Ideals in $(m+1)$-semigroups.

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Summary. - *The following memoir is concerned auith various idealtheoretic aspects of the theory of polyadic semigroups. Many of the results are generalizations of known theorems in the theory of ordinary or 2-semigroups.*

1. **Introduction.** - The existence of extensive theories of groups, semigroups, and $(m+1)$-groups has motivated the author [9], [10] to pursue the analogous study of $(m+1)$-semigroups. By an $(m+1)$-*semigroup* is meant an algebraic system $(A, [\cdots])$ with one $(m+1)$-ary operation

$$[\cdots]: A^{m+1} \rightarrow A$$

satisfying tho associative law

$$[[x_1 x_2 \dots x_{m+1}] x_{m+2} \dots x_{2m+1}] = [x_1 [x_2 x_3 \dots x_{m+2}] \dots x_{2m+1}]$$

$$= \dots = [x_1 x_2 \dots [x_{m+1} x_{m+2} \dots x_{2m+1}]]$$

for any set of elements $x_1, x_2, \dots, x_{2m+1} \in A$. An $(m+1)$-*group*, in particular, is an $(m+1)$-semigroup possessing the additional property that for each $a_1, \dots, a_{1-1}, a_{1+1}, \dots, a_{m+1}, b \in A$, a unique solution in the indeterminate x_i exists for the equation

$$[a_1 \dots a_{i-1} x_i a_{i+1} \dots a_{m1}] = b$$

for each $i = 1, 2, \dots, m+1$.

The following, for example, are $(m+1)$-semigroups:

1°. - Trivially, if $(S, \cdot)$ is an ordinary semigroup (i.e. a 2-semigroup), then $(S, [\cdots])$ is an $(m+1)$-semigroup with

$$[x_1 x_2 \dots x_{m+1}] = x_1 x_2 \dots x_{m+1}.$$

2°. - Let $S_1, S_2 \dots, S_m$ be any collection of m pairwise disjoint sets.

Consider the collection $\boldsymbol{F}(S_1, S_2, \ldots, S_m)$ of all partial or full functions

$$f\colon \bigcup_{i=1}^{m} S_i \rightarrow \bigcup_{i=1}^{m} S_i$$

such that $f(S_1) \subseteqq S_2$, $f(S_2) \subseteqq S_3$, ..., $f(S_m) \subseteqq S_1$. Note that if $f_1, f_2, \in \boldsymbol{F}(S_1, S_2, \ldots, S_m)$, then $f_1f_2(S_1) \subseteqq S_3$, $f_1f_2(S_2) \subseteqq S_4$, ..., $f_1f_2(S_m) \subseteqq S_2$ so that $f_1f_2 \notin \boldsymbol{F}(S_1, \ldots, S_m)$. On the other hand, if $f_1, f_2, \ldots, f_{m+1}$ are any $m+1$ elements belonging to $\boldsymbol{F}(S_1, \ldots, S_m)$ then $[f_1f_2\ f_{m+1}] = f_1f_2\ f_{m+1} \in \boldsymbol{F}(S_1, \ldots, S_m)$. Thus, $\boldsymbol{F}(S_1, \ldots, S_m)$ forms an $(m+1)$-semigroup under the operation of composition of any $m+1$ functions.

3°. - More generally, σ be an arbitrary permutation of 1, 2, ..., m. As in the preceding example, the collection $\boldsymbol{F}^{\sigma}(S_1, \ldots, S_m)$ of all functions

$$f\colon \bigcup_{i=1}^{m} S_i \rightarrow \bigcup_{i=1}^{m} S_i$$

such that $f(S_i) \subseteqq S_{\sigma(i)}$ for each $i = 1, 2, \ldots, m$, also constitute an $(m+1)$-semigroup under $(m+1)$-composition. Example 2°, for instance, is a special case of 3° when σ is the cyclic permutation $(12 \ldots m)$.

4°. - Let $R(k_1, k_2, \ldots, k_m)$ be the collection of all m-tuples of matrices $A = (A_1, A_2, \ldots, A_m)$ over a ring R, where A_i is k_i by k_{i+1}, $i = 1, \ldots, m-1$, and A_m is k_m by k_1. Then $R(k_1, k_2, \ldots, k_m)$ is an $(m+1)$-semigroup under the operation

$$[A^1A^2 \ldots A^{m+1}] = (A_1^1A_2^2 \ldots A_1^{m+1},\ A_2^1A_3^2 \ldots A_2^{m+1},\ \ldots,\ A_m^1A_1^2 \ldots A_m^{m+1}),$$

where

$$A^i = (A_1^i, A_2^i, \ldots, A_m^i),\ i = 1, 2, \ldots, m+1.$$

We do not intend to pursue in this communication the general theory of $(m+1)$-semigroups in all its various ramifications, but instead we shall devote our efforts mostly to certain ideal-theoretic results on the theory of $(m+1)$-semigroups. A large bulk of these results are extensions of those in ordinary semigroups [2], [5], and [8].

It will be convenient in our later discussions to adopt at this point a few simplifying conventions in notation. A sequence of symbols $x_1x_2 \ldots x_i$, whether they be sets or individual elements, will be abbreviated to x_1^i. With this convention, the above associative law may now be more compactly written as

$$[[x_1^{m+1}]x_{m+2}^{2m+1}] = [x_1[x_2^{m+2}]x_{m+3}^{2m+1}] = \,..\, = [x_1^m[x_{m+1}^{2m+1}]].$$

When, in addition, $x_1 = x_2 = \ldots = x_i = x$, then we will write $x_1 x_2 \ldots x_i = x^i = (x_j)^i$ for any $j = 1, 2, \ldots, i$.

Recursively, one may also define

$$x^{<0>} = x, \; x^{<n+1>} = [x^{<n>}x^m]$$

for every natural number n. The following exponential laws are then easily verified for $(m+1)$-semigroups:

(1) $(x^{<r>})^{<s>} = x^{<rsm+r+s>}$,

(2) $[x^{<r_1>}x^{<r_2>} \ldots x^{<r_{m+1}>}] = x^{<r_1+ \ldots + r_{m+1}+1>}$.

2. **Ideals in Surjective $(m+1)$-Semigroups.** - We commence by stating a few definitions. Any subset S of an $(m+1)$-semigroup A that forms an $(m+1)$-semigroup under the same operation in A will be called a *sub-$(m+1)$-semigroup*. In particular, a subset I of A is called an *$(i+1)$-ideal* iff

$$[A^i I A^{m-i}] \subseteq I,$$

$i = 0, 1, \ldots, m$. By convention, $[A^0 I A^m] = [IA^m]$, $[A^m I A^0] = [A^m I]$, and $[A^0 I A^0] = I$. An $(i+1)$-ideal for each $i = 0, 1, \ldots, m$ is simply called an *ideal.*

The smallest $(i+1)$-ideal of an $(m+1)$-semigroup A containing an element $a \in A$ (called the *principal $(i+1)$-ideal* generated by a) will be denoted by $(a)_{i+1}$. Constructively, this is given by

$$(a)_{i+1} = \bigcup_{n=0}^{\infty} X_n,$$

where $X_0 = \{a\}$, $X_{n+1} = [A^i X_n B^{m-i}]$. If A is *surjective,* i.e. $A^{<1>} = A$, then it may be written as

$$(a)_{i+1} = \bigcup_{n=1}^{\infty} [A^{\overline{ni}} a A^{\overline{n(m-i)}}] \cup [[A^m a] A^m],$$

where $[A^0 a A^0] = \{a\}$ and $\overline{ni}$, $\overline{n(m-i)}$ respectively denote ni, $n(m-i)$ reduced modulo m. While the union operation above is still applied indefinitely, it is easy to see that only a finite number of the terms that appear are actually distinct.

Note that an exception to the above statement occurs when we have $m=2$:

$$(a)_1 = \{a\} \cup [aA^2],\ (a)_2 = \{a\} \cup [AaA] \cup [A[AaA]A],$$

$$(a)_3 = \{a\} \cup [A^2 a].$$

If $S \subseteq A$, then the $(i+1)$-ideal generated by S is given by

$$(S)_{i+1} = \bigcup_{x \in S} (x)_{i+1}.$$

Corresponding remarks may be made for an ideal (a) generated by an element $a \in A$.

That these various notions of ideals are not independent is shown by the following

THEOREM 2.1. - *Let A be a surjective $(m+1)$-semigroup. If the g.c.d. of i and m divides that of j and m, then $(a)_{j+1} \subseteq (a)_{i+1}$ for each $a \in A$ and $(a)_{i+1}$ is a $(j+1)$-ideal of A.*

PROOF. - Suppose that (i, m) divides (j, m). To prove that $(a)_{j+1} \subseteq (a)_{i+1}$ it suffices to show that for each non-negative n,

$$nj \equiv ki \pmod{m}$$

for some natural number k. Consider the congruence equation

$$ix \equiv j \pmod{m}.$$

By number theory, this always possesses a solution $x = x_0$ since (i, m) divides j. Hence

$$nj \equiv (nx_0)i \pmod{m} \text{ and therefore}$$

COROLLARY 2.2. - *In a surjective $(m+1)$-semigroup A, $(a)_{i+1} = (a)_{j+1}$ for each $a \in A$ iff $(i, m) = (j, m)$.*

COROLLARY 2.3. - *Every $(i+1)$-ideal of a surjective $(m+1)$-semigroup is a $(j+1)$-ideal iff $(i, m) = (j, m)$.*

COROLLARY 2.4. - *If m is prime, then every $(i+1)$-ideal of a surjective $(m+1)$-semigroup is also a $(j+1)$-ideal for all $i, j = 1, 2, \ldots, m-1$.*

COROLLARY 2.5. - *Every $(i+1)$-ideal of a surjective $(m+1)$-semigroup is an $(m-i+1)$-ideal for each $i = 1, 2, \ldots, m-1$, and conversely.*

COROLLARY 2.6. - *Each $(i+1)$-ideal of a surjective $(m+1)$-semigroup is contained in some 2-ideal (and hence in some m-ideal), $i = 1, 2, \ldots, m-1$; moreover, every 2-ideal is an $(i+1)$-ideal for each $i = 1, 2, \ldots, m-1$.*

An element z of an $(m+1)$-semigroup A is called an $(i+1)$-*zero iff* $[A^i z A^{m-i}] = z$ and simply a *zero* (denoted by 0) iff it is an $(i+1)$-zero for all $i = 0, 1, \ldots, m$. An $(i+1)$-ideal (ideal) will be said to be *minimal* iff it contains properly no other $(i+1)$-ideal (ideal). When an $(m+1)$-semigroup possesses no ideals except itself and possibly the ideal consisting of the zero element, it is often called *simple*. If a simple $(m+1)$-semigroup is not isomorphic to an $(m+1)$-semigroup of order two with a zero element (i.e. a *two-element null $(m+1)$-semigroup*), then it is said to be *nullsimple.*

THEOREM 2.7. - *Every minimal $(i+1)$-ideal M $(i = 1, 2, \ldots, m-1)$ of a surjective $(m+1)$-semigroup A without zero element may be written in the form*

$$M = \cup \, [A^{\overline{ni}} x A^{\overline{n(m-i)}}] \cup [[A^m x] A^m]$$

for any $x \in A$, the union running over all non-negative integers n such that $ni \not\equiv 0$, $n(m-i) \not\equiv 0 \pmod{m}$. On the other hand, every minimal 1 — ideal *($(m+1)$-ideal) of an arbitrary $(m+1)$-semigroup (not necessarily surjective) is of the form $[xA^m]$ ($[A^m x]$), x being any element of the ideal.*

PROOF. - Let M be any minimal $(i+1)$-ideal of A, $i = 1, 2, \ldots, m-1$, and $x \in M$. Then for all n such that $ni \not\equiv 0$, $n(m-i) \not\equiv 0 \pmod{m}$ the union

$$I = \cup \, [A^{\overline{ni}} x A^{\overline{n(m-i)}}] \cup [[A^m x] A^m]$$

is an $(i+1)$-ideal. Moreover,

$$I \subseteq (x)_{i+1} \subseteq M$$

and hence by minimality of M one obtains $I = M$. The proof of the second part is very similar.

COROLLARY 2.8. - *Every minimal* 2*-ideal* (*m-ideal*) *of a surjective* $(m+1)$*-semigroup* A *without zero is a minimal ideal.*

PROOF. - By Corollary 2.6, it will suffice to only show that a minimal 2-ideal is both a 1-ideal and an $(m+1)$-ideal. By Proposition 2.7, we know that

$$M = \bigcup_{i=1}^{m} [A^i x A^{m-i}] \cup [[A^m x] A^m]$$

The relations $[MA^m] \subseteq M$ and $[A^m M] \subseteq M$ are easily verified.

3. Ideal Series in $(m+1)$-Semigroups and the Jordan-Hölder Theorem. - The sequence of theorems that leads to the JORDAN-HOLDER theorem for ideals in $(m+1)$-semigroups will be derived in this section. Conditions necessary and sufficient for the existence of a composition or chief series in an $(m+1)$-semigroup will be given. All these are extensions of results in ordinary semigroups found in [2] and [8].

Before continuing, however, it will be necessary to clarify a few things. Consider an $(m+1)$-semigroup A and the relation $\equiv$ defined on A by an ideal I of A such that

$$x \equiv y \quad (I)$$

when and only when both x and y belong to I or $x = y$. It is easily verified that $\equiv$ is an equivalence relation on A. Moreover, if $x_i \equiv y_i (I)$ for each $i = 1, 2, \ldots, m+1$, then $[x_1^{m+1}] \equiv [y_1^{m+1}]$ (I). This means that $\equiv$ is a congruence (relation) on A. The quotient $(m+1)$-semigroup $A/\equiv$ or A/I consists then of the disjoint classes I and all $\{x\}$ for $x \in A - I$. For convenience we will not distinguish between $\{x\}$ and x. Note also that I is the zero element in A/I.

THEOREM 3.1. - *If* I *is an ideal and* S *is a sub-*$(m+1)$*-semigroup of an* $(m+1)$*-semigroup* A, *then* $I \cap S \neq \emptyset$ *is an ideal of and* $I \cup S$ *is a sub-*$(m+1)$*-semigroup of* A *such that*

$$(I \cup S)/I \cong S/(I \cap S).$$

PROOF. - Note that

$$(I \cup S)^{<1>} = \cup \{[X_1^{m+1}] \colon X_1 = I \text{ or } X_1 = S\} \subseteq I \cup S$$

and therefore $I \cup S$ is a sub-$(m+1)$-semigroup of A.

From the relationships

$$[S^i(I \cap S)S^{m-i}] \subseteqq [S^i I S^{m-i}] \subseteqq I$$

and

$$[S^i(I \cap S)S^{m-i}] \subseteqq S^{<1>} \subseteqq S,$$

which holds for all $i = 0, 1, \ldots, m$, it follows also that $I \cap S$ is an ideal of S. In exactly the same manner, it can be shown that I is an ideal of $I \cup S$. Both $(I \cup S)/I$ and $S/(I \cap S)$ are well-defined quotient $(m+1)$-semigroups. Finally

$$(I \cup S)/I = (I \cup S - I) \cup \{I\} = (S - I) \cup \{I\}$$

$$\cong (S - I) \cup \{I \cap S\} = S - (I \cap S) \cup \{I \cap S\} = S/(I \cap S).$$

THEOREM 3.2. - *Let I be an ideal of an $(m+1)$-semigroup A and h: $A \longrightarrow A/I$ be the natural homomorphism of A onto A/I. Then h induces an isomorphism h^* on the lattice $\mathbf{L}$ of all ideals J of A containing I onto the lattice $\mathbf{L}^*$ of all ideals J/I of A/I. Moreover,*

$$(A/J)/(J/I) \cong A/J.$$

PROOF. - Observe that the natural homomorphism h is the mapping that sends each $x \in I$ to the set I and all others to their singletons. If J is an ideal of A containing I, then trivially $h(J) = (J - I) \cup \{I\} = J/I$ so that we may define h^*: $\mathbf{L} \longrightarrow \mathbf{L}^*$ by $h^*(J) = J/I$. *If* K is any ideal of A/I, then $h^{-1}(K) = J$ is clearly also an ideal of A containing $I = h^{-1}(\{I\})$ and therefore $h^*(J) = K$. If $I \subseteqq J \subset K$, where J and K are are ideals of A, then $J - I \subset K - I$ so that

$$J/I = (J - I) \cup \{I\} \subset (K - I) \cup \{I\} = K/I.$$

This shows that the mapping h^* is strictly inclusion preserving on the lattice of all ideals J in A containing I onto the lattice of all ideals J/I of A/I and therefore a lattice isomorphism. As such it is one-to-one and therefore

$$(A/I)/(J/I) = (A/I - J/I) \cup \{J/I\}$$

$$= ((A - I) \cup \{I\}) - ((J - I) \cup \{I\}) \cup \{J/I\}$$

$$= (A - J) \cup \{J/I\} \cong (A - J) \cup \{J\} = A/J.$$

COROLLARY 3.3. - *If* J^* *is an ideal of* A/J *such that* $S/I \supset J^* \supset I/I$, *then there exists an ideal* J *of* A *such that* $A \supset S \supset J \supset I$ *and* $J^* = J/I$.

THEOREM 3.4. - *If* S_1 *aud* S_2 *are sub-*$(m+1)$*-semigroups of an* $(m+1)$*-semigroup* A *and* I_1, I_2 *are ideals of* S_1, S_2 *respectively, then*

$$(I_1 \cup (S_1 \cap S_2))/(I_1 \cup (S_1 \cap I_2)) \cong (I_2 \cup (S_1 \cap S_2))/(I_2 \cup (I_1 \cap S_2)).$$

PROOF. - Since $I_1 \cup (S_1 \cap S_2) \subseteqq S_1$ and

$$[(I_1 \cup (S_1 \cap S_2))^i I_1 (I_1 \cup (S_1 \cap S_2))^{m-i}] \subseteqq [(S_1)^i I_1 (S_1)^{m-i}]$$

$\subseteqq I_1$, then I_1 is an ideal of $I_1 \cup (S_1 \cap S_2)$. Since I_2 is an ideal of S_2, then

$$[(S_1 \cap S_2)^i (S_1 \cap I_2)(S_1 \cap S_2)^{m-i}]$$

is contained both in S_1 and in I_2 for all $i = 0, 1, \dots, m$ and therefore in $S_1 \cap I_2$. Hence, $S_1 \cap I_2$ is an ideal of $S_1 \cap S_2$. From these we obtain, for all $i = 0, 1, \dots, m$,

$$[(I_1 \cup (S_1 \cap S_2))^i (I_1 \cup (S_1 \cap I_2))(I_1 \cup (S_1 \cap S_2))^{m-i}]$$

$$= \cup \{[X_1^{m+1}] : X_{i+1} = I_1 \text{ or } X_{i+1} = S_1 \cap I_2,\ X_j = I_1 \text{ or } X_j = S_1 \cap S_2$$

for all other $j \neq i\} \subseteqq I_1 \cup (S_1 \cap I_2)$, which shows that $I_1 \cup (S_1 \cap I_2)$ is an ideal of $I_1 \cup (S_1 \cap S_2)$. Now,

$$(I_1 \cup (S_1 \cap I_2)) \cup (S_1 \cap S_2) = I_1 \cup (S_1 \cap S_2)$$

and hence by Theorem 3.1,

$$(I_1 \cup (S_1 \cap S_2))/(I_1 \cup (S_1 \cap I_2)) \cong$$
$$(S_1 \cap S_2)/((I_1 \cup (S_1 \cap I_2)) \cap (S_1 \cap S_2)).$$

On the other hand,

$$(I_1 \cup (S_1 \cap I_2)) \cap (S_1 \cap S_2) = (I_1 \cap (S_1 \cap S_2)) \cup ((S_1 \cap I_2) \cap (S_1 \cap S_2))$$
$$= (I_1 \cap S_2) \cup (S_1 \cap I_2)$$

and hence

$$(I_1 \cup (S_1 \cap S_2))/(I_1 \cup (S_1 \cap I_2)) \cong (S_1 \cap S_2)/((I_1 \cap S_2) \cup (S_1 \cap I_2)).$$

In exactly the same way, one may show

$$(I_2 \cup (S_1 \cap S_2))/(I_2 \cup (I_1 \cap S_2)) \cong (S_1 \cap S_2)/((I_1 \cap S_2) \cup (S_1 \cap I_2)).$$

Whence the result.

We introduce a few more terms. By a series of an $(m+1)$-semigroup A is simply meant a sequence

$$A = A_0 \supseteqq A_1 \supseteqq \ldots \supseteqq A_r = \emptyset$$

of sub-$(m+1)$-semigroups of A such that for each $i = 0, 1, \ldots, r-1$, A_{i+1} is an ideal of A_i. The quotient $(m+1)$-semigroups

$$A_0/A_1, \ \ldots, \ A_{r-1}/A_r$$

are called the *factors* of the series. *A refinement* of a series is another series whose terms include those of the former. A series is said to be *proper* iff all the inclusion relations occurring in the series are proper. A *composition series* is a series which is proper and possesses no proper series refinements with more terms. A proper series of an $(m+1)$-semigroup A every term of which is an ideal of A and which possesses no proper series refiniment with the same property is called a *chief series.*

Theorem 3.5. - *Any two series of an* $(m+1)$-*semigroup* A *possesses refiniments with isomorphic factors.*

Proof. - Consider any two series

$$A = A_0 \supseteqq A_1 \supseteqq \ldots \supseteqq A_r = \emptyset,$$

$$A = B_0 \supseteqq B_1 \supseteqq \ldots \supseteqq B_s = \emptyset,$$

of A and their corresponding refinements

$$A = A_{00} \supseteqq A_{01} \supseteqq \ldots \supseteqq A_{0s} = A_{10} \supseteqq \ldots \supseteqq A_{rs} = \emptyset,$$

$$A = B_{00} \supseteqq B_{10} \supseteqq \ldots \supseteqq B_{r0} = B_{01} \supseteqq \ldots \supseteqq B_{rs} = \emptyset,$$

defined by

$$A_{ij} = A_{i+1} \cup (A_i \cap B_j) \quad \text{and} \quad B_{ij} = B_{j+1} \cup (A_i \cap B_j),$$

$$i = 0, 1, \ldots, r-1, \quad j = 0, 1, \ldots, s-1.$$

Then by Theorem 3.3, we obtain

$$A_{ij}/A_{i,j+1} \cong B_{ij}/B_{i+1,j}$$

for all $i = 0, 1, \ldots, r-1$ and $j = 0, 1, \ldots, s-1$. The proof is thus completed.

COROLLARY 3.6. - *Any two series of an $(m+1)$-semigroup A all of whose terms are ideals of A possesses isomorphic refinements all of whose terms are also ideals of A.*

COROLLARY 3.7. - *(Jordan-Holder Theorem). Any two composition series (chief series) of an $(m+1)$-semigroup have isomorphic refiniments.*

THEOREM 3.8. - *Any sub-$(m+1)$-semigroup I of an $(m+1)$-semigroup A which occurs as a term in some series of A and which satisfies the property $I^{<1>} = I$ is an ideal of A.*

PROOF. - By hypothesis, A possesses a series

$$A = I_0 \supseteq I_1 \supseteq \ldots \supseteq I_n = I \supset \emptyset.$$

The result is obvious when $n = 0$ or $n = 1$. Suppose then that n is greater than 1. Then it is sufficient to show that if $I = I_n$ is an ideal of I_i for any i greater than 1, then I is also an ideal of I_{i-1}. If I is an ideal of I_i, then

$$I = I^{<1>} = I^{<2>} \subseteq [[(I_i)^m I](I_i)^m] \subseteq [I(I_i)^m] \subseteq I$$

so that $I = [[(I_i)^m I](I_i)^m]$. Hence, for each $k = 0, 1, \ldots, m$,

$$[(I_{i-1})^k I (I_{i-1})^{m-k}] = [(I_{i-1})^k [[(I_i)^m I](I_i)^m](I_{i-1})^{m-k}]$$

$$= [[(I_{i-1})^k (I_i)^{m-k+1}](I_i)^{k-1} I (I_i)^{m-k-1}[(I_i)^{k+1}(I_{i-1})^{m-k}]]$$

$$\subseteq [(I_i)^k I (I_i)^{m-k}] \subseteq I.$$

Thus, I is also an ideal of I_{i-1}.

Since $I = I_n$ is an ideal of I_{n-1}, then by induction $I = I_n$ must also be an ideal of $I_0 = A$.

Notice that if an $(m+1)$-semigroup A has a composition series, then its last non-empty term in every composition series is a minimum ideal. For, if K is this last therm in the composition series, then $K^{<1>}$ is an ideal of K and hence $K = K^{<1>}$. By the preceding theorem, this means that K is an ideal of A. Since K is however minimal the conclusion follows.

THEOREM 3.9. - *An* $(m+1)$*-semigroup* A *possesses a composition series if and only if the following conditions hold:*

(1) *Any proper series of* A *is finite;*

(2) *Any properly ascending sequence of ideals*

$$J_1 \subset J_2 \subset \dots \subset J_n \subset \dots$$

of an ideal J *of* A *is finite.*

PROOF. - **Sufficiency.** - Suppose that conditions (1) and (2) hold. Write $A = A_0$. If A' is any ideal of A_0 and A_0/A' has no proper non-zero ideal, then let $A_1 = A'$. Otherwise, if A_0/A' has a proper non-zero ideal, then there exists, by Corollary 3.3, an ideal A'' of A_0 such that $A' \subset A'' \subset A_0$. Now, if A_0/A'' has no proper non-zero ideal, set $A_1 = A''$. Otherwise, we repeat the process indefinitely. By condition (2), one must eventually arrive after a number of steps to an ideal A_1 such that A_0/A_1 possesses no proper non-zero ideal.

The whole process is again repeted for A_1 until one obtains an ideal A_2 such that A_1/A_2 has no proper non-zero ideal. In this manner, a descending sequence of sub-$(m+1)$-semigroups

$$A = A_0 \supset A_1 \supset \dots \supset A_n \supset \dots$$

is obtained such that each A_{i+1} is an ideal of A_i and A_i/A_{i+1} possesses no proper non-zero ideal. By condition (1), such a sequence can only have a finite number of terms which thus form a composition series for A.

Necessity. - Assume that A has a composition series with n terms. Consider any properly descending series of sub-$(m+1)$-semigroups of A:

$$A = A_0 \supset A_1 \supset \dots \supset A_k .$$

By the JORDAN-HOLDER theorem, then k is less than or equal to n.

The condition (1) thus holds. Let J then be any ideal of A and

$$J_1 \subset J_2 \subset \dots \subset J_m = J$$

be any propertly ascending sequence of ideals of J. By Theorem 3.5, the series

$$A \supset J_m \supset J_{m-1} \supset \dots \supset J_1 \supset \emptyset$$

can be refined to a composition series of A. Hence, $m \leq n$ and condition (2) is thus satisfied.

In exactly the same manner as the preceding the following result can be easily demonstrated:

Theorem 3.10. - *An* $(m+1)$-*semigroup* A *possesses a chief series if and only if the following conditions are satisfied:*

(1) *Any properly descending sequence of ideals of* A *is of finite length*;

(2) *Any properly ascending sequence of ideals of* A *is of finite length.*

From our previous results, it is clear that if an $(m+1)$-semigroup possesses a chief series, then any proper series of ideals of A can be refined into a chief series of A. Similarly, if A has a composition series, then the same series of ideals of A can be refined into a composition series of A. This means that the length of any series of ideals of A is finite. Consequently, if A possesses a composition series, then it must also possess a chief series. It is known that a 2-semigroup may have a chief series without necessarily having a composition series. Since any 2-semigroup may be converted into an $(m+1)$-semigroup, the same must also be true of $(m+1)$-semigroups.

An $(m+1)$-semigroup will be called *semisimple* if and only if it possesses a chief series all of whose factors are null-simple.

Note that, in general, any factor of a chief series of an $(m+1)$-semigroup A is simple. For, by Corollary 3.3, if J^* is an ideal of the factor A_i/A_{i+1} such that

$$A_{i+1}/A_{i+1} \subset J^* \subset A_i/A_{i+1},$$

then there exists an ideal J of A such that

$$A_{i+1} \subset J \subset A_i \subset A,$$

contrary to assumption. More precisely, A_i/A_{i+1} is either nullsimple or isomorphic to a two-element null $(m+1)$-semigroup. For, either

$$(A_i/A_{i+1})^{<1>} = A_i/A_{i+1} \quad \text{or} \quad (A_i/A_{i+1})^{<1>} = A_{i+1}/A_{i+1}.$$

Obviously, in the first case we have null-simplicity, while in the second we obtain a two element null $(m+1)$-semigroup. For suppose $\bar{0}$, is the zero and $\bar{a}$ is any non-zero element of A_i/A_{i+1}. Then $\{\bar{0}, \bar{a}\}$ is a non-zero ideal of A_i/A_{i+1} and hence

$$A_i/A_{i+1} = \{\bar{0}, \bar{a}\}.$$

The following supplies a condition when a composition series is also a chief series:

THEOREM 3.11. - *If A is a semisimple $(m+1)$-semigroup, then any series of A is a composition series iff it is a chief series.*

PROOF. - Consider any chief series

$$A = A_1 \supset A_2 \supset \ldots \supset A_n \supset A_{n+1} = \emptyset$$

of an $(m+1)$-semigroup A. We know that if for any $i = 1, 2, \ldots, n-1$, there is an ideal I_i of A_i such that

$$A_i \supset I_i \supseteqq A_{i+1},$$

then $I_i = A_{i+1}$ since A_i/A_{i+1} is simple. Thus the above series is also a composition series.

Let now

$$A = A_1 \supset A_2 \supset \ldots \supset A_n \supset A_{n+1} = \emptyset$$

be any composition series of A. To show that it is also a chief series, it will suffice to show that $A_i^{<1>} = A_i$ for each $i = 1, 2, \ldots, n$ (by Theorem 3.8). One proceeds by backward finite induction. It is obvious that $A_n^{<1>} = A_n$. Suppose that $A_k^{<1>} = A_k$ for all $k \geqq i+1$, so that $A_{i+1}^{<1>} = A_{i+1}$ in particular. Then

$$A_i \supseteqq A_i^{<1>} \supseteqq A_{i+1}^{<1>} = A_{i+1}.$$

On the other hand, $A_i^{<1>}$ is an ideal of A_i and therefore

$$A_i^{<1>} = A_i \quad \text{or} \quad A_i^{<1>} = A_{i+1}.$$

The latter case implies that

$$(A_i/A_{i+1})^{<1>} = A_i^{<1>}/A_{i+1} = A_{i+1}/A_{i+1}$$

contrary to the nullsimplicity of A_i/A_{i+1}. Hence $A_i^{<1>} = A_i$ and our induction is complete.

Corollary 3.12. – *A semisimple* $(m+1)$*-semigroup is surjective.*

It should be noted that by virtue of the previous theorem semisimplicity may just as well be characterized in terms of composition series rather than its chief series.

Theorem 3.13. – *An* $(m+1)$*-semigroup* A *is semisimple if and only if both* A/I *and* I *are semisimple for each ideal* I *of* A.

Proof. – If I is an improper ideal the result is obvious. Suppose then that I is a proper ideal and A is semisimple.

The series $A \supset I$ possesses a composition series refiniment

$$A = A_0 \supset A_1 \supset \dots \supset A_r = I \supset \dots \supset A_n = \emptyset.$$

Thus,

$$A/I = A_0/I \supset A_1/I \supset \dots \supset A_r/I = I/I \supset \emptyset$$

is a composition series of A/I and by Theorem 3.2,

$$(A_i/I)/(A_{i+1}/I) \cong A_i/A_{i+1},$$

the last quotient $(m+1)$-semigroup being also nullsimple. Hence A/I is semisimple. Moreover,

$$I = A_r \supset A_{r+1} \supset \dots \supset A_n = \emptyset$$

is a composition series for I such that A_i/A_{i+1} $(i = r, \dots, n-1)$ is nullsimple and therefore A is semisimple.

Conversely, suppose that both I and A/I are semisimple.

If then

$$A/I = A_0^* \supset A_1^* \supset \dots \supset A_r^* = I/I$$

is a composition series for A/I, then by a previous result, there must exist a sequence of $(m+1)$-semigroups

$$A = A_0 \supset A_1 \supset \dots \supset A_r = I$$

such that A_{i+1} is an ideal of A_i and $A_i^* = A_i/I$. Moreover,

$$A_i/A_{i+1} \cong A_i^*/A_{i+1}^*$$

and by hypothesis this is nullsimple. If

$$I = A_r \supset A_{r+1} \supset \dots \supset A_n = \emptyset$$

is a composition series for I, then

$$A = A_0 \supset A_1 \supset \dots \supset A_r \supset \dots \supset A_{n-1} \supset A_n = \emptyset$$

is a composition series for A and all its factors are nullsimple.

THEOREM 3.14. - *An* $(m+1)$*-semigroup* A *which possesses a chief series is semisimple if and only if every ideal* I *of* A *satisfies the condition* $I^{<1>} = I$.

PROOF. - Let A be a semisimple $(m+1)$-semigroup and I be any ideal of A. By the previous thorem, then I is semisimple and hence surjective.

Conversely, let A possess the chief series

$$A = A_0 \supset A_1 \supset \dots \supset A_n = \emptyset$$

and suppose that all ideals of A satisfy the given condition. If any factor, say A_i/A_{i+1}, were a two-element null $(m+1)$-semigroup, then

$$A_i^{<1>} \subseteq A_{i+1} \subset A_i$$

contrary to hypothesis. Thus all factors of the series must be nullsimple.

COROLLARY 3.15. - *The collection of all ideals of a semisimple* $(m+1)$*-semigroup* A *forms a commutative* $(m+1)$*-semigroup.*

PROOF. - Let $I_i (i = 1, 2, \dots, m+1)$ be any $m+1$ ideals of A. Then

$$[I_1^{m+1}] \subseteq \bigcap_{i=1}^{m+1} I_i = [\bigcap_{i=1}^{m+1} I_i]^{<1>} \subseteq [I_1^{m+1}].$$

From this and the commutativity of the of the intersection operation, we obtain

$$[I_1^{m+1}] = [I_{\varnothing(1)}^{\varnothing(m+1)}],$$

for all permutations $\varnothing$ of the integers $1, 2, \ldots, m+1$.

Theorem 3.16. - *Let A be an $(m+1)$-semigroup possessing a chief series. The collection $\boldsymbol{X}$ of all ideals I of A such that A/I is semisimple has a minimum member, the ideal M contained in all members of $\boldsymbol{X}$.*

Proof. - Consider any pair J, $I \in \boldsymbol{X}$. From Theorem 3.1,

$$(I \cup J)/I \cong J/(I \cap J).$$

$(I \cup J)/I$ being an ideal of the semisimple $(m+1)$-semigroup A/I is itself semisimple. Thus $J/(I \cap J)$ is semisimple. Since $J \in \boldsymbol{X}$, then A/J is also semisimple. From the relation

$$(A/(I \cap J))/(J/(I \cap J)) \cong A/J$$

(see Theorem 3.2), it follows that $A/(I \cap J)$ is semisimple and therefore $I \cap J \in \boldsymbol{X}$. By Zorn's lemma, $\boldsymbol{X}$ possesses a minimal member M. For any $I \in \boldsymbol{X}$, then $I \cap M = M$ and therefore $M \subseteq I$. If M^* is another minimal element of $\boldsymbol{X}$, then $M^* = M^* \cap M = M$. The result is now clear.

4. Certain Structure Space of an $(m+1)$-Semigroup. - An ideal I in an $(m+1)$-semigroup A will be called *irreducible* iff for any pair of ideals J and K in A,

$$I \supseteq J \cap K \text{ implies either } I \supseteq J \text{ or } I \supseteq K.$$

An ideal is *complely prime* iff $[x_1^{m+1}] \in P$ implies $x_i \in P$ for some $i = 1, 2, \ldots, m+1$ It is *prime* iff for any set of ideals $I_1, I_2, \ldots, I_{m+1}$, if $P \supseteq [I_1^{m+1}]$, then $P \supseteq I_i$ for some $i = 1, 2, \ldots, m+1$.

Corollary 4.1. - *An ideal P of an $(m+1)$-semigroup A is completely prime iff $A - P$ is a sub-$(m+1)$-semigroup of A.*

This is a mere translation of the definition in contrapositive terms.

An $(m+1)$-semigroup is said to be commutive iff for every set of elements $x_1, x_2, \ldots, x_{m+1}$ and each permutation $\varnothing$ of $1, 2, \ldots, m+1$, we have

$$[x_1^{m+1}] = [x_{\varnothing(1)} x_{\varnothing(2)} \ldots x_{\varnothing(m+1)}] = [x_{\varnothing(1)}^{\varnothing(m+1)}].$$

Theorem 4.2. - *The following are equivalent conditions for a commutative* $(m+1)$*-semigroup* A:

(1) P *is a completely prime ideal of* A;

(2) *For any set of elements* $a_1, a_2, \ldots, a_{m+1} \in A$, if $P \supseteq [(a_1)(a_2) \ldots (a_{m+1})]$, *then* $P \supseteq (a_i)$ *for some* $i = 1, 2, \ldots, m+1$.

(3) P *is a prime ideal.*

Proof. - Assume (1) and $P \supseteq [(a_1)(a_2) \ldots a_{m+1})]$. Hence $[a_1^{m+1}] \in P$ and by (1) therefore $a_i \in P$ for some i. This means that $P \supseteq (a_i)$ for some i. To prove its converse, note first that $(a_i) = \{a_i\} \cup [A^m a_i]$ for each $i = 1, \ldots, m+1$. Let $[a_1^{m+1}] \in P$ and assume (2).

If now $[x_1^{m+1}] \in [(a_1)(a_2) \ldots (a_{m+1})]$, then by virtue of commutativity

$$[x_1^{m+1}] = [[a_1^{m+1}] y_i^m]$$

for some elements $y_1, \ldots, y_m \in A$ so that $[x_1^{m+1}] \in P$. This means $P \supseteq [(a_1)(a_2) \ldots \ldots (a_{m+1})]$ and hence by (2), $P \supseteq (a_i)$ for som $i = 1, \ldots, m+1$. Whence $a_i \in P$ for some $i = 1, 2, \ldots, m+1$.

That (3) implies (2) is clear. It thus remains to show that (1) implies (3). Suppose (1) holds and $P \supseteq [I_1^{m+1}]$ for any set of ideals $I_1, I_2, \ldots, I_{m+1}$ of A, but $P \not\supseteq I_j$ for all $j \neq i$. Then for some $x_j \in I_j$ $(j \neq i)$, $x_j \notin P$. For any $x_i \in I_i$, then $x_i \in P$ since $[x_1^{m+1}] \in P$. Whence it follows that $P \supseteq I_i$.

Corollary 4.3. - *Every prime ideal* P *in an* $(m+1)$*-semigroup* A *is irreducible.*

Proof. - Let $P \supseteq I \cap J$ for any pair of ideals in A.

Since $I \cap J \supseteq [I^k J^{m-k+1}]$ for any non-negative k, then, by the previous theorem 4.2 (3), we obtain $P \supseteq I$ or $P \supseteq J$.

Consider now any subfamily $\boldsymbol{I}$ of the family of all irreducible ideals in an $(m+1)$-semigroup A. For any $x \in A$, set

$$\boldsymbol{I}_x = \{I\colon\ I \in \boldsymbol{I} \text{ and } x \notin \boldsymbol{I}\}.$$

The topology generated by all these sets as subbase is the socalled *Stone-Gelfand topology* on $\boldsymbol{I}$.

Theorem 4.5. – *The closure of any subset* $\boldsymbol{S}$ *of* $\boldsymbol{I}$ *under its Stone-Gelfand topology is given by*

$$\overline{\boldsymbol{S}} = \{I \colon\ I \supseteq \bigcap_{J \in S} J,\quad I \in \boldsymbol{I}\}.$$

Proof. – Let $\boldsymbol{S}^*$ be equal to the right side of the above relation. If $\boldsymbol{I}_x$ is a neighborhood of $I \in \boldsymbol{S}^*$, then $x \notin I$ so that $x \notin \bigcap_{J \in S} J$. This means that for some ideal $J_0 \in S$,

$$J_0 \supseteq \bigcap_{J \in S} J \quad \text{and} \quad x \notin J_0$$

and also $J_0 \in \boldsymbol{I}_x$. Hence $\boldsymbol{I}_x \cap \boldsymbol{S} \neq \emptyset$, in other words, $I \in \overline{S}$. Whence $\boldsymbol{S}^* \subseteq \overline{S}$.

To prove the other inclusion, choose any irreducible ideal $I \notin \boldsymbol{S}^*$. If $\bigcap_{J \in S} J = \emptyset$, then $\boldsymbol{S}^* = \boldsymbol{I}$ and hence $\overline{\boldsymbol{S}} \subseteq \boldsymbol{S}^*$. If $\bigcap_{J \in S} J \neq \emptyset$, then $\bigcap_{J \in S} J - I \neq \emptyset$. For any $x \in \bigcap_{J \in S} J - I$, then $x \in J$ for all $J \in \boldsymbol{S}$ but $x \notin I$. This means that $I \in \boldsymbol{I}_x$ but $J \notin \boldsymbol{I}_x$ for all $J \in \boldsymbol{S}$. Therefore we have $\boldsymbol{I}_x \cap \boldsymbol{S} = \emptyset$ and $I \notin \overline{\boldsymbol{S}}$. This completes the proof.

Theorem 4.6. – *The mapping* $\boldsymbol{S} \to \overline{\boldsymbol{S}}$ *is a closure operation, that is to say,*

(1) $\boldsymbol{S} \subseteq \overline{\boldsymbol{S}}$;

(2) $\overline{\boldsymbol{S}} = \overline{\overline{\boldsymbol{S}}}$;

(3) $\boldsymbol{S}_1 \subseteq \boldsymbol{S}_2$ implies $\overline{\boldsymbol{S}}_1 \subseteq \overline{\boldsymbol{S}}_2$;

with the additional properties:

(4) $\overline{\{I_1\}} = \overline{\{I_2\}}$ implies $I_1 = I_2$;

(5) $\overline{\boldsymbol{S}_1 \cup \boldsymbol{S}_2} = \overline{\boldsymbol{S}}_1 \cup \overline{\boldsymbol{S}}_2$.

Proof. – (1) — (3) are clear.

(4). – By (1) $I_2 \in \overline{\{I_1\}}$ and therefore $\{I_2\} \supseteq \{I_1\}$.

Similarly, $\{I_1\} \supseteq \{I_2\}$. Whence $I_1 = I_2$.

(5). - From $S_1 \subseteq S_1 \cup S_2$ and $S_2 \subseteq S_1 \cup S_2$ we obtain $\bar{S}_1 \subseteq \overline{S_1 \cup S_2}$ and $\bar{S}_2 \subseteq \overline{S_1 \cup S_2}$ and hence $\bar{S}_1 \cup \bar{S}_2 \subseteq \overline{S_1 \cup S_2}$ by (3). If $P \notin \bar{S}_1 \cup \bar{S}^2$ so that $P \notin \bar{S}$ and $P \notin \bar{S}_2$, then $P \not\supseteq \bigcap_{J \in S_1} J$ and $P \not\supseteq \bigcap_{J \in S_2} J$. *If* $\bigcap_{J \in S_i} J = \emptyset$, then $\bar{S}_i = I$ and $\bar{S}_1 \cup \bar{S}_2 = \overline{S_1 \cup S_2}$. Suppose then that $\bigcap_{J \in S_i} J \neq \emptyset$, $i = 1, 2$. Since these are ideals, then

$$P \not\supseteq \bigcap_{J \in S_1} J \cap \bigcap_{J \in S_2} J = \bigcap_{J \in S_1 \cup S_2} J.$$

For, if otherwise, then by irreducibility of P, either

$$P \supseteq \bigcap_{J \in S_1} J \quad \text{or} \quad P \supseteq \bigcap_{J \in S_2} J,$$

contrary to assumption. Whence $P \notin \overline{S_1 \cup S_2}$. The proof is thus completed.

THEOREM 4.7. - *Any subset* $S \subseteq I$ *is dense in* I *iff*

$$\bigcap_{J \in S} J = \bigcap_{J \in I} J.$$

PROOF. - Let S be dense in I, i.e. $\bar{S} = I$, under the STONE-GELFAND topology. Thns,

$$\{I\colon I \supseteq \bigcap_{J \in S} J \quad \text{and} \quad I \in I\} = I,$$

which means that each $I \in I$ satisfies the condition $I \supseteq \bigcap_{J \in S} J$. Whence

$$\bigcap_{J \in I} J \supseteq \bigcap_{J \in S} J.$$

The other inclusion is obvious.

Conversely, suppose $I - \bar{S} \neq \emptyset$. Then there is some irreducible ideal $I \in I$ with $I \notin \bar{S}$. This means that for some I_x, $I \in I_x$ with $I_x \cap S = \emptyset$. In other words.

$$\bigcap_{J \in I} J \overset{\subset}{\neq} \bigcap_{J \in S} J,$$

a contradiction!

LEMMA 4.8. - *If* B *is any sub-*$(m+1)$*-semigroup of a commutative*

$(m+1)$-semigroup A disjoint from an ideal I of A, then there exists an ideal M of A maximal under the property of being disjoint from B. In addition, M is also prime.

Proof. - By Zorn's lemma, the existence of M is assured. It thus remains to show that $A-M$ is a sub-$(m+1)$-semigroup of A. One proceeds indirectly. Suppose $x_1, x_2, \ldots, x_{m+1} \in A-M$, but $[x_1^{m+1}] \in M$. Consider the ideals I_i generated by $M \cup \{x_i\}$, for each $i = 1, 2, \ldots, m+1$. Since M is maximal with respect to its disjointness from I, then $I \cap I_i \neq \emptyset$. Let $y_i \in I \cap I_i$. Then $[x_1^{i-1} y_i x_{i+1}^{m+1}] \in M$. To prove this, we shall proceed by induction on k, k being defined by,

$$I_i = \bigcup_{k=0}^{\infty} Y_k,$$

where $Y_0 = M \cup \{x_i\}$ and $Y_{n+1} = [A^{i-1} Y_n A^{m-i+1}]$. When $y_i \in Y_0$ the result is obvious. Suppose then that the result is true for all $y_i \in Y_k$ with $k \leqq n$. Consider now $y_i \in Y_{n+1} = [A^{i-1} Y_n A^{m-i+1}]$. Then $y_i = [z_1^{m+1}]$ where

$$z_i \in Y_n,\ z_1, \ldots, z_{m+1} \in A,\ [x_1^{i-1} z_i x_{i+1}^{m+1}] \in M.$$

Thus

$$[x_1^{i-1}[z_1^{m+1}]\, x_{i+1}^{i+1}] = [z_1^{i-1}[x_1^{i-1} z_i x_{i+1}^{m+1}]\, z_{i+1}^{m+1}] \in M.$$

By repeating the process on $[x_1^{i-1} y_i x_{i+1}^{m+1}] \in M$ instead of $[x_1^{m+1}] \in M$, we will eventually arrive to the conclusion that $[y_1^{m+1}] \in M$, which is a contradiction.

A prime ideal P is called a *minimal prime ideal belonging to the ideal* I iff $I \subseteqq P$ and no other prime ideal containing I is properly contained in P.

Theorem 4.8. - *A subset P of a commutative $(m+1)$-semigroup A is a minimal prime ideal belonging to an ideal I if and only if $A-P$ is a sub-$(m+1)$-semigroup of A maximal with respect to the property of being disjoint from I.*

Proof. - First, assume that $P \subseteqq A$ and $A-P$ is a sub-$(m+1)$-semigroup of A maximal with respect to its property of disjointness from I. By the preceding Lemma 4.7, then I is contained in a prime ideal M maximal with respeet to its being disjoint from $A-P$. This means $I \subseteqq M \subset P$ so that $A-P \subseteqq A-M$. On the other hand, $A-M$ is a sub-$(m+1)$-semigroup of A disjoint from I, by virtue of Corollary 4.1. Hence $A-M \subseteqq A-P$ and therefore $A-P = A-M$ or $M = P$.

Conversely, suppose P is a minimal ideal belonging to the ideal I of A. Then P is prime and hence $A - P$ is a sub-$(m+1)$-semigroup of A disjoint from I. By Zorn's lemma, then there is a maximal sub-$(m+1)$-semigroup B of A disjoint from I. By the preceding proof, then $A - B$ is a minimal prime ideal belonging to I so that $I \subseteqq A - B \subseteqq P$. Whence $A - B = P$ and $A - P = B$ is a maximal sub-$(m+1)$-semigroup of A disjoint from I.

By an application of Zorn's lemma and the preceding theorem, we easily derive the following

Corollary 4.9. - *If P is a prime ideal containing the ideal I of an $(m+1)$-semigroup A, then there exists a minimal prime ideal belonging to I contained in P.*

The *radical* $R(I)$ of an ideal I of an $(m+1)$-semigroup A is defined as

$$R(I) = \{x \colon x \in A \text{ and for some } n \geqq 0,\ x^{<n>} \in I\}.$$

An ideal I may be called *radical* iff $I = R(I)$. As in [9] we will say that A is a *strongly reversible* $(m+1)$-semigroup iff for each x_1, x_2 .. ..., $x_{m+1} \in A$, there exists non-negative integers $n, n_1, \ldots, n_m$ such that

$$[x_1^{m+1}]^{<n>} = [x_{\varnothing(1)}^{<n_{\varnothing(1)}>} x_{\varnothing(2)}^{<n_{\varnothing(2)}>} \ldots x_{\varnothing(m+1)}^{<n_{\varnothing(m+1)}>}]$$

for any permutation $\varnothing$ of $1, 2, \ldots, m+1$. Note that any commutative $(m+1)$-semigroup is strongly reversible. An $(m+1)$-semigroup A is said to be *homogenous* when and only when for each $a \in A$, the cyclic $(m+1)$-$[a]$ generated by a contains an idempotent, i.e an element e such that $e^{<1>} = e$. Note that a cyclic $(m+1)$-semigroup or an $(m+1)$-group need not possess an idempotent. The cyclic $(n+1)$-semigroup generated by a such that $a^{<1>} = a^{<3>}$, for instance, has no idempotent (see [9]).

Theorem 4.10. - *The radical $R(I)$ of an ideal I of a strongly reversible $(m+1)$-semigroup A is an ideal.*

Proof. - Let $a_{j+1} \in R(I)$ and $a_1, a_2, \ldots, a_{m+1} \in A$. Then for some integer s,

$$a_{j+1}{}^{<s>} \in I$$

and by strong reversibility, there exist integers $n, n_1, \ldots, n_{m+1}$ such that

$$[a_1^{m+1}]^{<n>} = [a_{\emptyset(1)}^{<n_{\emptyset(1)}>} a_{\emptyset(2)}^{<n_{\emptyset(2)}>} \dots a_{\emptyset(m+1)}^{<n_{\emptyset(m+1)}>}]$$

for any permutation $\emptyset$ of $1, 2, \dots, m+1$. Then

$$([a_1^{m+1}]^{<n>})^{<s>} = ([a_{j+1}^{<n_{j+1}>} \dots a_{j+1}^{<n_{j+1}>} \dots a_{m+1}^{<n_{m+1}>}])^{<s>}$$

$$= [(a_1^{<n_1>})^{<s>} \dots (a_{j+1}^{<n_{j+1}>})^{<s>} \dots (a_{m+1}^{<n_{m+1}>})^{<s>}]$$

$$= [a_1^{<s>})^{<n_1>} \dots (a_{j+1}^{<s>})^{<n_{j+1}>} \dots (a_{m+1}^{<s>})^{<n_{m+1}>}] \in I.$$

Whence $[a_1^{m+1}] \in R(I)$ and since j is arbitrary this shows that $R(I)$ is an ideal.

Theorem 4.11. - *If I is an ideal of a strongly reversible and homogenous $(m+1)$-semigroup A and E is the collection of all idempotents of I, then*

$$R(I) = \bigcup_{e \in E} S_e .$$

Proof. - If $a \in \bigcup_{e \in E} S_e$ so that $a \in S_e$ for some $e \in E$, then $a^{<s>} = e$ for some integer s. Thus $a \in R(I)$ and therefore $\bigcup_{e \in E} S_e \subseteq R(I)$. Conversely, suppose $a \in R(I)$ so nhat $a^{<s>} \in I$ for some integer s. Since for some non-negative t, $(a^{<s>})^{<t>} = e \in S_e$ for some $e \in E$, then $a \in \bigcup_{e \in E} S_e$. Whence the result.

Theorem 4.12. - *The intersection of any collection of prime ideals P_i, $i \in T$, of an $(m+1)$-semigroup is a radical ideal.*

Proof. - Let $I = \bigcap_{i \in T} P_i$. Clearly $I \subseteq R(I)$. For each $x \in R(I)$, there exists an integer $s \geqq 0$ such that

$$x^{<s>} \in I = \bigcap_{i \in T} P_i .$$

Hence $x^{<s>} \in P_i$ for each $i \in T$. But then, since P_i is a prime ideal, $x \in P_i$ for all $i \in T$. Hence $x \in I$. The final result is now clear.

Theorem 4.13. - *The radical $R(I)$ of any ideal I of a commutative $(m+1)$-semigroup is the intersection of all minimal prime ideals belonging to I.*

Proof. - For any prime ideal $P \supseteqq I$, more particularly, for any minimal prime ideal P belonging to I, if $x \in R(I)$, then $x^{<n>} \in I \subseteqq P$ for some non-negative integer n. Thus $R(I)$ is contained in the intersection of all minimal prime ideals belonging to I. Suppose that the preceding inclusion is proper. Then for some element x common to all minimal prime ideals belonging to I we have $x \notin R(I)$. Then the cyclic $(m+1)$-semigroup $[x]$ generated by x is disjoint from I. By Zorn's lemma there is a sub-$(m+1)$-semigroup B of A containing $[x]$ which is maximal with respect to its being disjoint from I. Hence, by Theorem 4.8, $A-B$ is a minimal prime ideal belonging to I with $x \notin A-B$. This is contradictory.

Lemma 4.14. - *A prime ideal P containing an ideal I of a commutative $(m+1)$-semigroup A is a minimal prime ideal belonging to I if and only if for all $y \in P$, there exists elements $x_1, x_2, \ldots, x_i \notin P$ with $i \leqq m$ such that*

$$[x_1^i y^{m-i} y^{<n>}] \in I$$

for some $n \geqq 0$.

Proof. - Suppose the above condition holds. Consider any prime ideal Q such that $I \subseteqq Q \subset P$ and choose $y \in P$ such that $y \notin Q$. Then, by hypothesis, there exists for some $i \leqq m$ elements $x_1, x_2, \ldots, x_i \notin P$ such that

$$[x_1^i y^{m-i} y^{<n>}] \in I$$

for some n. Since Q is prime, $y^{<n>} \notin Q$, $x_1, x_2, \ldots, x_i \notin Q$, then

$$[x_1^i y^{m-i} y^{<n>}] \notin Q.$$

This last statement is a contradiction.

Conversely, suppose P is a minimal prime ideal belonging to I. Then by Theorem 4.8, $A-P$ is a sub-$(m+1)$semigroup of A which is maximal with respect to its being disjoint from I. Choose any $y \in P$ and consider

$$B = (A-P) \cup \{[x_1^i y^{m-i} y^{<n>}] : \; i = 1, \ldots, m, \; x_1, \ldots, x_i \in A-P, \; n = 0, 1, 2, \ldots\}.$$

Then B is a sub-$(m+1)$-semigroup of A containing $A-P$. By maximality of $A - P$ with respect to its being disjoint from I, there must exist, therefore, some elements $x_1, \ldots, x_i \notin P$ such that $[x_1^i y^{m-1} y^{<n>}] \in I$.

Theorem 4.15. - *The structure space* $\boldsymbol{M}$ *of all minimal prime ideals belonging to an ideal* I *of a commutative* $(m+1)$*-semigroup* A *is a completely regular and totally disconnected topological space.*

Proof. - Theorem 4.5 (4) implies that $\boldsymbol{M}$ is a T_0-space. To prove the theorem it suffices then to show that the subbase members $\boldsymbol{M}_x$ are clopen. under the Stone-Gelfand topology. Recall that

$$\boldsymbol{M}_x = \{P : P \in \boldsymbol{M} \text{ and } x \notin P\}.$$

Naturally, this is open under the Stone-Gelfand topology. Consider an ideal $P \in \boldsymbol{M}$ such that $P \notin \boldsymbol{M}_y$. Then $y \in P$ and by Lemma 4.14, there exists $x_1, x_2, \ldots, x_i \notin P$ for some $i \leqq m$ such that

$$[x_1^i y^{m-i} y^n] \in I.$$

Hence,

$$\emptyset = \boldsymbol{M}_{[x_1^i y^{m-i} y^{<n>}]} = \boldsymbol{M}_{x_1} \cap \boldsymbol{M}_{x_2} \cap \ldots \cap \boldsymbol{M}_{x_i} \cap \boldsymbol{M}_y .$$

Whence

$$P \in \boldsymbol{M}_{x_1} \cap \boldsymbol{M}_{x_2} \cap \ldots \cap \boldsymbol{M}_{x_i} \subseteqq \boldsymbol{M} - \boldsymbol{M}_y$$

and therefore $\boldsymbol{M}_y$ is also closed.

Corollary 4.16. - *Th family of all minimal prime ideals (belonging to the ideal* (0)*) of a commutative* $(m+1)$*-semigroup with* 0 *under its Stone-Gelfand topology is a completely regular and totally disconnected space.*

A commutative $(m+1)$-semigroup all of whose elements are idempotent is designated as an $(m+1)$-*semilattice*. The particular $(m+1)$-semilattice of interest to us is the family of all subbase elements

$$\boldsymbol{P}_x = \{P : P \text{ a prime ideal, } x \notin P\},$$

under the operation defined by

$$[\boldsymbol{P}_{x_1}^{x_{m+1}}] = \boldsymbol{P}_{x_1} \cap \boldsymbol{P}_{x_2} \cap \ldots \cap \boldsymbol{P}_{x_{m+1}} = \boldsymbol{P}_{[x_1^{m+1}]}.$$

They obviously form an $(m+1)$-semilattice. A sub-$(m+1)$-semilattice of the former is given by the family of all $\boldsymbol{M}_x$ with $x \in A$, where

$$\boldsymbol{M}_x = \{M: \ M \text{ is a minimal prime ideal belonging to } (0), \ x \notin M\}.$$

To distinguish this last $(m+1)$-semilattice, it shall be called *the dual* $(m+1)$-*semilattice of the* $(m+1)$-*semigroup* A and will be denoted by $\boldsymbol{D}(A)$.

For each ideal I of an $(m+1)$-semigroup A and each subset S of A set

$$I[S] = \{y: \ y \in A, \ [yS^m] \subseteqq I\},$$

where by convention $I[x] = I[\{x\}]$. The radical of the ideal (0) which is the set of all (nilpotent) elements x such that $x^{<n>} = 0$ will be called for short the *nilradical* of A and is denoted by $N = R(0)$.

Lemma 4.17. - *For any subcollection* $\boldsymbol{P}$ *of prime ideals of a commutative* $(m+1)$-*semigroxp* A, *if* $I = \bigcap\limits_{P \in \boldsymbol{P}} P$, *then* $I[x] = \bigcap\limits_{P \in \boldsymbol{P}_x} P$ *for each* $x \in A$.

Proof. - In case $\boldsymbol{P}_x = \emptyset$, that is to say, if $x \in P$ for all $P \in \boldsymbol{P}$, then obviously

$$\bigcap_{P \in \boldsymbol{P}_x} P = A$$

and hence $I[x] \subseteqq \bigcap\limits_{P \in \boldsymbol{P}_x} P$. Consider then the case when $\boldsymbol{P}_x \neq \emptyset$. If $y \in I[x]$ and P is an arbitrary element of $\boldsymbol{P}_x$, i.e. any $P \in \boldsymbol{P}$ with $x \notin P$, then $[yx^m] \in I$. By definition of I this means $[yx^m] \in P$ for all $P \in \boldsymbol{P}$. Since for all $P \in \boldsymbol{P}_x$, $x \notin P$, this in turn implies that $y \in P$ for all $P \in \boldsymbol{P}_x$ and hence $y \in \bigcap\limits_{P \in \boldsymbol{P}_x} P$. Thus in any case,

$$I[x] \subseteqq \bigcap_{P \in \boldsymbol{P}_x} P.$$

Conversely, suppose $y \in \bigcap\limits_{P \in \boldsymbol{P}_x} P$. Thus $y \in P$ for all $P \in \boldsymbol{P}$ with $x \notin P$ and therefore (since P is an ideal) $[yx^m] \in P$. When $P \notin \boldsymbol{P}_x$ so that $x \in P$, then also $[yx^m] \in P$. Combining cases, then $[yx^m] \in P$ for all $P \in \boldsymbol{P}$. Whence $y \in I[x]$. The result now follows.

The *nilradical* N of an $(m+1)$-semigroup with 0 determines a congruence on A as follows. For each x, $y \in A$, define

$$x \equiv y(N) \text{ if and only if } N[x] = N[y].$$

It is easy to show that this is an equivalence relation. To show it is a congruence, suppose $x_i \equiv y_i(N)$ for each $i = 1, 2, \ldots, m+1$. This means $[z(x_i)^m] \in N$ iff $[z(y_i)^m] \in N$ for each $z \in A$, $i = 1, 2, \ldots, m+1$. Then

$$z \in N[[x_1^{m+1}]] \text{ iff } [z[x_1^{m+1}]^m] \in N$$

$$\text{iff } [[\ldots[[z(x_1)^m](x_2)^m] \ldots](x_{m+1})^m] \in N$$

$$\text{iff } [[\ldots[[z(x_1)^m](x_2)^m] \ldots](y_{m+1})^m] \in N$$

$$\text{iff } [[\ldots[[z(y_{m+1})^m](x_2)^m] \ldots](x_m)^m] \in N$$

$$\text{iff } [[\ldots[[z(y_{m+1})^m](x_2)^m] \ldots](y_m)^m] \in N$$

$$\text{iff } \ldots\ldots\ [[\ldots[[z(y_1)^m](y_2)^m] \ldots](y_{m+1})^m] \in N$$

$$\text{iff } [z[y_1^{m+1}]^m] \in N \text{ iff } z \in N[[y_1^{m+1}]]. \text{ Hence } [x_1^{m+1}] \equiv [y_1^{m+1}](N)$$

and therefore $\equiv$ is indeed a congruence. Let A/N be its quotient $(m+1)$-semigroup. Then

THEOREM 4.18. - *If A is a commutative $(m+1)$-semigroup, then $A/N \cong \boldsymbol{D}(A)$.*

PROOF. - Define a mapping $h\colon \boldsymbol{D}(A) \to A/N$ such that $h(\boldsymbol{M}_x) = x/N$, where x/N denotes the equivalence class containing x. By Theorem 4.8, note $N = \bigcap_{M \in \boldsymbol{M}} M$. Thus by choosing $I = N$ in the previous Lemma 4.17, we have

$$x/N = N[x] = \bigcap_{M \in \boldsymbol{M}_x} M.$$

This means that h is a well-defined function. Furthermore,

$$h([\boldsymbol{M}_{x_1}^{x_{m+1}}]) = h(\boldsymbol{M}_{[x_1^{m+1}]}) = [x_1^{m+1}]/N = [(x_1/N)(x_2/N) \ldots (x_{m+1}/N)]$$

$$= [h(\boldsymbol{M}_{x_1}) h(\boldsymbol{M}_{x_2}) \ldots h(\boldsymbol{M}_{x_{m+1}})],$$

so h is an epimorphism. If $x/N = y/N$ so that $\bigcap_{M \in \boldsymbol{M}_x} M = \bigcap_{M \in \boldsymbol{M}_y} M$, then $\overline{\boldsymbol{M}}_x = \overline{\boldsymbol{M}}_y$. Since each $\boldsymbol{M}_z$ is however clopen, then $\boldsymbol{M}_x = \boldsymbol{M}_y$ which shows that h is also a monomorphism. Whence $A/N \cong \boldsymbol{D}(A)$.

THEOREM 4.19. - *Let A be a commutative $(m+1)$-semigroup with 0 and $\boldsymbol{D} = \boldsymbol{D}(A)$ be its dual $(m+1)$-semilattice. Then the space $\boldsymbol{M}(A)$ of minimal* *prime ideals of A is homeomorphic with the space $\mathfrak{M}(\boldsymbol{D})$ of minimal prime ideals of $\boldsymbol{D}$ under their respective Stone-Gelfand topologies.*

PROOF. - Define $h: \boldsymbol{M}(A) \to \mathfrak{M}(\boldsymbol{D})$ by $h(P) = \mathfrak{M}_P$, where $\mathfrak{M}_P = \{\boldsymbol{M}_y : \boldsymbol{M}_y \in \boldsymbol{D},\ P \notin \boldsymbol{M}_y\}$. By a previous lemma, since P is a minimal prime ideal (belonging to (0)), then for each $y \in P$ and some $i \leq m$, there exist elements $x_1, x_2, \ldots, x_i \notin P$ such that $[x_1^i y^{m-i} y^{<n>}] = 0$ for some natural number n. This means that for $\boldsymbol{M}_y \in h(P)$ (i.e. $P \notin \boldsymbol{M}_y$), there exist $\boldsymbol{M}_{x_1}, \boldsymbol{M}_{x_2}, \ldots, \boldsymbol{M}_{x_i} \notin h(P)$ (i.e. $P \in \boldsymbol{M}_{x_k}$, $k = 1, \ldots, i$) such that

$$[\boldsymbol{M}_{x_1}^{x_i} (\boldsymbol{M}_y)^{m-i} \boldsymbol{M}_y^{<n>}] = \boldsymbol{M}_{[x_1^i y^{m-i} y^{<n>}]} = \emptyset$$

for some natural number n. By applying the same previous lemma, then $h(P) \in \mathfrak{M}(\boldsymbol{D})$ and h is well-defined. Suppose then that $P \neq Q$. Since the STONE-GELFAND topology in $\boldsymbol{M}(A)$ is T_0, this means that there exists $\boldsymbol{M}_{x_1}, \ldots, \boldsymbol{M}_{x_i} \in \boldsymbol{D}$ such that

$$P \in \boldsymbol{M}_{x_1} \cap \boldsymbol{M}_{x_1} \cap \boldsymbol{M}_{x_2} \cap \ldots \cap \boldsymbol{M}_{x_i},\ Q \notin \boldsymbol{M}_{x_1} \cap \boldsymbol{M}_{x_2} \cap \ldots \cap \boldsymbol{M}_{x_i}.$$

Thus for at least one $j = 1, 2, \ldots, i$, $P \in \boldsymbol{M}_{x_j}$ but $Q \notin \boldsymbol{M}_{x_j}$. Consequently, $h(P) \neq h(Q)$ and h is therefore a one-to-one mapping. To show that it is also onto, let $\mathfrak{M}_P \in \mathfrak{M}(\boldsymbol{D})$ and $P = \{x : x \in A,\ \boldsymbol{M}_x \in \mathfrak{M}_P\}$. If $x_i \in P$ and $i = 1, 2, \ldots, m+1$ and $x_1, x_2, \ldots, x_{m+1} \in A$, then

$$\boldsymbol{M}_{[x_1^{m+1}]} = \boldsymbol{M}_{x_1} \cap \boldsymbol{M}_{x_2} \cap \ldots \cap \boldsymbol{M}_{x_{m+1}} = [\boldsymbol{M}_{x_1}^{x_{m+1}}] \in \mathfrak{M}_P,$$

since $\mathfrak{M}_P$ is an ideal. Thus $[x_1^{m+1}] \in P$ and P is also an ideal of A. If $[x_1^{m+1}] \in P$, then

$$\boldsymbol{M}_{[x_1^{m+1}]} = [\boldsymbol{M}_{x_1}^{x_{m+1}}] \in \mathfrak{M}_P,$$

which in turn implies $\boldsymbol{M}_{x_i} \in \mathfrak{M}_P$ for some $i = 1, \ldots, m+1$, since $\mathfrak{M}_P$ is prime. Hence $x_i \in P$ for some $i = 1, \ldots, m+1$, and therefore is also prime. By a previous corollary, then there exists a minimal prime ideal P' of A contained in P. If $\boldsymbol{M}_x \in h(P')$, then $P' \in \boldsymbol{M}_x$ or $x \in P'$ and hence $x \in P$ or $\boldsymbol{M}_x \in \mathfrak{M}_P$. Whence $h(P') \subseteq \mathfrak{M}_P$. Since $h(P')$ is also a prime ideal in $\boldsymbol{D}$, then by a reapplication of the same previous corollary we obtain $h(P') = \mathfrak{M}_P$. The bicontinuity now follows from the obvious relation

$$h(\boldsymbol{M}_x) = h(\boldsymbol{M}(A)) \cap \{\mathfrak{M}_P : \mathfrak{M}_P \in \mathfrak{M}(\boldsymbol{D}),\ \boldsymbol{M}_x \notin \mathfrak{M}_P\}$$

and the fact that the topologies of both $\boldsymbol{M}(A)$ and $\mathfrak{M}(\boldsymbol{D})$ are extremally disconnected.

5. Ideals in Topological $(m+1)$**-Semigroups.** - This section deals with certain generalizations of propositions given by A. D. Wallace and his school for ordinary topological semigroups. By a *topological* $(m+1)$-*semigroup* we mean an algebraic $(m+1)$-semigroup endowed with a topology under which its $(m+1)$-ary operation is continuous. Thus, adjectives that modify subsets of a topological space may now be used to modify subsets of a topological $(m+1)$-semigroup too.

For any subset S of a topological $(m+1)$-semigroup A, let

$$[S]_n = \bigcup_{k=n}^{\infty} S^{<k>}.$$

Then

Theorem 5.1. - (1) $[S] = [S]_0$ *is the smallest sub-*$(m+1)$*-semigroup of* A *containing* S;

(2) $[\bar{S}_1^{m+1}] \subseteq \overline{[S_1^{m+1}]}$ *so that in particular, if* S *is a sub-*$(m+1)$*-semigroup of* A *then so is* $\bar{S}$.

(3) $[\bar{S}]$ *is the smallest closed* $(m+1)$*-semigroup of* A *containing* S.

Proof. (1). - Note

$$[S]^{<1>} = \bigcup_{k_i \geq 0} S^{<k_1 + \dots + k_{m+1} + 1>} = \bigcup_{k=1}^{\infty} S^{<k>} \subseteq [S].$$

If $S \subseteq T$ and $T^{<1>} \subseteq T$ (i.e. a sub-$(m+1)$-semigroup), then $S^{<1>} \subseteq T^{<1>} \subseteq T$ and hence $S^{<1>} \subseteq T^{<k>} \subseteq T$, in general. This means $[S] \subseteq T$ and therefore $[S]$ is the smallest sub-$(m+1)$-semigroup of A containing S.

(2) If $f\colon A^{m+1} \to A$ is the mapping such that

$$f(x_1, \dots, x_{m+1}) = [x_1^{m+1}],$$

then

$$[\bar{S}_1^{m+1}] = f(\bar{S}_1 \times \dots \times S_{m+1}) = f(\overline{S_1 \times \dots + S_{m+1}}) \subseteq$$

$$\subseteq \overline{f(S_1 \times \dots \times S_{m+1})} = \overline{[S_1^{m+2}]}.$$

Thus, if S is a sub-$(m+1)$-semigroup, then $\bar{S}^{<1>} \subseteq \overline{S^{<1>}} \subseteq \bar{S}$, so that $\bar{S}$ is also a sub-$(m+1)$-semigroup.

(3) From (2) it follows then that $\overline{[S]}$ is a sub-$(m+1)$-semigroup of A containing S. If $T^{<1>} \subseteq T$ and $S \subseteq T = \bar{T}$, then $[S] \subseteq T$ so that $\overline{[S]} \subseteq \bar{T} = T$ and $\overline{[S]}$ is the smallest closed (topological) sub-$(m+1$-semigroup of A containig S.

Theorem 5.2. - (*Gottschalk-Hedlund*). *Let* X_i, $i = 1, 2, \ldots, m+1$, *and* Y *be arbitrary topological spaces and*

$$f\colon X_1 \times X_2 \times \ldots \times X_{m+1} \rightarrow Y$$

be a continuous function. If C_i *is a compact subset of* X_i *for each* $i = 1, 2, \ldots, m+1$ *and* W *is a neighborhood of* $f(C_1 \times C_2 \times \ldots \times C_{m+1})$, *then there exist neighborhoods* U_i *of* C_i *for all* $i = 1, 2, \ldots, m+1$ *such that*

$$f(U_1 \times U_2 \times \ldots \times U_{m+1}) \subseteq W.$$

Proof. - The proof is by induction on m. If $m = 1$, the proposition reduces to a Lemma of Gottschalk and Hedlund (see page 3 of reference [4]). Suppose that the result has already been shown for any function on a cartesian product with $m = k$ components. Consider then any collection of $k+1$ compact subsets C_i of X_i $(i = 1, 2, \ldots, k+1)$ and a continuous function

$$f\colon X_1 \times X_2 \times \ldots \times X_{k+1} \rightarrow Y$$

together with any neighborhood W of $f(C_1 \times C_2 \times \ldots \times C_{k+1})$. Let g be the natural homeomorphism between $(X_1 \times \ldots \times X_k) \times X_{k+1}$ and $X_1 \times \ldots \times X \times X_{k+1}$ such that $g((x_1, \ldots, x_k), x_{k+1}) = (x_1, \ldots, x_k, x_{k+1})$. The composition fg is still a continuous function on $(X_1 \times \ldots \times X_k) \times X_{k+1}$ to Y and W is a neighborhood of $(fg)((C_1 \times \ldots \times C_k) \times C_{k+1})$. By Tychonoff's theorem $C_1 \times \ldots \times C_k$ is also compact and hence by applying the ordinary Gottschalk-Hedlund lemma, there exist open sets V and U_{k+1} containing $C_1 \times \ldots \times C_k$ and C_{k+1} respectively such that

$$fg(V \times U_{k+1}) \subseteq W.$$

Then applying our hypothesis of induction on the identity function defined on $C_1 \times \dots \times C_k$, there exists open sets U_i containing C_i such that

$$U_1 \times \dots \times U_k \subseteq V.$$

Thus,

$$fg((U_1 \times \dots \times U_k) \times U_{k+1}) = f(U_1 \times \dots \times U_k \times U_{k+1}) \subseteq W.$$

Our induction is then complete.

COROLLARY 5.3. (Wallace). – *If X_i are topological spaces containing the compact sets C_i $(i = 1, 2, \dots, m+1)$ and W is a neighborhood of $C_1 \times \dots \times C_{m+1}$ in the product space $X_1 \times \dots \times X_{m+1}$, then there exist neighborhoods U_i of C_i for each i such that*

$$U_1 \times U_2 \times \dots \times U_{m+1} \subseteq W.$$

THEOREM 5.4. – (1) *If C is a closed set, $S_1, \dots, S_{m+1}$ are arbitrary subsets of a Hausdorff topological $(m+1)$-semigroup A, then for each $i = 1, \dots, m$,*

$$\{x\colon [S_1^i x S_{i+2}^{m+1}] \subseteq C\} \text{ is closed.}$$

(2) *Under the same hypotheses, if S is an arbitrary subset of A and $C_1, \dots, C_{m+1}$ are compact subsets of A, then for any $i = 0, 1, \dots, m$,*

$$\{x\colon [C_1^i x C_{i+2}^{m+1}] \supseteq S\} \text{ is closed.}$$

PROOF. – (1). Let $y \in A$ such that $[S_1^i y S_{i+2}^{m+1}] \not\subseteq C$. *Then there* exist elements $s_j \in S_j$ $(j = 1, 2, \dots, m+1)$ such that

$$[s_1^i y s_{i+2}^{m+1}] \in A - C.$$

Since A is HAUSDORFF the sets $\{y\}$, $\{s_j\}$ $(j = 1, \dots, m+1)$ are compact and hence by Theorem 5.2, there exist open sets $U_1, \dots, U_{m+1}$ of A such that

$$[s_1^i y s_{i+2}^{m+1}] \in [s_1^i U_{i+1} s_{i+2}^{m+1}] \subseteq [U_1^{m+1}] \subseteq A - C.$$

This means that for each $z \in U_{i+1}$ we have

$$[s_1^i z s_{i+2}^{m+1}] \notin C \text{ and hence } [S_1^i z S_{i+2}^{m+1}] \not\subseteq C.$$

Thus $y \in U_{i+1} \subseteq A - \{x\colon [S_1^i x S_{i+2}^{m+1}] \subseteq C\}$ so that this last set is open. This proves our result.

(2) Let $y \in A$ such that $[C_1^i y C_{i+2}^{m+1}] \not\supseteq S$. Then for some $s \in S$, $s \notin [C_1^i y C_{i+2}^{m+1}]$ and therefore

$$[C_1^i y C_{i+2}^{m+1}] \subseteq A - \{s\}.$$

Again by Theorem 5.2, then there exists open sets $U_1, \ldots, U_{m+1}$ in A such that

$$[C_1^i y C_{i+2}^{m+1}] \subseteq [C_1^i U_{i+1} C_{i+2}^{m+1}] \subseteq [U_1^{m+1}] \subseteq A - \{s\}$$

so that $y \in U_{i+1} \subseteq A - \{x\colon [C_1^i x C_{i+2}^{m+1}] \supseteq S\}$ for some U_{i+1}. This means $\{x\colon [C_1^i x C_{i+2}^{m+1} \supseteq S\}$ is a closed set.

THEOREM 5.5. - *If* $\bar{S}_1, \bar{S}_2, \ldots, \bar{S}_{m+1}$ *are compact subsets of a Hausdorff* $(m+1)$*-semigroup* A, *then* $\overline{[S_1^{m+1}]} = [\bar{S}_1^{m+1}]$.

PROOF. - From Theorem 5.1, recall that $\overline{[S_1^{m+1}]} \supseteq [\bar{S}_1^{m+1}]$. Obviously,

$$[S_1^{m+1}] \subseteq [\bar{S}_1^{m+1}].$$

Since A is HAUSDORFF, the operation is continuous, and $\bar{S}_1 \times \ldots \times \bar{S}_{m+1}$ is compact (by TYCHONOFF'S theorem), then $[\bar{S}_1^{m+1}]$ is also compact and therefore closed. Hence $\overline{[S_1^{m+1}]} \subseteq [\bar{S}_1^{m+1}]$.

THEOREM 5.6. - *If* $\boldsymbol{T}_i$ $(i = 1, 2, \ldots, m+1)$ *are towers of compact sets in a Hausdorff* $(m+1)$*-semigroup* A, *then*

$$[(\bigcap_{S_1 \in T_1} S_1)(\bigcap_{S_2 \in T_2} S_2) \ldots (\bigcap_{S_{m+1} \in T_{m+1}} S_{m+1})] = \bigcap_{S_1 \in T_1} \ldots \bigcap_{S_{m+1} \in T_{m+1}} [S_1^{m+1}].$$

PROOF. - This follows from the following result in topology [12]:

LEMMA 5.7. - *Let* $f\colon X \longrightarrow Y$ *be a function and* T *a filter base of closed sets in* X. *If*

(1) *some* $B \in T$ *is compact and* $f^{-1}(y)$ *for each* $y \in Y$ *is closed, or*

(2) $f^{-1}(y)$ *for each* $y \in Y$ *is compact, then*

$$f(\bigcap_{A \in T} A) = \bigcap_{A \in T} f(A).$$

THEOREM 5.8. - *The following conditions for a subset S of an $(m+1)$-semigroup A are equivalent:*

(1) *S is an $(m+1)$-group under the same operation in A, that is to say, a sub-$(m+1)$-group;*

(2) *For all $i = 0, \dots, m$ and each set of elements*

$$x_1, \dots, x_i, x_{i+2}, \dots, x_{m+1} \in S, \; [x_1^i S x_{i+2}^{m+1}] = S;$$

(3) *For one $i = 1, \dots, m-1$ and each set of elements*

$$x_1, \dots, x_i, x_{i+2}, \dots, x_{m+1} \in S, \; [x_1^i S x_{i+2}^{m+1}] = S;$$

(4) *For all*

$$x_1, \dots, x_m \in S, \; [x_1^m S] = S = [S x_1^m];$$

(5) *For all*

$$x \in S, \; [x S^m] = S = [S^m x].$$

PROOF. - The implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$ are obvious. $(2) \Rightarrow (1)$ follows from the POST COSET Theorem. To comple the proof we now show $(5) \Rightarrow (2)$.

For each i and set of elements $x_1, \dots, x_i, x_{i+2}, \dots, x_{m+1} \in S$, we obtain the following through applications of (5), $S^{<1>} = S$, and the law of associativity:

$$[x_1^i S x_{i+2}^{m+1}] = [x_1^i S^{<2>} x_{i+2}^{m+1}] = [x_1^{i-1} [x_i S^m] S [S^m x_{i+2}] x_{i+3}^{m+1}] =$$

$$= [x_1^{i-1} S^3 x_{i+3}^{m+1}] = [x_i^{i \quad 1} S^{<1>} S S^{<1>} x_{i+3}^{m+1}] = [x_1^{i-2} [x_{i-1} S^m] S^3 [S^m x_{i+3}] x_{i+4}^{m+1}] =$$

$$= [x_1^{i-2} S^5 x_{i+4}^{m+1}] = \dots = S^{<1>} = S.$$

THEOREM 5.9. - (1) *If S is a non-empty subset of a Hausdorff $(m+1)$-semigroup A such that $\overline{[S]}$ is compact, then*

$$N = N(S) = \bigcup_{n=0}^{\infty} \overline{[S]}_n$$

is an ideal of the closed sub-$(m+1)$-semigroup $\overline{[S]}$.

Proof. - Note

$$[[S]_{n_1}[S]_{n_2} \dots [S]_{n_{m+1}}] = [S]_{n_1+\dots n_{m+1}+1}$$

and if $S_1, \dots, S_{m+1}$ are arbitrary subsets of $\overline{[S]}$, then $[\overline{S_1^{m+1}}] = \overline{[S_1^{m+1}]}$. Hence for each $i = 0, 1, \dots, m$, by Th. 5.7, 5.5

$$[\overline{[S]}^i N \overline{[S]}^{m-i}] = [\overline{[S]}^i \bigcap_{n=0}^{\infty} \overline{[S]_n}\, \overline{[S]}^{m-i}] = \bigcap_{n=0}^{\infty} [\overline{[S]}^i \overline{[S]_n}\, \overline{[S]}^{m-i}] =$$

$$= \bigcap_{n=0}^{\infty} \overline{[[S]^i [S]_n [S]^{m-i}]} = \bigcap_{n=0}^{\infty} \overline{[(\bigcup_{k=0}^{\infty} S^{<k>})^i (\bigcup_{k=n}^{\infty} S^{<k>}) (\bigcup_{k=0}^{\infty} S^{<k>})^{m-i}]}$$

$$= \bigcap_{n=0}^{\infty} \overline{\bigcup_{k=n+1}^{\infty} S^{<k>}} = \bigcap_{n=0}^{\infty} \overline{[S]_{n+1}} = \bigcap_{n=0}^{\infty} \overline{[S]_n} = N.$$

Hence N is an ideal of $\overline{[S]}$.

Theorem 5.10. - *If S is a commutative subset of a Hausdosff $(m+1)$-semigroup A, then $\bar{S}$ is also commutative; particularly, $\overline{[a]}$ is commutative for each $a \in A$.*

Proof. - Consider the function $f\colon A^{m+1} \to A \times A$ defined by

$$f(x_1, \dots, x_{m+1}) = ([x_1^{m+1}], [x_{\varnothing(1)}^{\varnothing(m+1)}])$$

for any permutation $\varnothing$ of $1, 2, \dots, m+1$. If D is the diagonal of $A \times A$, then note that a subset S of A is commutative iff $f(S \times \dots \times S) \subseteq D$.

On the other hand, D is a closed set. Since f is continuous, then $f^{-1}(D)$ is also closed. If S is therefore commutative, then $S \times \dots \times S \subseteq f^{-1}(D)$ so that $\overline{S \times \dots \times S} = \bar{S} \times \dots \times \bar{S} \subseteq f^{-1}(D)$. Hence $f(\bar{S} \times \dots \times \bar{S}) \subseteq D$ and $\bar{S}$ is also commutative.

Theorem 5.11. - *If a belongs to a Hausdorff $(m+1)$-semigroup A and $\overline{[a]}$ is compact, then $N(a)$ is a maximal sub-$(m+1)$-group and minimal ideal of $\overline{[a]}$.*

Proof. - It is obvious that every $[a]_n$ and hence every $\overline{[a]_n}$ is a sub-$(m+1)$-semigroup of $\overline{[a]}$. Since $[a]$ is commutative, then, by Theorem 5.10, $\overline{[a]}$ is also a commutative $(m+1)$-semigroup. Let $\{x\colon [xN^m] = N\} = H$ where $N = N(a)$. By Theorem 5.4, this is closed. Note also that for any non-negative integer n,

$$[a^{<n>}N^m] = [a^{<n>} \bigcap_{j=0}^{\infty} \overline{[a]}_j \dots \bigcap_{k=0}^{\infty} \overline{[a]}_k] = \bigcap_{j=0}^{\infty} \dots \bigcap_{k=0}^{\infty} [a^{<n>} \overline{[a]}_j \dots \overline{[a]}_k]$$

$$= \bigcap_{j=0}^{\infty} \dots \bigcap_{k=0}^{\infty} [a^{<n>} \overline{\bigcup_{i=j}^{\infty} a^{<i>}} \dots \overline{\bigcup_{i=k}^{\infty} a^{<i>}}] = \bigcap_{j=0}^{\infty} \dots \bigcap_{k=0}^{\infty} \overline{\bigcup_{i=n+j+\dots+k+1}^{\infty} a^{<i>}}$$

$$= \bigcap^{\infty} \overline{[a]}_{n+i+1} = N.$$

Thus, $[a] \subseteq H$ and also $\overline{[a]} \subseteq H$. In particular, $N(a) \subseteq H$ and hence $N(a)^{<1>} = N(a)$. Whence for any set of elements $x_1, \dots, x_m \in N(a) = N$,

$$[x_1^m N] = [x_1^m N^{<1>}] = [x_1^{m-1}[x_m N^m] N] = [x_1^{m-1} N^2] = [x_1^{m-1} N^{<1>} N]$$

$$= [x_1^{m-2}[x_{m-1} N^m] N^2] = \dots = [x_1 N^m] = N.$$

By Theorem 5.8, it follows then that $N = N(a)$ is an $(m+1)$-group. If G is any $(m+1)$-group and I is any ideal of G, then for each $x \in I$,

$$G = [G^i x G^{m-i}] \subseteq [G^i I G^{m-i}] \subseteq I \subseteq G.$$

Therefore $I = G$.

From Theorem 5.9, we know that $N(a)$ is an ideal of $\overline{[a]}$ for each $a \in A$. Suppose I is any ideal of $\overline{[a]}$. Then

$$[(N(a))^i I (N(a))^{m-i}] \subseteq I \cap N(a) \quad \text{and} \quad I \cap N(a) \neq \emptyset,$$

and hence $I \cap N(a)$ is also an ideal of $N(a)$ since

$$[(N(a))^i (I \cap N(a)) (N(a))^{m-i}] \subseteq I \cap N(a).$$

Thus $N(a) \cap I = N(a) \subseteq I$ and therefore $N(a)$ is a minimal ideal of $\overline{[a]}$.

Let G be any sub-$(m+1)$-group of $\overline{[a]}$. If $N(a) \cap G \neq \emptyset$ and $b \in N(a) \cap G$, then for each element $x \in G$ and each $i = 0, 1, \dots, m$, there exist elements $x_1, \dots, x_i, x_{i+2}, \dots, x_{m+1} \in G$ such that

$$[x_1^i b x_{i+2}^{m+1}] = x.$$

Since $N(a)$ is an ideal of $\overline{[a]}$, then $x \in N(a)$. Whence $G \subseteq N(a)$ and $N(a)$ is a maximal $(m+1)$-group in $\overline{[a]}$.

COROLLARY 5.12. - *Under the same hypotheses of Theorem* 5.11, *if* $\overline{[a]}$ *is compact and connected, then it is an* $(m+1)$*-group.*

PROOF. - Note that $\overline{[a]} = \{a\} \cup \overline{[a]}_1$. Since A is HAUSDORFF, then $\{a\}$ is closed like $\overline{[a]}_1$ and hence by connectedness $a \in \overline{[a]}_1$. If for some smallest integer p, $a = a^{<p>}$, then $\overline{[a]}$ is the $(m+1)$-group consisting of the elements $a, a^{<1>}, \ldots, a^{<p-1>}$. Otherwise, since $a \in \overline{[a]}_1 = [a]_1 \cup N(a)$, then $a \in N(a)$. Then $[a] \subseteq N(a)$ and $\overline{[a]} \subseteq N(a)$. Since $N(a) \subseteq \overline{[a]}$, then $N(a) = \overline{[a]}$.

THEOREM 5.13. - *The following conditions are equivalent for a compact Hausdorff* $(m+1)$*-semigroup* A:

(1) *For each* $x \in A$ *and each neighborhood* U *of* x, *there exists a natural number* n *such that* $x^{<n>} \in U$;

(2) *For each subset* S *of* A, $S^{<1>} \subseteq S = \bar{S}$ *implies* $S^{<1>} = S$;

(3) A *is a union of* $(m+1)$*-groups.*

PROOF. - (1) $\Rightarrow$ (2). Let $S^{<1>} \subseteq S = \bar{S}$ and suppose that there exists an element $x \in A - S^{<0>}$. Note

$$S \supseteq S^{<1>} \supseteq \ldots S^{<n>} \supseteq \ldots$$

and hence $A - S$ is an open set such that

$$A - S \subseteq A - S^{<1>} \subseteq \ldots \subseteq A - S^{<n>} \subseteq \ldots .$$

Thus $A - S$ is some neighborhood of x such that $x^{<n>} \notin A - S$ for all natural numbers n, contrary to (1).

(2) $\Rightarrow$ (3). Assume (2). Then for each $x \in A$,

$$\overline{[x]}^{<1>} \subseteq \overline{x^{<1>}} = \overline{[\bigcup_{k=0}^{\infty} x^{<k>} \bigcup_{k=0}^{\infty} x^{<k>} \ldots \bigcup_{k=0}^{\infty} x^{<k>}]} = \overline{\bigcup_{k=1}^{\infty} x^{<k>}}$$

$$= \overline{[x]}_1 \subseteq \overline{[x]} \quad \text{so that} \quad \overline{[x]}^{<1>} = \overline{[x]}_1 = \overline{[x]}.$$

Proceeding by induction, suppose that $\overline{[x]}_n = \overline{[x]}$. Then

$$\overline{[x]} = \overline{[x]}^{<1>} = [\overline{[x]}_n \overline{[x]}^m] \subseteq \overline{[[x]_n [x]^m]} =$$

$$= \overline{[\bigcup_{k=n}^{\infty} x^{<k>} \bigcup_{k=0}^{\infty} x^{<k>} \ldots \bigcup_{k=0}^{\infty} x^{<k>}]} = \overline{\bigcup_{k=n+1}^{\infty} x^{<k>}} = \overline{[x]}_{n+1} \subseteq \overline{[x]}.$$

Hence $\overline{[x]}_{n+1} = \overline{[x]}$ and induction is completed. Therefore $\overline{[x]} = \overline{[x]}_n = N(x)$. Applying Theorem 5.11, then $N(x)$ is an $(m+1)$-group. Every element of A thus belongs to such an $(m+1)$-group.

(3) $\Rightarrow$ (1). - Let $x \in A$ and G be the smallest closed sub-$(m+1)$-group in A containing x. Then $G \supseteqq \overline{[x]} \supseteqq N(x)$. Since A is compact and HAUSDORFF, then $\overline{[x]}$ is also compact and hence by Theorem 5.11, $N(x)$ is an $(m+1)$-group. Whence $G = N(x)$. This means that $x \in N(x)$ for all $x \in A$. Condition (1) now follows.

LEMMA 5.14. - *If C, C_{i+1} are arbitrary non-empty subsets of a connected Hausdorff $(m+1)$-semigroup A, then for each $i \neq j$ $(i, j = 0, 1, \ldots, m)$, both*

$$[A^i C_{i+1} A^{m-i}] \cup [A^j C_{j+1} A^{m-j}]$$

and

$$[A^i C_{i+1} A^{m-i}] \cup [[A^m C] A^m]$$

are connected.

PROOF. - Without loss of generality, one may assume that i is less than j. For each $x \in C_{i+1}$, $y \in C_{j+1}$ and each set of elements $x_1, \ldots, x_i$, $x_{i+2}, \ldots, x_j$, $x_{j+2}, \ldots, x_{m+1} \in A$,

$$[x_1^i x x_{i+2}^j \, y x_{j+2}^{m+1}] \in [A^i x A^{m-i}] \cap [A^j y A^{m-j}]$$

and hence the latter set is non-empty. Since for each $x \in C_{i+1}$ and $y \in C_{j+1}$ both $[A^i x A^{m-i}]$ and $[A^j y A^{m-j}]$ are connected, then their union is also connected. Moreover, since

$$[A^i x A^{m-i}] \subseteqq \bigcap_{y \in C_{j+1}} ([A^i x A^{m-i}] \cup [A^j y A^{m-j}])$$

so that the latter set is again non-empty, then

$$[A^i x A^{m-i}] \cup [A^j C_{j+1} A^{m-j}] = \bigcup_{y \in C_{j+1}} ([A^i x A^{m-i}] \cup [A^j y A^{m-j}]$$

is also connected. Continuing this argument, we conclude that

$$[A^i C_{i+1} A^{m+i}] \cup [A^j C_{j+1} A^{m-j}]$$

is connected. Taking $j = m$ and $C_{m+1} = [CA^m]$, we obtain the second conclusion.

Theorem 5.15. - *If I is an $(i+1)$-ideal ($i = 0, 1, \ldots, m$) of a topological Hausdorff $(m+1)$-semigroup A, then one and only one component of I is an $(i+1)$-ideal.*

Proof. - Consider

$$J = \bigcup_{n=1}^{\infty} [\overline{A^{ni} I A^{n(m-i)}}] \cup [[A^m I] A^m].$$

Then J is clearly a connected $(i+1)$-ideal of A (by Lemma 5.14) and J is contained in I. Let C be the component of I containing J. Thus

$$[A^i C A^{m-i}] \subseteq [A^i I A^{m-i}] \subseteq J \subseteq C.$$

that is to say, the component C of I is also an $(i+1)$-ideal. Note also that

$$[A^i [A^i C A^{m-i}] A^{m-i}] \subseteq [A^i C A^{m-i}] \subseteq C.$$

If C and C' are therefore any two $(i+1)$-ideal components of I, then

$$[A^i [C A^{i-1} C' A^{m-i}] A^{m-i}] \subseteq [A^i [A^i C' A^{m-i}] A^{m-i}] \subseteq [A^i C' A^{m-i}] \subseteq C'$$

and

$$[A^i [C A^{i-1} C' A^{m-i}] A^{m-i}] \subseteq [A^i C A^{i-1} C' A^{m-2i-1} [A^{m+1}]] \subseteq$$

$$\subseteq [A^i C A^{i-1} A^{m-2i} A] = [A^i C A^{m-i}] \subseteq C$$

so that

$$[A^i [C A^{i-1} C' A^{m-i}] A^{m-i}] \subseteq C \cap C' \neq \emptyset.$$

This implies then that $C = C'$ and I has exactly one $(i+1)$-ideal component.

Corollary 5.16. - *If I is an ideal of a Hausdorff $(m+1)$-semigroup A, then one and only one component of I is an ideal.*

Theorem 5.17. - *If C is a closed subset of a Hausdorff $(m+1)$-semigroup A, then the union $U_{i+1}(C)$ of all $(i+1)$-ideals of A contained in C is a closed $(i+1)$-ideal, $i = 0, \ldots, m$.*

Proof. - If $U_{i+1}(C) = \emptyset$, then the result is obvious. Othervise, if

$U_{i+1}(C) = U_{i+1} \neq \emptyset$, then by Theorem (5.1(2),

$$[A^i \overline{U_{i+1}} A^{m-i}] \subseteq [\overline{A^i U_{i+1} A^{m-i}}] \subseteq \overline{U}_{i+1} \subseteq \overline{C} = C.$$

Thus $\overline{U}_{i+1}$ is also an $(i+1)$-ideal contained in C and hence $\overline{U}_{i+1} \subseteq U_{i+1}$. The other inclusion is evident. Whence U_{i+1} is closed.

Corollary 5.18. *The union* $U(C)$ *of all ideals contained in a closed subset* C *of a Hausdorff* $(m+1)$*-semigroup is also a closed ideal.*

Theorem 5.19. - *If* O *is an open subset of a compact Hausdorff* $(m+1)$*-semigroup* A*, then the union* $U_{i+1}(O)$ *of all* $(i+1)$*-ideals of* A *contained in* O *is also an open* $(i+1)$*-ideal,* $i = 0, 1, \ldots, m$.

Proof. - It is easy to see that the union of any number of $(i+1)$-ideals is also an $(i+1)$-ideal. If $X_0 = \{x\}$, $X_{n+1} = [A^i X_n A^{m-i}]$ for all natural numbers n, since $U_{i+1}(O)$ is an $(i+1)$-ideal, then

$$\bigcup_{n=0}^{\infty} X_n \subseteq U_{i+1}(O) \subseteq O.$$

Since A is compact and Hausdorff so that $\{x\}$ is also compact, then by Theorem 5.2, there exist an open neighborhood U of x such that

$$K = \bigcup_{n=0}^{\infty} Y_n \subseteq O.$$

where $Y_0 = U$ and $Y_{n+1} = [A^i Y_n A^{m-i}]$ for each natural number n. It is easily verified that K is also an $(i+1)$-ideal of A and therefore is an element of $U_{i+1}(O)$. Therefore $x \in U \subseteq U_{i+1}(O)$ *and* $U_{i+1}(O)$ *is also open.*

Theorem 5.20. - *The union of all ideals contained in an open subset of a compact Hausdorff* $(m+1)$*-semigroup is an open ideal.*

Theorem 5.16. - *Any proper* $(i+1)$*-ideal of a compact Hausdorff* $(m+1)$*-semigroup* A *is contained in a maximal proper* $(i+1)$*-ideal and each such proper maximal* $(i+1)$*-ideal is open.*

Proof. - Let I be any proper $(i+1)$-ideal of A. The family $\mathbf{C}$ of all proper $(i+1)$-ideals of A containing I is a partially ordered set under inclusion every linearly ordered subfamily of which has its union as an upper bound. Hence, by Zorn's lemma the family $\mathbf{C}$ must possess a maximal member M. M is thus a proper maximal $(i+1)$-ideal of A.

Let $x \in A - M$ and consider $U_{i+1}(A - \{x\})$. If $y \in U_{i+1}(A - \{x\})$ so that for some $(i+1)$-ideal $I \subseteq A - \{x\}$, $y \in I$, then for any set of elements $x_1, \dots, x_i, x_{i+2}, \dots, x_{m+1} \in A$, we have

$$[x_1^i y x_{i+2}^{m+1}] \in I \subseteq U_{i+1}(A - \{x\}).$$

Moreover $M \subseteq U_{i+1}(A - \{x\}) \neq A$ and therefore $M = U_{i+1}(A - \{x\})$. M is therefore an open $(i+1)$-ideal.

COROLLARY 5.22. - *Any proper ideal of a compact Hausdorff $(m+1)$-semigroup A is contained in a maximal proper ideal and any such maximal ideal is open.*

COROLLARY 5.23. - *Any proper $(i+1)$-ideal of a compact Hausdorff $(m+1)$-semigroup is contained in an open proper maximal ideal, $i = 0, 1, \dots, m$.*

This follows from the fact that every $(i+1)$-ideal is contained in an ideal (see]10]) and the previous corollary.

From the fact that the closure of an ideal is also an ideal, the following result is easily derived:

COROLLARY 5.24. - *A maximal proper ideal of a compact Hausdorff and connected $(m+1)$-semigroup is dense.*

THEOREM 5.25. - *Every compact Hausdorff $(m+1)$-semigroup A possesses a minimal 1-ideal ((m + 1-ideal) and each such ideal is closed. If A is in addition surjective, then it has a minimal $(i+1)$-ideal for each $i = 1, 2, \dots, m-1$ and each such ideal is also closed.*

PROOF. - We shall only prove the first part, since the proof of the second part goes in exactly the same way.

Consider the collection of all closed $(i+1)$-ideals of A. This is non-empty since it contains A itself. By ZORN'S lemma, it must have a maximal tower $\boldsymbol{T}$. Then

$$M = \bigcap_{I \in T} I$$

is a minimal closed $(i+1)$-ideal of A. M is also a closed minimal $(i+1)$-ideal of A. To see this consider an $(i+1)$-ideal J contained in M with $x \in J$. Then

$$[A^i x A^{m-i}] \subseteq [A^i J A^{m-i}] \subseteq J \subseteq M$$

and hence by induction

$$[A^i[A^{ni}xA^{n(m-i)}[A^{m-i}] =]A^{(n+1)i}xA^{(n+1)(m-i)}] \subseteq [A^iJA^{m-i}] \subseteq J \subseteq M$$

for each natural number n. Hence

$$K = \bigcup_{n=1}^{\infty} [A^{ni}xA^{n(m-i)}] \cup [[A^m x]A^m] \subseteq J \subseteq M.$$

K is evidently an $(i+1)$-ideal. Since A is compact HAUSDORFF, its operation is a closed mapping; since there are actually only a finite number of distinct terms in the above union, and since each of the terms $[A^{ni}xA^{n(m-i)}]$ is closed, then K is also a closed $(i+1)$-ideal. From the minimality of M as a closed ideal, then $M = K$.

Now suppose N is any minimal $(i+1)$-ideal of A so that for each $x \in N$, we have the relation $[A^ixA^{m-i}] \subseteq N$. Then by exactly the same procedure as in the preceding K is an $(i+1)$-ideal contained in N and hence $K = N$. Thus any minimal $(i+1)$-ideal is always closed.

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