A general solution of a system of nonlinear ordinary differential equations $xy' = f(x, y)$ in the case when $f_y(0, 0)$ is the zero matrix.

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Summary. - *Let there be given a system of n nonlinear ordinary differential equations of the form $xy' = f(x, y)$. Here x is a complex variable, y is an n -column vector and $f(x, y)$ is an n -column vector whose components are holomorphic functions of (x, y) at $(0, 0)$ and vanish at $x = 0, y = 0$. In the case when the Jacobian matrix $f_y(x, y)$ is reduced to the zero matrix at $(0, 0)$, we shall study how to construct an analytic expression for a general solution under Assumptions I and II.*

Introduction.

§ 1. - Assumptions and Main Result.

1°. - The singular points $x = 0$ of a system of nonlinear ordinary differential equations which, in vectorial form, can be written as

$$(A) \quad xy' = f(x, y) \quad \left(' = \frac{d}{dx} \right)$$

are usually said to be of BRIOT-BOUQUET type. Here x is a complex independent variable, y is an n -dimensional column vector with elements $\{y_j\}$, $f(x, y)$ is an n -dimensional column vector whose components are holomorphic functions of (x, y) in a domain of the form

$$|x| \leq a, \quad \|y\| = \max_{j=1}^n |y_j| \leq b$$

and $f(0, 0)$ is the zero vector, a and b being positive constants.

Such singular points have been studied by diverse authors since C. BRIOT and J. C. BOUQUET. However, it has not yet been studied when all the eigenvalues of the matrix $f_y(0, 0)$ are zero except for the one-dimensional case. In 1937, M. HUKUHARA [1, 4] studied first this case for $n = 1$. He obtained an analytic expression for a general solution and, moreover, investigated the

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property of the analytic function defined by this analytic expression in the vicinity of the singular point $x = 0$. After eighteen years, HUKUHARA's result played an important role in the theory of T. KIMURA [6], which comes in close contact with the excellent work of J. MALMQUIST [7]. So, it is of some interest, I think, to try to extend HUKUHARA's result concerning the construction of an analytic expression for a general solution to the higher dimensional case.

In this paper we shall study the case when the Jacobian matrix $f_y(x, y)$ is the zero matrix at $x = 0, y = 0$. However, in the higher dimensional case, it is very difficult to give an analytic meaning to a formal transformation which formally changes the given equation into an equation with a simpler form, unless the vector $f(0, y)$ has a particular form. In 1949, M. HUKUHARA [2] studied a problem on constructing a formal transformation for a system of two differential equations of the form

$$xy' = y^m F(x, y)y,$$

but he left the study of analytic meanings of this formal transformation undone. Here m is a two-row vector $\{m_1, m_2\}$ with nonnegative integers m_j not simultaneously zero; $y^m = y_1^{m_1} y_2^{m_2}$; $F(x, y)$ is a two-by-two diagonal matrix and is non-singular at $x = 0, y = 0$.

2°. - The starting point of this study was a problem on giving an analytic meaning to the formal transformation that M. HUKUHARA left undone.

Before introducing our fundamental assumptions, we shall explain the notation. For an n -column vector y with elements $\{y_j\}$ and for an n -row vector m with elements $\{m_j\}$, the symbol y^m stands for the scalar expression

$$y^m = y_1^{m_1} y_2^{m_2} \dots y_n^{m_n};$$

$\arg y$ denotes an n -column vector with elements $\{\arg y_j\}$; my is the inner product and $\mathbf{L}_n(y)$ is an n -by- n diagonal matrix with diagonal elements $\{y_j\}$.

We introduce the following two assumptions:

ASSUMPTION I. - The n -vector $f(0, y)$ has the form

$$(1.1) \quad f(0, y) = y^m \mathbf{L}_n(f(y))y,$$

where $f(y)$ is an n -column vector whose components are holomorphic functions of y in $\|y\| \leq b$ and m is an n -row vector with elements $\{m_j\}$ with nonnegative integers m_j not simultaneously zero.

ASSUMPTION II. - All the components of the n -column vector $\frac{1}{m\alpha} \alpha$, where $\alpha = f(0)$, have positive real parts.

We observe by virtue of Assumption I that the Jacobian matrix $f_y(x, y)$ is the zero matrix at $(0, 0)$. In the case of $n = 1$, it is easily seen that these two assumptions are automatically satisfied.

Assumption II is equivalent to assuming that

$$(1.2) \quad \left\| \arg \frac{\alpha}{m\alpha} \right\| \leq \frac{\pi}{2} - 3\varepsilon$$

for a sufficiently small positive number ε .

3°. – The purpose of this paper is to prove the following theorem:

THEOREM A. – *Suppose that Assumptions I and II are satisfied.*

Then there exists a transformation of the form

$$(T) \quad y = \Phi(x, z)$$

by which equation (A) is transformed into an equation of the form

$$(B) \quad xz' = z^n \mathbf{I}_n(g(z))z$$

with the properties that:

(i) *The n -column vector $\Phi(x, z)$ admits a uniformly convergent expansion of the form*

$$(E) \quad \Phi(x, z) = z + \mathbf{I}_n(z)P(z) + \sum_{l=1}^{\infty} x^l P_l(z)$$

in a domain of the form

$$(D) \quad |x| < a'', \quad 0 < \|z\| < b'', \quad \underline{\Theta} < \arg z^n < \bar{\Theta},$$

where the angles $\underline{\Theta}$ and $\bar{\Theta}$ are given by

$$(S) \quad \underline{\Theta} = -\arg m\alpha - \frac{3\pi}{2} + 5\varepsilon, \quad \bar{\Theta} = -\arg m\alpha + \frac{3\pi}{2} - 5\varepsilon$$

and a'' , b'' are some positive constants.

(ii) *$P(z)$ is an n -column vector whose components are holomorphic and bounded functions of z for $\|z\| < b''$.*

(iii) *$P_l(z)$ are n -column vectors whose components are holomorphic and bounded functions of z for*

$$0 < \|z\| < b'', \quad \underline{\Theta} < \arg z^n < \bar{\Theta}$$

and, moreover, admit asymptotic expansions in powers of z as z tends to zero through the sector $\underline{\Theta} < \arg z^m < \bar{\Theta}$.

(iv) $g(z)$ is an n -column vector of the form

$$(1.3) \quad g(z) = \alpha + \sum_{q \in \mathcal{S}} z^q \beta_q,$$

where $\mathcal{S}$ is a finite set of n -row vectors q such that

$$(1.4) \quad (q - m)\alpha = 0$$

and β_q are n -column constant vectors satisfying

$$(1.5) \quad (p - m)\beta_q = 0 \quad \text{for any } p \in \mathcal{S}.$$

4°. - By virtue of this theorem we see that equation (A) has a general solution of the form $\Phi(x, \tilde{Z}(x))$ whenever the values of x and $\tilde{Z}(x)$ satisfy the inequalities

$$|x| < \alpha'', \quad 0 < \|\tilde{Z}(x)\| < b'', \quad \underline{\Theta} < \arg \tilde{Z}(x)^m < \bar{\Theta},$$

where $\tilde{Z}(x)$ is a holomorphic solution of equation (B). We can obtain the explicit form of a general solution of (B) by a parametric representation.

The author would like to emphasize that his recent work [5] plays an essential role in the proof of Theorem A and HUKUHARA'S result for one-dimensional case is deduced from Theorem A as a corollary.

§ 2. - Notation and Definitions.

For an n -column vector y with elements $\{y_j\}$ and an n -row vector p with elements $\{p_j\}$, the symbol y^p stands for the scalar expression

$$(2.1) \quad y^p = y_1^{p_1} y_2^{p_2} \dots y_n^{p_n};$$

py is the inner product and $|p| = p_1 + p_2 + \dots + p_n$.

For an n -column vector function $P(y)$ and for an n -column vector y , $\frac{\partial P(y)}{\partial y}$ or $P_{,j}(y)$ is the Jacobian matrix of $P(y)$ with respect to y .

e_j is an n -dimensional unit-vector whose j^{th} component is equal to 1 and $\mathbf{1}$ is an n -column vector whose components are all equal to 1.

For an n -column vector y with elements $\{y_j\}$, the symbol $\mathbf{1}_n(y)$ denotes an n -by- n diagonal matrix with diagonal elements $\{y_j\}$, and $\mathbf{1}_n$ is the n -by- n unit-matrix.

For an n -column vector γ with elements $\{\gamma_j\}$ and for a scalar w , w^γ is an n -column vector with elements $\{w^{\gamma_j}\}$, so that $\mathbf{I}_n(w^\gamma)$ denotes an n -by- n diagonal matrix with diagonal elements $\{w^{\gamma_j}\}$.

The norm of an n -column vector y with elements $\{y_j\}$ is given by $\|y\| = \max_{j=1}^n |y_j|$.

For n -column vectors v and $\tilde{v}$ with elements $\{v_j\}$ and $\{\tilde{v}_j\}$ respectively, the symbol $[v]$ denotes an n -column vector with elements $\{|v_j|\}$ and the vectorial inequality $[v] < [\tilde{v}]$ means that $|v_j| < |\tilde{v}_j|$ for each index j . In particular, if $v_j \geq 0$ for every j , we write simply v instead of $[v]$.

An n -vector $\varphi(w, z)$ whose components are holomorphic and bounded functions of (w, z) in a domain

$$(2.2) \quad 0 < |w| < c, \quad \underline{\Theta} < \arg w < \bar{\Theta}, \quad \|z\| < b$$

will be called to have *Property- $\mathcal{Q}$* with respect to z in (2.2), if the following two conditions are satisfied:

(U₁) $\varphi(w, z)$ admits a uniformly convergent expansion of the form

$$\varphi(w, z) = \sum_p z^p \varphi_p(w)$$

whenever the values of w and z belong to (2.2).

(U₂) The coefficients $\varphi_p(w)$ are n -column vectors whose components are holomorphic and bounded functions of w for

$$0 < |w| < c, \quad \underline{\Theta} < \arg w < \bar{\Theta}$$

and, moreover, admit asymptotic expansions in powers of w as w tends to zero through the sector $\underline{\Theta} < \arg w < \bar{\Theta}$.

By definition it is almost evident that, if $b^{|m|} \leq c$, the function $\varphi(z^m, z)$ admits an asymptotic expansion in powers of z as z tends to zero through

$$(2.3) \quad 0 < \|z\| < b, \quad \underline{\Theta} < \arg z^m < \bar{\Theta}.$$

In the case of $n = 1$, the set of points z satisfying the inequalities (2.3) makes an angular domain whose vertex is the origin, but in the higher dimensional case the point z satisfying (2.3) is located on the product space of the RIEMANN surfaces of $\log z_1, \dots, \log z_n$.

§ 3. - Outline of the Proof of Theorem A.

1°. - By virtue of Assumption I, the n -vector $f(0, y)$ has the form $y^m \mathbf{1}_n(f(y))y$. In order to simplify $f(0, y)$ by applying a suitable transformation,

we shall first prove the following theorem:

THEOREM 1. - *Under Assumptions I and II, there exists a transformation of the form*

$$(T_0) \quad y = u + \mathbf{1}_n(u)P(u), \quad (P(0) = 0),$$

where $P(u)$ is an n -column vector whose components are holomorphic functions of u for $\|u\| \leq b_0$, such that equation (A) is transformed into an equation of the form

$$(B_0) \quad xu' = g(x, u) \equiv u^m G(u)u + \sum_{l=1}^{\infty} x^l g_l(u),$$

with the properties that:

(i) *The power series in the right-hand member of (B_0) is uniformly convergent for $|x| \leq a_0$, $\|u\| \leq b_0$, a_0 and b_0 being positive constants.*

(ii) *$G(u)$ is an n -by- n diagonal matrix, $\mathbf{1}_n(g(u))$, whose diagonal components are polynomials of u :*

$$(3.1) \quad g(u) = \alpha + \sum_{q \in \mathcal{S}} u^q \beta_q,$$

where $\mathcal{S}$ is a finite set of vectors q such that

$$(3.2) \quad (q - m)\alpha = 0$$

and β_q are n -column vectors satisfying

$$(3.3) \quad (p - m)\beta_q = 0 \quad \text{for any } p \in \mathcal{S}.$$

(iii) *$g_l(u)$ are n -column vector functions holomorphic in u for $\|u\| \leq b_0$.*

The proof of this theorem will be given in Chapter I.

2°. - We shall prove the following theorem concerning a formal transformation:

THEOREM 2. - *Equation (B_0) admits a formal transformation of the form*

$$(T_1) \quad u \sim z + \sum_{l=1}^{\infty} x^l Q_l(z)$$

such that equation (B₀) is formally transformed into the equation

$$(B) \quad xz' = z^n \mathbf{L}_n(g(z))z.$$

The n -column vectors $Q_l(z)$ have the form

$$(3.4.l) \quad Q_l(z) = \varphi_l(z^m, z),$$

where $\varphi_l(w, z)$ are n -column vector functions having Property- $\mathfrak{A}$ with respect to z in

$$(3.5) \quad 0 < |w| < c_1, \quad \underline{\Theta} < \arg w < \bar{\Theta}, \quad \|z\| < b_1.$$

The angles $\underline{\Theta}$ and $\bar{\Theta}$ are given by

$$(3.6) \quad \underline{\Theta} = -\arg m\alpha - \frac{3\pi}{2} + 5\varepsilon, \quad \bar{\Theta} = -\arg m\alpha + \frac{3\pi}{2} - 5\varepsilon$$

and c_1, b_1 are positive constants and satisfy $b_1^{|m|} \leq c_1$.

This theorem will be proved in Chapter II.

3°. - In Chapter III we shall prove the following theorem:

THEOREM 3. - The formal transformation (T₁) is uniformly convergent for

$$(3.7) \quad |x| < a'', \quad 0 < \|z\| < b'', \quad \underline{\Theta} < \arg z^m < \bar{\Theta}$$

such that the sum $\tilde{\Phi}(x, z)$ is a holomorphic function of (x, z) in the domain (3.7) and that the Jacobian matrix $\tilde{\Phi}_x'(x, z)$ is reduced to the n -by- n unit-matrix at $(0, 0)$.

4°. - By combining transformation (T₁) with transformation (T₀), we obtain a transformation from y to z , say $y = \Phi(x, z)$.

It is readily seen that the n -column vector $\Phi(x, z)$ admits a uniformly convergent expansion of the form (E) that appeared in Theorem A whenever the values of x and z belong to a domain of the form (D). This completes the proof of Theorem A.

5°. - In the last chapter we shall discuss about a parametric representation of a general solution of the simplified equation (B).

CHAPTER I.

Proof of Theorem 1.**§4. - Equation Satisfied by the n -Vectors $P(u)$ and $g(u)$.**

By virtue of Assumption I equation (A) is written as

$$(A) \quad xy' = f(x, y) \equiv y^m \mathbf{1}_n(f(y))y + \sum_{l=1}^{\infty} x^l f_l(y),$$

where $f_l(y)$ are n -column vectors holomorphic in y for $\|y\| \leq b$.

We consider a transformation of the form

$$(T_0) \quad y = u + \mathbf{1}_n(u)P(u),$$

where $P(u)$ is assumed to be a holomorphic n -vector function of u in the vicinity of $u=0$ and to vanish at $u=0$. We write the transformed equation as equation (B₀) appearing in Theorem 1.

Differentiating both sides of (T₀) with respect to x , we have

$$xy' = xu' + \mathbf{1}_n(xu')P(u) + \mathbf{1}_n(u) \frac{\partial P(u)}{\partial u} xu',$$

where xy' and xu' must be replaced by the right-hand members of equations (A) and (B₀) respectively. By picking up the terms only independent of x , the above equation yields

$$(4.1) \quad y^m \mathbf{1}_n(f(y))y + [\dots] = u^m G(u)u + \mathbf{1}_n(u^m G(u)u)P(u) + \\ + \mathbf{1}_n(u) \frac{\partial P(u)}{\partial u} u^m G(u)u + [\dots].$$

Since

$$u + \mathbf{1}_n(u)P(u) = \mathbf{1}_n(u)(\mathbf{1} + P(u)) = \mathbf{1}_n(\mathbf{1} + P(u))u,$$

where $\mathbf{1}$ is an n -column vector whose components are all equal to 1, a substitution of (T₀) for y into $y^m \mathbf{1}_n(f(y))y$ implies that

$$y^m \mathbf{1}_n(f(y))y = u^m (\mathbf{1} + P(u))^m \mathbf{1}_n(f(u + \mathbf{1}_n(u)P(u)))(u + \mathbf{1}_n(u)P(u)) \\ = u^m (\mathbf{1} + P(u))^m \mathbf{1}_n(f(u + \mathbf{1}_n(u)P(u))) \mathbf{1}_n(\mathbf{1} + P(u))u.$$

Observe furthermore that

$$\begin{aligned}\mathbf{1}_n(u^m G(u)u)P(u) &= u^m \mathbf{1}_n(P(u))G(u)u, \\ \mathbf{1}_n(u) \frac{\partial P(u)}{\partial u} G(u)u &= \mathbf{1}_n\left(\frac{\partial P(u)}{\partial u} G(u)u\right)u.\end{aligned}$$

It follows then from (4.1) that $P(u)$ and $G(u)$ satisfy the matrix equation

$$\begin{aligned}(4.2) \quad (\mathbf{1} + P(u))^m \mathbf{1}_n(f(u + \mathbf{1}_n(u)P(u))) \mathbf{1}_n(\mathbf{1} + P(u)) &= \\ &= (\mathbf{1}_n + \mathbf{1}_n(P(u)))G(u) + \mathbf{1}_n\left(\frac{\partial P(u)}{\partial u} G(u)u\right).\end{aligned}$$

This equation shows that *the matrix $G(u)$ must be of diagonal form.*

Hence we write $G(u)$ as $\mathbf{1}_n(g(u))$. By comparing the diagonal components of (4.2), we see that $P(u)$ and $g(u)$ must satisfy the equation

$$\begin{aligned}(4.3) \quad (\mathbf{1} + P(u))^m \mathbf{1}_n(\mathbf{1} + P(u))f(u + \mathbf{1}_n(u)P(u)) &= \\ &= (\mathbf{1}_n + \mathbf{1}_n(P(u)))g(u) + \frac{\partial P(u)}{\partial u} \mathbf{1}_n(u)g(u).\end{aligned}$$

§ 5. - Determination of the n -Vector $g(u)$ (Part I).

The conclusion of Theorem 1 means that, if we choose $P(u)$ a holomorphic n -vector function of u in the vicinity of $u = 0$ in a suitable way, then $g(u)$ is reduced to a polynomial of u .

To prove this fact, let

$$(5.1) \quad f(u) = \alpha + \sum_{|q|=1}^{\infty} u^q \alpha_q$$

be TAYLOR'S expansion of $f(u)$, where α_q are constant n -column vectors. We assume at this moment that $P(u)$ and $g(u)$ are expressed by power series of u :

$$(5.2) \quad P(u) \sim \sum_{|q|=1}^{\infty} u^q D_q, \quad g(u) \sim \beta + \sum_{|q|=1}^{\infty} u^q \beta_q,$$

where D_q and β_q are constant n -column vectors. It is easily seen that $\beta = \alpha$.

We shall prove that, *if we choose the coefficients $\{D_q\}$ in a suitable way, $g(u)$ is reduced to a polynomial of u .*

To do this, we must look for algebraic equations which determine the vectors D_q and β_q . Observe, by a simple calculation, that

$$(5.3) \quad \frac{\partial P(u)}{\partial u} \sim \Sigma u^q D_q q \mathbf{1}_n(u)^{-1}, \quad \frac{\partial P(u)}{\partial u} \mathbf{1}_n(u) \sim \Sigma u^q D_q q,$$

where $\mathbf{1}_n(u)^{-1}$ is the inverse matrix of $\mathbf{1}_n(u)$. If we substitute (5.2) for $\{P(u), g(u)\}$ into both sides of equation (4.3), the resulting equation becomes a formal identity between power series of u . Equating the coefficients of the terms u^q , for each q , appearing in both sides of this formal identity, we have an equation of the form

$$(5.4.q) \quad m D_q \alpha + F_q(\alpha_p, D_p) = \beta_q + D_q(q\alpha) + G_q(\beta_p, D_p).$$

Here $F_q(\alpha_p, D_p)$ is a linear form of n -column vectors α_p for $|p| \leq |q|$ with polynomial coefficients of D_p for $|p| < |q|$ and $G_q(\beta_p, D_p)$ is a bilinear form of n -column vectors β_p and D_p for $|p| < |q|$.

Assume that the n -vectors $D_{q'}$ and $\beta_{q'}$ have been already determined for $|q'| < |q|$ so that they satisfy equations (5.4.q'). Then

$$H_q = F_q(\alpha_p, D_p) - G_q(\beta_p, D_p)$$

is considered as a *known* n -column vector.

Now we consider the linear equation

$$(5.5.q) \quad (q\alpha)D_q - (mD_q)\alpha = H_q - \beta_q.$$

As can be easily seen, this equation can be written as

$$L D_q = H_q - \beta_q,$$

where

$$(5.6) \quad L = \begin{bmatrix} q\alpha - m_1\alpha_1 & -m_2\alpha_1 & \dots & \dots & -m_n\alpha_1 \\ -m_1\alpha_2 & q\alpha - m_2\alpha_2 & -m_3\alpha_2 & \dots & -m_n\alpha_2 \\ \dots & \dots & \dots & \dots & \dots \\ -m_1\alpha_n & \dots & \dots & -m_{n-1}\alpha_n & q\alpha - m_n\alpha_n \end{bmatrix}.$$

We assert that:

LEMMA 1. - *We have*

$$(5.7) \quad \det L = (q\alpha)^{n-1}(q - m)\alpha.$$

This lemma will be proved in Section 8.

Since Assumption II is supposed to be satisfied, we have

$$q\alpha \neq 0 \quad \text{if} \quad |q| \geq 1.$$

Hence the determinant of L is equal to zero if and only if q satisfies the equation

$$(5.8) \quad (q - m)\alpha = 0.$$

We denote the set of such vectors q by $\mathfrak{S}$. It is clear by virtue of Assumption II that $\mathfrak{S}$ is a finite set.

Thus we have the following proposition:

PROPOSITION 1.1. - *For each $q \in \mathfrak{S}$, we can determine the value of the vector D_q in such a way that the value of the vector β_q is reduced to zero. Since $\mathfrak{S}$ is a finite set, $g(u)$ is reduced to a polynomial of u .*

§ 6. - Determination of the n -Vector $g(u)$ (Part II).

We have just proved that $g(u)$ is reduced to a polynomial of u . However, in order to have a parametric representation of a general solution of equation (B) appearing in Theorem A, we have to determine the coefficients β_q for $q \in \mathfrak{S}$ in a particular way.

We assert that:

LEMMA 2. - *If $q \in \mathfrak{S}$ and $m_j \neq 0$ for some index j , say $j = \kappa$, the determinant of the $(n-1)$ -by- $(n-1)$ matrix $L(\kappa)$, which is obtained from L by removing the components of the κ^{th} row and the κ^{th} column, is equal to $m_\kappa \alpha_\kappa (m\alpha)^{n-2} \neq 0$.*

We shall prove this lemma in Section 8.

Since $|m| \geq 1$, such an index κ certainly exists. We can assume without loss of generality that $\kappa = 1$. Lemma 2 shows that the rank of the matrix L for $q \in \mathfrak{S}$ is equal to $n-1$. Moreover, as can be easily verified, the matrix L for $q \in \mathfrak{S}$ satisfies

$$(6.1) \quad mL = 0.$$

We can assume without loss of generality that the n -vector α is expressed by a direct sum of ν column vectors $\alpha^{(i)}$ with dimension n_i as follows:

$$(6.2) \quad \alpha = \alpha^{(1)} \oplus \alpha^{(2)} \oplus \dots \oplus \alpha^{(\nu)}$$

with the property that two components of α are commensurable if and only if both of them belong to the same $\alpha^{(i)}$ for some index i . Then, for each

index k , there exist a non-zero complex constant γ_k and an n_k -column vector $\mu^{(k)}$ such that

$$\alpha^{(k)} = \gamma_k \mu^{(k)}$$

and the components of $\mu^{(k)}$ are all positive integers. The quantity γ_k is uniquely determined under the condition that the greatest common divisor of the components of $\mu^{(k)}$ is equal to 1.

Let

$$p = p^{(1)} \oplus p^{(2)} \oplus \dots \oplus p^{(v)}, \quad n = m^{(1)} \oplus m^{(2)} \oplus \dots \oplus m^{(v)}$$

be the partitionings of the n -vectors p and m corresponding to (6.2). Then the equation $(p - m)\alpha = 0$ becomes equivalent to v equations:

$$(6.3.k) \quad (p^{(k)} - m^{(k)})\mu^{(k)} = 0 \quad (k = 1, 2, \dots, v).$$

Noticing that $m_1 \geq 1$ and $m^{(1)}\mu^{(1)} \neq 0$, we set

$$(6.4.q) \quad \beta_q = \frac{mH_q}{m^{(1)}\mu^{(1)}} \mu^{(1)} \oplus 0 \oplus \dots \oplus 0.$$

Then it is easy to see that $m(H_q - \beta_q) = 0$. Hence it is concluded from (6.1) that

$$(6.5) \quad \text{rank } L = \text{rank } (L, H_q - \beta_q) = n - 1,$$

where $(L, H_q - \beta_q)$ is an n -by- $(n+1)$ matrix. As is well known, equation (5.5.q) has a solution.

Thus we have the following proposition:

PROPOSITION 1.2. - *We can determine the value of the vector D_q for each $q \in \mathfrak{S}$ as a solution of equation (5.5.q) in such a way that the vector β_q has the form (6.4.q). Furthermore, the vector β_q satisfies the equation $(p - m)\beta_q = 0$ for any $p \in \mathfrak{S}$. The n -vector $g(u)$ has the form*

$$(6.6) \quad g(u) = \alpha + \sum_{q \in \mathfrak{S}} u^q \beta_q.$$

§ 7. - Determination of the n -Vector $P(u)$.

We consider equation (4.3) where the n -vector $g(u)$ is a polynomial of u given by (6.6). Then equation (4.3) is a partial differential equation which determines the n -vector $P(u)$. This equation is written as

$$(7.1) \quad \frac{\partial P}{\partial u} \mathbf{1}_n(g(u))u = h(u; P),$$

where $h(u; P)$ is an n -column vector whose components are holomorphic functions of (u, P) for

$$\|u\| \leq b_1, \quad \|P\| \leq c_1, \quad (b_1(1 + c_1) \leq b)$$

and vanish at $(0, 0)$. Moreover, equation (7.1) has a formal solution of the form

$$(7.2) \quad P \sim \sum_{|q|=1}^{\infty} u^q D_q.$$

We shall prove the following proposition:

PROPOSITION 1.3. - *The formal solution (7.2) is uniformly convergent for $\|u\| \leq b_0$ and the sum $P(u)$ is a solution of partial differential equation (7.1) for $\|u\| \leq b_0$, where b_0 is a positive constant.*

1°. - Let $U(x)$ be a holomorphic general solution of the equation

$$(7.3) \quad x \frac{du}{dx} = \mathbf{1}_n \left(\frac{g(u)}{m\alpha} \right) u.$$

We assert that:

(i) *If we replace u by $U(x)$ in (7.2), the power series*

$$(7.4) \quad P \sim \sum_{|q|=1}^{\infty} U(x)^q D_q$$

is a formal solution of the ordinary differential equation

$$(7.5) \quad x \frac{dP}{dx} = \frac{1}{m\alpha} h(U(x); P).$$

(ii) *If the power series (7.4) is uniformly convergent for $\|U(x)\| \leq b_0$, then the sum $P(U(x))$ is a holomorphic solution of (7.5) for $\|U(x)\| \leq b_0$ and $P(u)$ is a holomorphic solution of (7.1) for $\|u\| \leq b_0$.*

Indeed, denote by $Q(u)$ an n -column vector which is expressed by the power series (7.2). Since $Q(u)$ is a formal solution of partial differential equation (7.1), we have a formal identity of the form

$$\frac{\partial Q(u)}{\partial u} \mathbf{1}_n(g(u))u = h(u; Q(u)).$$

On the other direction, we have a formal identity of the form

$$\begin{aligned} x \frac{dQ(U(x))}{dx} &= \frac{\partial Q(U(x))}{\partial U(x)} x \frac{dU(x)}{dx} = \frac{\partial Q(U(x))}{\partial U(x)} \mathbf{1}_n \left(\frac{g(U(x))}{m\alpha} \right) U(x) = \\ &= \frac{1}{m\alpha} h(U(x); Q(U(x))). \end{aligned}$$

These relations show that the power series (7.4) is a formal solution of equation (7.5). This proves the first portion of the assertion. The second portion of the assertion is almost evident.

2°. - In the equation (7.3) the following facts are to be noticed:

(i) $x=0$ is a BRIOT-BOUQUET type singular point.
(ii) The Jacobian matrix of the right-hand member of (7.3) with respect to u is reduced to an n -by- n diagonal matrix, $\mathbf{1}_n \left(\frac{\alpha}{m\alpha} \right)$, at $u=0$ and the real parts of the diagonal components are all positive.

(iii) The j^{th} component of the n -vector appearing in the right-hand member of (7.3) contains the j^{th} component of the *unknown vector* as a factor.

From these facts it is concluded that:

A general solution, $U(x)$, of equation (7.3) is written as

$$(7.6) \quad U(x) = \Phi(V(x)), \quad V(x) = \mathbf{1}_n(x^{\frac{\alpha}{m\alpha}})C''$$

whenever $\|V(x)\| \leq \delta_1$. Here $\Phi(v)$ is an n -column vector such that $\mathbf{1}_n(v)^{-1}\Phi(v)$ is a holomorphic function of v for $\|v\| \leq \delta_1$ and the Jacobian matrix $\Phi_v(0)$ is the n -by- n unit-matrix; C'' is an arbitrary constant n -column vector and δ_1 is a positive constant.

We can assume without loss of generality that

$$\|\Phi(v)\| \leq b_1 \quad \text{for} \quad \|v\| \leq \delta_1.$$

3°. - It is easily seen that the power series

$$(7.4)^* \quad P \sim \sum_{|q|=1}^{\infty} V(x)^q D_q^*,$$

which is obtained by rearranging the power series $\sum_{|q|=1}^{\infty} \Phi(V(x))^q D_q$ in powers of $V(x)$, is a formal solution of the equation

$$(7.5)^* \quad x \frac{dP}{dx} = \frac{1}{m\alpha} h^*(V(x); P),$$

where

$$h^*(v; P) \equiv h(\Phi(v); P).$$

$h^*(v; P)$ is an n -column vector function holomorphic in (v, P) for

$$\|v\| \leq \delta_1, \quad \|P\| \leq c_1.$$

Instead of proving uniform convergence of the formal solution (7.4), we want to prove uniform convergence of the formal solution (7.4)*. However, an equation of the form (7.5)* was studied by M. IWANO [5]. By applying Theorem B in Section 11 in [5] to equation (7.5)*, it is concluded that:

The formal solution (7.4) is uniformly convergent for $\|V(x)\| \leq \delta'_1$ ($\leq \delta_1$) and the sum $P^*(V(x))$ is a solution of equation (7.5)*.*

Let $\Phi^{-1}(v)$ be the inverse function of $\Phi(v)$. Then there exists a positive constant b_0 such that $\Phi^{-1}(v)$ is a holomorphic function of v for $\|v\| \leq b_0$ and satisfies there $\|\Phi^{-1}(v)\| \leq \delta'_1$.

We define the n -column vector $P(u)$ by

$$(7.7) \quad P(u) = P^*(\Phi^{-1}(u)).$$

Then it is clear that $P(U(x))$ is a holomorphic solution of equation (7.5) whenever $\|U(x)\| \leq b_0$ and admits there the uniformly convergent expansion (7.4). As we have already proved, $P(u)$ is a holomorphic solution of the partial differential equation (7.1) provided that u belongs to the domain $\|u\| \leq b_0$.

Thus the n -vector $P(u)$ has been determined as a holomorphic solution of partial equation (7.1) for $\|u\| \leq b_0$.

§ 8. - Complete Proof of Theorem 1.

In order to complete the proof of Theorem 1, we have only to prove Lemma 1 in Section 5 and Lemma 2 in Section 6.

1°. - We shall prove first Lemma 1. We consider the following determinant:

$$\mathfrak{D}(m, \alpha) = \begin{vmatrix} b - m_1\alpha_1 & -m_2\alpha_1 & \dots & \dots & -m_n\alpha_1 \\ -m_1\alpha_2 & b - m_2\alpha_2 & -m_3\alpha_2 & \dots & -m_n\alpha_2 \\ \dots & \dots & \dots & \dots & \dots \\ -m_1\alpha_n & -m_2\alpha_n & \dots & -m_{n-1}\alpha_n & b - m_n\alpha_n \end{vmatrix}.$$

Clearly, $|L| = \mathfrak{D}(m, \alpha)$ with $b = q\alpha$. We shall prove that

$$(8.1) \quad \mathfrak{D}(m, \alpha) = b^{n-1}(b - m\alpha).$$

where M_k are row vectors and A^k are column vectors such that

$$M_k = (m_{k+1}, \dots, m_n) \quad (A^k)^T = (\alpha_{k+1}^k, \dots, \alpha_n^k) \quad \text{with } \alpha_j^k = \alpha_j^{k-1} - \alpha_k^{k-1}.$$

Since, by the definition of α_j^k , we have

$$\alpha_k^{k-1} = \alpha_k^{k-2} - \alpha_{k-1}^{k-2} = \alpha_k^{k-3} - \alpha_{k-1}^{k-3} = \dots = \alpha_k - \alpha_{k-1},$$

it follows then that

$$\begin{aligned} \mathfrak{D}(M, \alpha) &= b^n - (|m| \alpha_1 + |M_1| \alpha_2^1 + \dots + |M_{n-1}| \alpha_n^{n-1}) b^{n-1} = \\ &= b^{n-1}(b - m\alpha). \end{aligned}$$

This proves Lemma 1.

2°. - We shall prove Lemma 2. Since α is supposed to be equal to 1, the matrix $L(1)$ is written as, using $q\alpha = m\alpha$,

$$L(1) = \begin{pmatrix} m\alpha - m_2\alpha_2 & -m_3\alpha_2 & \dots & \dots & -m_n\alpha_2 \\ -m_2\alpha_3 & m\alpha - m_3\alpha_3 & -m_4\alpha_3 & \dots & -m_n\alpha_3 \\ \dots & \dots & \dots & \dots & \dots \\ -m_2\alpha_n & \dots & \dots & -m_{n-1}\alpha_n & m\alpha - m_n\alpha_n \end{pmatrix}.$$

This matrix has a form similar to the original matrix L except that the size of $L(1)$ is $(n-1)$ -by- $(n-1)$. Hence we see that the determinant of $L(1)$ is equal to $\mathfrak{D}(M_1, A^1)$ with $b = m\alpha$.

By virtue of (8.1) we have

$$\begin{aligned} |L(1)| &= (m\alpha)^{n-2}(m\alpha - M_1 A^1) \\ &= (m\alpha)^{n-2} m_1 \alpha_1 \neq 0. \end{aligned}$$

Thus Lemma 2 has been proved.

CHAPTER II.

Proof of Theorem 2.

§ 9. - Equation Satisfied by the n -Vectors $Q(z)$.

We consider a formal transformation of the form

$$(T_1) \quad u \sim z + \sum_{l=1}^{\infty} x^l Q_l(z),$$

where $Q_l(z)$ are n -column vectors. We assume that the equation formally derived from equation (B₀) by formal transformation (T₁) has the form

$$(B) \quad xz' = z^m \mathbf{I}_n(g(z))z.$$

We shall look for partial differential equations which determine the n -vectors $Q_l(z)$.

Differentiating (T₁) term by term with respect to x and replacing xz' by the right-hand member of (B), we have the equation

$$(9.1) \quad xu' \sim z^m \mathbf{I}_n(g(z))z + \sum_{l=1}^{\infty} x^l \left\{ lQ_l(z) + \frac{\partial Q_l(z)}{\partial z} z^m \mathbf{I}_n(g(z))z \right\}.$$

On the other hand, if we substitute (T₁) for u into equation (B₀), we have an equation of the form, after an elementary calculation,

$$(9.2) \quad xu' \sim z^m \mathbf{I}_n(g(z))z + \sum_{l=1}^{\infty} x^l \{ G(z)Q_l(z) + R_l(z) \},$$

where

$$(9.3) \quad G(z) = \frac{\partial}{\partial z} (z^m \mathbf{I}_n(g(z))z)$$

and $R_l(z)$ is a linear form of *known* n -column vector functions holomorphic in z for $\|z\| \leq b_0$ with polynomial coefficients of $Q_h(z)$ for $h < l$ and, in particular,

$$R_1(z) = g_1(z).$$

From (9.1) and (9.2) it follows immediately that the equations satisfied by $Q_l(z)$ are given by the linear partial differential equations

$$(9.4.l) \quad z^m \frac{\partial Q}{\partial z} \mathbf{I}_n(g(z))z = -lQ + G(z)Q + R_l(z).$$

We shall prove that the matrix $G(z)$ vanishes at $z = 0$, namely

$$(9.5) \quad G(0) = 0.$$

By an elementary calculation we have

$$\begin{aligned} G(z) &= (m_1 z^{m-e_1} \mathbf{I}_n(g(z))z, \dots, m_n z^{m-e_n} \mathbf{I}_n(g(z))z) + \\ &+ z^m \left(\frac{\partial}{\partial z_1} \mathbf{I}_n(g(z)) \cdot z, \dots, \frac{\partial}{\partial z_n} \mathbf{I}_n(g(z)) \cdot z \right) + \end{aligned}$$

$$\begin{aligned}
 & + z^m \left(\mathbf{1}_n(g(z)) \frac{\partial z}{\partial z_1}, \dots, \mathbf{1}_n(g(z)) \frac{\partial z}{\partial z_n} \right) = \\
 & = z^m \mathbf{1}_n(g(z)) z(m_1 z^{-e_1}, \dots, m_n z^{-e_n}) + \\
 & + z^m \left(\mathbf{1}_n(z) \frac{\partial g(z)}{\partial z_1}, \dots, \mathbf{1}_n(z) \frac{\partial g(z)}{\partial z_n} \right) + z^m \mathbf{1}_n(g(z)),
 \end{aligned}$$

where e_j is an n -row unit-vector whose j^{th} component is equal to 1.

Hence we have the identity

$$(9.6) \quad G(z) = z^m \left(\mathbf{1}_n(g(z)) z m \mathbf{1}_n(z)^{-1} + \mathbf{1}_n(z) \frac{\partial g(z)}{\partial z} + \mathbf{1}_n(g(z)) \right).$$

However, since $|m| \geq 1$, this identity shows that $G(0) = 0$.

§ 10. - Fundamental Lemma.

Before solving the partial differential equation (9.4.1), we must establish a lemma which will play a fundamental role in the determination of the n -vectors $Q_i(z)$.

Let

$$(10.1) \quad x = X(w), \quad z = Z(w)$$

be a holomorphic solution of the differential equations

$$(10.2) \quad w^2 \frac{dx}{dw} = \frac{x}{mg(z)}, \quad w \frac{dz}{dw} = \mathbf{1}_n \left(\frac{g(z)}{mg(z)} \right) z$$

such that

$$(10.3) \quad X(w_1) = x_1, \quad Z(w_1) = z^1,$$

where (x_1, w_1, z^1) is an arbitrary point in a domain of the form

$$(10.4) \quad |x| \leq \alpha_0, \quad 0 < |w| \leq c_0, \quad \underline{\Theta} \leq \arg w \leq \bar{\Theta}, \quad \|z\| \leq b_0.$$

We see by inspection that equations (10.1) define a holomorphic solution $z = \tilde{Z}(x)$ of equation (B) if and only if $Z(w)^m = w$. However, as can be easily verified, $Z(w)^m - w$ satisfies the linear differential equation

$$(10.5) \quad w \frac{dY}{dw} = Y.$$

Hence $Z(w)^m \equiv w$ is equivalent to $(z^1)^m = w_1$.

Thus it is concluded that: *Equations (10.1) define a solution, $\tilde{Z}(x)$, of equation (B) if and only if $(z^1)^m = w_1$ and we have $\tilde{Z}(x) = Z(X^{-1}(x))$, where $X^{-1}(x)$ is the inverse function of $X(x)$.*

We now consider a system of differential equations of the form

$$(10.6) \quad w^2 \frac{dQ}{dw} = \frac{1}{mg(Z(w))} h(X(w), w, Z(w); Q).$$

Here $h(x, w, z; Q)$ is an n -column vector such that the components are holomorphic and bounded functions of (x, w, z, Q) in

$$(10.7) \quad |x| \leq \alpha_0, \quad 0 < |w| \leq c_0, \quad \underline{\Theta} \leq \arg w \leq \bar{\Theta}, \quad \|z\| \leq b_0, \quad \|Q\| \leq d_0,$$

α_0, c_0, b_0 and d_0 are positive constants and the angles $\underline{\Theta}$ and $\bar{\Theta}$ are the same as those that appeared in Theorem A.

Then the lemma which we are going to establish can be stated as follows:

LEMMA 3. - *Assume that equation (10.6) has a solution $\zeta(X(w), w, Z(w))$ whenever the values of $X(w), w, Z(w)$ belong to a domain of the form*

$$(10.8) \quad |x| < \alpha'_0, \quad 0 < |w| < c_0, \quad \underline{\Theta} < \arg w < \bar{\Theta}, \quad \|z\| < b'_0.$$

Here $\zeta(x, w, z)$ is an n -column vector function holomorphic in (x, w, z) for (10.8). Assume furthermore that $(b'_0)^{|m|} \leq c'_0$.

Then we have the following conclusions:

(i) $\zeta(x, \tilde{Z}(x)^m, \tilde{Z}(x))$ is a solution of the equation

$$(10.9) \quad x \frac{dQ}{dx} = h(x, \tilde{Z}(x)^m, \tilde{Z}(x); Q)$$

if the values of x and $\tilde{Z}(x)$ belong to

$$(10.10) \quad |x| < \alpha'_0, \quad 0 < \|z\| < b'_0, \quad \underline{\Theta} < \arg z^m < \bar{\Theta}.$$

(ii) If we put $\eta(x, z) = \zeta(x, z^m, z)$, then the n -vector $\eta(x, z)$ is a holomorphic solution of the partial differential equation

$$(10.11) \quad x \frac{\partial Q}{\partial x} + z^m \frac{\partial Q}{\partial z} \mathbf{1}_n(g(z))z = h(x, z^m, z; Q)$$

provided that the values of x and z are in (10.10).

(iii) In the case when $h(x, v, z; Q)$ does not depend on x , if equation (10.6) has a solution of the form $\zeta(v, Z(v))$ for $0 < |v| < c'_0$, $\Theta < \arg v < \bar{\Theta}$, $\|Z(v)\| < b'_0$, then $\zeta(z^m, z)$, considered as a function of z , is a solution of

$$(10.12) \quad z^m \frac{\partial Q}{\partial z} \mathbf{1}_n(g(z))z = h(x, z^m, z; Q)$$

for $0 < \|z\| < b'_0$, $\Theta < \arg z^m < \bar{\Theta}$.

To prove this lemma, it is to be noticed that: Since $\{X(v), Z(v)\}$ is considered as a holomorphic general solution of equations (10.2), the assumption of the lemma implies that $\zeta(x, v, z)$ is a solution of the partial differential equation

$$(10.13) \quad x \frac{\partial Q}{\partial x} + v^2 m g(z) \frac{\partial Q}{\partial v} + v \frac{\partial Q}{\partial z} \mathbf{1}_n(g(z))z = h(x, v, z; Q)$$

whenever the values of (x, v, z) belong to (10.8). Namely, relation (10.13) with $Q = \zeta(x, v, z)$ is an identity.

Since $\tilde{Z}(x)$ is a solution of (B), we have

$$x \frac{d\tilde{Z}(x)^m}{dx} = \tilde{Z}(x)^{2m} m g(\tilde{Z}(x)).$$

Then a simple calculation shows that, ζ denoting $\zeta(x, \tilde{Z}(x)^m, \tilde{Z}(x))$,

$$(10.14) \quad \begin{aligned} x \frac{d\zeta}{dx} &= x \frac{\partial \zeta}{\partial x} + \frac{\partial \zeta}{\partial (\tilde{Z}(x)^m)} x \frac{d\tilde{Z}(x)^m}{dx} + \frac{\partial \zeta}{\partial \tilde{Z}(x)} x \frac{d\tilde{Z}(x)}{dx} = \\ &= x \frac{\partial \zeta}{\partial x} + (\tilde{Z}(x)^{2m} m g(\tilde{Z}(x))) \frac{\partial \zeta}{\partial (\tilde{Z}(x)^m)} + \tilde{Z}(x)^m \frac{\partial \zeta}{\partial \tilde{Z}(x)} \mathbf{1}_n(g(\tilde{Z}(x))) \tilde{Z}(x). \end{aligned}$$

However we observe that equation (10.13) with $Q = \zeta(x, v, z)$ is an identity provided (x, v, z) belongs to (10.8). Hence we see from (10.14) that $\zeta(x, \tilde{Z}(x)^m, \tilde{Z}(x))$ is a solution of ordinary differential equation (10.9) whenever the values of x and $\tilde{Z}(x)$ satisfy

$$|x| < a'_0, \quad \|\tilde{Z}(x)\| < b'_0, \quad \Theta < \arg \tilde{Z}(x)^m < \bar{\Theta}, \quad 0 < |\tilde{Z}(x)^m| < c'_0$$

and, consequently, belong to (10.10). This proves (i).

By definition we have

$$\begin{aligned} x \frac{\partial \eta}{\partial x} + \tilde{Z}(x)^m \frac{\partial \eta}{\partial \tilde{Z}(x)} \mathbf{1}_n(g(\tilde{Z}(x))) \tilde{Z}(x) &= x \frac{d\eta(x, \tilde{Z}(x))}{dx} = \\ &= x \frac{d}{dx} \zeta(x, \tilde{Z}(x)^m, \tilde{Z}(x)). \end{aligned}$$

However, by virtue of (i), the last expression is equal to $h(x, Z(x)^m, \tilde{Z}(x); \eta(x, \tilde{Z}(x)))$, if the values of x and $\tilde{Z}(x)$ are in (10.10). This proves (ii) because $\tilde{Z}(x)$ is considered as a general solution of equation (B).

The proof of (iii) can be carried out in a same way and we leave to the reader the proof.

§ 11. - Determination of the n -Vector $Q_1(z)$.

The equation which determines $Q_1(z)$ is given by

$$(11.1) \quad z^n \frac{\partial Q}{\partial z} \mathbf{1}_n(g(z))z = -Q + G(z)Q + g_1(z).$$

Owing to Assertion (iii) of Lemma 3 in Section 10, instead of solving partial differential equation (11.1) directly, we study the ordinary differential equation

$$(11.2) \quad w^2 \frac{dQ}{dw} = \frac{1}{mg(Z(w))} (-Q + G(Z(w))Q + g_1(Z(w))).$$

Since $Z(w)$ is a general solution of the second equation of (10.2), equation (11.2) is equivalent to the system

$$(11.3) \quad w^2 \frac{dQ}{dw} = \frac{1}{mg(z)} (-Q + G(z)Q + g_1(z)), \quad w \frac{dz}{dw} = \mathbf{1}_n \left(\frac{g(z)}{mg(z)} \right) z$$

under the condition that z is a general solution of the second equation.

To study these equations the following facts must be noticed:

The first equation has the properties that:

- (i) $w = 0$ is an irregular type singular point.
- (ii) The Jacobian matrix of the right-hand member with respect to Q is reduced to an n -by- n nonsingular diagonal matrix $-(m\alpha)^{-1} \mathbf{1}_n$ at $(0, 0, 0)$.

The second equation has the properties that:

- (i) $w = 0$ is a regular type singular point.
- (ii) The right-hand member does depend on neither Q nor w .
- (iii) The Jacobian matrix of the right-hand member with respect to z is reduced to the diagonal matrix $\mathbf{1}_n \left(\frac{\alpha}{m\alpha} \right)$ at $z = 0$, and the diagonal components have all positive real parts.
- (iv) For each index j , the j^{th} component of the right-hand member contains the j^{th} component of the unknown vector z as factor.

From these facts we see that equations (11.3) have a form similar to equations which were studied by M. IWANO [5]. Let Θ and $\bar{\Theta}$ be the angles that appeared in Theorem A in Section 3. Then Theorem 1 in Section 3 in [5] states that:

Equations (11.3) have a solution $\{ \Phi^1(w, V(w)), \Psi(V(w)) \}$ whenever the values of w and $V(w)$ stay in a domain of the form

$$(11.4) \quad 0 < |w| < c_1, \quad \underline{\Theta} < \arg w < \bar{\Theta}, \quad \|v\| < b_1$$

for suitably chosen positive constants b_1 and c_1 .

Here $V(w) \equiv \mathbf{I}_n(w^{\frac{\alpha}{m\alpha}})C''$ with an arbitrary constant n -column vector C'' ; $\Phi^1(w, v)$ is an n -column vector function with Property- $\mathfrak{A}$ with respect to v in (11.4); $\Psi(v)$ is an n -column vector such that $\mathbf{I}_n(v)^{-1}\Psi(v)$ is a holomorphic function of v for $\|v\| < b_1$ and the Jacobian matrix $\Psi_v(v)$ is equal to the n -by- n unit-matrix at $v = 0$; Moreover, $\Psi(V(w))$ is a general solution of the second equation of (11.3).

Let $\Psi^{-1}(z)$ be the inverse function of $\Psi(z)$. Then we can assume without loss of generality that $\Psi^{-1}(z)$ is a holomorphic function of z for $\|z\| < b_1$ and satisfies there $\|\Psi^{-1}(z)\| < b_1$. We can suppose moreover that the quantity $b_1^{|m|}$ does not exceed c_1 .

Then, by using Lemma 3 in Section 10, we immediately have the following conclusion;

PROPOSITION 2.1. - *If we define the n -vector $\varphi_1(w, z)$ by*

$$(11.5) \quad \varphi_1(w, z) = \Phi^1(w, \Psi^{-1}(z)),$$

$\varphi_1(w, Z(w))$ is a solution of equation (11.2) whenever the values of w and $Z(w)$ belong to

$$(11.6) \quad 0 < |w| < c_1, \quad \underline{\Theta} < \arg w < \bar{\Theta}, \quad \|z\| < b_1$$

and $\varphi_1(w, z)$ is an n -column vector function with Property- $\mathfrak{A}$ with respect to z in (11.6).

If we define the n -vector $Q_1(z)$ by

$$(11.7) \quad Q_1(z) = \varphi_1(z^m, z),$$

$Q_1(z)$ is a solution of the partial differential equation (11.1) whenever z satisfies an inequality of the form

$$(11.8) \quad 0 < \|z\| < b_1, \quad \underline{\Theta} < \arg z^m < \bar{\Theta}$$

and admits an asymptotic expansion in powers of z as z tends to zero through (11.8).

Thus the n -vector $Q_1(z)$ has been determined.

§ 12. - Determination of the n -Vectors $Q_l(z)$ ($l \geq 2$).

Since the reasonings to determine the n -vectors $Q_l(z)$ for $l \geq 3$ are almost similar to those for the n -vector $Q_2(z)$, we shall show only how to determine the n -vector $Q_2(z)$.

We consider the linear partial differential equation (9.4.2) which determines $Q_2(z)$. In this case, the non-homogeneous term, $R_2(z)$, depends actually on the n -vector $Q_1(z) \equiv \varphi_1(z^n, z)$. We write therefore it as

$$(12.1) \quad R_2(z) = \mathcal{R}_2(z, Q_1(z)).$$

By assumption $\mathcal{R}_2(z, Q)$ is a holomorphic vector function of (z, Q) for

$$\|z\| \leq b_0, \quad \|Q\| < \infty.$$

If we define the n -vector $\mathcal{Q}_2(w, z)$ by

$$(12.2) \quad \mathcal{Q}_2(w, z) = \mathcal{R}_2(z, \varphi_1(w, z)),$$

$\mathcal{R}_2(w, z)$ obviously has Property- $\mathcal{M}$ with respect to z in (11.6).

Corresponding to the partial differential equation (9.4.2), we consider the ordinary differential equation

$$(12.3) \quad w^2 \frac{dQ}{dw} = \frac{1}{mg(Z(w))} (-2Q + G(Z(w))Q + \mathcal{Q}_2(w, Z(w)))$$

or, what is the same thing, the system

$$(12.4) \quad w^2 \frac{dQ}{dw} = \frac{1}{mg(z)} (-2Q + G(z)Q + \mathcal{Q}_2(w, z)), \quad w \frac{dz}{dw} = \mathbf{I}_n \left(\frac{g(z)}{mg(z)} \right) z$$

under the condition that z is a general solution of the second equation.

It is not so difficult to verify that the first equation of (12.4) has a formal solution of the form

$$Q \sim \Sigma V(w)^q \Phi_q(w), \quad V(w) \equiv \mathbf{I}_n (w^{\frac{\alpha}{m\alpha}}) C''.$$

Here $\Phi_q(w)$ are n -column vectors whose components are holomorphic and bounded functions of w for

$$0 < |w| < c_1, \quad \underline{\Theta} < \arg w < \bar{\Theta}$$

and admit asymptotic expansions in powers of w as w tends to zero through the sector $\underline{\Theta} < \arg w < \bar{\Theta}$.

By applying Theorem B in Section 11 in [5] and by using Lemma 3 in Section 10, we have the following conclusion:

PROPOSITION 2.2. - *Equations (12.4) have a solution $\{ \Phi^2(w, V(w)), \Psi(V(w)) \}$ provided that the values of w and $V(w)$ belong to domain (11.4). Here $\Psi(v)$ is the same as before and $\Phi^2(w, v)$ is an n -column vector function with Property- $\mathcal{Q}$ with respect to v in (11.4).*

If we define $\varphi_2(w, z)$ by

$$(12.5) \quad \varphi_2(w, z) = \Phi^2(w, \Psi^{-1}(z)),$$

$\varphi_2(w, Z(w))$ is a solution of equation (12.3) provided that the values of w and $Z(w)$ remain in (11.6) and $\varphi_2(w, z)$ has Property- $\mathcal{Q}$ with respect to z in (11.6).

If we define the n -vector $Q_2(z)$ by

$$(12.6) \quad Q_2(z) = \varphi_2(z^m, z),$$

$Q_2(z)$ is a holomorphic solution of partial differential equation (9.4.2) if z belongs to (11.8) and is asymptotically developable in powers of z as z tends to zero through (11.8).

In this way we can determine the n -vectors $Q_i(z) \equiv \varphi_i(z^m, z)$ as solutions of the partial differential equations (9.4.4) in such a way that $\varphi_i(w, z)$ are functions with Property- $\mathcal{Q}$ with respect to z in (11.6).

Thus Theorem 2 has been proved.

CHAPTER III.

Proof of Theorem 3.

§ 13. - Problem A and Proof of Theorem 3.

In order to prove uniform convergence of the formal transformation (T₁) appearing in Theorem 2 in Section 3, we put

$$(13.1) \quad P_N(x, w, z) = z + \sum_{l=1}^{N-1} \alpha^l \varphi_l(w, z),$$

where $\varphi_l(w, z)$ have Property- $\mathcal{Q}$ with respect to z in

$$(13.2) \quad 0 < |w| < c_1, \quad \underline{\Theta} < \arg w < \bar{\Theta}, \quad \|z\| < b_1.$$

Let b_0 be the constant appearing in Theorem 1 in Section 3. Then, for a positive constant $d_N^0 < b_0$, we choose positive constants a_N^0, b_N^0, c_N^0 so that

$$\|P_N(x, w, z)\| \leq b_0 - d_N^0$$

for

$$(13.3) \quad |x| \leq a_N^0, \quad 0 < |w| \leq c_N^0, \quad \underline{\Theta} < \arg w < \bar{\Theta}, \quad \|z\| \leq b_N^0.$$

Let $\{X(w), Z(w)\}$ be a holomorphic solution of equations (10.2) such that

$$X(w_1) = x_1, \quad Z(w_1) = z^1,$$

where x_1, w_1, z^1 are arbitrary values belonging to

$$|x| \leq a_N^0, \quad 0 < |w| \leq c_N^0, \quad \underline{\Theta} < \arg w < \bar{\Theta}, \quad \|z\| \leq b_N^0.$$

Since w and $Z(w)$ can be regarded as functions of x by the equation $x = X(w)$, $x \frac{d}{dx} P_N(x, w, Z(w))$ becomes a function of $(x, w, Z(w))$. If we write this function as

$$x \frac{d}{dx} P_N(x, w, Z(w)) = Q_N(x, w, Z(w)),$$

$Q_N(x, w, z)$ is a polynomial of x whose coefficients have Property- $\mathcal{Q}\ell$ with respect to z in (13.2).

1°. - We make the change of variable

$$(13.4) \quad u = \eta + P_N(x, \tilde{Z}(x)^m, \tilde{Z}(x))$$

to equation (B₀) appearing in Theorem 1 in Section 3. Here $\tilde{Z}(x)$ is a holomorphic solution of equation (B) appearing in Theorem 2 in Section 3 such that $\tilde{Z}(x_1) = z^1$, where x_1 and z^1 are arbitrary values in

$$|x| < a_N^0, \quad 0 < \|z\| < b_N^0, \quad \underline{\Theta} < \arg z^m < \bar{\Theta}.$$

The equation satisfied by η is written by

$$(13.5) \quad x\eta' = h_N(x, \tilde{Z}(x)^m, \tilde{Z}(x); \eta),$$

where

$$h_N(x, w, z; \eta) = g(x, \eta + P_N(x, w, z)) - Q_N(x, w, z).$$

Hence $h_N(x, w, z; \eta)$ is a holomorphic and bounded function of (x, w, z, η) in

$$(13.6) \quad |x| < a_N^0, \quad 0 < |w| < c_N^0, \quad \underline{\Theta} < \arg w < \bar{\Theta}, \quad \|z\| < b_N^0, \quad \|\eta\| < d_N^0.$$

Since equation (13.5) has a formal solution of the form

$$(13.7) \quad \eta \sim \sum_{l=N}^{\infty} x^l \varphi_l(\tilde{Z}(x)^m, \tilde{Z}(x)),$$

it is easily verified that $h_N(x, w, z; \eta)$ satisfies inequalities of the form

$$(13.8) \quad \|h_N(x, w, z; \eta)\| \leq A \|\eta\| + B |x|^N,$$

$$(13.9) \quad \|h_N(x, w, z; \eta^1) - h_N(x, w, z; \eta^2)\| \leq A \|\eta^1 - \eta^2\|$$

provided that the values of the arguments belong to (13.6). Here A is a positive constant independent of N while B may depend on N .

2°. – We shall prove that *the uniform convergence of the formal transformation (T₁) results from the solution of the following problem:*

PROBLEM A. – *Equation (13.5) has a unique solution $S_N(x, \tilde{Z}(x))$ satisfying the condition*

$$(13.10) \quad \eta = O(|x|^N),$$

whenever the values of x and $\tilde{Z}(x)$ belong to a domain of the form

$$(13.11)_N \quad |x| < a_N'', \quad 0 < \|z\| < b_N'', \quad \underline{\Theta} < \arg z^m < \bar{\Theta}.$$

Here $S_N(x, z)$ is an n -column vector whose components are holomorphic and bounded functions of (x, z) for (13.11)_N, and a_N'' and b_N'' are positive constants.

Indeed,

$$(13.12) \quad \tilde{\Phi}_N(x, \tilde{Z}(x)) = P_N(x, \tilde{Z}(x)^m, \tilde{Z}(x)) + S_N(x, \tilde{Z}(x))$$

is a solution of equation (B₀) if $(x, \tilde{Z}(x))$ belongs to the domain (13.11)_N. In order to assert that this solution is independent of N , let $N' > N$ be any integer. Then

$$(13.13) \quad \tilde{\Phi}_{N'}(x, \tilde{Z}(x)) - P_N(x, \tilde{Z}(x)^m, \tilde{Z}(x))$$

is a solution of equation (13.5) and moreover satisfies the condition $\eta = O(|x|^N)$ provided that $(x, \tilde{Z}(x))$ belongs to the common part of the domains (13.11)_{N'} and (13.11)_N. However, since such a solution is unique, the solution (13.13) must coincide with the solution $S_N(x, \tilde{Z}(x))$. Thus we have the identity $\tilde{\Phi}_{N'}(x, \tilde{Z}(x)) \equiv \tilde{\Phi}_N(x, \tilde{Z}(x))$ whenever $(x, \tilde{Z}(x))$ is in the common part mentioned above. This allows us to conclude that the solution (13.12) is independent of N .

We denote this solution by $\tilde{\Phi}(x, \tilde{Z}(x))$.

Then, by analytic continuation, the function $\tilde{\Phi}(x, z)$ is defined in a domain of the form

$$(13.14) \quad |x| < a'', \quad 0 < \|z\| < b'', \quad \underline{\Theta} < \arg z^m < \bar{\Theta}$$

with $a'' = \sup_N a_N''$ and $b'' = \sup_N b_N''$. Clearly the function $\tilde{\Phi}(x, z)$ admits the asymptotic expansion (T₁). On the other hand, $x=0$ is an interior point of the domain in which $\tilde{\Phi}(x, z)$ is defined. Hence, by virtue of CAUCHY'S theorem, $\tilde{\Phi}(x, z)$ admits a uniformly convergent expansion in powers of x . By the uniqueness of asymptotic expansions, this convergent expansion must coincide with the asymptotic expansion (T₁).

Thus the proof of uniform convergence of the formal transformation (T₁) has been completed. This completes the proof of Theorem 3.

§ 14. - Solution of Problem A.

Corresponding to equation (13.5), we consider the equation

$$(14.1) \quad w^2 \frac{d\eta}{dw} = \frac{1}{mg(Z(w))} h_N(X(w), w, Z(w); \eta)$$

or, what is the same thing, the equation

$$(14.2) \quad w^2 \frac{d\eta}{dw} = \frac{1}{mG(V(w))} H_N(X(w), w, V(w); \eta)$$

with

$$(14.3) \quad \begin{cases} G(v) = g(\Psi(v)), \\ H_N(x, w, v; \eta) = h_N(x, w, \Psi(v); \eta). \end{cases}$$

Here $V(w)$ and $\Psi(v)$ are the functions that appeared in Section 11. We can assume without loss of generality that the inverse function $\Psi^{-1}(z)$ of $\Psi(z)$ is a holomorphic function of z in $\|z\| \leq b_N''$ and satisfies there $\|\Psi^{-1}(z)\| \leq b_N''$.

Instead of solving Problem A directly, we want to solve the following problem:

PROBLEM A*. - Equation (14.2) has a unique solution $\Xi_N(X(w), w, V(w))$ satisfying the condition

$$(14.4) \quad \|\Xi_N(X(w), w, V(w))\| = O(\|X(w)\|^N),$$

whenever the values of $X(w)$, w , $V(w)$ stay in a domain of the form

$$(14.5)_N \quad |x| < a_N'', \quad 0 < |w| < c_N'', \quad \underline{\Theta} < \arg w < \bar{\Theta}, \quad \|v\| < b_N'.$$

Here $\Xi_N(x, w, v)$ is an n -column vector function holomorphic and bounded in (x, w, v) for $(14.5)_N$.

If we assume that Problem A* has been solved,

$$(14.6) \quad \xi_N(X(w), w, Z(w)) \equiv \Xi_N(X(w), w, \Psi^{-1}(Z(w)))$$

is a solution of equation (14.1) satisfying the condition $\eta = O(|X(w)|^N)$ provided that the values of $X(w), w, Z(w)$ belong to

$$(14.7) \quad |x| < a_N'', \quad 0 < |w| < c_N'', \quad \underline{\Theta} < \arg w < \bar{\Theta}, \quad \|z\| < b_N''.$$

We choose b_N'' so small that $(b_N'')^{|m|} \leq c_N''$. Then, by applying Assertion (i) of Lemma 3 in Section 10, we can conclude that *the function*

$$(14.8) \quad S_N(x, \tilde{Z}(x)) \equiv \xi_N(x, \tilde{Z}(x)^m, \tilde{Z}(x))$$

is a solution of Problem A.

However, in order to solve Problem A*, we must replace the domain $(14.5)_N$ by a domain of the form

$$(14.9)_N \quad \begin{cases} |x| < a_N^*, & [v] < b_N^* \chi(\arg w), \\ 0 < |w| < c_N^* \omega(\arg w), & \underline{\Theta} < \arg w < \bar{\Theta}. \end{cases}$$

where $\chi(\varphi)$ is an n -column vector with elements $\{\chi_k(\varphi)\}$; $\chi_k(\varphi)$ and $\omega(\varphi)$ are strictly positive valued and continuous functions of φ for $\underline{\Theta} \leq \varphi \leq \bar{\Theta}$; a_N^*, b_N^*, c_N^* are suitably chosen positive constants.

Obviously a domain of the form $(14.5)_N$ is equivalent to a domain of the form $(14.9)_N$.

We define the functions $\omega(\varphi)$ and $\chi_k(\varphi)$ by

$$(14.10) \quad \omega(\varphi) = \exp \int_{\theta_0}^{\varphi} \cot A(\tau) d\tau,$$

$$(14.11.k) \quad \chi_k(\varphi) = \exp \left\{ (\operatorname{Re} \mu_k) \int_{\theta_0}^{\varphi} \cot A(\tau) d\tau + (\operatorname{Im} \mu_k)(\theta_0 - \varphi) \right\},$$

$$\left(\mu_k = \operatorname{Re} \mu_k + \sqrt{-1} \operatorname{Im} \mu_k, \quad \mu_k = \frac{\alpha_k}{m\alpha} \right).$$

Here θ_0 is a fixed value in $\underline{\Theta} \leq \varphi \leq \bar{\Theta}$ and

$$(14.12) \quad A(\varphi) = \begin{cases} \varphi - \theta_+ + 4\varepsilon, & \theta_+ + \frac{\pi}{2} - 4\varepsilon \leq \varphi \leq \bar{\Theta}, \\ \frac{\pi}{2}, & \theta_- - \frac{\pi}{2} + 4\varepsilon \leq \varphi \leq \theta_+ + \frac{\pi}{2} - 4\varepsilon, \\ \varphi - \theta_- + \pi - 4\varepsilon, & \underline{\Theta} \leq \varphi \leq \theta_- - \frac{\pi}{2} + 4\varepsilon \end{cases}$$

with

$$(14.13) \quad \theta_- = -\arg m\alpha - \frac{\pi}{2}, \quad \theta_+ = -\arg m\alpha + \frac{\pi}{2}.$$

Since

$$\underline{\Theta} = -\arg m\alpha - \frac{3\pi}{2} + 5\varepsilon, \quad \bar{\Theta} = -\arg m\alpha + \frac{3\pi}{2} - 5\varepsilon,$$

the function $A(\varphi)$ satisfies the inequality

$$(14.14) \quad \varepsilon \leq A(\varphi) \leq \pi - \varepsilon \quad \text{for} \quad \underline{\Theta} \leq \varphi \leq \bar{\Theta}$$

and is continuous for $\underline{\Theta} \leq \varphi \leq \bar{\Theta}$. From this we see immediately that the functions $\omega(\varphi)$ and $\chi_k(\varphi)$ thus defined have the above specified property.

§ 15. - Solution of Problem A* (Part I).

1°. - We consider the family $\mathcal{F} = \{ \psi(x, w, v) \}$ of n -column vectors $\psi(x, w, v)$ which are holomorphic and bounded functions of (x, w, v) in a domain of the form

$$(15.1)_N \quad \begin{cases} |x| < a_N^*, & [v] < b_N^* \chi(\arg w), \\ 0 < |w| < c_N^* \omega(\arg w), & \underline{\Theta} < \arg w < \bar{\Theta} \end{cases}$$

and satisfy there an inequality of the form

$$(15.2) \quad \|\psi(x, w, v)\| \leq K |x|^N,$$

where K is a certain positive constant and satisfies

$$(15.3) \quad K(a_N^*)^N \leq d_N^0.$$

Since $\{0\} \in \mathcal{F}$, the family $\mathcal{F}$ is not empty. We define the n -vector $\Psi(x, n, v)$ by

$$(15.4) \quad \Psi(x_1, n_1, v^1) = \int_0^{n_1} \mathcal{H}_N(X(w), n, V(w)) dw,$$

where

$$(15.5) \quad \mathcal{H}_N(x, n, v) = \frac{1}{n^2 m G(v)} H_N(x, n, v; \psi(x, n, v)).$$

Here $\{X(w), V(w)\}$ is a holomorphic solution of the equations

$$(15.6) \quad n^2 \frac{dx}{dn} = \frac{x}{m G(v)}, \quad n \frac{dv}{dn} = \mathbf{I}_n \left(\frac{\alpha}{m \alpha} \right) v$$

such that

$$X(n_1) = x_1, \quad V(n_1) = v^1,$$

where x_1, n_1, v^1 are arbitrary values in the domain $(15.1)_N$. The integration is to be carried out along a curve Γ_{n_1} , joining the point n_1 with the origin, which is defined below:

We denote by (r, θ) and (ρ, φ) the polar coordinates of the point n_1 and the variable point n on the curve Γ_{n_1} respectively.

If $\theta_+ - 2\varepsilon < \theta$ or $\theta < \theta_- + 2\varepsilon$, the curve Γ_{n_1} consists of a curvilinear part Γ' :

$$(15.7) \quad \rho = r \exp \int_0^\varphi \cot A(\tau) d\tau, \quad \theta_+ - 2\varepsilon \leq \varphi \leq \theta \quad \text{or} \quad \theta \leq \varphi \leq \theta_- + 2\varepsilon$$

and of a rectilinear part Γ'' :

$$0 \leq \rho \leq r \exp \int_0^\varphi \cot A(\tau) d\tau, \quad \varphi = \theta_+ - 2\varepsilon \quad \text{or} \quad \theta_- + 2\varepsilon.$$

If $\theta_- + 2\varepsilon \leq \theta \leq \theta_+ - 2\varepsilon$, the curve Γ_{n_1} consists of a rectilinear part Γ'' only:

$$0 \leq \rho \leq r, \quad \varphi = \theta.$$

Since $A(\varphi)$ is given by (14.11), we see by inspection that the curve Γ_{n_1} is entirely contained in the domain

$$(15.8) \quad 0 < |n| < c_N^* \omega(\arg n), \quad \underline{\Theta} < \arg n < \bar{\Theta}$$

except for the origin.

It will be shown later that the integral (15.4) is convergent. Then the mapping $\mathfrak{C}: \psi(x, w, v) \rightarrow \Psi(x, w, v)$ has a well defined meaning.

2°. – The following proposition is evident:

I. *The family $\mathfrak{F}$ is a closed, convex and normal set.*

To solve Problem A*, we shall prove the following propositions:

II. *The integral (15.4) along the curve Γ_{w_1} is convergent.*

III. *$\Psi(x, w, v)$ belongs to $\mathfrak{F}$, namely*

III'. *We have the inequality*

$$(15.9) \quad \|\Psi(x, w, v)\| \leq K |x|^N.$$

III''. *The n -vector $\Psi(x, w, v)$ is a holomorphic and bounded function of (x, w, v) in $(15.1)_N$.*

IV. *$\mathfrak{C}$ is a continuous mapping with respect to the topology of uniform convergence on compact subset.*

If we assume that these propositions have been established, it is known by a fixed-point theorem (See M. HUKUHARA [3]) that there exists a member $\Psi(x, w, v)$ of the family $\mathfrak{F}$ which corresponds to a fixed point of the mapping $\mathfrak{C}$. We shall prove moreover that

V. *The n -vector $\Psi(X(w), w, V(w))$ is a solution of equation (14.2).*

VI. *A solution of equation (14.2) satisfying the condition*

$$(15.10) \quad \eta = O(|X(w)|^N)$$

is unique.

If we denote this solution by $\Xi_N(X(w), w, V(w))$, we have a solution of Problem A*.

§ 16. – Solution of Problem A* (Part II).

Since $\Psi(0) = 0$, the first relation of (14.3) implies that $G(0) = g(\Psi(0)) = \alpha$. Observe that $h_N(x, w, z; \eta)$ is defined in (13.6). From (13.8), (13.9) and (14.3) we can derive the following inequalities

$$(16.1) \quad \frac{1}{2} |m\alpha| \leq |mG(v)| \leq 2 |m\alpha|,$$

$$(16.2) \quad |v^2 mG(v)|^{-1} \|H_N(x, w, v; \eta)\| \leq 2 |m\alpha|^{-1} |w|^{-2} (A \|\eta\| + B |x|^N),$$

$$(16.3) \quad \|H_N(x, w, v; \eta^1) - H_N(x, w, v; \eta^2)\| \leq A \|\eta^1 - \eta^2\|$$

whenever the values of the arguments belong to

$$(16.4) \quad \begin{cases} |x| < \alpha_N^*, & [v] < b_N^* \chi(\arg w), & \|\eta\| < d_N^0, \\ 0 < |w| < c_N^* \omega(\arg w), & \underline{\Theta} < \arg w < \bar{\Theta} \end{cases}$$

provided that

$$\alpha_N^* \leq \alpha_N^0, \quad b_N^* \max_{\varphi} \|\chi(\varphi)\| \leq b_N^0, \quad c_N^* \max_{\varphi} \omega(\varphi) \leq c_N^0.$$

1°. PROOF OF II. - We assume that the following lemma has been established:

LEMMA 4. - *As w moves on the curve Γ_{w_1} , we have*

$$(16.5) \quad [V(w)] < b_N^* \chi(\arg w), \quad \underline{\Theta} < \arg w < \bar{\Theta},$$

$$(16.6) \quad \frac{d|X(w)|}{ds} \geq \frac{\sin \varepsilon}{2|m\alpha|} |w|^{-2} |X(w)|.$$

Here s is the arc length of the curve Γ_{w_1} measured from the origin to the variable point.

This lemma will be proved in Section 19.

Since (16.6) shows that $|X(w)|$ is a monotone increasing function of s and (16.5) shows that the value of $V(w)$ always remains in

$$[v] < b_N^* \chi(\arg w), \quad \underline{\Theta} < \arg w < \bar{\Theta},$$

it follows that $\mathcal{H}_N(X(w), w, V(w))$ is a holomorphic function of w on the curve Γ_{w_1} provided that $K(a_N^*)^N \leq d_N^0$. Moreover, owing to (16.2), we have

$$(16.7) \quad \begin{aligned} \|\Psi(x_1, w_1, v^1)\| &\leq \int_0^{s_1} \|\mathcal{H}_N(X(w), w, V(w))\| ds \leq \\ &\leq 2|m\alpha|^{-1}(AK+B) \int_0^{s_1} |w|^{-2} |X(w)|^N ds, \end{aligned}$$

where s_1 is the arc length of the curve Γ_{w_1} .

However, by virtue of (16.6), the function $|X(w)|$ tends to 0 exponentially as w approaches the origin along the curve Γ_{w_1} . This proves that integral (15.4) is convergent.

2°. PROOF OF III'. - It will be sufficient to prove that

$$(16.8) \quad \|\Psi(x_1, w_1, v^1)\| \leq K |x_1|^N.$$

However we have by (16.6)

$$(16.9) \quad \int_0^{s_1} |w|^{-2} |X(w)|^N ds \leq \frac{2 |m\alpha|}{N \sin \varepsilon} |x_1|^N.$$

By using this inequality, it follows from (16.7) that

$$\|\Psi(x_1, w_1, v^1)\| \leq \frac{4(AK + B)}{N \sin \varepsilon} |x_1|^N.$$

Hence we see that inequality (16.8) is an immediate consequence of the inequality

$$4(AK + B) \leq KN \sin \varepsilon.$$

We first choose N so large that

$$(16.10) \quad 8A < N \sin \varepsilon$$

and next take K large enough to have

$$4B(N \sin \varepsilon - 4A)^{-1} \leq K.$$

Then we get the inequality to be required, but we must take α_N^* so small to have (15.3).

3°. - PROOF OF III''. - Inequality (16.8) shows that integral (15.4) is uniformly convergent with respect to (x_1, v^1) for each w_1 . Namely, the n -vector $\Psi(x, w_1, v)$ is a holomorphic function of (x, v) at (x_1, v^1) for each w_1 . Hence, to have Proposition III'', it will be sufficient to prove that the n -vector $\Psi(x_1, w, v^1)$ is a holomorphic function of w at $w = w_1$. However, the last assertion is an immediate consequence of the lemma below.

LEMMA 5. - *For any w_0 sufficiently near to w_1 , the relation*

$$(16.11) \quad \Psi(x_1, w_1, v^1) = \int_0^{w_0} \mathcal{H}_N(X(w), w, V(w)) dw + \int_{w_0}^{w_1} \mathcal{H}_N(X(w), w, V(w)) dw$$

holds. Here the first and the second integrations must be carried out respectively along the curve Γ_{w_0} and the segment $\overline{w_0 w_1}$ joining w_0 with w_1 .

We shall prove this lemma. Since $X(w)$ and $V(w)$ are continuous functions of w , the integrand of the first integral of (16.11) is a holomorphic function of w for $w \in \Gamma_{w_0}$ except for $w = 0$ and the integrand of the second integral is a holomorphic function of w for $w \in \Gamma_{w_1}$. Therefore the integrals appearing in the right-hand member of (16.11) have meaning.

Let γ_0 and γ_1 be the points where the curves Γ_{w_0} and Γ_{w_1} respectively intersect with a circle $|w| = \delta$ of small radius. Then the relation (16.11) is an immediate consequence of the assertion that:

$$(16.12) \quad \left\| \int_{\gamma_0}^{\gamma_1} \mathcal{K}_N(X(w), w, V(w)) dw \right\| \rightarrow 0 \quad \text{as } \delta \rightarrow 0.$$

Here the integration must be carried out on the circle $|w| = \delta$.

However, due to (16.2), the integral appearing in (16.12) does not exceed the expression

$$\frac{2(AK + B)}{|m\alpha|} (\theta_+ - \theta_- - 4\varepsilon) \max_{w \in \mathcal{K}} \left\{ |w|^{-2} |X(w)|^N \right\},$$

where $\mathcal{K}$ is a closed set of w satisfying $|w| \leq \delta$, $|\arg m\alpha + \arg w| \leq \frac{\pi}{2} - 2\varepsilon$. As we have already seen, the function $|X(w)|$ tends to zero exponentially as w approaches the origin through the sector $|\arg m\alpha + \arg w| \leq \frac{\pi}{2} - 2\varepsilon$. Hence we have (16.12).

4°. PROOF OF IV. - Let $\{\psi^k\}_{k=1}^\infty$ be any sequence of the family $\mathcal{F}$ which converges to $\psi \in \mathcal{F}$. Then it is sufficient to prove that the corresponding sequence $\{\Psi^k\}_{k=1}^\infty$ converges to the corresponding function $\Psi \in \mathcal{F}$. However, by virtue of inequality (16.3), $H_N(x, w, v; \eta)$ satisfies LIPSCHITZ'S condition with respect to η . Hence the proposition is almost evident.

5°. PROOF OF V. - Since $\{X(w), V(w)\}$ is a holomorphic solution of equations (15.6) satisfying the initial conditions $X(w_1) = x_1$ and $V(w_1) = v^1$, it is convenient to write the functions $X(w)$ and $V(w)$ as

$$X(w) = \mathfrak{X}(w, w_1, x_1), \quad V(w) = \mathfrak{V}(w, w_1, v^1).$$

In order to prove the proposition, it is sufficient to prove that:

$$(16.13) \quad \frac{d}{dw_0} \Psi(x_0, w_0, v^0) = \mathcal{K}_N(x_0, w_0, v^0),$$

where x_0 and v^0 are considered as functions of w_0 given by

$$x_0 = \mathfrak{X}(w_0, w_1, x_1), \quad v^0 = \mathfrak{Y}(w_0, w_1, v^1).$$

By definition, we have

$$\begin{aligned} \frac{d}{dw_0} \Psi(x_0, w_0, v^0) &= \mathcal{H}_N(x_0, w_0, v^0) + \\ &+ \int_0^{w_0} \frac{d}{dw_0} \mathcal{H}_N(\mathfrak{X}(w, w_0, x_0), w, \mathfrak{Y}(w, w_0, v^0)) dw. \end{aligned}$$

However, we have

$$\begin{aligned} (16.14) \quad & \frac{d}{dw_0} \mathcal{H}_N(\mathfrak{X}(w, w_0, x_0), w, \mathfrak{Y}(w, w_0, v^0)) = \\ &= \frac{\partial \mathcal{H}_N(\mathfrak{X}, w, \mathfrak{Y})}{\partial \mathfrak{X}} \{ \mathfrak{X} \}_{dw_0} + \frac{\partial \mathcal{H}_N(\mathfrak{X}, w, \mathfrak{Y})}{\partial \mathfrak{Y}} \{ \mathfrak{Y} \}_{dw_0}, \end{aligned}$$

where, for example,

$$\{ \mathfrak{X} \}_{dw_0} = \frac{\partial \mathfrak{X}(w, w_0, x_0)}{\partial w_0} + \frac{\partial \mathfrak{X}(w, w_0, x_0)}{\partial x_0} \frac{\partial \mathfrak{X}(w_0, w_1, x_1)}{\partial w_0}.$$

It is easily seen that, for any constant γ , $\{ \mathfrak{X}(\gamma, w, x), \mathfrak{Y}(\gamma, w, v) \}$ is an integral of equations (15.6). Hence the expression of the right-hand member of (16.14) is zero identically and we have (16.13).

6°. PROOF OF VI. - Assume that there exist two solutions satisfying the same conditions. Then the difference $\zeta(X(w), w, V(w))$ of these two solutions satisfies an inequality of the form

$$(16.15) \quad \|\zeta(X(w), w, V(w))\| \leq \tilde{K} |X(w)|^N$$

for a certain nonnegative finite constant $\tilde{K}$, whenever the values of $X(w)$, w , $V(w)$ belong to the domain $(15.1)_N$. Moreover, thanks to inequality (16.3) multiplied by $|w^2 m G(v)|^{-1}$, we have

$$\|\zeta(x_1, w_1, v^1)\| \leq \frac{2A}{|m\alpha|} \int_0^{s_1} |w|^{-2} \|\zeta(X(w), w, V(w))\| ds.$$

Let K^* be the least lower bound of constants $\tilde{K}$ satisfying (16.15). Then it is sufficient to show that $K^* = 0$. Assume the contrary. By using (16.15) and (16.9), it is readily verified that

$$\|\zeta(x_1, w_1, v^1)\| \leq \frac{4AK^*}{N \sin \epsilon} |x_1|^N \leq \frac{K^*}{2} |x_1|^N$$

by (16.10). However, since (x_1, w_1, v^1) is an arbitrary point in the domain (15.1)_N, the above inequality contradicts the hypothesis that K^* is the least lower bound of $\tilde{K}$. Hence the quantity K^* must be zero.

CHAPTER IV.

Analytic Integration of Equation (B).

§ 17. - Parametric Representation of Solution.

Let $\{X(w), Z(w)\}$ be a holomorphic solution of the equations

$$(17.1) \quad w^2 \frac{dw}{dz} = \frac{x}{mg(z)}, \quad w \frac{dz}{dw} = \mathbf{1}_n \left(\frac{g(z)}{mg(z)} \right) z, \\ g(z) = \alpha + \sum_{q \in \mathbb{S}} \beta_q z^q,$$

such that $X(w_1) = x_1$ and $Z(w_1) = z^1$. Here x_1, w_1, z^1 are arbitrary values such that

$$|x_1| < a'', \quad 0 < |w_1| < c'', \quad \Theta < \arg w_1 < \bar{\Theta}, \quad \|z^1\| < b''$$

with $(b'')^{|m|} \leq c''$. Then, as it was already proved in Section 10, the equations $x = X(w), z = Z(w)$ define a solution, $\tilde{Z}(x)$, of equation (B) if and only if we have $(z^1)^m = w_1$. This solution satisfies obviously the initial condition $\tilde{Z}(x_1) = z^1$.

We assert that:

Let $\tilde{Z}(x)$ be any solution of the equation (B). Then the function $\tilde{Z}(x)^{p-m}$ is independent of x for every $p \in \mathbb{S}$.

Indeed, since $\tilde{Z}(x)$ is a solution of equation (B), we have

$$x \frac{d\tilde{Z}(x)^{p-m}}{dx} = \tilde{Z}(x)^{p-m} (p-m) \mathbf{1}_n(\tilde{Z}(x))^{-1} x \frac{d\tilde{Z}(x)}{dx} = \\ = \tilde{Z}(x)^p (p-m) g(\tilde{Z}(x)) = \\ = \tilde{Z}(x)^p \{ (p-m)\alpha + \sum_{q \in \mathbb{S}} (p-m)\beta_q \tilde{Z}(x)^q \} = 0.$$

The last relation is derived from (1.4) and (1.5). This proves the assertion.

Owing to this assertion, the n -column vector α^* defined by

$$(17.2) \quad \alpha^* = \sum_{q \in \mathfrak{S}} \tilde{Z}(x)^{q-m} \beta_q$$

does not depend on x . Then *the solution $z = \tilde{Z}(x)$ is equivalent to the solution, $\{x = X(w), z = Z(w)\}$, of the equations*

$$(B') \quad w^2 \frac{dx}{dw} = \frac{x}{m\alpha + m\alpha^*w}, \quad w \frac{dz}{dw} = \mathbf{1}_n \left(\frac{\alpha + \alpha^*w}{m\alpha + m\alpha^*w} \right) z$$

such that $x = x_1, z = z^1$ at $w = w_1$ with $w_1 = (z^1)^m$.

By an elementary calculation, we can integrate equations (B') by quadratures and we have the following conclusion:

A solution $z = \tilde{Z}(x)$ of equation (B) such that $\tilde{Z}(x_1) = z^1$ has a parametric representation of the form $x = X(w), z = Z(w)$, where

$$(17.3) \quad \begin{cases} X(w) = c \left(1 + \frac{m\alpha}{m\alpha^*w} \right)^{\frac{m\alpha^*}{(m\alpha)^2}} \exp \left(-\frac{1}{m\alpha w} \right), \\ Z(w) = \mathbf{1}_n \left(w^{\frac{\alpha}{m\alpha}} \right) \mathbf{1}_n \left((m\alpha + m\alpha^*w)^{\frac{\alpha^*}{m\alpha^*} - \frac{\alpha}{m\alpha}} \right) C. \end{cases}$$

Here the constant c and the n -column vector C must be so chosen that

$$X(w_1) = x_1, \quad Z(w_1) = z^1 \quad \text{and} \quad C^m = 1.$$

§ 18. Function $X(w)$ and its Inverse Function.

If we want to express $Z(w)$ as a function of x , we must eliminate w from equations (17.3). To do this it is convenient to use a certain transcendental function introduced by M. HUKUHARA [1].

Following HUKUHARA, we put

$$(18.1) \quad x = \exp \left(-\frac{m\alpha^*}{(m\alpha)^2} X \right), \quad w = \frac{m\alpha}{m\alpha^*} W.$$

Then the first equation of (B') is transformed into

$$(18.2) \quad \frac{dW}{dX} = 1 + \frac{1}{W}.$$

Let $W = \mathcal{A}(X)$ be a function defined implicitly by

$$X = W - \log(W + 1)$$

and consider the branch of $\mathcal{A}(X)$ such that $\mathcal{A}(X) - X - \log X$ vanishes at $X = \infty$. Then we have the following conclusion:

The general solution $\tilde{Z}(x)$ of equation (B) can be written as

$$(18.3) \quad \tilde{Z}(x) = \mathbf{1}_n \left(\mathcal{A}(X + \tilde{c})^{-\frac{x}{m\alpha}} \right) \mathbf{1}_n \left(\left(m\alpha + \frac{m\alpha}{\mathcal{A}(X + \tilde{c})} \right)^{\frac{\alpha^*}{m\alpha^*} - \frac{\alpha}{m\alpha}} \right) \tilde{C},$$

where

$$\tilde{C}^m = \frac{m\alpha}{m\alpha^*}, \quad X = -\frac{(m\alpha)^2}{m\alpha^*} \log x.$$

Here $\tilde{c}$ and $\tilde{C}$ must be so chosen that $\tilde{Z}(x_1) = z^1$.

It is to be noticed that, when w tends to zero along a segment in the sector $|\arg m\alpha + \arg w| \leq \frac{\pi}{2} - 2\varepsilon$, the function $X(w)$ defined by the first equation of (17.3) decreases monotonously in absolute value for sufficiently small $|w|$ and tends to 0 exponentially. Then the nature of the function $x = X(w)$ and its inverse function is clarified by the theorem below.

THEOREM 4. - *Let x_1 and z^1 be arbitrary values in*

$$|x| < \alpha', \quad 0 < \|z\| < b', \quad \Theta < \arg z^m < \bar{\Theta}.$$

We choose a value of the constant c so that the first equation of (17.3) is satisfied at $x = x_1$ and $w = w_1$ with $w_1 = (z^1)^m$.

Then, if w tends to zero along a curve L_{w_1} , which starts from $w = w_1$ and approaches $w = 0$ from the interior of the sector $|\arg m\alpha + \arg w| \leq \frac{\pi}{2} - 2\varepsilon$, the point $x = X(w)$ describes a spiral-shaped curve $L_{x_1}^$, turning around the origin in the complex x -plane, and approaches $x = 0$.*

The converse is also true.

§ 19. - Estimation of the Growth of the Functions $X(w)$ and $Z(w)$ on the Curve T_{w_1} .

As we have already seen in Section 11, the function $Z(w)$ can be expressed as

$$(19.1) \quad Z(w) = \Psi(V(w)), \quad V(w) = \mathbf{1}_n \left(w^{\frac{\alpha}{m\alpha}} \right) C''.$$

Then $x = X(w)$ is a solution of the equation

$$(19.2) \quad w^2 \frac{dx}{dw} = \frac{x}{mG(V(w))}, \quad G(v) \equiv g(\Psi(v)).$$

Let x_1, w_1, v^1 be arbitrary values in the domain

$$\begin{cases} |x| < a_N^*, [v] < b_N^* \chi(\arg w), \\ 0 < |w| < c_N^* \omega(\arg w), \underline{\Theta} < \arg w < \bar{\Theta}, \end{cases}$$

where $\omega(\varphi)$ and $\chi(\varphi)$ are given by (14.10) and (14.11.k) respectively. We choose the constant C'' so that $V(w_1) = v^1$.

Then we have the following lemma which was stated as Lemma 4 without proof in Section 16:

LEMMA 4. - *As w moves on the curve Γ_{w_1} , we have*

$$(19.3) \quad [V(w)] < b_N^* \chi(\arg w), \quad \underline{\Theta} < \arg w < \bar{\Theta},$$

$$(19.4) \quad \frac{d|X(w)|}{ds} \geq \frac{\sin \varepsilon}{2|m\alpha|} |w|^{-2} |X(w)|.$$

We shall prove first inequality (19.3). To prove this, it is sufficient to show that, for each index k , we have

$$(19.3.k) \quad |V_k(w)| < b_N^* \chi_k(\arg w), \quad \underline{\Theta} < \arg w < \bar{\Theta}$$

on the curve Γ_{w_1} .

Observe that

$$V_k(w) = v_k^1 \left(\frac{w}{w_1} \right)^{\mu_k}, \quad \mu_k = \frac{\alpha_k}{m\alpha}$$

and the curve Γ_{w_1} consists of two parts Γ' and Γ'' in general.

Since

$$|V_k(w)| = |v_k^1| \left| \frac{w}{w_1} \right|^{\operatorname{Re} \mu_k} e^{-\operatorname{Im} \mu_k \cdot (\arg w - \arg w_1)},$$

$|V_k(w)|$ is a monotone increasing function of $|w|$ on the rectilinear part Γ'' . Hence if inequality (19.3.k) holds at the point from which the rectilinear part Γ'' starts, this inequality continues to hold on Γ'' .

For w on the curvilinear part Γ' , $|w|$ is a function of $\arg w$ given by (15.7). Hence we have

$$\left| \left(\frac{w}{w_1} \right)^{\mu_k} \right| = \exp \left\{ (\operatorname{Re} \mu_k) \int_{\theta}^{\varphi} \cot A(\tau) d\tau + (\operatorname{Im} \mu_k)(\theta - \varphi) \right\},$$

where $\varphi = \arg w$ and $\theta = \arg w_1$. On the other hand, (19.3.k) holds at $\arg w = \theta$. So, we have

$$|v_k^1| < b_N^* \exp \left\{ (\operatorname{Re} \mu_k) \int_{\theta_0}^{\theta} \cot A(\tau) d\tau + (\operatorname{Im} \mu_k)(\theta_0 - \theta) \right\}.$$

From the above two inequalities we see by inspection that inequality (19.3.k) continues holding for w on the curvilinear part Γ' .

Thus inequality (19.3) has been proved.

To prove inequality (19.4), observe that $X(w)$ is a solution of equation (19.2). So, we have

$$\begin{aligned} (19.5) \quad \frac{1}{|X(w)|} \frac{d|X(w)|}{ds} &= \operatorname{Re} \left(\frac{1}{X(w)} \frac{dX(w)}{dw} \frac{dw}{ds} \right) = \\ &= \frac{1}{\rho^2} \operatorname{Re} \left(\frac{1}{m\alpha} \frac{m\alpha}{mG(V(w))} e^{-2i\varphi} \frac{dw}{ds} \right), \quad \rho = |w|. \end{aligned}$$

On the curvilinear part Γ' , ρ is a function of φ given by (15.7). A simple calculation implies that

$$(19.6) \quad \frac{dw}{ds} = + e^{i(A(\varphi)+\varphi)} \text{ or } - e^{i(A(\varphi)+\varphi)}, \quad i = \sqrt{-1}$$

according as $\theta_+ - 2\varepsilon \leq \varphi \leq \theta$ or $\theta \leq \varphi \leq \theta_- + 2\varepsilon$.

From this it follows that

$$(19.7) \quad \frac{1}{|X(w)|} \frac{d|X(w)|}{ds} = \pm \frac{1}{\rho^2} \operatorname{Re} \left(\frac{e^{i(A(\varphi)-\varphi)}}{m\alpha} \cdot \frac{m\alpha}{mG(V(w))} \right)$$

according as $\theta_+ - 2\varepsilon \leq \varphi \leq \theta$ or $\theta \leq \varphi \leq \theta_- + 2\varepsilon$.

For a preassigned positive constant ε ($\sin \varepsilon \leq \cos 5\varepsilon$), we can choose b_N^* so as to have

$$(19.8) \quad \frac{1}{2} \leq \left| \frac{m\alpha}{mG(V(w))} \right| \leq 2, \quad \left| \arg \frac{m\alpha}{mG(V(w))} \right| \leq \varepsilon$$

for

$$\|V(w)\| \leq b_N^* \|\chi(\arg w)\|, \quad \theta_- \leq \arg w \leq \bar{\theta}.$$

On the other hand, by the definition of $A(\varphi)$, it is easily found that

$$\begin{aligned} -\frac{\pi}{2} + 4\varepsilon &\leq A(\varphi) - \varphi - \arg m\alpha \leq 2\varepsilon && \text{for } \theta_+ - 2\varepsilon \leq \varphi \leq \bar{\theta}, \\ \pi - 2\varepsilon &\leq A(\varphi) - \varphi - \arg m\alpha \leq \frac{3\pi}{2} - 4\varepsilon && \text{for } \bar{\theta} \leq \varphi \leq \theta_- + 2\varepsilon. \end{aligned}$$

By using these inequalities together with (19.8), we can derive inequality (19.4) from (19.7).

On the rectilinear part Γ'' , ρ is independent of φ and $\rho = s = |w|$. Since $dw/ds = e^{i\varphi}$, equality (19.5) implies that

$$\frac{1}{|X(w)|} \frac{d|X(w)|}{ds} = \frac{1}{\rho^2} \operatorname{Re} \left(\frac{e^{-i\varphi}}{m\alpha} \frac{m\alpha}{mG(V(w))} \right).$$

If we notice that, for $w \in \Gamma''$,

$$|\arg m\alpha + \varphi| \leq \frac{\pi}{2} - 2\varepsilon,$$

we have inequality (19.4) by using (19.8).

Thus inequality (19.4) has been proved. This completes the proof of Lemma 4.

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