A characterization of the bilinear sums associated with the classical second variation.

by

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1. Introduction. — There are two principal reasons why the study of the classical second variation should be subordinated to the more inclusive study of bilinear ⁽¹⁾ and quadratic functionals over abstract spaces. With such a broadening the generalized Euler equations include a vast array of integro-differential equations and permit the introduction into the corresponding spectral theory of new methods incidental to the variational theory in the large. In a deeper sense such a technical quadratic analysis of a characteristic solution gives the first indication of how the topological indices should be attached to the extremal in the general case where the quadratic analysis fails.

The independent variables will be functionals u and v in IL_2 , that is in the class of indefinite L -integrals of functionals in L_2 , with $u(s)$ and $v(s)$ defined over $[0, 1]$. The functionals S will be sums of bilinear ⁽¹⁾ functionals with values $S(u, v)$ defined for $u, v \in IL_2$. The quadratic functional with values $S(u, u)$ will be denoted by $\bar{S}$. Two functionals S and $\bar{S}$ will be termed of the *same type* if there exists a transformation

$$(1.1) \quad u(s) = \varphi(s)\bar{u}(s), \quad v(s) = \varphi(s)\bar{v}(s), \quad [\bar{u}, \bar{v} \in IL_2]$$

of the class IL_2 into itself in which φ is in IL_2 and $\varphi(s) \neq 0$ over $[0, 1]$ while under (1.1), $\bar{S}(\bar{u}, \bar{v}) = S(u, v)$.

Let $E(u, v)$ be a symmetric form bilinear in the two sets of variables $[u(0), u(1)]$ and $[v(0), v(1)]$. We term E an *end form*. Our starting point is the

⁽¹⁾ A functional F is bilinear in our sense over the product $A \times B$ of two «pseudo-normed» linear spaces A and B , if F has values $F(x, y)$ defined for each $x \in A$ and $y \in B$, is additive and homogeneous over A for fixed y and over B for fixed x , and satisfies a relation

$$|F(x, y)| \leq |x|_A |y|_B M \quad [M \text{ a constant}].$$

Here $|x|_A$ is a norm of x in the ordinary sense (Cf. BANACH, p. 53) except that we do not demand that u be the null element in the linear space A when $|u|_A = 0$. For this reason $|u|_A$ is termed a *pseudo-norm* over A . We suppose that $|y|_B$ is a pseudo-norm over B in the same sense. The space L_2 is the linear space of measurable functions x defined over $[0, 1]$ with pseudo-norms

$$\left[\int_0^1 x^2(s) ds \right]^{\frac{1}{2}}.$$

bilinear sum

$$(1.2) \quad J(u, v; P, Q, R, E) = \int_0^1 [R(s)\dot{u}(s)\dot{v}(s) + Q(s)(u(s)\dot{v}(s) + v(s)\dot{u}(s)) + P(s)u(s)v(s)]ds + E(u, v)$$

associated with the classical variation with J essentially non-singular in a sense to be defined later in this section. $J(u, v; P, Q, R, E)$ is the sum of functionals of the form

$$(1.3) \quad \int_0^1 R(s)x(s)y(s)ds, \quad [x, y \in L_2]$$

$$(1.4) \quad \int_0^1 Q(s)u(s)y(s)ds,$$

$$(1.5) \quad \int_0^1 P(s)u(s)v(s)ds + E(u, v), \quad [u, v \in C]$$

after obvious substitutions. The functionals (1.3), (1.4) and (1.5) are bilinear over

$$(1.6) \quad L_2 \times L_2, \quad C \times L_2, \quad C \times C,$$

respectively. (Cf. GOLDSTINE where functionals bilinear over $C \times C$ are considered).

The most general functional F bilinear over $A \times B$, where $A \times B$ is any one of the products (1.6), is representable as a LEBESGUE-STIELTJES integral (L - S -integral)

$$(1.7) \quad F(x, y, k) = \int_0^1 x(s)ds \int_0^1 y(t)dk(s, t) = \int_0^1 y(s)ds \int_0^1 x(t)dk(t, s)$$

for (x, y) in $A \times B$, where k is canonical (Cf. § 2) relative to the square $[0, 1] \times [0, 1]$ ⁽²⁾ and has a finite variation $h(A, B; k)$ defined in § 2. For the general case see MT 2 and 3. For the first results when $A = B = C$, see FRÉCHET. A k which is canonical relative to $[0, 1] \times [0, 1]$ with $h(A, B; k) < \infty$ is termed a d -function (*distribution function*) relative to $A \times B$. Let p, q, r be d -functions relative to $C \times C$, $C \times L_2$, and $L_2 \times L_2$ respectively.

As a generalization of J we consider bilinear sums $S(p, q, r)$ with values

$$(1.8) \quad S(u, v; p, q, r) = F(\dot{u}, \dot{v}; r) + F(u, \dot{v}; q) + F(v, \dot{u}; q) + F(u, v; p) \\ \text{[for } u, v \in IL_2].$$

We suppose that $r = r^*$ and $p = p^*$ where

$$r^*(s, t) = r(t, s), \quad p^*(s, t) = p(t, s).$$

As a functional over $IL_2 \times IL_2$, $S(p, q, r)$ has infinitely many other representations $S(p', q', r')$ such that

$$(1.8)' \quad S(u, v; p, q, r) = S(u, v; p', q', r') \quad [u, v \in IL_2].$$

See MT 4 and § 11.

⁽²⁾ References to MORSE and TRANSUE are denoted by MT and numbered from 1 to 5.

We seek necessary and sufficient conditions on the « extremals » of $\widehat{S}(p, q, r)$ that

$$(1.9) \quad S(u, v; p, q, r) = J(u, v; P, Q, R, E) \quad [u, v \in IL_2]$$

for suitable choice of P, Q, R and end form E .

The extremal properties $(\alpha), (\beta), (\gamma)$. We naturally admit only those functionals P, Q, R which are measurable over $[0, 1]$. In § 6 it is shown that necessary and sufficient conditions that the integrals (1.3), (1.4) and (1.5) exist over the respective products (1.6) are that $|R(s)|$ be bounded p. p. (presque partout), that Q be in L_2 and P in L . When J is given, these conditions are assumed. When J is derived from $S(p, q, r)$ as in § 10, these conditions are established.

To define extremal property (α) , let $[a, b]$ be any subinterval of $[0, 1]$. Set

$$(1.10) \quad F_a^b(x, y, k) = \int_a^{b-} x(s) d_s \int_a^{b-} y(t) d_t k(s, t) = \int_a^{b-} y(s) d_s \int_a^{b-} x(t) d_t k(t, s)$$

for $b < 1$, with $b-$ replaced by 1 when $b = 1$. The limits $a, b-$ indicate integration over the interval $[a, b]$, including a but excluding b . Set

$$(1.11) \quad S_a^b(u, v; p, q, r) = F_a^b(\dot{u}, \dot{v}; r) + F_a^b(u, \dot{v}; q) + F_a^b(v, \dot{u}; q) + F_a^b(u, v; p)$$

for u, v in $IL_2[a, b]$ (that is in the class of indefinite integrals over $[a, b]$ of functionals integrable square over $[a, b]$). Let $J_a^b(u, v; P, Q, R, E)$ similarly denote the right member of (1.2) with the limits of integration 0, 1 replaced by a, b , and E replaced by an end form E_a^b in $[u(a), u(b)]$ and $[v(a), v(b)]$ to be defined in § 5. See Theorem 5.2.

A generalized Euler equation for $\widehat{S}_a^b$ will be defined in § 4. An extremal of $\widehat{S}_a^b$ will be any solution of this equation for which u is in $IL_2(a, b)$. When $S \equiv J$ over $IL_2 \times IL_2$ the Euler equations of $\widehat{S}$ in « normal » form take a classic form. In case $1/R$ and Q/R are in L_2 , J is termed *essentially non-singular*. The extremals of $\widehat{J}$ have the following property (α) . When J is essentially non-singular the properties (β) and (γ) also hold. Cf. § 6.

(α) . Each extremal u of $\widehat{J}$, with $u(s)$ taken over $[a, b]$ defines an extremal $^{(3)}$ of $\widehat{J}_a^b$.

(β) . For each point c in $[0, 1]$ there is at least one extremal u_c such that $u_c(c) = 1$.

(γ) . There is at least one pair of extremals u, v whose Wronskian $u(s)v(s) - \dot{u}(s)\dot{v}(s) = W(s)$ exists with $W(s) \neq 0$ p.p. on $[0, 1]$, but not every $u \in IL_2$ is an extremal.

That neither (α) , (β) , nor (γ) holds in general when $\widehat{J}$ is replaced by $\widehat{S}(p, q, r)$ will be shown by example in § 12. The properties (α) , (β) , (γ) , read

⁽³⁾ An extremal u of $\widehat{J}$ is defined over $[0, 1]$ since $J = J_0^1$ by definition. It is understood that $[a, b]$ in (α) is an arbitrary subinterval of $[0, 1]$.

with $\widehat{S}(p, q, r)$ replacing $\widehat{J}$, will be shown to be independent. This becomes more transparent when it is seen that the Euler equations of $\widehat{S}(p, q, r)$ include integro-differential equations of very general types.

Our final theorems will be given for the case in which $q(s, t) \equiv 0$. Little if any generality is lost by this restriction. In case $q(s, t) \equiv 0$ we shall write $S(u, v; p, 0, r) = S(u, v; p, r)$. It is easy to show that in the essentially non-singular case every J admits a representation $S(p, r)$ after a suitable transformation (1.1). Even when no transformation is used, $S(p, q, r)$ still admits a representation of the form $S(p', r')$ provided only that the t -derivative of $q(1, t)$, $0 \leq t \leq 1$, gives the values p.p. of a function of bounded variation (See § 11). Since the t -derivative of $q(1, t)$ over $[0, 1]$ necessarily defines a function in L_2 , our restriction is slight.

We say that $S(p, r)$ satisfies (α) , (β) , (γ) , if (α) , (β) , (γ) hold when $\widehat{S}$ is read in place of $\widehat{J}$.

A first theorem follows.

THEOREM 1.1. - *A bilinear sum $S(p, r)$ which satisfies (α) , (β) , (γ) equals a suitably chosen essentially non-singular J for every u, v in IL_2 .*

A d -function k will be said to be a triangle d -function if $k = k^*$, and if

$$k(s, t) = k(s, 1) \quad [0 \leq s \leq t \leq 1].$$

In MT 5 it has been shown that an admissible bilinear sum $S(p, r)$ has a unique representation $S(p', r')$ in which p' is a triangle d -function. With p' a triangle d -function, $S(p', r')$ is termed a *normal representation*. Our final theorem then reads as follows.

THEOREM 1.2. - *A necessary and sufficient condition that a bilinear sum $S(p, r)$ equal a suitably chosen essentially non-singular J for every u, v in IL_2 is that the corresponding normal representation of $S(p, r)$ satisfy (α) , (β) , (γ) .*

We have here a characterization of an essentially non-singular J in the class of bilinear sums $S(p, r)$ in terms of extremals.

2. The representation of bilinear functionals. — In this section we recall those results of MT 2 and 3 which will be useful in the present study. The general pseudo-normed vector spaces A and B of MT 2 are here taken as the space C or the space L_2 with corresponding simplifications of the general theory. In MT 2 a normed linear space A_d has been associated with A . The space A_d is defined as follows when $A = C$ or L_2 .

The space A_d . Let g be a functional mapping R_1 into R_1 . In the terminology of MT 2, g is canonical relative to the interval $[a, b]$ if $g(s) = 0$ for $s \leq a$ and $g(s) = g(b)$ for $s \geq b$, and if g is left continuous except at most at the point $s = b$. We admit partitions π of $[a, b]$ with vertices

$$(2.1) \quad a = s_0 < s_1 < \dots < s_{r(\pi)} = b,$$

and with intervals

$$(2.2) \quad [s_0, s_1), \quad [s_1, s_2), \dots, \quad [s_{r(\pi)-1}, b],$$

closed at the left end point in all cases, and closed at the right end point only in the case of the last interval. A step-function (s-function) *associated* with π and *belonging* to $[a, b]$ is a function η mapping R_i into R_i , with $\eta(s) = 0$ for $s \in [a, b]$, and $\eta(s)$ a constant η_r on the r th of the intervals (2.2).

When A is the space C of functionals continuous over $[0, 1]$, A_a shall be the normed linear space of functionals g canonical relative to $[0, 1]$, with finite total variation $T(g)$ and norm

$$(2.3) \quad \|g\|_{A_a} = T(g).$$

This is a norm and not merely a pseudo-norm. When $A = L_2$, A_a shall be the normed linear space of all functionals g canonical relative to E with finite HELLINGER integrals (Cf. HOBSON, vol. I, p. 673),

$$(2.4) \quad I_2(g) = \int_0^1 \frac{[dg]^2}{ds},$$

and norm

$$(2.5) \quad \|g\|_{A_a} = [I_2(g)]^{\frac{1}{2}}.$$

When g is canonical but not in A_a we understand that $\|g\|_{A_a} = \infty$. In this case (2.3) and (2.5) still hold for $A = C$ and L_2 respectively.

In characterizing elements in A_a in the next lemma use will be made of the difference

$$\Delta_r(g) = g(s_r) - g(s_{r-1}), \quad [r = 1, \dots, r(\pi)]$$

corresponding to the r th interval of an admissible partition π of $[0, 1]$. Throughout this paper use will be made of the convention of tensor algebra, whereby a repeated index r implies a summation over the range of the index.

LEMMA 2.1. - A necessary and sufficient condition that a functional g which is canonical relative to E be in A_a is that

$$(2.6) \quad \sup_{\eta} |\eta_r \Delta_r(g)| < \infty, \quad [\|\eta\|_A \leq 1]$$

where the sup is taken over all s-functions associated with partitions of $[0, 1]$ for which $\|\eta\|_A \leq 1$. The sup in (2.6) equals $\|\eta\|_{A_a}$ whether finite or not. (Cf. MT 2, § 5).

An s-function η belonging to $[0, 1]$ is not in C in general. Nevertheless it will be convenient to write

$$\|\eta\|_C = \max_s |\eta(s)|.$$

The norm of η in L_2 is well defined since η is in L_2 .

The A-B-variation $h(A, B; k)$ of k . Let the product space $R_2 = R_1 \times R_1$ be referred to Cartesian coordinates (s, t) . Let k map R_2 into R_1 . Let E' and E'' respectively represent the interval $[0, 1]$ on the s - and t -axes. We say that k

is canonical relative to $E' \times E''$ if ⁽⁴⁾ $k(s, \cdot)$ is canonical relative to E'' for each fixed s and $k(\cdot, t)$ is canonical relative to E' for each fixed t . Let π' be an admissible partition of E' defined by vertices

$$0 = s_0 < s_1 < s_2 < \dots < s_{r(\pi')} = 1,$$

and let π'' be a similar partition of E'' with vertices

$$0 = t_0 < t_1 < t_2 < \dots < t_{n(\pi'')} = 1.$$

Let η and ζ be s -functions associated respectively with π' and π'' . Set

$$\Delta_{r,n}(k) = k(s_r, t_n) - k(s_{r-1}, t_n) - k(s_r, t_{n-1}) + k(s_{r-1}, t_{n-1})$$

for $r = 1, \dots, r(\pi')$ and $n = 1, \dots, n(\pi'')$. If $A \times B$ is any one of the products

$$(2.7) \quad C \times C, \quad C \times L_2, \quad L_2 \times C, \quad L_2 \times L_2,$$

the A - B -variation of a functional k which is canonical relative to $E' \times E''$ is well defined (summing as to r and n) as

$$(2.7)' \quad h(A, B; k) = \sup_{\eta, \zeta} \eta_r \zeta_n \Delta_{r,n}(k), \quad [r = 1, \dots, r(\pi'); n = 1, \dots, n(\pi'')]$$

taking the sup over all η and ζ respectively associated with arbitrary admissible partitions π' and π'' of E' and E'' with $|\eta|_A \leq 1$, $|\zeta|_B \leq 1$. The A - B -variation of k may be infinite. When k is canonical relative to $E' \times E''$ and $h(A, B; k) < \infty$, k is termed a h -function (distribution function) relative to $A \times B$.

To state the fundamental properties of $h(A, B; k)$ set

$$(2.8)' \quad k(s_r, t) - k(s_{r-1}, t) = \Delta_r^{(1)}[k(\cdot, t)],$$

$$(2.8)'' \quad k(s, t_n) - k(s, t_{n-1}) = \Delta_n^{(2)}[k(s, \cdot)],$$

the superscript (1) or (2) indicating that the first or second argument respectively is differenced. Let the functional mapping R_1 into R_1 whose value at t is $\Delta_r^{(1)}[k(\cdot, t)]$ be denoted by $\Delta_r^{(1)}(k)$, and the functional whose value at s is $\Delta_n^{(2)}[k(s, \cdot)]$ be denoted by $\Delta_n^{(2)}(k)$. The following lemma results from MT 2, (7.8) and (7.5) noting that $M_A(s)$ and $M_B(t)$ may be taken as 1.

LEMMA 2.2. - For k canonical relative to $E' \times E''$,

$$h(A, B; k) = \sup_{\eta} |\eta_r \Delta_r^{(1)}(k)|_{B_d} = \sup_{\zeta} |\zeta_n \Delta_n^{(2)}(k)|_{A_d},$$

taking the sups over all admissible s -functions η and ζ for which $|\eta|_A \leq 1$ and $|\zeta|_B \leq 1$. For arbitrary s and t ,

$$(2.9) \quad |k(s, \cdot)|_{B_d} \leq h(A, B; k),$$

$$(2.10) \quad |k(\cdot, t)|_{A_d} \leq h(A, B; k).$$

⁽⁴⁾ If F is a function over a product $A \times B$ of space A and B , with the value $F(u, v)$ at $u \in A$, $v \in B$, then $F(u, \cdot)$ shall denote the function over B with values $F(u, v)$ for fixed u and arbitrary $v \in B$. The function $F(\cdot, v)$ is similarly defined.

The relations in this lemma are valid even when one or more of the members are infinite. In the case in which the space $B = C$, (2.9) implies that

$$(2.11) \quad T[k(s, \cdot)] \leq h(A, C; k),$$

and a similar consequence of (2.10) appears when $A = C$. When $B = L_2$, (2.9) implies that the HELLINGER integral

$$(2.12) \quad I_2[k(s, \cdot)] \leq h^*(A, B; k),$$

and so is finite with $h(A, B; k)$ for each fixed s . In particular, $k(s, \cdot)$ is then absolutely continuous for each fixed s . No a priori assumption as to the measurability of $k(s, \cdot)$ or $k(\cdot, t)$ for fixed s and t respectively has been made or required. These functionals are (at the least) of bounded variation when $h(A, B; k) < \infty$ and $A \times B$ is one of the products (2.7).

Given a canonical function k we shall set $k(t, s) = k^*(s, t)$. Then k^* is again canonical relative to $E' \times E''$ with k . It follows from the definition of A - B -variation that

$$(2.13) \quad h(A, B; k) = h(B, A; k^*).$$

The principal theorem of MT 2 on the representation of bilinear functionals follows. For the purposes of this study admissible products $A \times B$ are those listed in (2.7).

THEOREM. 2.1. - *A functional Φ bilinear over an admissible product $A \times B$ has a value for $x \in A$ and $y \in B$ given by an L-S-integral*

$$(2.14) \quad \Phi(x, y) = \int_0^1 x(s) d_s \int_0^1 y(t) d_t k(s, t) = \int_0^1 y(s) d_s \int_0^1 x(t) d_t k(t, s),$$

where k is canonical relative to $E' \times E''$, $h(A, B; k) < \infty$, and

$$(2.15) \quad h(A, B; k) = \sup_{x, y} \frac{|\Phi(x, y)|}{|x|_A |y|_B} \quad [|x|_A > 0, |y|_B > 0].$$

Conversely if k is canonical relative to $E' \times E''$ and $h(A, B; k) < \infty$, then the integrals in (2.14) define a functional Φ bilinear over $A \times B$ for which (2.15) holds. Given Φ bilinear over $A \times B$ there is but one d-function k relative to $A \times B$ for which (2.15) holds.

It follows from the definition of $h(A, B; k)$ that

$$h(A, B; k + k') \leq h(A, B; k) + h(A, B; k')$$

for any two functionals k and k' which are canonical relative to $E' \times E''$. If $h(A, B; k) < \infty$, it is clear that

$$h(A, B; ck) = |c| h(A, B; k)$$

for any constant c .

In the notation of § 1, let p, q, r be d -functions relative to $C \times C, C \times L_2$, and $L_2 \times L_2$ respectively. In accordance with Lemma 2.2 we note the following implications:

$$(2.16) \quad h(C, C; p) < \infty \rightarrow p(s, \cdot) \in C_d; \quad p(\cdot, t) \in C_d,$$

$$(2.17) \quad h(C, L_2; q) < \infty \rightarrow q(s, \cdot) \in L_{2d}; \quad q(\cdot, t) \in C_d,$$

$$(2.18) \quad h(L_2, L_2; r) < \infty \rightarrow r(s, \cdot) \in L_{2d}; \quad r(\cdot, t) \in L_{2d};$$

provided p, q, r are assumed canonical relative to $E' \times E''$.

Transformations Φ_y from B to A_d . Let k be a d -function relative to $A \times B$. Given $y \in B$, the L - S -integral

$$\int_0^1 y(t) d_t k(s, t) = \Phi_y(s)$$

defines a functional $\Phi_y \in A_d$ as affirmed in § 9 of MT 2, Lemma 9.4. Particular implications are as follows. Assuming p, q, r , canonical relative to $E' \times E''$,

$$(2.19)' \quad h(C, C; p) < \infty; \quad u \in C; \quad \Phi_u(s) = \int_0^1 u(t) d_t p(s, t) \rightarrow [\Phi_u \in C_d].$$

$$(2.19)'' \quad h(C, L_2; q) < \infty; \quad u \in C; \quad \Phi_u(s) = \int_0^1 u(t) d_t q(t, s) \rightarrow [\Phi_u \in L_{2d}],$$

$$(2.19)''' \quad h(C, L_2; q) < \infty; \quad y \in L_2; \quad \Phi_y(s) = \int_0^1 y(t) d_t q(s, t) \rightarrow [\Phi_y \in C_d],$$

$$(2.19)'''' \quad h(L_2, L_2; r) < \infty; \quad y \in L_2; \quad \Phi_y(s) = \int_0^1 y(t) d_t r(s, t) \rightarrow [\Phi_y \in L_{2d}].$$

Recall that a functional g mapping R_i into R_i is in C_d if and only if it is canonical relative to $[0, 1]$ and of bounded variation; and in L_{2d} if and only if it is canonical relative to $[0, 1]$ and has a finite HELLINGER integral $I_2(g)$. Finally $I_2(g) < \infty$ if and only if g is in IL_2 , or, if one pleases, if and only if g is a. c. and $|g|^2$ integrable.

We need the relation

$$(2.20) \quad h(C, C; k) \leq h(A, B; k)$$

for each admissible product $A \times B$. It follows from the definition of $h(A, B; k)$ in (2.7)' that

$$(2.21) \quad |\eta_r \zeta_n \Delta_{r,n}(k)| \leq |\eta|_A |\zeta|_B h(A, B; k).$$

The spaces A and B are L_2 or C . In either case

$$|\eta|_A \leq |\eta|_C, \quad |\zeta|_B \leq |\zeta|_C,$$

so that

$$(2.22) \quad |\eta_r \zeta_n \Delta_{r,n}(k)| \leq |\eta|_C |\zeta|_C h(A, B; k).$$

It follows from the definition of $h(C, C; k)$ that (2.20) holds.

Left continuous d-functions. It has been shown in MT 1, Lemma 5.1 that a d -function k for which $h(C, C; k) < \infty$ has the following property. If (a, b) is any point of $E' \times E''$, then with $s > 0$ and $t > 0$,

$$(2.23) \quad \lim_{(s, t) \rightarrow (0, 0)} k(a - s, b - t) = k_{ab}$$

exists. We shall say that k is *left continuous* at (a, b) if

$$(2.23') \quad \lim_{(s, t) \rightarrow (0, 0)} k(a - s, b - t) = k(a, b) \quad [(s, t) \neq (0, 0)]$$

for any approach of (s, t) to $(0, 0)$ for which $s \geq 0, t \geq 0$. The d -functions k which we are admitting are, by definition of a d -function, left continuous in each argument separately at each point not on the lines $s = 1$ and $t = 1$, and hence, using (2.23), left continuous in the above sense at each point (s, t) not on the lines $s = 1$ and $t = 1$. The following lemma will be useful in § 9.

LEMMA 2.3. - *With each admissible d -function k there is associated a unique d -function $\bar{k}$ which is left continuous at each point in R_2 and such that $\bar{k}(s, t) = k(s, t)$ for $s < 1, t < 1$. Moreover*

$$(2.24) \quad h(C, C; \bar{k}) \leq h(C, C; k).$$

The function $\bar{k}$ is defined by requiring that $\bar{k}(s, t) = k(s, t)$ for $s < 1, t < 1$, that $\bar{k}(s, t) = k_{st}$ for $s = 1$ or $t = 1$, and extending $\bar{k}$ for $s > 1$ or $t > 1$ so as to be canonical relative to $E' \times E''$. Cf. Lemma 6.1, MT 2.

The $\bar{k}$ so defined is unique. That it satisfies (2.24) is seen as follows. With $0 < \lambda < 1$, set

$$k[\lambda s, \lambda t] = k^{(\lambda)}(s, t) \quad [(s, t) \in E' \times E''].$$

It is clear from the definition of $h(C, C; k)$ that

$$h(C, C; k) \geq h(C, C; k^{(\lambda)}).$$

Observe that

$$\lim_{\lambda \rightarrow 1-} k^{(\lambda)}(s, t) = \bar{k}(s, t) \quad [(s, t) \in E' \times E''].$$

From the lower semi-continuity of the variation $h(C, C; k)$ of a d -function (MT 1, Lemma 3.2),

$$\lim_{\lambda \rightarrow 1-} h(C, C; k^{(\lambda)}) \geq h(C, C; \bar{k}).$$

Hence (2.24) holds as stated.

Extensions of terminology. It is clear that the interval $[0, 1]$ can be replaced by an arbitrary interval $[a, b]$ and that the square $E' \times E''$ can be replaced by $[a, b] \times [a, b]$. The spaces C and L_2 will then be replaced by spaces $C[a, b]$ and $L_2[a, b]$ of functionals continuous over $[a, b]$ and integrable square over $[a, b]$ with the appropriate pseudo-norms. Functionals k

mapping R_2 into R_1 and canonical relative to $[a, b] \times [a, b]$ will be defined by obvious extension of terminology as well as d -functions relative to products

$$C[a, b] \times C[a, b], \quad C[a, b] \times L_2[a, b], \quad L_2[a, b] \times L_2[a, b].$$

Functionals bilinear over these products will have L - S -integral representations similar to those already described, with the limits a, b replacing 0, 1.

3. The bilinear sum S . — Let p, q, r be d -functions relative to $C \times C$, $C \times L_2$, and $L_2 \times L_2$ as previously, and let S and $\bar{S}$ be defined as in § 1, for $u, v \in IL_2$.

We shall suppose that $r = r^*$ and $p = p^*$. Cf. (2.13). No generality will thereby be lost for our purposes. For we are principally concerned with the quadratic sum $\bar{S}$, and any simplification which does not alter the values of $\bar{S}$ is admissible. In case $r \neq r^*$ we would make use of the relations

$$(3.1) \quad F(\dot{u}, \dot{v}; r) = F(\dot{v}, \dot{u}; r^*), \quad F(\dot{u}, \dot{u}; r) = F(\dot{u}, \dot{u}; r^*).$$

Then for $2\bar{r} = r + r^*$,

$$F(\dot{u}, \dot{u}; \bar{r}) = F(\dot{u}, \dot{u}; r),$$

and $\bar{r}$ is symmetric, that is $\bar{r} = \bar{r}^*$. It must be noted that r^* is a d -function relative to $L_2 \times L_2$ when r is [Cf. (2.13)]. Similar remarks hold for p relative to $C \times C$. One can thus replace r and p by symmetric d -functions without altering any value of $\bar{S}$. With $r = r^*$ and $p = p^*$ note that

$$(3.2) \quad S(u, v; p, q, r) = S(v, u; p, q, r).$$

A transformation of S . We shall obtain a formula for $S(u, v; p, q, r)$ which is the analogue of a classical transformation of the second variation. The validity of this formula depends definitely upon the right to change the order of integration in the repeated L - S -integrals used.

For brevity set

$$(3.3) \quad \int_0^1 u(t) d_t p(s, t) = h_u(s),$$

$$(3.4) \quad \int_0^1 u(t) d_t q(s, t) = f_u(s),$$

recalling that these integrals exist, that h_u and f_u , for fixed u are in C_a , and in particular are of bounded variation and canonical. Cf. (2.19). From (3.3) and (3.4) respectively,

$$(3.5) \quad F(u, v; p) = \int_0^1 v(s) d_s h_u(s),$$

$$(3.6) \quad F(v, \dot{u}; q) = \int_0^1 v(s) d_s f_u(s).$$

where use has been made of the right to interchange the order of integration in the repeated integral defining $F(u, v; p)$. We shall integrate the right member of (3.6) by parts using a formula given by Saks, p. 102. In applying this formula one must note that

$$f_u(1+) = f_u(1), \quad f_u(0-) = f_u(0).$$

Saks' formula then gives

$$\begin{aligned} (3.7) \quad F(v, u; q) &= [v(s)f_u(s)]_0^1 - \int_0^1 \dot{v}(s)f_u(s)ds = \\ &= [v(s)f_u(s)]_0^1 - \int_0^1 \dot{v}(s)ds \int_0^s f_u(a)da. \end{aligned}$$

Similarly,

$$\begin{aligned} (3.8) \quad F(u, v; p) &= [v(s)h_u(s)]_0^1 - \int_0^1 \dot{v}(s)h_u(s)ds, \\ &= [v(s)h_u(s)]_0^1 - \int_0^1 \dot{v}(s)ds \int_0^s h_u(a)da. \end{aligned}$$

Note also that after change of order of integration,

$$(3.9)' \quad F(u, \dot{v}; q) = \int_0^1 \dot{v}(s)ds \int_0^1 u(t)d_t q(t, s),$$

$$(3.9)'' \quad F(\dot{u}, \dot{v}; r) = \int_0^1 \dot{v}(s)ds \int_0^1 \dot{u}(t)d_t r(s, t).$$

For $u \in IL_2$ and arbitrary s set

$$\begin{aligned} (3.10) \quad \phi(u, s; p, q, r) &= \int_0^1 \dot{u}(t)d_t r(s, t) + \int_0^1 u(t)d_t q(t, s) - \\ &- \int_0^s d\alpha \int_0^1 \dot{u}(t)d_t q(\alpha, t) - \int_0^s d\alpha \int_0^1 u(t)d_t p(\alpha, t). \end{aligned}$$

Equations (3.7) to (3.10) give the following:

THEOREM 3.1. - For u and $v \in IL_2$,

$$(3.11) \quad S(u, v; p, q, r) = \int_0^1 \dot{v}(s)ds \phi(u, s; p, q, r) + [v(s)\{f_u(s) + h_u(s)\}]_0^1.$$

It should be noted that the first two integrals in ϕ are in $L_2 a$, the second two integrals in IC_a (that is integrals of a functional in C_a), and that f_u and h_u are in C_a .

COROLLARY 3.1. - If $v(0) = v(1) = 0$, then

$$(3.12) \quad S(u, v; p, q, r) = \int_0^1 \dot{v}(s)ds \phi(u, s; p, q, r).$$

We term ϕ the *Euler operator* belonging to $\widehat{S}$. Given $\widehat{S}(p, q, r)$, the image of u in IL_2 under ϕ is $\phi(u, \cdot; p, q, r)$.

4. **The first variation of $\widehat{S}$.** - Given $u \in IL_2$, the first variation of $\widehat{S}$ based on u will be defined.

Let η be a function mapping $[0, 1]$ into R_1 , of class C' over $[0, 1]$. Let e be a positive constant. Dropping p, q, r , observe that

$$\begin{aligned} \widehat{S}(u + e\eta) &= S(u + e\eta, u + e\eta) = \\ &= \widehat{S}(u) + 2eS(\eta, u) + e^2\widehat{S}(\eta), \\ (4.0) \quad \left. \frac{\partial}{\partial e} \widehat{S}(u + e\eta) \right|_{e=0} &= 2S(\eta, u). \end{aligned}$$

DEFINITION. - For fixed $u \in IL_2$, $2S(\eta, u)$ is termed the first variation of $\widehat{S}(u)$ based on u and evaluated for the variation η .

It follows from Corollary 3.1 that when $\eta(0) = \eta(1) = 0$,

$$(4.1) \quad 2S(\eta, u) = 2 \int_0^1 \eta(s) d\phi(u, s).$$

The following lemma is well known in case the integral is a RIEMANN-STIELTJES integral. For the following variant a proof is indicated. For the reader acquainted with the fact that the L - S -integral here used equals the R - S -integral, no proof is necessary.

LEMMA 4.0. - If g is of bounded variation over $[0, 1]$, defined over a neighborhood of $[0, 1]$ in R_1 and left continuous except at most at $s = 1$ and there right continuous, if further the L - S -integral

$$(4.2) \quad \int_0^1 z(s) dg(s) = 0$$

for every $z \in C$, then $g(s)$ is constant over $[0, 1]$.

Let k^a be the characteristic function of $[0, a)$ when $0 < a < 1$ and of $[0, a]$ when $a = 1$. Let e be such that $e > 0$, $a - e \geq 0$, and let k_e^a map $[0, 1]$ continuously into R_1 with $k_e^a(s) = k^a(s)$ except on the interval $(a - e, a)$ when $0 < a < 1$. Take $k_e^a(s)$ linear in s over $[a - e, a]$ when $0 < a < 1$. Then $k_e^a(s) \rightarrow k^a(s)$ for each $s \in [0, 1]$ as $e \rightarrow 0$. Now z can be taken as k_e^a in (4.2), and by the theorem on bounded convergence

$$\lim_{e \rightarrow 0} \int_0^1 k_e^a(s) dg(s) = \int_0^1 k^a(s) dg(s) = 0.$$

Because of the left continuity of g at $s = a$ when $a \neq 1$, and right continuity when $a = 1$,

$$\int_0^1 k^a(s) dg(s) = g(a) - g(0) = 0, \quad [0 < a \leq 1]$$

thus establishing the lemma.

The form of the integral in (4.1) suggests the following lemma.

LEMMA. 4.1. - *If g satisfies the hypotheses of Lemma 4.0 and if the L-S-integral*

$$(4.3) \quad \int_0^1 \dot{\eta}(s) dg(s) = 0$$

for every η of class C' on $[0, 1]$ with $\eta(0) = \eta(1) = 0$, then g is linear over $[0, 1]$.

Set $L(s) = sg(1) + (1-s)g(0)$. Then

$$(4.4) \quad \int_0^1 \dot{\eta}(s) d[g(s) - L(s)] = \int_0^1 \dot{\eta}(s) dg(s) - [\eta(s)\dot{L}(s)]_0^1 = 0$$

for every admissible η . Moreover,

$$\int_0^1 c d[g(s) - L(s)] = c[g(s) - L(s)]_0^1 = 0$$

for every constant c . Any $z \in C$ can be represented in the form

$$z(s) = c + \dot{\eta}(s),$$

where η is in C' and $\eta(0) = \eta(1) = 0$. In fact, given z , set

$$\eta(s) = \int_0^s [z(t) - c] dt,$$

and choose c so that $\eta(1) = 0$. With this choice of c and η ,

$$(4.5) \quad \int_0^1 z(s) d[g(s) - L(s)] = \int_0^1 \dot{\eta}(s) d[g(s) - L(s)] = 0$$

by virtue of (4.4). Thus the first integral in (4.5) vanishes for every $z \in C$. It follows from Lemma 4.0 that $g(s) - L(s)$ is constant over $[0, 1]$. Since $g(0) - L(0) = 0$, we conclude that $g(s) = L(s)$ over $[0, 1]$, thus establishing the lemma.

DEFINITION. - *A functional $u \in \Pi L_2$ such that the first variation $2S(\eta, u) = 0$ for every $\eta \in C'$ for which $\eta(0) = \eta(1) = 0$ is called an extremal of $\widehat{S}$.*

THEOREM 4.1. - *A necessary and sufficient condition that a $u \in \Pi L_2$ be an extremal of $\widehat{S}$ is that*

$$(4.6) \quad \psi(u, s) = s\psi(u, 1)$$

for every $s \in [0, 1]$.

The condition (4.6) is sufficient that u be an extremal of $\widehat{S}$. For it follows from Corollary 3.1 that

$$S(\eta, u) = \int_0^1 \dot{\eta}(s) d_s \psi(u, s) = 0$$

when (4.6) holds, when η is in C' and $\eta(0) = \eta(1) = 0$.

The condition (4.6) is necessary. In fact, if u is an extremal of $\widehat{S}$, then by definition of an extremal and Corollary 3.1,

$$(4.7) \quad 0 = S(\eta, u) = \int_0^1 \eta(s) d_s \psi(u, s)$$

for every $\eta \in C'$ for which $\eta(0) = \eta(1) = 0$. Lemma 4.1 can be applied, for $\psi(u, \cdot)$ is continuous over R_1 and of bounded variation over $[0, 1]$. It follows from Lemma 4.1 that $\psi(u, \cdot)$ is linear over $[0, 1]$. But $\psi(u, 0) = 0$, so that (4.6) must hold.

Referring to (3.10) set

$$(4.8) \quad \psi_1(u, s) = \int_0^1 u(t) d_t r(s, t) + \int_0^1 u(t) d_t q(t, s),$$

$$(4.9) \quad \psi_2(u, s) = \int_0^s da \int_0^1 u(t) d_t q(a, t) + \int_0^s da \int_0^1 u(t) d_t p(a, t),$$

and recall that $\psi(u, s) = \psi_1(u, s) - \psi_2(u, s)$. Recall also that $\psi_1(u, \cdot)$ is in L_{2a} and that $\psi_2(u, \cdot)$ is in IC_a whether u is an extremal or not. Hence the following corollary of the theorem:

COROLLARY 4.1. - *If u is an extremal of $\widehat{S}$ $\psi_1(u, \cdot)$ as well as $\psi_2(u, \cdot)$ is in IC_a .*

The following lemma will be useful in § 10.

LEMMA 4.2. - *If f is of bounded variation over the closed interval $[0, 1]$ and left continuous except at most at $s = 1$ and if the L-S-integral*

$$(4.10) \quad \int_0^1 \eta(s) df(s) = 0$$

for every η of class C with $\eta(0) = \eta(1) = 0$, then $f(s)$ is constant over the open interval $(0, 1)$.

Let $0 < a < b < 1$. In terms of the functions k_e^a of the proof of Lemma 4.0, the difference

$$k_e^b(s) - k_e^a(s) = \eta(s) \quad [0 \leq s \leq 1]$$

defines an η admissible in the present lemma if e is sufficiently small. As in the proof of Lemma 4.0, it follows that

$$\int_0^1 [k^b(s) - k^a(s)] df(s) = 0.$$

This integral equals $f(b) - f(a) = 0$ because of the left continuity of f . This establishes the lemma.

LEMMA 4.3. - *The preceding lemma is still true if (4.10) holds merely for η of class C' with $\eta(0) = \eta(1) = 0$.*

An η of class C with $\eta(0) = \eta(1) = 0$ can be approximated by an $\bar{\eta}$ of class C' with $\bar{\eta}(0) = \bar{\eta}(1) = 0$ and $|\eta - \bar{\eta}|_C$ less than a prescribed positive constant. The hypothesis in Lemma 4.3 thus implies the hypothesis in Lemma 4.2, and the conclusion of Lemma 4.2 follows.

5. **A functional S derived from J .** — Given the functionals P, Q, R of § 1, we shall exhibit symmetric d -functions p_P, q_Q, r_R relative to the product spaces

$$(5.1) \quad C \times C, \quad C \times L_2, \quad L_2 \times L_2,$$

such that

$$(5.2)' \quad \int_0^1 P(s)u(s)v(s)ds = F(u, v; p_P) \quad [u, v \in C]$$

$$(5.2)'' \quad \int_0^1 Q(s)u(s)y(s)ds = F(u, y; q_Q)$$

$$(5.2)''' \quad \int_0^1 R(s)x(s)y(s)ds = F(x, y; r_R) \quad [x, y \in L_2].$$

The proposed d -functions have the values

$$(5.3) \quad p_P(s, t) = \int_0^s P(a)da, \quad q_Q(s, t) = \int_0^s Q(a)da, \quad r_R(s, t) = \int_0^s R(a)da$$

for $0 \leq s \leq t \leq 1$. Equations (5.3) together with the condition of symmetry determine p_P, q_Q , and r_R over $E' \times E''$. Extended over R_2 so as to be canonical relative to $E' \times E''$, these functions are uniquely determined. They are continuous over R_2 . That they are d -functions relative to the respective products (5.1) is established by the following lemma.

LEMMA 5.1. — With p_P, q_Q, r_R defined as above

$$(5.4) \quad h(C, C; p_P) \leq \|P\|_L, \quad h(C, L_2; q_Q) \leq \|Q\|_{L_2}, \quad h(L_2, L_2; r_R) \leq m.$$

where m is the « essential » maximum of R .

If η and ζ are s -functions associated with admissible partitions of $[0, 1]$.

$$\zeta_n \Delta_n^{(2)}[p_P(s, \cdot)] = \int_0^1 \zeta(t) d_t p_P(s, t) = \int_0^s \zeta(t) P(t) dt,$$

$$\eta, \Delta, [\zeta_n \Delta_n^{(2)}(p_P)] = \int_0^1 \eta(s) d_s \int_0^s \zeta(t) P(t) dt,$$

$$(5.5) \quad \eta, \zeta_n \Delta_{rn}(p_P) = \int_0^1 \eta(s) \zeta(s) P(s) ds.$$

Similarly

$$(5.6) \quad \eta, \zeta_n \Delta_{rn}(q_Q) = \int_0^1 \eta(s) \zeta(s) Q(s) ds,$$

$$(5.7) \quad \eta, \zeta_n \Delta_{rn}(r_R) = \int_0^1 \eta(s) \zeta(s) R(s) ds.$$

On taking the sups of the members of (5.5), (5.6), and (5.7), subject to the respective conditions

$$|\eta|_C \leq 1, \quad |\zeta|_C \leq 1; \quad |\eta|_C \leq 1, \quad |\zeta|_{L_2} \leq 1; \quad |\eta|_{L_2} \leq 1, \quad |\zeta|_{L_2} \leq 1;$$

appropriately applying SCHWARZ'S lemma to the right members of (5.6) and (5.7), relations (5.4) result.

THEOREM 5.1. - *The functionals p_P, q_Q, r_R are d -functions relative to the respective products (5.1) and are such that equations (5.2) hold and*

$$S(u, v; p_P, q_Q, r_R) = J(u, v; P, Q, R, 0)$$

for every u, v in IL_2 .

The proof of (5.2)' will be given. The proofs of (5.2)'' and (5.2)''' are similar and can be omitted. The definition of $p_P = p$ implies that, for a given s , $p_2(s, t) = P(t)$ for ⁽⁵⁾ almost all t such that $0 \leq t \leq s \leq 1$, while $p_2(s, t) = 0$ similarly $0 \leq s \leq t \leq 1$. Hence

$$\int_0^1 v(t) d_1 p(s, t) = \int_0^s v(t) p_2(s, t) dt = \int_0^s v(t) P(t) dt.$$

Equation (5.2)' follows from the definition of $F(u, v; p_P)$.

A representation of the end form E. It is clear that the end form E is a functional bilinear over $C \times C$. According to a theorem of MT 5, there then exist unique d -functions p_E, r_E , of which p_E is of triangle type (See § 1 for definition) such that

$$(5.8) \quad E(u, v) = S(u, v; p_E, r_E)$$

for every u, v in IL_2 . We shall find r_E and p_E . We shall set $r_E(s, t) = c_0 st$ on $E' \times E''$ leaving c_0 to be determined. Let $p_E(s, t) = c_1$, an undetermined constant, at each point (s, t) of $(0, 1] \times (0, 1]$, except $(1, 1)$. Let $p_E(1, 1) = c_1 + c_2$ where c_2 is another constant to be determined. Then

$$F(u, v; r_E) = c_0[u(0) - u(1)][v(0) - v(1)],$$

$$F(u, v; p_E) = c_1 u(0)v(0) + c_2 u(1)v(1).$$

Write $E(u, v)$ in the form

$$E(u, v) = Au(0)v(0) + B[u(0)v(1) + u(1)v(0)] + Cu(1)v(1).$$

Then (5.8) holds if and only if

$$A = c_0 + c_1, \quad B = c_0, \quad C = c_0 + c_2.$$

⁽⁵⁾ Given a distribution function k , we shall denote the partial derivative of k with respect to its second argument by $k_2(s, t)$ at those points (s, t) where this partial derivative exists. At all other points (s, t) , set

$$(a) \quad k_2(s, t) = \sup \left[k\left(s, t + \frac{1}{n}\right) - k(s, t) \right] n \quad [n = 1, 2, \dots].$$

Then (a) holds at each point (s, t) without exception. Since k is measurable (See MT 4, § 2), k_2 is measurable. The functional k_1 is similarly defined relative to the first argument of k .

We have the solutions

$$c_0 = B, \quad c_1 = A - B, \quad c_2 = C - B,$$

so that (5.8) holds for p_E and r_E so determined. We accordingly have the relation

$$(5.9) \quad J(u, v; P, Q, R, E) = S(u, v; p_P + p_E, q_Q, r_R + r_E)$$

for every u, v in IL_2 .

DEFINITION. - A bilinear sum $S(p, q, r)$ in which

$$(5.10) \quad p = p_P + p_E, \quad q = q_Q, \quad r = r_R + r_E,$$

will be said to be derived from $J(P, Q, R, E)$.

If one sets $S_b^a(u, v; p_E, r_E) = E_a^b(u, v)$ and observes that E_a^b is an end form in $[u(a), u(b)]$ and $[v(a), v(b)]$, the following theorem results.

THEOREM 5.2. - If $S(p, q, r)$ is derived from $J(P, Q, R, E)$, then

$$(5.11) \quad S_a^b(u, v; p, q, r) = J_a^b(u, v; P, Q, R, 0) + E_a^b(u, v)$$

for every u, v in $IL_2(a, b)$.

It is clear that

$$\int_a^b P(s)u(s)v(s)ds = F_a^b(u, v; p_P)$$

is a more general form for (5.2)', and that (5.2)'' and (5.2)''' similarly extended lead to the relation

$$S_a^b(u, v; p_P, q_Q, r_R) = J_a^b(u, v; P, Q, R, 0),$$

so that (5.11) follows from (5.9) and the definition of $E_a^b(u, v)$.

6. The extremals of $\tilde{J}$. — The functions P, Q, R of (1.2) are required to be measurable and satisfy the conditions

$$(6.1) \quad \begin{aligned} (\alpha_1) \quad & P \in L, \\ (\alpha_2) \quad & Q \in L_2, \\ (\alpha_3) \quad & |R| \leq M \text{ p. p.} \end{aligned} \quad [M \text{ a constant}].$$

These conditions on P, Q, R , are justified by the following lemma.

LEMMA 6.1. - In order that the L-integrals

$$(6.2)' \quad \int_0^1 P(s)u(s)v(s)ds, \quad [(u, v) \in C \times C]$$

$$(6.2)'' \quad \int_0^1 Q(s)u(s)y(s)ds, \quad [(u, y) \in C \times L_2]$$

$$(6.2)''' \quad \int_0^1 R(s)x(s)y(s)ds, \quad [(x, y) \in L_2 \times L_2]$$

exist for measurable P, Q, R , it is necessary and sufficient that the conditions (6.1) be satisfied.

The conditions (6.1) are sufficient that the integrals exist as is seen by an appropriate use of the SCHWARZ Lemma. We shall accordingly prove the conditions (6.1) necessary.

Proof of (a₁). - The mere existence of the L -integral (6.2)' when $u(s) = v(s) = 1$ implies that P is in L .

Proof of (a₂). - The assumption that the integral (6.2)'' exist whenever $u(s) \equiv 1$ and y is in L_2 , implies that Q is in L_2 . See BANACH, p. 85.

Proof of (a₃). - Let z be an arbitrary function in L . Set

$$(6.3) \quad x(s) = |z(s)|^{\frac{1}{2}}, \quad y(s) = |z(s)|^{\frac{1}{2}} \operatorname{sign} z(s).$$

Then x and y are in L_2 , and the integral

$$(6.4) \quad \int_0^1 R(s)x(s)y(s)ds = \int_0^1 R(s)z(s)ds$$

exists, the latter for every $z \in L$. According to BANACH, p. 85, $|R|$ is bounded p. p. .

In order to treat the Euler equations of $\widehat{J}$, it will be useful to establish a lemma of general character. This lemma is motivated as follows. If f is in L , then

$$\frac{d}{dt} \int_0^t f(a)da = f(t) \quad [0 < t < 1]$$

p. p. in $[0, 1]$. Suppose that z is in C . Then

$$(6.5) \quad \frac{d}{dt} \int_0^t z(a)f(a)da = z(t)f(t)$$

p. p. in $[0, 1]$, but the set K_z such that (6.5) holds for $t \in K_z$ might conceivably vary with z so generally that the maximal set E such that (6.5) holds for $t \in E$ and for every $z \in C$, might have a measure less than 1. That this is not the case is affirmed in the following lemma.

LEMMA 6.2. - For $z \in C$ and $f \in L$,

$$(6.6) \quad \frac{d}{dt} \int_0^t z(a)f(a)da = z(t)f(t)$$

at every point $t \in [0, 1]$ of the set ω at which

$$(6.7) \quad \frac{d}{dt} \int_0^t f(a)da = f(t), \quad \frac{d}{dt} \int_0^t |f(a)| da = |f(t)|.$$

The proof begins with statement (a).

(a). If for some point b in $[0, 1]$, $z(b) = 0$, and if the second relation in (6.7) holds for $t = b$, then (6.6) holds for $t = b$.

To establish (a), observe that

$$(6.8) \quad \left| \int_b^{b+\Delta t} z(a)f(a)da \right| \leq \max |z(a)| \left| \int_b^{b+\Delta t} f(a) da \right|, \quad [\Delta t \neq 0]$$

where $\max |z(a)|$ is taken over the range $[b, b + \Delta t]$ or $[b + \Delta t, b]$ according as $\Delta t > 0$ or $\Delta t < 0$. Since this max tends to zero as $|\Delta t|$ tends to zero, (a) follows from (6.8) on dividing both members of (6.8) by $|\Delta t|$ and letting $|\Delta t|$ tend to zero.

Consider the general case, and choose $b \in \omega$. Write

$$(6.9) \quad \int_0^t z(a)f(a)da = \int_0^t z(b)f(a)da + \int_0^t [z(a) - z(b)]f(a)da,$$

and apply (a) to the final integral in (6.9). According to (a) the derivative of this integral is then zero when $t = b$. At such a point (6.6) accordingly holds.

DEFINITION. - With f in L , let the subset of points of $[0, 1]$ at which (6.7) holds be termed the proper set of f .

The first variation of $\widehat{J}$. Given $u \in IL_2$ over $[0, 1]$ and η of class C' over $[0, 1]$, the first variation of $\widehat{J}$ based on u is by definition the e -derivative of $\widehat{J}(u + e\eta)$ evaluated when $e = 0$. On setting

$$(6.10) \quad \Phi(u, s) = \int_0^s \{ R(t)\dot{u}(t) + Q(t)u(t) - \int_0^t [Q(a)\dot{u}(a) + P(a)u(a)]da \} dt,$$

one finds that when $\eta(0) = \eta(1) = 0$,

$$(6.11) \quad \left. \frac{d}{de} \widehat{J}(u + e\eta) \right|_{e=0} = \int_0^1 \eta(s) d_s \Phi(u, s).$$

For fixed $u \in IL_2$, $\Phi(u, \cdot)$ is in IL_2 .

DEFINITION. - An extremal of $\widehat{J}$ is a functional $u \in IL_2$ such that the first variation (6.11) of $\widehat{J}$ based on u vanishes for every $\eta \in C'$ for which $\eta(0) = \eta(1) = 0$.

THEOREM 6.1. - A necessary and sufficient condition that a $u \in IL_2$ be an extremal of $\widehat{J}$ is that $\Phi(u, s) = s\Phi(u, 1)$ for every $s \in [0, 1]$.

The proof of Theorem 6.1 is (except for notation) identical with that of Theorem 4.1.

COROLLARY 6.1. - A necessary and sufficient condition that a $u \in IL_2$ be an extremal of $\widehat{J}$ is that p. p. in $[0, 1]$,

$$(6.12) \quad R(t)\dot{u}(t) + Q(t)u(t) - \int_0^t [Q(a)\dot{u}(a) + P(a)u(a)]da = k_u,$$

where k_u is a constant independent of t but dependent on u .

DEFINITION. - An extremal of $\tilde{J}_a^b$ is a u in $IL_2[a, b]$ such that

$$\left. \frac{d}{de} \tilde{J}_a^b(u + e\eta) \right|_{e=0} = 0$$

for every $\eta \in C[a, b]$ with $\eta(a) = \eta(b) = 0$.

THEOREM 6.2. - If a bilinear sum $\tilde{S}$ is « derived » from a bilinear sum J , then the extremals u of $\tilde{S}$ with $u(s)$ taken over $[a, b]$ define extremals of $\tilde{S}_a^b$.

Since the addition of an end form such as $-E_a^b$ does not affect the extremals of $\tilde{S}_a^b$, it follows from (5.11) that the extremals of $\tilde{S}$ with $u(s)$ taken over $[a, b]$, define extremals of $\tilde{S}_a^b$ provided the extremals of $\tilde{J}(P, Q, R, 0)$ similarly define extremals of $\tilde{J}_a^b(P, Q, R, 0)$. By an easy extension of Corollary 6.1 a functional $u \in IL_2[a, b]$ is an extremal of $\tilde{J}_a^b$ if and only if

$$(6.12a) \quad R(t)\dot{u}(t) + Q(t)u(t) - \int_a^t [Q(s)u(s) + P(s)u(s)]ds = k_u(a)$$

for almost all t in $[a, b]$. The left member of (6.12a) differs from the left member of (6.12), the Euler operator for $\tilde{J}$, by

$$\int_0^a [Q(s)\dot{u}(s) + P(s)u(s)]ds,$$

a number independent of t in (6.12a). Hence a $u \in IL_2$ which satisfies (6.12) also satisfies (6.12a) for almost all $t \in [a, b]$, and for a suitable choice of $k_u(a)$. The theorem follows.

The existence of extremals of $\tilde{J}$ in the essentially non-singular case. For each extremal u of $\tilde{J}$, set

$$(6.13) \quad z(t) = \int_0^t [Q(a)u(a) + P(a)u(a)]d\tau + k_u, \quad [0 \leq t \leq 1]$$

where k_u is the constant associated with u in (6.12). Then z is absolutely continuous, and

$$(6.14) \quad z(t) = R(t)\dot{u}(t) + Q(t)u(t) \quad [\text{p. p. in } [0, 1]]$$

by virtue of (6.12). In this case for almost all $t \in [0, 1]$

$$(6.15)' \quad \dot{u}(t) = a_{11}(t)u(t) + a_{12}(t)z(t), \quad [\text{from (6.14)}]$$

$$(6.15)'' \quad \dot{z}(t) = a_{21}(t)u(t) + a_{22}(t)z(t), \quad [\text{from (6.13)}]$$

where

$$(6.16) \quad \begin{aligned} a_{11}(t) &= -\frac{Q(t)}{R(t)}, & a_{12}(t) &= \frac{1}{R(t)}, \\ a_{21}(t) &= P(t) - \frac{Q^2(t)}{R(t)}, & a_{22}(t) &= \frac{Q(t)}{R(t)}. \end{aligned}$$

Under the hypothesis that J is essentially non-singular $a_{ij}(t)$ is in L ($i, j = 1, 2$).

Notation. Let ω be the intersection of the proper sets of a_{11} , a_{12} , a_{21} , a_{22} . With ω so defined the following lemma holds.

LEMMA 6.3. - *If u is an extremal of $\tilde{J}$, and z is defined by (6.13), then (6.15) holds for each $t \in \omega$.*

It follows from the absolute continuity of u and z and from (6.15) that

$$(6.17) \quad u(t) = \int_0^t [a_{11}(s)u(s) + a_{21}(s)z(s)]ds + u(0),$$

$$(6.17)'' \quad z(t) = \int_0^t [a_{21}(s)u(s) + a_{22}(s)z(s)]ds + z(0),$$

for each $t \in [0, 1]$. Lemma 6.2 applied to (6.17) implies the validity of (6.15) for each $t \in \omega$.

A « solution » of (6.15) is a pair (u, z) of functionals which are absolutely continuous on $[0, 1]$ and such that (6.15) holds p. p. on $[0, 1]$. For such a solution (6.17) holds and it follows from Lemma 6.2 that (6.15) holds for every $t \in \omega$. The following statement of the relation between solutions (u, z) of (6.15) and extremals u of $\tilde{J}$ is useful.

LEMMA 6.4. - *Under the transformation (6.13) in which an extrema u of $\tilde{J}$ and associated constant k_u determine z , there is a one-to-one correspondence between solutions (u, z) of (6.15) and sets $[u, k_u]$ of extremals u of $\tilde{J}$ and associated constants k_u .*

Given an extremal u of $\tilde{J}$ and associated constant k_u , z is uniquely determined by (6.13) and (u, z) is a solution of (6.15). Conversely, given a solution (u, z) of (6.15), (6.15)' implies (6.14). Moreover (6.15)'' implies (6.17)', which in turn implies (6.12) with $k_u = z(0)$ on replacing z in (6.17)'' by the right member of (6.14). Thus u is an extremal of $\tilde{J}$ and the constant $k_u = z(0)$ is the associated k_u . The pair $[u, k_u]$ is thus uniquely determined by (u, z) subject to (6.13).

The existence of solutions of (6.15). MCSHANE, § 69.2, gives existence theorems which apply to the system (6.15) provided the a_{ij} are in L . That the a_{ij} are in L has been seen. With MCSHANE one infers the following. For $t_0 \in [0, 1]$ and arbitrary constants (u_0, z_0) there exists a unique (MCSHANE, § 69.2) solution (u, z) of (6.15) of the form

$$(6.18) \quad u = U(t, t_0, u_0, z_0), \quad z = Z(t, t_0, u_0, z_0), \quad [0 \leq t \leq 1]$$

for which

$$(6.18)'' \quad u_0 = U(t_0, t_0, u_0, z_0), \quad z_0 = Z(t_0, t_0, u_0, z_0),$$

where U and Z are continuous over the product domain $[0, 1] \times [0, 1] \times R_1 \times R_1$ of the variables (t, t_0, u_0, z_0) .

In the case at hand a somewhat sharper statement is possible in terms of the set ω .

THEOREM 6.3. - *If $\mathcal{J}$ is essentially non-singular, then for $t_0 \in \omega$ and arbitrary constants u_0 and $\dot{u}_0$ there is a unique extremal u of $\widehat{\mathcal{J}}$ for which*

$$(6.18) \quad u(t_0) = u_0, \quad \dot{u}(t_0) = \dot{u}_0.$$

Let the values of P, R, Q, α_{ij} at t_0 be denoted by $P_0, Q_0, R_0, \alpha_{ij}^0$. Set

$$(6.19) \quad z_0 = Q_0 u_0 + R_0 \dot{u}_0.$$

Then $U(\cdot, t_0, u_0, z_0) = u$ is the desired extremal as we shall see. Clearly $u(t_0) = u_0$. It remains to prove that $\dot{u}(t_0) = \dot{u}_0$.

Since u is the first component of a solution (u, z) of (6.15) of the form (6.18)' satisfying (6.18)'', it follows from (6.15) that

$$\begin{aligned} \dot{u}(t_0) &= \alpha_{11}^0 u(t_0) + \alpha_{12}^0 z(t_0) = \alpha_{11}^0 u_0 + \alpha_{12}^0 z_0, & [\text{using (6.18)'}] \\ &= -\frac{Q_0 u_0}{R_0} + \frac{1}{R_0} [Q_0 u_0 + R_0 \dot{u}_0] = \dot{u}_0 & [\text{using (6.19)}]. \end{aligned}$$

The above extremal u is unique among extremals which satisfy (6.18). To establish this, let $\bar{u}$ be an arbitrary extremal which satisfies (6.18). The extremal $\bar{u}$ determines a solution $(\bar{u}, \bar{z})$ of (6.15) as stated in Lemma 6.4. For this solution (6.15)' holds so that

$$\dot{\bar{u}}_0 = -\frac{Q_0}{R_0} u_0 + \frac{1}{R_0} \bar{z}(t_0), \quad [\text{using (6.18)}]$$

and hence

$$\bar{z}(t_0) = Q_0 u_0 + R_0 \dot{\bar{u}}_0 = z_0 \quad [\text{using (6.19)}].$$

Thus the solution $(\bar{u}, \bar{z})$ is determined by the same initial values u_0, z_0 when $t = t_0$ as determine (u, z) . It follows that $\bar{u}(s) = u(s)$ over $[0, 1]$.

LEMMA 6.5. - *When $\widehat{\mathcal{J}}$ is essentially non-singular, there exists a pair of extremals u, v of $\widehat{\mathcal{J}}$ whose Wronskian $u(t)\dot{v}(t) - \dot{u}(t)v(t) = W(t)$ exists and fails to vanish at each point t of ω .*

Let t_0 be a point of ω . By virtue of Theorem 6.3 there exist extremals u and v such that

$$(6.20) \quad \begin{aligned} u(t_0) &= 1, & \dot{u}(t_0) &= 0, \\ v(t_0) &= 0, & \dot{v}(t_0) &= 1. \end{aligned}$$

For such a pair $W(t_0) = 1$. If $W(a) = 0$ for some $a \in \omega$, there would exist an extremal z with values

$$(6.21) \quad z(t) = c_1 u(t) + c_2 v(t) \quad [|c_1| + |c_2| \neq 0]$$

such that $z(a) = \dot{z}(a) = 0$. Because of the uniqueness of such an extremal, as affirmed in Theorem 6.3, we conclude that $z(t) = 0$ for every $t \in [0, 1]$. It follows that $c_1 u(t) + c_2 v(t) = 0$, and hence that $W(t) = 0$ for every $t \in \omega$, contrary to the condition $W(t_0) = 1$. We conclude that $W(t) \neq 0$ at no point $t \in \omega$.

LEMMA 6.6. - When P, Q, R are measurable and satisfy conditions (6.1) and the extremals of $\widehat{J}$ have properties (β) and (γ) of § 1, it is impossible that $R(t)$ vanish p. p. in $[0, 1]$.

If $R(t)$ vanished p. p. in $[0, 1]$, then the EULER equation (6.12) would imply that

$$(6.22) \quad Q(t)u(t) = \int_0^t [Q(t)\dot{u}(t) + P(t)\dot{u}(t)]dt + k_u$$

p. p. in $[0, 1]$. By property (β) of extremals there exists an extremal u_c such that $u_c(c) = 1$ at a prescribed point c of $[0, 1]$. It follows from (6.22) that Q is equivalent in the sense of LEBESGUE to an absolutely continuous function. We can suppose Q is then absolutely continuous. From (6.22) it would follow that

$$(6.23) \quad \dot{Q}(t)u_c(t) = P(t)u_c(t)$$

p. p. in $[0, 1]$, and hence $\dot{Q}(t) = P(t)$ p. p.. Then every $u \in IL_2$ would be an extremal contrary to property (γ) of extremals. We infer that $R(t)$ does not vanish p. p. in $[0, 1]$.

The following theorem converts the condition that a J be essentially non-singular into a condition involving properties of extremals of $\widehat{J}$.

THEOREM 6.4. - If P, Q, R are measurable and satisfy conditions (6.1), necessary and sufficient conditions that J be essentially non-singular are that the extremals of $\widehat{J}$ satisfy (β) and (γ) of § 1.

When J is essentially non-singular, the extremals of $\widehat{J}$ have the properties (β) and (γ) . In particular, the extremal $U(t, t_0, u_0, z_0) = u(t)$ with $t_0 = c$, $u_0 = 1$, $z_0 = 0$ satisfies the condition $u(c) = 1$. Thus the extremals of $\widehat{J}$ satisfy (β) . Not every $u \in IL_2$ is an extremal by virtue of Theorem 6.3. The extremal of $\widehat{J}$ satisfy (γ) in accordance with Lemma 6.5

It remains to show that J is essentially non-singular when the extremals of $\widehat{J}$ satisfy (β) and (γ) ; that is, we must show that $1/R$ and Q/R are in L_2 .

Proof that $1/R$ is in L_2 . In accordance with (γ) there exist at least two extremals u and v of $\widehat{J}$ whose Wronskian $W(t) \neq 0$ p. p. in $[0, 1]$. Let z_u be the absolutely continuous function over $[0, 1]$ defined by the right member of (6.13). Let z_v be similarly defined. The Euler equation (6.12) implies the relations

$$(6.24) \quad \begin{aligned} z_u(t) &= R(t)\dot{u}(t) + Q(t)u(t) \\ z_v(t) &= R(t)\dot{v}(t) + Q(t)v(t), \end{aligned}$$

p. p. in $[0, 1]$. Set $u(t)z_v(t) - v(t)z_u(t) = F(t)$. On using (6.13) to find $\dot{z}_u$ and $\dot{z}_v$, one sees that $\dot{F}(t) = 0$ p. p.. Since F is absolutely continuous,

$$u(t)z_v(t) - v(t)z_u(t) = R(t)W(t) = D \quad [D \text{ a constant}]$$

p. p. in $[0, 1]$. Now $W(t) \neq 0$ p. p. in $[0, 1]$ by (γ), so that by the preceding lemma on R , $D \neq 0$. Thus

$$(6.25) \quad \frac{1}{R} = \frac{W(t)}{D} \quad \{p. p. \text{ in } [0, 1]\}$$

and $1/R$ is in L_2 .

Proof that Q/R is in L_2 . Since $1/R$ is in L_2 , the Euler equation (6.12) can be written, p. p. in $[0, 1]$,

$$\frac{Q(t)}{R(t)} u(t) = -\dot{u}(t) + \frac{1}{R(t)} \int_0^t [Q(t)\dot{u}(t) + P(t)u(t)]dt + \frac{k_u}{R(t)}.$$

If u is taken as an extremal u_c such that $u(c) = 1$, we see Q/R is integrable square in a neighborhood of $s = c$. Since $[0, 1]$ can be covered by a finite set of such neighborhoods, Q/R must be in L_2 over $[0, 1]$.

This completes the proof of the theorem.

According to a definition given in § 1, a bilinear sum S is of the type of a bilinear sum $\bar{S}$ if $S(u, v) = \bar{S}(\bar{u}, \bar{v})$ under a transformation

$$(6.26) \quad \begin{aligned} u(s) &= \varphi(s)\bar{u}(s), \\ v(s) &= \varphi(s)\bar{v}(s) \end{aligned} \quad [\bar{u}, \bar{v} \in IL_2]$$

in which φ is in IL_2 and never vanishes. Observe that the inverse transformation is of the same character since $1/\varphi$ is in IL_2 . With this understood, we state the theorem:

THEOREM 6.5. - *Each essentially non-singular $J(P, Q, R, E)$ is of the type of an essentially non-singular bilinear sum of the form*

$$\int_0^1 [\bar{R}(s)\dot{\bar{u}}(s)\dot{\bar{v}}(s) + \bar{P}(s)\bar{u}(s)\bar{v}(s)]ds + \bar{E}(\bar{u}, \bar{v}).$$

In terms of the P, Q, R used to define J , one needs a $\varphi \in IL_2$ never vanishing and such that

$$\frac{\dot{\varphi}(s)}{\varphi(s)} = -\frac{Q(s)}{R(s)}.$$

Such a φ is

$$\varphi(s) = e^{-\int \frac{Q(s)}{R(s)} ds}.$$

We recall that Q/R is in L_2 by hypothesis, so that φ is in IL_2 . Under the transformation (6.26) thereby defined, $J(u, v; P, Q, R, E) = J(\bar{u}, \bar{v}; \bar{P}, 0, \bar{R}, \bar{E})$ where

$$\bar{R}(s) = \varphi^2(s)R(s), \quad \bar{P}(s) = R(s)\dot{\varphi}^2(s) + 2Q(s)\varphi(s)\dot{\varphi}(s) + P(s)\varphi^2(s),$$

while $E(\bar{u}, \bar{v}) = E(u, v)$, under the linear transformation of the end values defined at $s = 0$ and $s = 1$ by (6.26). Observe that $|R(s)|$ is essentially bounded with R and that $\bar{P}$ is in L with P . Understanding that $\bar{Q}(s) \equiv 0$, we see that $1/\bar{R}$ and $\bar{Q}/\bar{R}$ are in L_2 . Thus $J(\bar{P}, \bar{Q}, \bar{R}, \bar{E})$ is admissible and essentially non-singular.

7. The functional $S_a^b(p, r)$. — This functional is defined by setting

$$(7.1) \quad S_a^b(u, v; p, r) = F_a^b(u, v; p) + F_a^b(\dot{u}, \dot{v}; r).$$

The following lemma is needed.

LEMMA 7.1. — For $u, v \in \Pi_{L_2}[a, b]$ the integrals $F_a^b(u, v; p)$ and $F_a^b(\dot{u}, \dot{v}; r)$ exist and define functionals bilinear over

$$(7.2) \quad C[a, b] \times C[a, b], \quad L_2[a, b] \times L_2[a, b],$$

respectively, such that

$$(7.3) \quad F_a^b(u, v; p) = F_a^b(v, u; p), \quad F_a^b(\dot{v}, \dot{u}; r) = F_a^b(\dot{u}, \dot{v}; r).$$

We begin with $F_a^b(u, v; p)$. Let u and v be extended over $[0, 1]$ so as to be constant on $[0, a]$ and $[b, 1]$. Let $\bar{u}$ and $\bar{v}$ denote the extended functionals. When $0 < b < 1$,

$$(7.4) \quad \int_0^{1-} v(t) d_t p(s, t) = v(a)p(s, a) + \int_a^{b-} v(t) d_t p(s, t) + v(b)[p(s, 1) - p(s, b)].$$

This relation shows that the integral on the right is of bounded variation for $s \in [a, b]$ and left continuous, since $p(\cdot, t)$ has this character together with the integral on the left of (7.4). Cf. (2.19).

Hence the integral $F_a^b(u, v; p)$ exists. Let $f_v(s)$ equal the right member of (7.4). Then

$$F(\bar{u}, \bar{v}; p) = \int_0^{a-} \bar{u}(s) df_v(s) + \int_a^{b-} \bar{u}(s) df_v(s) + \int_b^1 \bar{u}(s) df_v(s),$$

and using the first term in (7.4) in all three of these integrals, then the second term etc. $F(\bar{u}, \bar{v}; p)$ equals

$$(7.4)' \quad \begin{aligned} & u(a)v(a)p(a, a) + v(a) \int_a^{b-} u(s) d_s p(s, a) + u(b)v(a)[p(1, a) - p(b, a)] + \\ & + u(a) \int_a^{b-} v(t) d_t p(a, t) + F_a^b(u, v; p) + u(b) \int_a^{b-} v(t) d_t [p(1, t) - p(b, t)] + \\ & + u(a)v(b)[p(a, 1) - p(a, b)] + v(b) \int_a^{b-} u(s) d_s [p(s, 1) - p(s, b)] + u(b)v(b)[p(1, 1) - p(b, b)]. \end{aligned}$$

From this it follows that F_a^b is bilinear over $C[a, b] \times C[a, b]$. For the terms in (7.4)' other than $F_a^b(u, v; p)$, define functionals bilinear over $C[a, b] \times C[a, b]$, including $F(\bar{u}, \bar{v}; p)$ which depends implicitly upon (u, v) . One sees also that

$$F_a^b(u, v; p) = F_a^b(v, u; p),$$

by virtue of the condition $p = p^*$ and the relation $F(\bar{u}, \bar{v}; p) = F(\bar{v}, \bar{u}; p)$. In case $b = 1$,

$$(7.4)'' \quad \begin{aligned} F(\bar{u}, \bar{v}; p) &= u(a)v(a)p(a, a) + v(a) \int_a^1 u(s) d_s p(s, a) + \\ &+ u(a) \int_a^1 v(t) d_t p(a, t) + F_a^1(u, v; p), \end{aligned}$$

and the conclusions are similar.

With $u, v \in IL[a, b]$ and extended in definition over $[0, 1]$ as above, consider the integral $\varphi_v(s)$ with values,

$$\begin{aligned}
 (7.5) \quad & \int_0^{1-} v(t) d_r(s, t) = \int_0^{1-} v(t) r_2(s, t) dt = \\
 & = \int_a^b v(t) r_2(s, t) dt = \int_a^{b-} v(t) r_2(s, t) dt = \\
 & = \int_a^b v(t) d_r(s, t) = \int_a^{b-} v(t) d_r(s, t).
 \end{aligned}$$

The integral φ_v is in L_2 by (2.19)''' so that $F_b^a(\dot{u}, \dot{v}; r)$ exists and

$$\int_0^{1-} u(s) d\varphi_v(s) = \int_a^{b-} u(s) d\varphi_v(s) = \int_a^b u(s) d\varphi_v(s).$$

That is,

$$F(\dot{u}, \dot{v}; r) = F_a^b(\dot{u}, \dot{v}; r).$$

The remainder of the lemma follows.

An extremal of $\widehat{S}_a^b$ is by definition a $u \in IL[a, b]$ such that

$$\left. \frac{d}{de} \widehat{S}_a^b(u + e\eta) \right|_{e=0} = 0$$

for every η in $C[a, b]$ with $\eta(a) = \eta(b) = 0$. To obtain the Euler equation for $\widehat{S}_a^b$ a lemma on integration by parts is needed.

LEMMA 7.2 - If f is absolutely continuous over $[a, b]$ and g of bounded variation over $[a, b]$, defined over an interval $[a - e, b]$ with $e > 0$, and left continuous over this interval, then

$$(7.6) \quad \int_a^{b-} f(s) dg(s) = f(b)g(b) - f(a)g(a) - \int_a^b g(s) f'(s) ds.$$

We apply a lemma in SAKS, p. 102, first extending f so as to be continuous over a neighborhood of $[a, b]$ and extending g so as to be continuous at $s = b$. We note that the validity of (7.6) is independent of these extensions. However with these extensions, SAKS' lemma applies and implies that the integral

$$(7.7) \quad \int_a^b f(s) dg(s)$$

equals the right member of (7.6). With g continuous at $s = b$ the integral (7.7) equals the left member of (7.6), and (7.6) follows.

The preceding lemma could be reduced to a known lemma on RIEMANN-STIELTJES integration by parts. See SANSONE, p. 435, equation (6). We have kept within the theory of L - S -integration to avoid confusion.

Note. In the application it will be noted that g will satisfy the hypotheses of the lemma, but will not in general be continuous at $s = b$. As pointed out, (7.6) holds regardless of this.

A transformation of S_a^b . As in § 3 for any $u \in IL_2[a, b]$, set

$$(7.8)' \quad \int_a^b u(t) d_t p(s, t) = h_{au}^b(s), \quad [b < 1]$$

$$(7.8)'' \quad \int_a^b u(t) d_t p(s, t) = h_{au}^b(s), \quad [b = 1].$$

When $b < 1$, we see from (7.4) that the integral (7.8)' exists, is of bounded variation for fixed u and $s \in [a, b]$, and left continuous for $s < 1$. Moreover,

$$\begin{aligned} F_a^b(u, v; p) &= F_a^b(v, u; p) = & [\text{by (7.3)}] \\ &= \int_a^{b-} v(s) d_s h_{au}^b(s) = & [b < 1] \\ (7.9) \quad &= [v(s) h_{au}^b(s)]_a^b - \int_a^{b-} v(s) h_{au}^b(s) ds = & [\text{by Lemma 7.2}] \\ &= [v(s) h_{au}^b(s)]_a^b - \int_a^{b-} v(s) d_s \int_0^s h_{au}^b(\alpha) d(\alpha). \end{aligned}$$

Relation (7.9) also holds with $b-$ replaced by 1, when $b = 1$, using SAKS' lemma, p. 102, directly, and noting that $h_{au}^b(1+) = h_{au}^b(1)$. According to Lemma 7.1,

$$F_a^b(u, v; r) = F_a^b(v, u; r).$$

For $u \in IL_2[a, b]$ and arbitrary s , set

$$(7.10) \quad \Psi_a^b(u, s; p, r) = \int_a^{b-} u(t) d_t p(s, t) - \int_a^s d\alpha \int_a^{b-} u(t) d_t p(\alpha, t), \quad [b < 1]$$

with $b-$ replaced by 1 when $b = 1$. The preceding equations then give the following theorem.

THEOREM 7.1. - For $u, v \in IL_2[a, b]$ and $[a, b] \in [0, 1]$,

$$(7.11) \quad S_a^b(u, v; p, r) = \int_a^{b-} v(s) d_s \Psi_a^b(u, s; p, r) + [v(s) h_{au}^b(s)]_a^b.$$

It should be recalled that the first integral on the right of (7.10) gives the values for $s \in R_1$ of a function in L_{2a} , as (7.5) shows, while the second integral on the right of (7.10) is the integral of a function in C_a . Cf. (7.4). In effect, $\Psi_a^b(u, \cdot; p, r)$ maps R_1 continuously into R_1 . When $v(a) = v(b) = 0$, (7.11) implies that

$$S_a^b(u, v; p, r) = \int_a^{b-} v(s) d_s \Psi_a^b(u, s; p, r).$$

We can accordingly apply Lemma 4.1, and infer the following:

THEOREM 7.2. - *A necessary and sufficient condition that a $u \in \Pi_2[a, b]$ be an extremal of $\bar{S}_a^b$ is that $\phi_a^b(u, s; p, r)$ be linear in s over $[a, b]$.*

A transformation of $\phi_a^b(u, s)$. Set

$$(7.12) \quad \Theta(s, t) = \int_0^s p(\alpha, t) d\alpha \quad [s \in R_1].$$

LEMMA 7.3. - *For fixed t , $\Theta(\cdot, t)$ is absolutely continuous over any bounded interval, while for fixed s , $\Theta(s, \cdot)$ is of bounded variation and canonical relative to $[0, 1]$.*

That $\Theta(\cdot, t)$ is absolutely continuous over any bounded interval follows from the definition of $\Theta(\cdot, t)$ as an integral.

It follows from the fact that $p(\alpha, \cdot)$ is left continuous except at most at point $t = 1$, that $\Theta(s, \cdot)$ is similarly left continuous. For the theorem on bounded convergence implies that for $a \neq 1$,

$$\lim_{t \rightarrow a-} \int_0^s p(\alpha, t) d\alpha = \int_0^s p(\alpha, a) d\alpha.$$

The right continuity of $\Theta(s, \cdot)$ at $t = 1$ similarly follows. For fixed s , it is trivial that $\Theta(s, t)$ is constant for $t \leq 0$ and $t \geq 1$ since $p(s, t)$ is so constant. To show finally that $\Theta(s, \cdot)$ is of bounded variation over $[0, 1]$, let ζ be a step-function associated with an admissible partition of $[0, 1]$. Then

$$|\zeta_n \Delta_n^{(2)}[\Theta(s, \cdot)]| \leq \int_0^s |\zeta_n \Delta_n^{(2)}[p(\alpha, \cdot)]| d\alpha.$$

On taking the sup of both members over all ζ with $|\zeta|_C \leq 1$,

$$\begin{aligned} T[\Theta(s, \cdot)] &\leq \int_0^s T[p(\alpha, \cdot)] d\alpha \leq \\ &\leq sh(C, C; p) \end{aligned} \quad [\text{by (2.11)}].$$

This completes the proof of the lemma.

LEMMA 7.4. - *For $0 < b < 1$, and $u \in \Pi_2[a, b]$,*

$$(7.13) \quad \int_a^s d\alpha \int_a^{b-} u(t) d_t p(\alpha, t) = \int_a^{b-} u(t) d_t [\Theta(s, t) - \Theta(a, t)]$$

with $b-$ replaced by 1 when $b = 1$.

Let $H(s)$ be the left member of (7.13). On performing the inner integration in (7.13) by parts in accordance with Lemma 7.2,

$$(7.14) \quad H(s) = \int_a^s [u(t)p(\alpha, t)]_{t=a}^{t=b} d\alpha - \int_a^s d\alpha \int_a^b p(\alpha, t) u(t) dt \quad [\text{for } b < 1].$$

In the final integral in (7.14) the order of integration can be inverted (See FUBINI, TONELLI) so that

$$(7.15) \quad H(s) = \left[u(t) \int_a^s p(\alpha, t) d\alpha \right]_{t=a}^t - \int_a^b u(t) \left[\int_a^s p(\alpha, t) d\alpha \right] dt.$$

On the other hand the right member $K(s)$ of (7.13) takes the form

$$K(s) = [u(t) \{ \Theta(s, t) - \Theta(a, t) \}]_{t=a}^{t=b} - \int_a^b u(t) [\Theta(s, t) - \Theta(a, t)] dt \quad [\text{for } b < 1].$$

Reference to the definition of Θ shows that $H(s) = K(s)$ when $b < 1$. When $b = 1$, a similar use of the lemma in SAKS, p. 102, establishes (7.13). Hence we have the following lemma.

LEMMA 7.5. - *The Euler operator $\phi_a^b(u, s)$ can be written in the form*

$$(7.16) \quad \phi_a^b(u, s; p, r) = \int_a^b u(t) d_t r(s, t) - \int_a^{b-} u(t) d_t [\theta(s, t) - \theta(a, t)]$$

when $0 < b < 1$, with $b-$ replaced by 1 when $b = 1$.

8. Implications of extremal property (β). — Recall that extremal properties (α), (β), (γ) are as follows.

(α). *Each extremal u of $\widehat{S}$, with $u(s)$ taken over any subinterval $[a, b]$ of $[0, 1]$ is an extremal of $\widehat{S}_a^b$.*

(β). *Corresponding to each point $c \in [0, 1]$ there is at least one extremal u_c such that $u_c(c) = 1$.*

(γ). *There is at least one pair of extremals u and v of $\widehat{S}$ whose Wronskian $u(s)\dot{v}(s) - \dot{u}(s)v(s) = W(s)$ exists and is not zero, p. p. on $[0, 1]$, while not every $u \in IL_2$ is an extremal of $\widehat{S}$.*

That neither (α), (β), nor (γ) holds in general when J is replaced by S will be shown by examples in § 11. The properties (α), (β), (γ) will also be shown to be independent of each other. In this section we shall obtain certain consequences of the assumption that the extremals of $\widehat{S}$ have property (β).

By a rational subinterval of $[0, b]$ is meant any subinterval of $[0, b]$ whose end coordinates are rational multiples of b . The need for dealing with rational subintervals of $[0, b]$ arises from the fact that the choice (in § 10) of a subinterval of $[0, b]$ limits an auxiliary parameter s to a subset $\tau(s)$ of $[b, 1]$ of measure $1 - b$. The limitation on s imposed by the totality of rational subintervals of $[0, b]$ remains a limitation of s to a subset of $[b, 1]$ still measure $1 - b$.

The following lemma seeks to imply the constancy of $g(s)$ over $[0, b]$ under relatively weak hypotheses. A collection $[f]$ of functionals f in $C[0, b]$ will be said to be *homogeneous* if for every $f \in [f]$, kf is also in $[f]$ for each constant k , and *locally non-null* over $[0, b]$ if for each point of $[0, b]$ there is at least one $f \in [f]$ such that $f(c) = 1$.

LEMMA 8.1. - *Let g be of bounded variation over $[0, b]$ and left continuous for $s \leq b$. Let $[f]$ be a homogeneous and locally non-null collection of functionals f continuous over $[0, b]$. If for every rational subinterval of $[0, b]$ of*

the form $[a, a']$,

$$(8.1) \quad \int_a^{a'-} f(s) dg(s) = 0,$$

then $g(s)$ is constant over $[0, b]$.

We begin by proving the following:

(i). Let ϵ and δ be arbitrary positive constants. There then exist $f_1, \dots, f_n$ in $[f]$ and a partition of $[0, b]$ into rational subintervals,

$$(8.2) \quad [0, a_1), \quad [a_1, a_2), \dots, [a_{n-1}, b),$$

each of length less than δ such that for s in the r th interval I_r of (8.2),

$$|f_r(s) - 1| < \epsilon.$$

Let c be an arbitrary point of $[0, b]$. By hypothesis there exist an $f \in [f]$ such that $f(c) = 1$. Since f is continuous over $[0, b]$ there exists a neighborhood N_c of c relative to $[0, b]$ such that $|f(s) - 1| < \epsilon$ for s in N_c .

It follows from the HEINE-BOREL theorem that $[0, b]$ is the union of a finite set of such neighborhoods. These neighborhoods can be taken less than δ in length, and intervals I_r satisfying (i) together with the associated $f_r \in [f]$ clearly exist.

(ii). Let ϵ be a positive constant and F a functional in $C[0, b]$. There then exists a partition of $[0, b]$ into rational subinterval I_r ($r = 1, \dots, n$) of the form (8.2) together with functionals $F_r \in [f]$ such that

$$(8.3) \quad |F(s) - F_r(s)| < \epsilon \quad [s \in I_r, r = 1, \dots, n].$$

Let M be the absolute bound of F over $[0, 1]$. Let δ in (i) be chosen so that in each subinterval of $[0, b]$ of length less than δ the oscillation of F is less than $\epsilon/2$. Let ϵ in (i) be chosen as less than $\epsilon/2M$. For this choice of ϵ and δ , we refer to the intervals I_r and $f_r \in [f]$ of (i) and set $F(a_r) = c_r$. With r not summed and for $s \in I_r$,

$$\begin{aligned} |F(s) - c_r f_r(s)| &\leq |c_r - c_r f_r(s)| + |F(s) - c_r| \leq \\ &\leq M |1 - f_r(s)| + |F(s) - c_r| \leq \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2}. \end{aligned}$$

Statement (ii) is accordingly satisfied by the intervals I_r of (i) and functionals $F_r = c_r f_r$.

Proof of Lemma 8.1. Let g be extended for $s > b$ so that $g(s) = g(b)$. So extended g is continuous at $s = b$. That $g(s)$ is constant over $[0, b]$ will follow when it has been shown that

$$(8.4) \quad \int_0^{b-} F(s) dg(s) = 0$$

for every $F \in C[0, b]$. To that end let ε be a positive constant and F_r and I_r ($r = 1, \dots, n$) the functionals in $[f]$ and intervals affirmed to exist in (ii). Let φ_ε be a functional over $[0, b]$ such that for $s \in I_r$,

$$\varphi_\varepsilon(s) = F_r(s) \quad [r = 1, \dots, n].$$

Then

$$\int_0^{b-} \varphi_\varepsilon(s) dg(s) = 0.$$

However, as $\varepsilon \rightarrow 0$, $\varphi_\varepsilon(s) \rightarrow F(s)$ for each $s \in [0, b]$. Moreover this convergence is bounded so that the limit integral (8.4) vanishes. Since

$$\int_0^{b-} F(s) dg(s) = \int_0^b F(s) dg(s) = 0,$$

it follows from Lemma 4.0 that $g(s)$ is constant over $[0, b]$. This completes the proof of Lemma 8.1.

The following lemma is an immediate consequence of Lemma 8.1.

LEMMA 8.2. — Let g be of bounded variation over $[0, b]$ and left continuous for $s \leq b$. If the extremals u of $\widehat{S}$ have the property (β) and if

$$\int_a^{a'-} u(s) dg(s) = 0$$

for every rational subinterval $[a, a']$ of $[0, b]$ and every extremal u of $\widehat{S}$, then $g(s)$ is constant over $[0, b]$.

In Lemma 8.1, let $[u]$ be the class of all extremals of $\widehat{S}$. The hypotheses of Lemma 8.2 ascribe to $[u]$ the attributes required of $[f]$ in Lemma 8.1, and the constancy of $g(s)$ over $[0, b]$ follows.

9. Some special d -functions. — Under the hypotheses of Theorem 1.1 reduction of S to a J involves certain special d -functions r and p . In this section we shall define these d -functions and obtain formulae for the corresponding bilinear functionals.

LEMMA 9.1. — Let ρ be the functional which is canonical relative to $E' \times E''$ and such that

$$(9.1) \quad \rho(s, t) = st \quad [(s, t) \in E' \times E''].$$

Then ρ is a d -function relative to $L_2 \times L_2$ and for u, v in L_2 ,

$$(9.2) \quad F(\dot{u}, \dot{v}; \rho) = [u(1) - u(0)][v(1) - v(0)].$$

It is immediate that

$$F(u, \dot{v}; \rho) = \int_0^1 u(s) ds \int_0^1 v(t) dt.$$

from which (9.2) follows. For s -functions η and ζ belonging to E' and E'' respectively

$$\eta, \zeta_n \Delta_{r,n}(\rho) = \int_0^1 \eta(s) ds \int_0^1 \zeta(t) dt,$$

from which it follows that $h(L_2, L_2; \rho) = 1$.

Let φ , mapping R_1 into R_1 , be of bounded variation and canonical relative to $[0, 1]$. Let $\pi^{(\varphi)}$ with values $\pi^{(\varphi)}(s, t)$, mapping R_2 into R_1 , be defined as follows. Let $\pi^{(\varphi)}(s, t) = 0$ for $s < 1$ and $t < 1$, and let

$$(9.3) \quad \pi^{(\varphi)}(1, t) = \pi^{(\varphi)}(t, 1) = \varphi(t) \quad [0 \leq t \leq 1].$$

Let $\pi^{(\varphi)}$ be extended over R_2 so as to be canonical relative to $E' \times E''$. We term $\pi^{(\varphi)}$ an *edge function* determined by φ .

LEMMA 9.2. - *The edge function $\pi^{(\varphi)}$ is a d-function relative to $C \times C$ and*

$$(9.4) \quad F(u, v; \pi^{(\varphi)}) = v(1) \int_0^1 u(s) d\varphi(s) + u(1) \int_0^1 v(s) d\varphi(s) - u(1)v(1)\varphi(1).$$

That $h(C, C; \pi^{(\varphi)}) < \infty$ follows at once from the definition of this variation and the hypothesis that φ is of bounded variation over $[0, 1]$. To establish (9.4) we introduce the functional f_v with values

$$(9.5) \quad \begin{aligned} f_v(s) &= \int_0^1 v(t) d_t \pi^{(\varphi)}(s, t) = v(1)\varphi(s), & [\text{for } s < 1]. \\ &= \int_0^1 v(t) d\varphi(t) & [\text{for } s = 1]. \end{aligned}$$

Then

$$(9.6) \quad \begin{aligned} F(u, v; \pi^{(\varphi)}) &= \int_0^{1-} u(s) df_v(s) + u(1)[f_v(1) - f_v(1-)] \\ &= v(1) \int_0^{1-} u(s) d\varphi(s) + u(1) \int_0^1 v(t) d\varphi(t) - u(1)v(1)\varphi(1) \end{aligned}$$

from which (9.4) follows.

LEMMA 9.3. - *The extremals of $\widehat{F}(\pi^{(\varphi)})$ include an extremal u for which $u(1) = 1$, if and only if φ is constant over the open interval $(0, 1)$.*

The first variation of $\widehat{F}(\pi^{(\varphi)})$ has the form

$$(9.7) \quad 2F(\eta, u; \pi^{(\varphi)}) = u(1) \int_0^1 \eta(s) d\varphi(s)$$

for $\eta(0) = \eta(1) = 0$. If there exists an extremal u such that $u(1) = 1$, then

$$(9.8) \quad \int_0^1 \eta(s) d\varphi(s) = 0$$

for every $\eta \in C'$ with $\eta(0) = \eta(1) = 0$. According to Lemma 4.3, $\varphi(s)$ must then be constant over $(0, 1)$. Conversely, if φ is constant over $(0, 1)$, the first variation (9.6) based on any $u \in IL_2$ vanishes. The lemma follows.

LEMMA 9.4. - In case φ is canonical relative to $[0, 1]$ and $\varphi(s) = k$, a constant, over $(0, 1)$, then

$$(9.9) \quad \begin{aligned} F(u, v; \pi^{(\varphi)}) &= d_{00}u(0)v(0) + d_{01}u(0)v(1) \\ &\quad + d_{10}u(1)v(0) + d_{11}u(1)v(1), \end{aligned}$$

where $d_{00} = 0$, $d_{01} = d_{10} = k$, $d_{11} = \varphi(1) - 2k$.

Relation (9.9) follows readily from (9.4).

LEMMA 9.5. - Let $\bar{p}$ be left continuous over R_2 (Cf. Lemma 2.3) and a triangle d-function relative to $C \times C$. (Cf. § 1). Then for u, v in IL_2 ,

$$(9.10) \quad F(u, v; \bar{p}) = u(1)v(1)\bar{p}(1, 1) - \int_0^1 \bar{p}(1, s) d[u(s)v(s)].$$

To establish (9.10), observe that

$$\begin{aligned} \int_0^1 v(t) d_t \bar{p}(s, t) &= \int_0^s v(t) d_t \bar{p}(1, t) = v(s)\bar{p}(1, s) - \int_0^s \dot{v}(t) \bar{p}(1, t) dt, \\ F(u, v; \bar{p}) &= \int_0^1 u(s) d_s [v(s) \bar{p}(1, s)] - \int_0^1 u(s) \dot{v}(s) \bar{p}(1, s) ds = \\ &= u(1)v(1)\bar{p}(1, 1) - \int_0^1 [\dot{u}(s)v(s) + u(s)\dot{v}(s)] \bar{p}(1, s) ds, \end{aligned}$$

using SAKS, p. 102. Thus (9.10) holds as stated.

LEMMA 9.6. - If σ is a d-function relative to $L_2 \times L_2$ of triangle type,

$$(9.11) \quad F(\dot{u}, \dot{v}; \sigma) = \int_0^1 \sigma_2(1, t) \dot{u}(t) \dot{v}(t) dt.$$

Relation (9.11) follows immediately from the definitions involved.

10. **Proof of Theorem 1.1.** - In its explicit form this theorem is as follows.

THEOREM 10.1 - A bilinear sum $S(p, r)$ with values

$$S(u, v; p, r) = F(\dot{u}, \dot{v}; r) + F(u, v; p), \quad [\text{for } u, v \in IL_2]$$

in which r and p are symmetric d-functions relative to $L_2 \times L_2$ and $C \times C$ respectively such that the extremals of $\tilde{S}(p, r)$ satisfy (α) , (β) , (γ) of § 8, equals an essentially non-singular bilinear sum $J(P, Q, R, E)$ for suitable choice of P, Q, R and end form E .

The proof of this theorem will depend upon several lemmas, as follows.

LEMMA 10.1. - Under the hypotheses of Theorem 10. 1,

$$(10.1) \quad t[r(s, t) - r(s, 1)] = s[r(t, t) - r(t, 1)]$$

for $0 \leq s \leq t \leq 1$.

Each extremal of $\widehat{S}$, with $u(s)$ taken over the subinterval $[0, b]$ of $[0, 1]$, is by hypothesis an extremal of S_0^b . The associated EULER operator $\Psi_0^b(u, s)$ given by (7.16) is, by Theorem 7.2, linear in s over $[0, b]$. In case $0 < b < 1$, we infer that

$$(10.2) \quad \int_0^{b+} \dot{u}(t) d_t r(s, t) - \int_0^{b-} \dot{d}_t \Theta(s, t) = s k_u(b), \quad [0 \leq s \leq b]$$

where $k_u(b)$ is independent of s . That there is no term $h_u(b)$ on the right of (10.2) independent of s , follows on setting $s = 0$ in (10.2) and recalling that $r(0, t) \equiv \Theta(0, t) \equiv 0$. Given s and u , for almost all b on $(0, 1)$,

$$(10.3)' \quad \frac{d}{db} \int_0^{b+} \dot{u}(t) d_t r(s, t) = \dot{u}(b) r_2(s, b),$$

$$(10.3)'' \quad \frac{d}{db} \int_0^{b-} \dot{u}(t) d_t \Theta(s, t) = u(b) \Theta_2(s, b).$$

Relation (10.3)' follows on writing $r_2(s, t) dt$ in place of $d_t r(s, t)$, as is permissible since $r(s, \cdot)$ is absolutely continuous. Equation (10.3)'' follows from the relation (Cf. Lemma 7.2),

$$\int_0^{b-} \dot{u}(t) d_t \Theta(s, t) = u(b) \Theta(s, b) - \int_0^{b+} \dot{u}(t) \Theta(s, t) dt,$$

on differentiating the right member as to b . There accordingly exists a subset $E(s)$ of $[0, 1]$ of unit L -measure such that (10.3) holds for $b \in E(s)$. For $b \in E(s)$ and $b \geq s \geq 0$, (10.2) implies that for the given extremal u

$$(10.4) \quad \dot{u}(b) r_2(s, b) - u(b) \Theta_2(s, b) = s \dot{k}_u(b).$$

The extremals of $\widehat{S}$ enjoy property (γ) of Theorem 10.1 by hypothesis. There then exists a subset ω_1 of $[0, 1]$ of unit L -measure and a pair of extremals u and v of $\widehat{S}$ such that

$$(10.5) \quad W(b) = \begin{vmatrix} \dot{u}(b) & u(b) \\ \dot{v}(b) & v(b) \end{vmatrix} \neq 0 \quad [b \in \omega_1].$$

For this u and v , equation (10.4) and the corresponding equation

$$(10.6) \quad v(b) r_2(s, b) - v(b) \Theta_2(s, b) = s \dot{k}_v(b),$$

hold for b in a suitable subset $E_1(s)$ of $[0, 1]$ of unit L -measure and for $0 \leq s \leq b$. We can suppose $E(s) \subset \omega_1$. Equations (10.4) and (10.6) can then be solved for $r_2(s, b)$ and $\Theta_2(s, b)$, because $W(b) \neq 0$. On replacing b by t in this solution, one has

$$(10.7) \quad r_2(s, t) = s \lambda(t), \quad \Theta_2(s, t) = s \mu(t), \quad [0 \leq s \leq t \in E_1(s)]$$

introducing $\lambda(t)$ and $\mu(t)$.

From (10.7), taken with a fixed $s > 0$, we see that λ is integrable square over any closed subinterval of $(0, 1]$ since $r_2(s, \cdot)$ is in L_2 . Cf. (2.18). Since $r(s, \cdot)$ is absolutely continuous, (10.7) implies that

$$(10.8) \quad r(s, t) - r(s, 1) = s\zeta(t), \quad [0 \leq s \leq t \leq 1; 0 < t]$$

introducing the integral $\zeta(t)$ for $t > 0$. On setting $s = t \neq 0$ in (10.8), one finds that

$$(10.9) \quad t\zeta(t) = r(t, t) - r(t, 1).$$

Multiplying (10.8) by $t > 0$,

$$(10.10) \quad t[r(s, t) - r(s, 1)] = st\zeta(t) = s[r(t, t) - r(t, 1)].$$

Relation (10.1) is thus established as stated except for $t = s = 0$. It is obviously valid for $s = t = 0$

LEMMA 10.2 - Under the hypotheses of Theorem 10.1.

$$(10.11) \quad (t-1)[r(s, t) - r(1, t)] = (s-1)[r(t, t) - r(1, t)] \quad [0 \leq t \leq s \leq 1].$$

For each extremal u of $\widehat{S}$, for each subinterval $[a, 1]$ of $[0, 1]$, and for $0 \leq a \leq s \leq 1$,

$$(10.12) \quad \int_a^1 \dot{u}(t) dr(s, t) - \int_a^1 u(t) d_t \theta(s, t) = (s-1)A_u(a) + B_u(a),$$

using (7.16) and Theorem 7.2, and introducing $A_u(a)$ and $B_u(a)$. That A_u and B_u are differentiable as to a for almost all $a \in [0, 1]$ and for a fixed u is a consequence of the corresponding α -differentiability of the integrals in (10.12) for fixed u and s . For a given u and two fixed positive values of s , say $0 < s_1 < s_2 < 1$, (10.12) can be solved for $A_u(a)$ and $B_u(a)$ for $0 \leq a \leq s_1$ in terms of the integrals in (10.12). One thereby establishes the α -differentiability of A_u and B_u , p. p. in $(0, 1)$, for fixed u and s .

As in the proof of the preceding lemma, there exists a subset $\tau(s)$ of unit L -measure such that for $a \in \tau(s)$, $0 \leq a \leq s \leq 1$, and for the extremals u and v of (γ) , one can differentiate the integrals in (10.12) as to a , as well as $A_u(a)$ and $B_u(a)$, and for each such a similarly differentiate the integrals in (10.12) obtained by replacing u by v . One thus obtains equations of the form

$$\begin{aligned} -\dot{u}(a)r_2(a, s) + u(a)\theta_2(a, s) &= (s-1)\dot{A}_u(a) + \dot{B}_u(a), \\ -\dot{v}(a)r_2(a, s) + v(a)\theta_2(a, s) &= (s-1)\dot{A}_v(a) + \dot{B}_v(a), \end{aligned}$$

which can be solved for $r_2(s, a)$ in the form

$$(10.13) \quad r_2(s, a) = (s-1)\alpha(a) + \beta(a), \quad [0 \leq a \leq s \leq 1; a \in \tau(s)]$$

introducing $\alpha(a)$ and $\beta(a)$. On setting $s = 1$ in (10.13), one sees that β is in L_2 with $r_2(1, \cdot)$. Bearing in mind the limitation that $a \leq s \leq 1$, (10.13) also shows that α is in L_2 over any closed subinterval of $[0, 1]$.

It follows from (10.13) that

$$(10.14) \quad r(s, t) = (s-1)h(t) + k(t), \quad [0 \leq t \leq s \leq 1; t < 1]$$

on integrating (10.13) as to a from 0 to t . On setting $s = 1$ in (10.14), one sees that $k(t) = r(1, t)$. On setting $s = t < 1$, one finds that

$$(10.15) \quad (t-1)h(t) = r(t, t) - r(1, t).$$

Using (10.14),

$$\begin{aligned} (t-1)[r(s, t) - r(1, t)] &= (t-1)[(s-1)h(t)] \\ &= (s-1)[r(t, t) - r(1, t)] \quad [\text{from (10.15)}]. \end{aligned}$$

Thus (10.11) holds except possibly when $s = t = 1$. But for $s = t = 1$, the validity of (10.11) is trivial, and the proof of the lemma is complete.

LEMMA 10.3. - *Under the hypotheses of Theorem 10.1,*

$$(10.16)' \quad r(s, t) - r(s, 1) = cst - cs, \quad [0 \leq s \leq t \leq 1]$$

$$(10.16)'' \quad r(s, t) - r(1, t) = cst - ct, \quad [0 \leq t \leq s \leq 1]$$

for some constant c .

Set $Z(t) = r(t, t) - r(1, t)$. From (10.1) and (10.11) respectively and the relation $r = r^*$,

$$(10.17) \quad r(t, s) - r(t, 1) = \frac{t}{s} Z(s), \quad [0 < t \leq s \leq 1]$$

$$(10.18) \quad r(s, t) - r(1, t) = \frac{s-1}{t-1} Z(t), \quad [0 \leq t \leq s < 1]$$

so that

$$\frac{t}{s} Z(s) = \frac{s-1}{t-1} Z(t) \quad [0 < t \leq s < 1].$$

For $0 < t < 1$, it follows that $Z(t)/(t-1)$ is independent of t so that for some constant c ,

$$(10.19) \quad Z(t) = ct(t-1) \quad [0 \leq t \leq 1].$$

Equation (10.16)'' follows from (10.17) and (10.19), while (10.16)' then follows from the relation $r = r^*$.

The decomposition of r and p . Under the hypotheses of Theorem 10.1, the given d -function r thus admits a decomposition of the form

$$(10.20) \quad r(s, t) = \sigma(s, t) + c\rho(s, t),$$

where ρ is the d -function of Lemma 9.1 with $\rho(s, t) = st$ on $E' + E''$, and c is constant determined by the relation

$$(10.21) \quad r(t, t) - r(t, 1) = ct(t-1) \quad [0 < t < 1]$$

of Lemma 10.3. Relations (10.16) imply that

$$(10.22) \quad \sigma(s, t) = r(s, 1) - cs, \quad [0 \leq s \leq t \leq 1]$$

$$10.23) \quad \sigma(s, t) = r(1, t) - ct \quad [0 \leq t \leq s \leq 1].$$

From (10.22) and (10.23) it follows that σ is of triangle type. The d -function p admits a decomposition

$$(10.24) \quad p(s, t) = \bar{p}(s, t) + \pi^{(\varphi)}(s, t),$$

where $\bar{p}$ is the left continuous d -function such that $p(s, t) = \bar{p}(s, t)$ for $s < 1$, $t < 1$, and $\pi^{(\varphi)}$ is the edge function determined by

$$\varphi(s) = p(s, 1) - \bar{p}(s, 1).$$

LEMMA 10.4. - *Under the hypotheses of Theorem 10.1, the left continuous d -function $\bar{p}$ determined by p is of triangle type.*

With σ defined by (10.20),

$$(10.25) \quad S_a^b(u, v; \sigma, p) = S_a^b(u, v; r, p) - cF_a^b(u, v; \rho)$$

$$(10.26) \quad = S_a^b(u, v; r, p) - c[u(a) - u(b)][v(a) - v(b)],$$

so that $\widehat{S}_a^b(\sigma, p)$ and $\widehat{S}_a^b(r, p)$ differ at most by an end form, and so have the same extremals. In particular, the extremals of $\widehat{S}(\sigma, p)$ must satisfy (α) , (β) , (γ) with those of $\widehat{S}(r, p)$.

Let u then be an extremal of $\widehat{S}(\sigma, p)$. Given b with $0 \leq b < 1$, $\psi_b^1(u, s; \sigma, p)$ must be linear in s over $[b, 1]$ since (α) is satisfied by $\widehat{S}(\sigma, p)$ by hypothesis. Hence for $0 \leq b \leq s \leq 1$,

$$(10.27) \quad \int_b^1 u(t) d_t \sigma(s, t) - \int_b^s d\alpha \int_0^1 u(t) d_t p(\alpha, t) = sk_u(b) + h_u(b),$$

introducing $k_u(b)$ and $h_u(b)$. In accordance with (10.23), (10.22)

$$\int_b^1 u(t) d_t \sigma(s, t) = \int_b^s u(t) \sigma_2(1, t) dt \quad [0 \leq b \leq s \leq 1].$$

There accordingly exists a subset $\tau(b)$ of $[b, 1]$ of L -measure $1 - b$ such that for $s \in \tau(b)$ the s -derivatives of the integrals in (10.27) exist. Differentiation of (10.27) for $s \in \tau(b)$ then yields the relation

$$\dot{u}(s) \sigma_2(1, s) - \int_b^1 u(t) d_t p(s, t) = k_u(b) \quad [s \in \tau(b)].$$

Let a be any rational submultiple of b with $0 \leq a \leq b$, and let τ^* be the intersection of all sets $\tau(a)$ for fixed b and such a . The set τ^* will be a subset of $[b, 1]$ of measure $1 - b$. For any rational submultiple a of b with $0 \leq a \leq b$,

$$(10.28) \quad \dot{u}(s) \sigma_2(1, s) - \int_a^1 u(t) d_t p(s, t) = k_u(a) \quad [s \in \tau^*].$$

For two such a 's, say a and a' with $a < a'$, (10.28) implies that

$$\int_a^{a'-} u(t) d_t p(s, t) = k_u(a) - k_u(a').$$

and hence for s and s' in τ^* ,

$$(10.29) \quad \int_a^{a'-} u(t) d_t [p(s, t) - p(s', t)] = 0.$$

From Lemma 8.2 we draw the conclusion that

$$(10.30) \quad p(s, t) - p(s', t) = k, \quad [0 \leq t \leq b; s, s' \in \tau^*]$$

where k is independent of $t \in [0, b]$. Since $t = 0$ is admissible in (10.30) we infer that $k = 0$. The points s, s' in τ^* are everywhere dense in $[b, 1]$ so that (10.30) and the left continuity of $p(\cdot, t)$ imply that

$$(10.31) \quad p(s, t) = \bar{p}(1, t) \quad [0 \leq t \leq b < s < 1].$$

Since b was originally chosen as *any* constant such that $0 \leq b < 1$, (10.31) holds with $0 \leq t < s < 1$. That one may admit $0 \leq t = s < 1$ follows from (10.31) and the relations

$$(10.32) \quad \lim_{t \rightarrow s-} p(s, t) = p(s, s),$$

$$(10.33) \quad \lim_{t \rightarrow s-} \bar{p}(1, t) = \bar{p}(1, s).$$

Thus (10.31) holds for $0 \leq t \leq s < 1$. That the lemma holds as stated follows from the definition of $\bar{p}$ and the relation $\bar{p} = \bar{p}^*$.

LEMMA 10.5. - *Under the hypotheses of Theorem 10.1,*

$$(10.34) \quad S(u, v; r, \bar{p}) = \int_0^1 R(s) \dot{u}(s) \dot{v}(s) ds + \int_0^1 Q(s) [\dot{u}(s) v(s) + u(s) \dot{v}(s)] ds + E_i(u, v),$$

where

$$(10.35) \quad R(s) = r_2(1, s) - c, \quad Q(s) = -\bar{p}(1, s), \quad [0 \leq s \leq 1]$$

where c is a constant such that

$$r(t, t) - r(t, 1) = ct(t - 1), \quad [0 < t < 1]$$

and

$$(10.36) \quad E_i(u, v) = c[u(1) - u(0)][v(1) - v(0)] + u(1)v(1)\bar{p}(1, 1).$$

Moreover every extremal u of $\widehat{S}(r, p)$ is an extremal of $\widehat{S}(r, \bar{p})$.

It has been seen that r admits the decomposition

$$r(s, t) = \sigma(s, t) + \text{cst} \quad [(s, t) \in E' \times E'']$$

of (10.20), so that

$$S(u, v; r, \bar{p}) = F(\dot{u}, \dot{v}; \sigma) + F(\dot{u}, \dot{v}; c\rho) + F(u, v; \bar{p}).$$

But

$$F(\dot{u}, \dot{v}; \sigma) = \int_0^1 \sigma_2(1, s) \dot{u}(s) \dot{v}(s) ds \quad [\text{by Lemma 9.6}]$$

$$= \int_0^1 [r_2(1, s) - c] \dot{u}(s) \dot{v}(s) ds, \quad [\text{by (10.23)}]$$

$$F(\dot{u}, \dot{v}; c\rho) = c[u(1) - u(0)][v(1) - v(0)], \quad [\text{by Lemma 9.1}]$$

$$F(u, v; p) = u(1)v(1)\bar{p}(1, 1) - \int_0^1 \bar{p}(1, s) [\dot{u}(s)v(s) + u(s)\dot{v}(s)] ds \quad [\text{by Lemma 9.5}].$$

From these equations, (10.34) follows subject to (10.35) and (10.36).

That every extremal u of $\widehat{S}(r, p)$ is an extremal of $\widehat{S}(r, \bar{p})$ may be seen as follows. Since $p(s, t) = \bar{p}(s, t)$ for $s < 1$ and $t < 1$, the extremals of $\widehat{S}_0^b(r, \bar{p})$ must be identical with those of $\widehat{S}_0^b(r, p)$ for $0 < b < 1$, and hence by property (α) of $\widehat{S}(r, p)$ must include the extremals u of $\widehat{S}(r, p)$. The Euler operator of $\widehat{S}_0^b(r, \bar{p})$ takes the form

$$(10.37) \quad \int_0^s [R(t)\dot{u}(t) + Q(t)u(t)]dt - \int_0^s d\alpha \int_0^2 Q(t)\dot{u}(t)dt$$

for $0 \leq b \leq 1$. If the operator (10.37), for a fixed u , is linear in s on $[0, b]$, for each b with $0 < b < 1$, it is clearly linear in s on $[0, 1]$. Thus the extremals u of $\widehat{S}(r, p)$ must be extremals of $\widehat{S}(r, \bar{p})$.

LEMMA 10.6. - *Under the hypotheses of Theorem 10.1, the functional φ with values*

$$\varphi(s) = p(s, 1) - \bar{p}(s, 1)$$

is constant over $(0, 1)$.

It has been seen that p admits the decomposition $p = \bar{p} + \pi^{(\varphi)}$ of (10.24). In particular,

$$(10.38) \quad S(\eta, u; r, p) = S(\eta, u; r, \bar{p}) + F(\eta, u; \pi^{(\varphi)}).$$

If u is an extremal of $\widehat{S}(r, p)$, it is also an extremal of $\widehat{S}(r, \bar{p})$ by the preceding lemma, and hence by (10.38) an extremal of $\widehat{F}(\pi^{(\varphi)})$. Since the extremals of $\widehat{S}(r, p)$ satisfy (β), there is an extremal u such that $u(1) = 1$. It follows from Lemma 9.3 that $\varphi(s)$ is constant on $(0, 1)$.

Summary and proof of Theorem 10.1. We are supposing that the extremals of the quadratic functional $\widehat{S}$ defined by the bilinear sum

$$(10.39) \quad S(u, v; p, r) = F(\dot{u}, \dot{v}; r) + F(u, v; p)$$

satisfy (α), (β), (γ) of § 8. Let $\bar{p}$ be the left continuous d -function such that $p(s, t) = \bar{p}(s, t)$ for $s < 1$, $t < 1$, and set

$$(10.40) \quad \varphi(s) = p(s, 1) - \bar{p}(s, 1).$$

Then for a uniquely determined constant c ,

$$(10.41) \quad \begin{aligned} r(s, t) &= r(s, 1) + cs(t - 1), & \bar{p}(s, t) &= \bar{p}(s, 1) & [0 \leq s \leq t \leq 1] \\ r(s, t) &= r(1, t) + ct(s - 1), & \bar{p}(s, t) &= \bar{p}(1, t), & [0 \leq t \leq s \leq 1] \end{aligned}$$

and φ is constant on the open interval $(0, 1)$. The following theorem includes Theorem 10.1.

THEOREM 10.2. - *When the extremals of $\widehat{S}(r, p)$ satisfy (α), (β), (γ),*

$$(10.42) \quad S(u, v; p, r) = J(u, v; P, Q, R, E) \quad [\text{for } u, v \in IL_2]$$

where P, Q, R, E may be chosen as follows. Let $\bar{p}$ be the left continuous d -function determined by p . Then $p(s, 1) - \bar{p}(s, 1) = k$, a constant on $(0, 1)$. Let a

be determined by (10.41). For $0 \leq s \leq 1$, one may take

$$(10.43) \quad \begin{aligned} P(s) &\equiv 0, & Q(s) &= -\bar{p}(1, s), & R(s) &= r_2(1, s) - c, & [0 \leq s \leq 1] \\ E(u, v) &= c[u(1) - u(0)][v(1) - v(0)] + k[u(0)v(1) + u(1)v(0)] + \\ &\quad + [p(1, 1) - 2k]u(1)v(1), \end{aligned}$$

and J is essentially non-singular.

Since $p = \bar{p} + \pi^{(\varphi)}$, $S(u, v; r, p) = S(u, v; r, \bar{p}) + F(u, v; \pi^{(\varphi)})$. Relation (10.42) follows from the representation (10.34) of $S(u, v; r, \bar{p})$ on noting that

$$(10.44) \quad E(u, v) = E_1(u, v) + F(u, v; \pi^{(\varphi)}).$$

The formula (10.43) is a consequence of (10.44) on using (10.36) to evaluate $E_1(u, v)$ and (9.9) to evaluate $F(u, v; \pi^{(\varphi)})$, noting that

$$\bar{p}(1, 1) + \varphi(1) = \bar{p}(1, 1) + [p(1, 1) - \bar{p}(1, 1)] = p(1, 1).$$

That the functional J is essentially non-singular follows from Theorem 6.4. This completes the proof of Theorem 10.2 and Theorem 10.1.

11. Proof of Theorem 1.2. — Let a representation $S(p', r')$ in which p' is of triangle type be termed *normal*. The following is established in MT 5.

LEMMA 11.0. — *Any admissible bilinear sum $S(p, r)$ possesses a unique normal representation $S(p', r')$. Explicitly, $p'(s, 1) = p(s, 1)$ for $s \in E'$ and*

$$(11.1) \quad r'(s, t) = r(s, t) + \int_0^s \int_0^t [p(\alpha, \beta) - p'(\alpha, \beta)] d\alpha d\beta$$

for $(s, t) \in E' \times E''$.

The following lemma will be useful in establishing Theorem 1.2.

LEMMA 11.1 — *Let $S(p, r)$ be a bilinear sum in which $r(s, t) = \sigma(s, t) + cst$ for some constant c and for $(s, t) \in E' \times E''$, where p and σ are d -functions of triangle type relative to $C \times C$ and $L_2 \times L_2$ respectively. Then the extremal of $\widehat{S}(p, r)$ satisfy (α) of § 8.*

Let $[a, b]$ be a subinterval of $[0, 1]$. The Euler operator for $\widehat{S}_a^b(p, r)$ has the form [cf. (7.10)]

$$\int_a^s \dot{u}(t) d_t \sigma(1, t) + cs \int_a^b \dot{u}(t) dt - \int_a^s d\alpha \int_a^z \dot{u}(t) d_t p(1, t).$$

For a fixed $u \in IL_2$ this function changes at most by a linear function of s when $[a, b]$ is changed to another subinterval $[a', b']$ of $[0, 1]$. The lemma follows.

Theorem 1.2 stated more explicitly follows.

THEOREM 11.1 — *In order that an admissible bilinear sum $S(p, r)$ equal a bilinear sum $J(P, Q, R, E)$ for suitable choice of P, Q, R and the end form E , it is necessary and sufficient that the unique normal representation $S(p', r')$ of $S(p, r)$ in which p' is a triangle d -function satisfy $(\alpha), (\beta), (\gamma)$.*

That the condition of the theorem is sufficient is an immediate consequence of Theorem 1.1 or Theorem 10.1. The proof that the condition is necessary is as follows.

Since $J(P, Q, R, E)$ represents $S(p, r)$ by hypothesis, $S(p_P + p_E, q_Q, r_R + r_E)$ equally represents $S(p, r)$ in accordance with (5.9). By virtue of MT 5, this is possible only if the t -derivative of $q_Q(1, t)$ taken over $[0, 1]$ is L -equivalent to a function of bounded variation over $[0, 1]$, and hence L -equivalent to a $g \in C_d$. Hence Q is L -equivalent over $[0, 1]$ to a $g \in C_d$ so that in J ,

$$\begin{aligned} \int_0^1 Q(s)[\dot{u}(s)v(s) + \dot{v}(s)u(s)]ds &= \int_0^1 g(s)d[u(s)v(s)] = \\ &= u(1)v(1)g(1) - \int_0^1 u(s)v(s)dg(s) \quad [\text{cf. SAKS, p. 102}]. \end{aligned}$$

Let π_g be the d -function relative to $C \times C$ of triangle type, with $\pi_g(s, 1) = g(s)$ over $[0, 1]$. Then

$$\int_0^1 u(s)v(s)dg(s) = F(u, v; \pi_g),$$

and hence with $E(u, v)$ given by (5.8)

$$S(u, v; p, r) = S(u, v; p_P - \pi_g, r_R) + E_1(u, v),$$

where $E_1(u, v) = E(u, v) + u(1)v(1)g(1)$. Thus

$$S(u, v; p, r) = S(u, v; p_P - \pi_g + p_{E_1}, r_R + r_{E_1}).$$

Since $p_P - \pi_g + p_{E_1}$ is of triangle type and $S(p', r')$ is the unique representation in which p' is of triangle type, we infer that

$$p' = p_P - \pi_g + p_{E_1}, \quad r' = r_R + r_{E_1}.$$

But $r_{E_1}(s, t) = c_1 st$ over $E' \times E''$ for suitable choice of c_1 , and it follows from Lemma 11.1 that $S(p', r')$ satisfies (α) . That (β) and (γ) are satisfied by $S(p', r')$ follows from the fact that $S(p', r') = J(P, Q, R, E)$.

This completes the proof of the theorem.

COROLLARY 11.1 - *If any representation $S(p, r)$ satisfies (α) , (β) , (γ) , the normal representation $S(p', r')$ satisfies (α) , (β) , (γ) .*

From Theorem 10.1, $S(p, r) = J(P, Q, R, E)$ for suitable choice of P, Q, R, E , and it follows then from Theorem 11.1 that $S(p', r')$ satisfies (α) , (β) , (γ) .

12. Remarks and counter examples. — If $S(p, r)$ satisfies (α) , (β) , (γ) , other representations $S(\widehat{p}, \widehat{r})$ always exist *which do not satisfy* (α) . To see this, let π be a d -function relative to $C \times C$ so chosen as to vanish on the edges of $E' \times E''$ and so that the left continuous d -function determined by

$p + \pi$ is not of triangle type. Infinitely many such choices of π are clearly possible. Set

$$(12.1) \quad \sigma(s, t) = \int_0^s \int_0^t \pi(\alpha, \beta) d\alpha d\beta, \quad [(s, t) \in E' \times E'']$$

extending σ over R_2 so as to be canonical relative to $E' \times E''$. It is readily seen that $h(L_2, L_2; \sigma) \leq \max |\pi(\alpha, \beta)|$ so that σ is a d -function relative to $L_2 \times L_2$. Moreover, we shall show that

$$(12.2) \quad S(u, v; p, r) = S(u, v; p + \pi, r - \sigma) \quad [u, v \in IL_2].$$

Relations (12.2) will follow from Lemma 11.0 once it has been shown that the normal d -functions p', r' of Lemma 11.0, determined as in Lemma 11.0 by p, r , are the same when determined by $p + \pi, r - \sigma$. Since π vanishes on the edges of $E' \times E''$, it is clear that the normal p' is the same in both cases. The normal r' determined by p, r is given by (11.1), while the normal r'' determined by $p + \pi, r - \sigma$ has the values over $E' \times E''$,

$$r(s, t) - \sigma(s, t) + \int_0^s \int_0^t [p(\alpha, \beta) + \pi(\alpha, \beta) - p'(\alpha, \beta)] d\alpha d\beta = r'(s, t)$$

in accordance with (11.1) and (12.1). Thus (12.2) holds. But $S(p + \pi, r - \sigma)$ cannot satisfy (α) since the left continuous d -function determined by $p + \pi$ is not of triangle type. Cf. Lemma 10.4.

Counter examples will suffice to prove the following theorem.

THEOREM 12.1. - *The conditions (α) , (β) , (γ) on the extremals of $\bar{S}(p, r)$ are independent.*

(α) . That (α) is independent of (β) and (γ) is shown by setting up a $S(p, r)$ which satisfies (β) and (γ) , but not (α) .

To that end let p be the d -function of triangle type for which $p(1, t) = t$ on E'' . Let r be a sum $r' + r''$, where r' is a triangle d -function such that $r'(1, t) = t$ on E' while r'' is a d -function such that $r''(s, t) = s^2 t^2$ over $E' \times E''$. For this choice of p and r , the Euler equation of $\bar{S}_0^b(p, r)$ is

$$(12.3) \quad \Psi_0^b(u, s) = \int_0^b u(t) d_t r(s, t) - \int_0^s dx \int_0^b u(t) d_t p(x, t) = A_u(b)s + B_u(b),$$

introducing A_u and B_u . This becomes

$$(12.4) \quad \int_0^s u(t) dt + 2s^2 \int_0^b t u(t) dt - \int_0^s dx \int_0^x u(t) dt = A_u(b)s + B_u(b).$$

We seek an analytic extremal u . For such an extremal s -differentiation of (12.4) gives the condition

$$(12.5) \quad u(s) + 4s \int_0^b t u(t) dt - \int_0^s u(t) dt = A_u(b),$$

$$\ddot{u}(s) - u(s) = -4 \int_0^b t \dot{u}(t) dt.$$

Equation (12.5), and hence (12.4), has the solutions

$$u(s, A, B) = Ae^s + Be^{-s} + K_b(A, B),$$

where A and B are arbitrary constants and $K_b(A, B)$ has the unique value

$$(12.6) \quad K_b(A, B) = 4 \int_0^b t[Ae^t - Be^{-t}]dt.$$

To show that (β) is satisfied, note that $K_i(1, 0) = 4 \int_0^1 te^t dt > 0$, and that therefore the extremal of $\widehat{S}(p, r)$ for which $A = 1$, $B = 0$ is such that

$$(12.7) \quad u(s, 1, 0) = e^s + K_i(1, 0) > 0 \quad [0 \leq s \leq 1].$$

The given representation of $S(p, r)$ is in normal form in the sense of Theorem 11.1, so that, if (α) is satisfied by any representation, it must be satisfied by the given representation. It would accordingly be necessary that the extremal of $\widehat{S}(p, r)$ given by (12.7), taken on the interval $0 \leq s \leq b$, be an extremal of $S_0^b(p, r)$. For this, it is found to be necessary that

$$4 \int_0^1 te^t dt = K_i(1, 0) = K_b(1, 0) = 4 \int_0^b te^t dt,$$

which is impossible if $0 < b < 1$. Hence (α) is satisfied by no representation of $S(p, r)$.

To see that (γ) is satisfied, note first that there is at least one $u \in IL_2$ which is not an extremal of $\widehat{S}(p, r)$, namely a u such that $u(s) \equiv 1$ over $[0, 1]$. To verify the Wronskian condition, note that the extremals

$$u(s, 1, 0) = e^s + K_i(1, 0), \quad u(s, 0, 1) = e^{-s} + K_i(0, 1),$$

have the Wronskian

$$W(s) = -2 - K_i(1, 0)e^{-s} - K_i(0, 1)e^s.$$

Clearly $W(s) \neq 0$ p. p. on $[0, 1]$. Thus (γ) is satisfied as well as (β) , while (α) never holds.

(β) . That (β) is independent of (α) and (γ) is shown by a bilinear sum $S(p, r)$ with p, r so chosen that

$$(12.8) \quad S(u, v; p, r) = u\left(\frac{1}{2}\right)v\left(\frac{1}{2}\right) \quad [u, v \in IL_2].$$

One can take $r(s, t) \equiv 0$, and

$$\begin{aligned} p(s, t) &= 0, & \left[s \leq \frac{1}{2} \text{ or } t \leq \frac{1}{2} \right] \\ p(s, t) &= 1, & \left[s > \frac{1}{2} \text{ and } t > \frac{1}{2} \right] \end{aligned}$$

It is then clear that (12.8) holds. The first variation of $\widehat{S}(p, r)$ based on u takes the form $2\eta\left(\frac{1}{2}\right) u\left(\frac{1}{2}\right)$, so that $u \in IL_2$ is an extremal of $\widehat{S}(p, r)$ if and only if $u\left(\frac{1}{2}\right) = 0$. Thus (β) is not satisfied while (γ) clearly is. To test the satisfaction of (α) , observe that

$$S_a^b(u, v; p, r) = F_a^b(u, v; p) = e(a, b) u\left(\frac{1}{2}\right) v\left(\frac{1}{2}\right),$$

where $e(a, b) = 1$ or 0 according as the interval $[a, b]$ includes $s = 1/2$ or not. If $e(a, b) = 1$, the extremals of $\widehat{S}$ are exactly those of $\widehat{S}_a^b$. If $e(a, b) = 0$, the extremals of $\widehat{S}$ are a subclass of the extremals of $\widehat{S}_a^b$. In both cases (α) is satisfied. Thus (β) is independent of (α) and (γ) .

(γ) . That (γ) is independent of (α) and (β) is shown by the null functional $\widehat{S}(p, r)$ in which $p(s, t) = r(s, t) \equiv 0$. For every $u \in IL_2$ is an extremal of this functional and of $\widehat{S}_a^b(p, r)$, so that (α) and (β) are satisfied. However (γ) is not satisfied, and so is independent of (α) and (β) ⁽⁵⁾.

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