, *is that the space be totally decomposable.*

We now introduce the *restricted curvature tensor* $R_A{}^B{}_{CD}$ by means of the integrability conditions

(3.8)
$$X^A{}_{;\,BC} - X^A{}_{;\,CB} = R_D{}^A{}_{BC} X^D.$$

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(9) Of course $g_{AB|C} \equiv 0$.

It can be shown that (3.8) will be satisfied if we define $R_D{}^A{}_{BC}$ according to the following scheme:

$$(3.9)\quad \begin{cases} R_i{}^j{}_{kl} = \dfrac{\partial}{\partial z^l}\left\{ {j \atop ik} \right\} - \dfrac{\partial}{\partial z^k}\left\{ {j \atop il} \right\} + \left\{ {v \atop ik} \right\}\left\{ {j \atop vl} \right\} - \left\{ {v \atop il} \right\}\left\{ {j \atop vk} \right\}, \\ R_{i^*}{}^{j^*}{}_{k^*l^*} = (R_i{}^j{}_{kl})^*, \qquad R_i{}^j{}_{kl^*} = \dfrac{\partial}{\partial z^{l^*}}\left\{ {j \atop ik} \right\}, \\ R_i{}^j{}_{l^*k} = - R_i{}^j{}_{kl^*}, \qquad R_{i^*}{}^{j^*}{}_{k^*l} = (R_i{}^j{}_{kl^*})^*, \\ R_{i^*}{}^{j^*}{}_{lk^*} = - R_{i^*}{}^{j^*}{}_{k^*l} \end{cases}$$

*with the remaining ten expressions being zero.*

$R_A{}^B{}_{CD}$ is indeed a tensor by virtue of Lemma 2(*b*).

We remark that $R_i{}^j{}_{kl}$ and $R_{i^*}{}^{j^*}{}_{k^*l^*}$ take the form of the curvature tensor in the case of a totally decomposed space, while $R_i{}^j{}_{kl^*}$ may be regarded as some form of « interaction » curvature term. However, it is not true that the vanishing of $R_i{}^j{}_{kl^*}$ characterises a totally decomposed space. This may be seen by considering the special case of $n = 2$ with the metric

$$ds^2 = f(z^1)g(z^{1^*})[(dz^1)^2 + (dz^2)^2] + f^*g^*[dz^{1^*})^2 + (dz^{2^*})^2],$$

where $f$ and $g$ are arbitrary function of $z^1$ and $z^{1^*}$ respectively. This is an example of a non-totally decomposed space for which $R_i{}^j{}_{kl^*} = 0$. We shall return to this condition later.

Many, but not all, of the properties usually associated with a curvature tensor are enjoyed by $R_A{}^B{}_{CD}$. From (3.9) we obviously have

$$(3.10)\qquad R_A{}^B{}_{CD} = - R_A{}^B{}_{DC},$$

together with

$$(3.11)\qquad R_A{}^D{}_{BC} + R_C{}^D{}_{AB} + R_B{}^D{}_{CA} = 0.$$

Furthermore the Bianchi identities are satisfied, viz.

$$(3.12)\qquad R_A{}^B{}_{CD;E} + R_A{}^B{}_{EC;D} + R_A{}^B{}_{DE;C} = 0.$$

On the other hand, if we define

$$(3.13)\qquad R_{ABCD} = g_{BE}R_A{}^E{}_{CD},$$

we have, in general,

$$\left\{ \begin{array}{l} R_{ABCD} \neq - R_{BACD}, \\ R_{ABCD} \neq R_{CDAB}, \end{array} \right. \tag{3.14}$$

although we do have equality for $R_{ijkl}$ and $R_{i^*j^*k^*l^*}$. By its very construction it is clear that $R_A{}^B{}_{CD}$ may be derived from $H_A{}^B{}_{CD}$ (defined by (3.5) or (3.6)) by omitting from the latter all (additive) tensorial terms.

We briefly return to (3.14) to investigate under what conditions equality holds.

LEMMA 4: *The following conditions are equivalent:*

(a) $$R_{ABCD} = - R_{BACD};$$

(b) $$R_{ABCD} = R_{CDAB};$$

(c) $$R_{ijhk^*} = 0.$$

The equivalence of (b) and (c) is obvious since (b) implies that

$$R_{ijkl^*} = R_{kl^*ij}\,,$$

and the right hand side is zero by (3.9). To prove the equivalence of (a) and (c) we see that from (a) and (3.9) we have

$$\begin{aligned} R_{ij\,l^*} &= \quad R_{kjil^*} \\ &= - R_{jkil^*} \\ &= - R_{ikjl^*}\,, \end{aligned}$$

together with

$$\begin{aligned} R_{ijkl^*} &= - R_{jikl^*} \\ &= - R_{kijl^*} \\ &= R_{ikjl^*}\,. \end{aligned}$$

A comparison of these sets of equations gives (c), which proves the lemma.

In view of (3.14) there exist three essentially different ways of contracting $R_A{}^B{}_{CD}$. We therefore introduce the tensors $R_{AB}$, $V_{AB}$ and $K^C{}_D$ defined by

$$\left\{ \begin{array}{l} R_{AB} = R_A{}^C{}_{BC}\,, \\ V_{AB} = R_C{}^C{}_{AB}\,, \\ \text{and} \\ K^A{}_B = g^{CD} R_C{}^A{}_{BD} \end{array} \right. \tag{3.15}$$

respectively.

By means of (3.9), $R_{AB}$ may be written out explicitly as

$$\text{(3.16)}\quad \begin{cases} R_{ij} = R_i{}^h{}_{jh}, & R_{i^*j^*} = (R_{ij})^*, \\ R_{hk^*} = -\dfrac{\partial}{\partial z^{k^*}} \left\{ {l \atop hl} \right\}, & R_{h^*k} = (R_{hk^*})^*, \end{cases}$$

while $V_{AB}$ takes on the form

$$\text{(3.17)}\quad \begin{cases} V_{ij} = 0, \qquad V_{i^*j^*} = 0, \\ -V_{ij^*} = V_{j^*i} = R_{ij^*} + R_{j^*i}. \end{cases}$$

If we define

$$\text{(3.18)}\quad R^A{}_B = g^{AC} R_{CB},$$

then $K^A{}_B$ is given by the following scheme

$$\text{(3.19)}\quad \begin{cases} K^h{}_k = -R^h{}_k, & K^{h^*}{}_{k^*} = (K^h{}_k)^*, \\ K^h{}_{k^*} = -g^{ij} \dfrac{\partial}{\partial z^{k^*}} \left\{ {h \atop ij} \right\}, & K^{h^*}{}_k = (K^h{}_{k^*})^*. \end{cases}$$

From (3.16)-(3.19) it is evident that $R_{AB}$, $V_{AB}$ and $K^A{}_B$ are intimately related. The major difference between them lies in the $R_{jh^*}$ and $K^j{}_{h^*}$ terms. We observe that if $R_{AB} = 0$ then $V_{AB} = 0$ together with $K^A{}_B = 0$ up to $K^h{}_{j^*}$.

We further remark that in general $R_{AB}$ is not symmetric in its lower indices (although, naturally, $R_{ij}$ and $R_{i^*j^*}$ are symmetric).

We define the scalar curvatures $R$, $V$ and $K$ by

$$\text{(3.20)}\quad \begin{cases} R = g^{AB} R_{AB}, \\ V = g^{AB} V_{AB}, \\ \text{and} \\ K = K^A{}_A \end{cases}$$

respectively. If we substitute (3.16)-(3.19) in (3.20) we find that

$$\text{(3.21)}\quad \begin{cases} R = R^i{}_i + (R^i{}_i)^*, \\ V = 0, \\ \text{and} \\ K = -R. \end{cases}$$

The interesting feature resulting from (3.21) is the fact that *the scalar curvatures are real.*

We now ruturn to the Bianchi identities (3.12) in order to introduce the counterpart of the Einstein tensor. In (3.12) we set $B = D$ and multiply $g^{AC}$ to find

(3.22)
$$(g^{AC}R_A{}^B{}_{CB})_{;E} + (g^{AC}R_A{}^B{}_{EC})_{;B} + (g^{AC}R_A{}^B{}_{BE})_{;C}$$
$$= g^{AC}{}_{;E}\, R_A{}^B{}_{CB} + g^{AC}{}_{;B}R_A{}^B{}_{EC} + g^{AC}{}_{;C}R_A{}^B{}_{BE}.$$

By means of (3.10), (3.15), (3.18) and (3.20) the left hand side of (3.22) may be written in the form

$$(R\delta^B_E + K^B{}_E - R^B{}_E)_{;B}.$$

Turning to the right hand side of (3.22) we find that in spite of the non-validity of the strict form of Ricci's Lemma, the last two terms on the right hand side vanish. Thus (3.22) reduces to

(3.23)
$$G^B{}_{E;B} = -g^{AC}{}_{;E}R_{AC},$$

where $G^B{}_E$ is defined by

(3.24)
$$G^B{}_E = R^B{}_E - K^B{}_E - R\delta^B{}_E,$$

and represents *the counterpart of the Einstein tensor.*

Although the divergence of $G^B{}_E$, viz. $G^B{}_{E;B}$, is not identically zero, it is zero whenever $R_{ij} = 0$.

Furthermore, if we define

(3.25)
$$G_{CE} = g_{BC}G^B{}_E,$$

and

$$G_{CE;}{}^C = g^{BC}G_{BE;C},$$

it follows that

(3.26)
$$G_{CE;}{}^C = G^C{}_{E;C}.$$

However, the right hand side of (3.23) satisfies

$$g^{AC}{}_{;E}R_{AC} = -g_{AC;E}R^{AC},$$

where

$$R^{AC} = g^{BC} R^{A}{}_{B},$$

so that (3.23) may be written in the form

$$(3.27) \qquad G_{CE;}{}^{C} = G^{C}{}_{E;\,C} = g_{AC;\,E} R^{AC} = -\, g^{AC}{}_{;\,E} R_{AC}\,.$$

We define

$$G = G^{A}{}_{A},$$

so that, by virtue of (3.20), (3.21) and (3.24), we have

$$G = 2(1 - n)R.$$

Hence $G$ *is also real.*

We remark that if $G^{A}{}_{B} = 0$ then $R^{i}{}_{j} = 0$ and $R^{i^*}{}_{j} = K^{i^*}{}_{j}$.

We now state an unexpected result in the form of

THEOREM 2. - *Under trasformations of the type*

$$g_{ij} \rightarrow f_{ij} = \lambda g_{ij}$$

*where* $\lambda = \lambda(z^{h^*})$, *the restricted curvature tensor* $R_{A}{}^{B}{}_{CD}$ *is invariant.*

The proof of this theorem follows immediately it is realised that $\left\{ {i \atop jk} \right\}$ and $\left\{ {i^* \atop j^*k^*} \right\}$ are invariant under $g_{ij} \rightarrow \lambda(z^{h^*}) g_{ij}$.

As an application of Theorem 2 we return to the remarks following (3.9) concerning the vanishing of $R_{i}{}^{j}{}_{hk^*}$. We conclude that *it is impossible to characterise totally decomposable spaces by imposing conditions on* $R_{A}{}^{B}{}_{CD}$ *alone.*

If we reconsider the geodesic equation (2.19), viz.

$$(3.28) \qquad \ddot{z}^{i} + \left\{ {i \atop jk} \right\} \dot{z}^{j} \dot{z}^{k} - \frac{\dot{L}}{L} \dot{z}^{i} = g^{ij} F_{jh^*} \dot{z}^{h^*},$$

in the light of Theorem 2, we have the following state of affairs.

Under transformations of the type $g_{ij} \rightarrow \lambda(z^{h^*}) g_{ij}$ the left hand side of (3.28) is invariant, whereas the right hand side is not. Consequently *we may have spaces with the same curvature properties but with different geodesics.* In fact under the transformation $g_{ij} \rightarrow \lambda(z^{h^*}) g_{ij}$ we find that

$$g^{i} F_{ih^*} \rightarrow \frac{\lambda^*}{\lambda} g^{li} F_{ih^*} + \left(\frac{\lambda^*}{\lambda} - 1\right) g^{li} \frac{\partial g_{ij}}{\partial z^{h^*}} \dot{z}^{j} + \frac{1}{\lambda} \left[\frac{1}{2} \frac{\partial \lambda^*}{\partial z^{i}} g^{i}{}_{h^* j^*} \dot{z}^{j^*} - \frac{\partial \lambda}{\partial z^{h^*}} \dot{z}^{l}\right].$$

Hence, if we demand that $g^{li}F_{ih^*} \longrightarrow g^{li}F_{ih^*}$ we must have

$$\lambda = \lambda^* = \text{constant}.$$

In view of Theorem 2 it is evident that we should seek conditions for deciding whether or not a given space is totally decomposable up to a transformation of the type $g_{ij} \rightarrow \lambda(z^{h^*})g_{ij}$, i.e. whether or not there exist quantities $\lambda = \lambda(z^{h^*})$ and $f_{ij} = f_{ij}(z^h)$ for which

$$g_{ij} = \lambda f_{ij}.$$

In $g_{ij}$ can be decomposed in this way, we shall call $C_n^R$ *almost totally decomposable* (10). A problem which is closely related to this (although not generally equivalent to it) is that of finding necessary and sufficient conditions for $g_{hk}/g_{ij}$ to be a function of $z^l$ alone (assuming $g_{ij} \neq 0$). If we consider the (non-tensorial) quantity $\gamma^{ij}_{hk}$ defined by

$$\gamma^{ij}_{hk} = g_{hk}/g_{ij}, \tag{3.29}$$

then

$$\frac{\partial \gamma^{ij}_{hk}}{\partial z^{l^*}} = \frac{1}{g_{ij}} \frac{\partial g_{hk}}{\partial z^{l^*}} - \frac{g_{hk}}{(g_{ij})^2} \frac{\partial g_{ij}}{\partial z^{l^*}}, \qquad (\textit{no summation}!).$$

For this to vanish we must have

$$g_{ij} \frac{\partial g_{hk}}{\partial z^{l^*}} = g_{hk} \frac{\partial g_{ij}}{\partial z^{l^*}},$$

or,

$$\frac{\partial g_{hk}}{\partial z^{l^*}} = g_{hk}\eta_{l^*}, \tag{3.30}$$

where $\eta_{l^*}$ is a vector given by

$$\eta_{l^*} = \frac{1}{n} g^{hk} g_{hk;\,l^*}. \tag{3.31}$$

Furthermore, since

$$\left\{ \begin{matrix} j^* \\ hk \end{matrix} \right\} = -\frac{1}{2} g^{l^*j^*} \frac{\partial g_{hk}}{\partial z^{l^*}},$$

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(10) Such spaces should not be confused with almost complex spaces, e.g. Yano [7].

(3.30) is equivalent to

$$\left\{ {j^* \atop hk} \right\} = -\frac{1}{2} g_{hk}\eta^{j^*}, \tag{3.32}$$

where

$$\eta^{j^*} = g^{l^*j^*}\eta_{l^*}.$$

It is easily seen that (3.32) is valid if and only if

$$\left\{ {i \atop jl^*} \right\} = \frac{1}{2} \delta^i_j \eta_{l^*}. \tag{3.33}$$

We have thus proved

THEOREM 3. - *A necessary and sufficient condition for* $g_{hk}/g_{ij}$ *to be independent of* $z^{l^*}$ *(assuming* $g_{ij} \neq 0$*) is that there exists a vector* $\eta_{l^*}$ *for which*

$$\left\{ {i \atop jl^*} \right\} = \frac{1}{2} \delta^i_j \eta_{l^*}.$$

*In this case*

$$\eta_{l^*} = \frac{1}{n} g^{hk} g_{hk;\, l^*}.$$

When (3.33) is substituted in the identity

$$\frac{\partial g_{hk}}{\partial z^{l^*}} = -\left\{ {h \atop r\ l^*} \right\} g^{rk} - \left\{ {k \atop r\ l^*} \right\} g^{hr},$$

we find

$$\frac{\partial g_{hk}}{\partial z^{l^*}} = -\, g^{hk}\eta_{l^*}. \tag{3.34}$$

Obviously (3.30), (3.32), (3.33) and (3.34) are all equivalent conditions. If we substitute them in $R_h{}^j{}_{kl^*}$ we have

$$R_h{}^j{}_{kl^*} = \delta^j_h \frac{\partial \eta_{l^*}}{\partial z^k} + \delta^j_k \frac{\partial \eta_{l^*}}{\partial z^h} - g^{ij} g_{kh} \frac{\partial \eta_{l^*}}{\partial z^i}.$$

Thus we conclude that *under condition* (3.33) $R_h{}^j{}_{kl^*}$ *vanishes if and only if*

$$\partial \eta_{l^*}/\partial z^i = 0. \tag{3.35}$$

In view of our suggested interpretation of $R_{h}{}^{j}{}_{kl^*}$ as an interaction curvature, one might suspect that the existence of $\eta_{l^*}$ satisfying (3.33) together with (3.35) (the latter being equivalent to $R_{h}{}^{j}{}_{kl^*}=0$) would ensure that $C^R_n$ is almost totally decomposable. We shall investigate this assertion.

If (3.33) is satisfied then (3.29) implies that $g_{hk}$ may be written in the form

$$g_{hk}=\varphi(z^l,\ z^{l^*})f_{hk}(z^i). \tag{3.36}$$

From (3.31) and (3.36) we find

$$\eta_{l^*}=\frac{1}{\varphi}\,\frac{\partial\varphi}{\partial z^{l^*}},$$

which, when substituted in (3.35), shows that $\varphi$ may be decomposed into the form

$$\varphi(z^h,\ z^{j^*})=\lambda(z^{j^*})\rho(z^h), \tag{3.37}$$

where $\lambda$ and $\rho$ are functions of $z^{j^*}$ and $z^h$ respectively.

By substituting (3.37) in (3.36), and absorbing $\rho$ in $f_{hk}$, we have

THEOREM 4. – *A necessary and sufficient condition for a given space to be almost totally decomposable is that there exists a vector $\eta_{l^*}$ satisfying*

$$\left\{ {i \atop j\ l^*} \right\}=\frac{1}{2}\,\delta^i_j\eta_{l^*},$$

*together with*

$$\frac{\partial\eta_{l^*}}{\partial z^i}=0.$$

*Furthermore, the space is totally decomposable if and only if $\eta_{l^*}=0$.*

The last part of this theorem is a consequence of (3.30) and the result following Lemma 3.

In a sequel to this paper [11] some applications of the above general theory will be discussed.

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(11) LOVELOCK [3].

## LITERATURE


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