On the Quartic Combinant of a pencil of Quadric Surfaces.

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Sunto. - *Un covariante biquadratico di un fascio di quadriche in* S_3*, già ottenuto analiticamente dall'autore, vien quì discusso ed approfondito dal punto di vista geometrico.*

1. In the course of a recent investigation (1) into the combinants associated with a pencil of quadric surfaces, I drew attention to a quartic surface whose relation to the geometry of the pencil appeared hitherto to have escaped notice. The algebraic origin of this surface may be described briefly as follows. If $S=0$ and $S'=0$ are the equations of two quadric surfaces, then the expression $\lambda_0 S+\lambda_1 S'$ is an algebraic form linear in the binary variables λ_0, λ_1 and quadratic in the quaternary coordinate variables x_0, x_1, x_2, x_3. If this form is denoted, in the CLEBSCH-ARONHOLD symbolic notation, by a set of equivalent expressions

$$a_x{}^2\alpha_\lambda \equiv b_x{}^2\beta_\lambda \equiv c_x{}^2\gamma_\lambda \equiv \dots ,$$

then the form

$$(acde)(bcde)(\alpha\gamma)(\beta\delta)a_x b_x \varepsilon_\lambda$$

is a covariant of the original form, which can be shown not to vanish identically. This form can be written as $\lambda_0 Q+\lambda_1 Q'$, where Q and Q' are certain covariant quadrics of S and S'. The quartic combinant is the resultant $SQ'-S'Q$ of the two forms $\lambda_0 S+\lambda_1 S'$, $\lambda_0 Q+\lambda_1 Q'$, with respect to λ_0 and λ_1.

If Σ and Σ' denote the contravariants of S and S' and if Δ, Θ, Φ, Θ', Δ' denote, as usual, the invariants of the two quadrics, then the two fundamental quadratic covariants of S and S' are the coefficients of $k\Delta$ and $k^2\Delta'$ in the point equation of the envelope $\Sigma+k\Sigma'=0$. If these covariants are denoted by d and d, then it was shewn in the paper (1) that

$$Q=\Phi S+3\Theta S'-6d', \qquad Q'=3\Theta' S+\Phi S'-6d.$$

If the pencil of quadrics $S-\lambda S'=0$, assumed to be general, is taken in the canonical form

$$\sum_{i=0}^{3}(k_i-\lambda)x_i{}^2=0, \tag{1}$$

then the equation of the quadric $Q-\lambda Q'=0$ has the form

$$\sum_{i=0}^{3}(q_i-\lambda q'_i)x_i{}^2=0, \tag{2}$$

(1) « Proc. Cambridge Phil. Soc. », 43 (1947), p. 475.

where

(3) $$q_i - \lambda q'_i = D_i \sum_{j \neq i} \frac{k_j - \lambda}{k_i - k_j},$$

and

(4) $$D_i = \prod_{j \neq i} (k_i - k_j).$$

The equation of the quartic combinant is then seen to be

(5) $$SQ' - S'Q = 3 \sum_{i=0}^{3} D_i x_i^4 = 0.$$

It is the purpose of this paper to make the existence of this combinant obvious on geometrical grounds. The basis of the discussion is the theorem proved in the next section.

2. It is well known [2] that if we have a g_4^1 on a twisted cubic curve Γ, then the tetrahedra whose vertices are the various sets of the g_4^1 are all self-polar with respect to a quadric K. We may, however, obtain a second quadric related to the g_4^1 in the following way. If P is a point of Γ, and G a set of the g_4^1, then the polar triad of P with respect to G is cut out on Γ by the polar plane $\pi(P, G)$ of P with respect to the tetrahedron whose vertices are G. If P is fixed and G describes the g_4^1, these planes cut out a g_3^1 on Γ and accordingly belong to a pencil whose axis we may denote by $l'(P)$. If, on the other hand, G remains fixed while P varies, the planes $\pi(P, G)$ equally cut out a g_3^1 on Γ, and so form a pencil with an axis which we shall call $l(G)$. Since $l'(P)$ and $l(G)$ lie in the plane $\pi(P, G)$ they intersect for all P and all G; and accordingly the two sets of lines are the two systems of generators on a quadric K'. The theorem which we shall prove is that, if the g_4^1 is such that it contains the hessian of one of its sets (and therefore contains the hessian of each of its sets), then the quadrics K and K' coincide.

We choose a tetrahedron of reference for a set of homogeneous coordinates $y_0, \ldots, y_3$ to coincide with the set G of the g_4^1, and choose the unit point in such a manner that the first polars of the tetrad G are cut out on Γ by the pencil of planes

(6) $$\sum_{i=0}^{3} (k_i - \lambda) y_i = 0.$$

The pole of the plane (6) with respect to the tetrahedron G is then the point with coordinates

(7) $$y_i = \frac{1}{k_i - \lambda},$$

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[2] See W. F. MEYER, *Apolarität und rationale Curven*, (Tubingen, 1883), p. 206, where the dual theorem is proved, or H. F. BAKER, *Principles of Geometry*, vol. 3, (Cambridge, 1923), p. 139.

and Γ is the locus of this point as λ varies. The tetrad G is then the set of points with parameters given by

$$f(\lambda) \equiv \prod_{i=0}^{3} (k_i - \lambda) = 0, \tag{8}$$

and the g_4^1 is determined on Γ by G and the set H given by the vanishing of the hessian $h(\lambda)$ of $f(\lambda)$. The set H consists of those points of Γ at which the tangent meets the line $l(G)$. Since the PLÜCKER coordinates of the tangent to Γ at the point (7) are given by

$$p_{ij} = \frac{k_i - k_j}{(k_i - \lambda)^2 (k_j - \lambda)^2},$$

while the dual line coordinates of $l(G)$ are given by

$$\pi_{ij} = k_i - k_j, \tag{9}$$

the set H is determined by

$$\sum_{j \neq i} \frac{(k_i - k_j)^2}{(k_i - \lambda)^2 (k_j - \lambda)^2} = 0,$$

and so we may write

$$h(\lambda) \equiv \Sigma\ \pi_{ij}^2 (k_l - \lambda)^2 (k_m - \lambda)^2, \tag{10}$$

where, with a convention to be retained in the following work, the suffixes i, j, l, m form a permutation of the numbers 0, 1, 2, 3.

If, now, $\varphi(\lambda)$ is a cubic in λ, then from the identity

$$\frac{\varphi(\lambda)}{f(\lambda)} \equiv - \sum_{i=0}^{3} \frac{\varphi(k_i)}{f'(k_i)(k_i - \lambda)}$$

we see that the triad of points $\varphi(\lambda) = 0$ is cut out on Γ by the plane with coordinates

$$u_i = - \frac{\varphi(k_i)}{f'(k_i)}. \tag{11}$$

Now the polar triad of the point of Γ with parameter μ with respect to H is given by

$$\varphi(\lambda) = \Sigma\ \pi_{ij}^2 (k_l - \lambda)^2 (k_m - \lambda)(k_m - \mu) = 0.$$

Since

$$f'(k_i) = - \prod_{j \neq i} (k_j - k_i) = \pi_{ij} \pi_{li} \pi_{mi},$$

and

$$\varphi(k_i) = \Sigma\ \pi_{ij}^2 \pi_{li}^2 \pi_{mi} (k_m - \mu),$$

this polar triad is cut out by the plane with coordinates

$$u_i = \Sigma\ \pi_{ji} \pi_{li} (k_m - \mu).$$

Thus, using (9), the line $l(H)$ is the intersection of the pencil of planes

$$\sum_{i=0}^{3} (q_i - \lambda q'_i) y_i = 0, \tag{12}$$

where q_i and q'_i are defined by (3); and (12) is the polar plane of the point (7) with respect to the tetrahedron H. The quadric K' is the locus of the common line of the planes (6) and (12) as λ varies, and a simple calculation shows that its equation reduces to

$$\sum_{i=0}^{3} D_i y_i^2 = 0, \tag{13}$$

where D_i is given by (4). The quadric K' thus has G as a self-polar tetrahedron. Since the hessian of any tetrad of the g_4^1 itself belongs to the g_4^1, it follows that all these tetrahedra are self-polar with respect to K'. Thus K and K' are identical, as was to be proved.

3. A comparison of equations (1), (2) and (5) with (6), (12) and (13) shows that if we consider the (8, 1) correspondence between the spaces in which $(x_0, \dots x_3)$ and $(y_0, \dots, y_3)$ are coordinates, defined by the equations

$$x_i^2 = y_i, \tag{14}$$

then the planes of the pencils, in the y-space, whose axes are the lines $l(G)$ and $l(H)$ correspond to the quadrics in the x-space belongin to the pencils defined respectively by S, S' and Q, Q'; and that the quadric K corresponds to the combinant quartic $SQ' - S'Q = 0$. The relevance of the transformation (14) to the geometry of the quadric pencil may be made more explicit in the following way. Consider a space U of nine dimensions whose points map linearly the quadric envelopes of the space X in which $(x_0, \dots, x_3)$ are coordinates. The degenerate quadric envelopes correspond to the points of a symmetric determinantal primal V_8^4, and the envelopes which degenerate into repeated points are represented by the points of a V_3^8, triple on V_8^4. A prime $\prod$ of U represents a linear ∞^8 system of quadric envelope, which are apolar to a quadric locus S, and if Σ is the envelope of the tangent planes of S then $\prod$ is the polar prime, with respect to V_8^4, of the point of U which maps Σ.

Since the tangential equation of the quadric surface $S - \lambda S' = 0$ involves λ to the third degree, it is easily seen that the points of U which correspond to the envelopes of the quadrics of the pencil defined by S and S', assumed general, lie on a twisted cubic Γ contained in a three-dimensional space Y. If a quadric of the pencil is a cone, its tangential equation reduces to the square of the equation of the vertex of the cone. Thus Γ contains four points of V_3^8, and Y meets V_8^4 in a quartic surface with four non-coplanar triple points, which must degenerate into the four faces of the tetrahedron G determined by these points. To each quadric of the pencil there corresponds, in U, the polar prime of a definite point P of Γ with respect to V_8^4, and this prime meets Y in the polar plane of P with respect to the tetrahedron G. The configuration defined in Y, consisting of the polar planes of the points of Γ with respect to an inscribed tetrahedron G, is thus exactly that described in the last section.

Moreover, the (8, 1) correspondence between the spaces X and Y arises naturally from this point of view. For the points of Y correspond to the quadric envelopes in X which have a common self-polar tetrahedron with S and S', and the planes of Y correspond to quadric loci in X with the same self-polar tetrahedron. In particular, the planes passing through a point of Y represent a net of quadric loci, all apolar to the envelope corresponding to the point, and this net has eight base-points in X, which are precisely the points corresponding to the given point of Y.