Axiomatics of Newtonian Cosmology.

by RUDOLF KURTH (Atlanta, Georgia, U. S. A.)

Summary. - *Four axioms are assumed which, essentially, state the existence of a frequency function, a qualitative law cf gravitation, the conservation of mass (or probability), Newton's law of motion and the « local » homogeneity of the Universe* (§ 2). *Theorems I and II yield a complete survey over all the universes which satisfy these axioms and an additional regularity hypothesis* (§ 3). *Some kinematical and thermodynamical features of these universes are discussed in* § 4.

§. 1 - **Introduction.**

Cosmology based on NEWTONIAN dynamics encounters the difficulty that NEWTON'S law of gravitation and the « World Postulate » (« Cosmological Principle ») are incompatible. According to this postulate the mass-density is supposed to be independent of the position vector x (in the euclidean 3-space). Therefore the gravitational potential, $V(x, t)$ (where t denotes the time variable), should be independent of x also. The potential $V(x, t)$ and the mass-density $\rho(x \mid t)$, however, are related by POISSON'S equation

$$\Delta V(x, t) = 4\pi G \rho(x \mid t)$$

(where Δ is the LAPLACE operator and G is the constant of gravitation) — which implies that $V(x, t)$ is a non-constant function of x. (Cf. [5], [8], [9]).

This inconsistency could be removed, for example, by the following modification of POISSON'S law :

$$\Delta V(x, t) = 4\pi G. \ [\rho(x \mid t) - \bar{\rho}(t)]$$

where $\bar{\rho}(t)$ is the « mean density of the universe at the time t ». It admits universes of which both the mass-density $\rho(x \mid t)$ and the gravitional potential $V(x, t)$ are independent of the position x. In such a universe the mass-elements move according to GALILEO'S law of inertia.

It is, however, by no means necessary to specify the modified law of gravitation : instead, it is sufficient to postulate that the gravitational force is uniquely determined by the mass-distribution of the universe. Then a mass-density which is independent of the position x implies a gravitational potential independent of x. It is the purpose of this note to discuss the consequences of such a qualitative law of gravitation and a strictly local world postulate.

§ 2. - Axioms.

The term « universe » is taken as an undefined primitive notion. Its properties will formally be described by a number of definitions and axioms. The following spaces are assigned to it:

(i) the « real space, X, i. e., the set of all « position vectors » x. « Vector » means « column vector ». The number of dimensions, n, of X is not specified since nearly all results are independent of the particular value of n;

(ii) the associate « velocity space » U, i. e., the set of all « velocity vectors » u; both X and U are n-dimensional vector spaces;

(iii) the « phase space » $\Gamma = U \times X$;

(iv) the « time-axis » T, i. e., the set of all real numbers t;

(v) T', an open interval of the time axis which contains the time zero, $t = 0$;

(vi) the cartesian products $\Gamma \times T$ and $\Gamma \times T'$.

To any universe a « frequency function » $f(u, x \mid t)$ is assigned which is defined on $\Gamma \times T'$. One intuitive interpretation of it reads: let M be any measurable bounded subset of Γ; then

$$\iint_M f(u, x \mid t)\, du\, dx$$

is the mass contained in M at the time t. Thus a universe is pictured by a continuous material substratum which fills the phase space and has the mass-density $f(u, x \mid t)$ in $\Gamma \times T'$.

AXIOM I. The frequency function, $f(u, x \mid t)$, of a universe is defined and non-negative on $\Gamma \times T'$, and is positive for at least one value of (u, x) and $t = 0$. It has continuous derivatives of first order with respect to the components of (u, x, t) everywhere in $\Gamma \times T'$. The moments of the orders 0, 1 and 2,

$$\int_U f(u, x \mid t)\, du\,,$$

$$\int_U u f(u, x \mid t)\, du\,,$$

$$\int_U u_i u_j f(u, x \mid t)\, du\,, \qquad 1 \leq i, j \leq n\,,$$

exist and have continuous derivatives with respect to the components of (x, t) everywhere in $\Gamma \times T'$.

The existence of these moments is needed for the following definitions of the « mass-density » $\rho(x \mid t)$, the « mean velocity » $\bar{u}(x, t)$, the « residual velocities » v, the « pressure (tensor) » $P(x, t) = \{ P_{ij}(x, t) \}$ and the « temperature (tensor) » $\Theta(x, t) = \{ \Theta_{ij}(x, t) \}$ of the universe:

$$\left.\begin{aligned} &\rho(x \mid t) = \int_U f(u, x \mid t)\, du\,, \\ &\bar{u}(x \mid t) = \int_U u \cdot f(u, x \mid t)\, du : \rho(x \mid t)\,, \\ &v = \{ v_i \} = u - \bar{u}(x, t)\,, \\ &P_{ij}(x, t) = \int_{R^n} v_i v_j f(\bar{u}(x, t) + v, x \mid t)\, dv\,, \\ &\Theta(x, t) = P(x, t) : \rho(x \mid t) \end{aligned}\right\} \quad \text{for } \rho(x \mid t) > 0$$

and $x \in X$, $t \in T'$.

AXIOM II (law of gravitation). There is a function $V(x, t)$ (the « gravitational potential ») which is defined and has continuous derivatives with respect to $x_1, \ldots, x_n$ everywhere on $\Gamma \times T'$. It is uniquely determined by the mass-density $\rho(x \mid t)$ up to an additive function of t only.

Further we shall assume that the law of conservation of mass and NEWTON'S equations of motion hold. In stellar dynamics it is shown (cf., e.g., [6]) that both these laws imply LIOUVILLE'S equation which we shall adopt as:

AXIOM III (Liouville's equation).

$$\frac{\partial f}{\partial t} + \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} u_j - \sum_{j=1}^{n} \frac{\partial f}{\partial u_j} \frac{\partial V}{\partial x_j} = 0$$

everywhere in $\Gamma \times T'$.

Relationships of this kind will also be written by means of the following matrix notation:

$$f_t + f_{x^*} u - f_{u^*} V_x = 0\,.$$

Asterisks will always indicate the interchange of rows and columns.

AXIOM IV (world postulate, homogeneity hypothesis). The « frequency function of the residual velocities v », $f(\bar{u}(x, t) + v, x \mid t)$, is independent of the position x.

Thus, where it is appropriate, it will be denoted by $g(v \mid t)$.

The world postulate states, roughly, that the universes considered are homogeneous in the following sense: the picture which any observer « moving with the material of the universe » obtains about his immediate neighbourhood (in the « real space ») is independent of his position x.

These four axioms will be taken as granted throughout the following discussion.

§. 3. - **Theorems.**

THEOREM I. Suppose that for the frequency function $g(v \mid t)$ of the residual velocities v all the integrals

$$\int_{R^n} \frac{\partial}{\partial v_i} [g(v \mid 0) v_j] \, dv \quad \text{and} \quad \int_{R^n} \frac{\partial}{\partial v_i} [g(v) \mid 0) v_j v_k v_l] \, dv$$

$(i, j, k, l = 1, 2, \dots, n)$ exist and vanish. (This is, for example, the case if $g(v \mid 0)$ vanishes for all sufficiently large values of $|v|$). Then there are a positive constant ρ_0, a constant $n \times n$ matrix C, a constant vector $c(0)$ and a normalized frequency function $h(v)$ such that, for $t \in T'$,

$$\text{(i)} \quad f(u, x \mid t) = g(v \mid t) = \rho_0 \cdot h[(J + Ct) v] \\ = \rho_0 \cdot h[(J + Ct) u - Cx - c(0)]$$

where J denotes the $n \times n$ unit matrix;

$$\text{(ii)} \quad \rho(x \mid t) = \rho_0 \cdot |\det(J + Ct)|^{-1},$$

$$\text{(iii)} \quad \bar{u}(x, t) = (J + Ct)^{-1} Cx + (J + Ct)^{-1} c(0);$$

$$\text{(iv)} \quad \Theta(x, t) = (J + Ct)^{-1} \Theta(0, 0) (J + C^* t)^{-1}.$$

$h(v)$ has moments of up to the second order, and those of the first order vanish.

The proof is accomplished by the following steps.

LEMMA 1.

$$\text{(i)} \quad \rho(x \mid t) = \rho(0 \mid t),$$

$$\text{(ii)} \quad \Theta(x, t) = \Theta(0, t),$$

$$\text{(iii)} \quad P(x, t) = P(0, t),$$

$$\text{(iv)} \quad V_x(x, t) = 0,$$

$$\text{(v)} \quad f_t + f_{x^*} u = 0$$

everywhere on $\Gamma \times T'$.

PROOF. The homogeneity hypothesis (Axiom IV) implies (i), (ii) and (iii). Since $\rho(x \mid t)$ is invariant under the group of translations of the sytem of coordinates, $V(x, t)$ is invariant also because of the « law of gravitation » (II): this implies (iv). Hence, and by LIOUVILLE's equation (III), (v).

LEMMA 2. The mean velocity $\bar{u}(x, t)$ satisfies the equation

$$\bar{u}(x, t) = c(x - t \cdot \bar{u}(x, t))$$

where $c(x)$ is the mean velocity at the moment $t = 0$:

$$\bar{u}(x, 0) = c(x).$$

PROOF. Multiply LIOUVILLE's equation (cf. Lemma 1(v)) by u and integrate with respect to u: thus,

$$\rho \cdot (\bar{u}_t + \bar{u}_{x^*} \bar{u}) + \left\{ \sum_{j=1}^{n} \frac{\partial P_{ij}}{\partial x_j} \right\} = 0.$$

(Cf., e.g., [6]). By Lemma 1, P is independent of x. Hence, and since $\rho(x \mid t) > 0$ (by Axiom I),

$$\bar{u}_t + \bar{u}_{x^*} \bar{u} = 0.$$

The corresponding characteristic equations,

$$\begin{cases} \dfrac{dx}{dt} = \bar{u}, \\[2ex] \dfrac{d\bar{u}}{dt} = 0, \end{cases}$$

have the $2n$ independent integrals $\bar{u}$, $x - \bar{u} \cdot t$. (Cf., e.g., [2]). Hence the statement of Lemma 2.

LEMMA 3. $f(u, x \mid t) = \rho_0 \cdot h[u - c(x - ut)]$ where $h(v)$ is the normalized frequency function of the residual velocities v at the time $t = 0$ and ρ_0 is the densitity at $t = 0$. (« Normalized » means: $\int_{R^n} h(v)\,dv = 1$).

PROOF. LIOUVILLE's equation (Lemma 1(v)) has the $2n$ independent integrals u, $x - ut$. Hence,

$$f(u, x \mid t) = \varphi(u, x - ut)$$

for some function φ. By the homogeneity hypothesis (IV), the function $\varphi[\bar{u}(x, t) + v, x - \bar{u}(x, t) \cdot t - v \cdot t]$ is independent of x. Differentiation with respect to the components of x at the time $t = 0$ therefore yields

$$\frac{\partial}{\partial x_i} \varphi(u, x) + \sum_{j=1}^{n} \frac{\partial}{\partial u_j} \varphi(u, x) \frac{\partial}{\partial x_i} c_j(x) = 0, \qquad i = 1, 2, \dots, n.$$

For $i = 1$ the resulting equation has the $2n - 1$ independent integrals $x_2, x_3, \dots, x_n, u_1 - c_1(x), u_2 - c_2(x), \dots, u_n - c_n(x)$. Thus φ is a function of these integrals only. Similarly, when $i = 2$, it appears as a function of $x_1, x_3, \dots, x_n, u - c(x)$ only, etc. Hence, it is a function of $u - c(x)$ only; i.e.,

$$\varphi(u, x) = \rho_0 \cdot h(u - c(x))$$

where $h(v)$ is a normalized frequency function and ρ_0 the density at $t = 0$. Thus,

$$f(u, x \mid t) = \varphi(u, x - ut) = \rho_0 \cdot h[u - c(x - ut)].$$

LEMMA 4. Let

$$z = x - \bar{u}(x, t) \cdot t$$

and

$$e(z) = c(z) - c_{z^*}(0) z .$$

Then,

$$h_{v^*}(v) e_{z^*}(z) v = 0$$

for all vectors v and z.

PROOF. By Lemma 2,

$$\begin{cases} \bar{u} = c(z), \\ x = z + c(z) \cdot t. \end{cases}$$

By the homogeneity hypothesis (IV), the frequency function of the residual velocities v, i.e., by Lemma 3, the function $h[\bar{u} + v - c(x - vt - \bar{u} \cdot t)]$ is independent of x. Therefore, the function $h[c(z) + v - c(z - vt)]$ (obtained by substituting z for x) is independent of z, and so is its derivative with respect to t at $t = 0$; i.e., $h_{v^*}(v) c_{z^*}(z) v$ is independent of z. Hence,

$$h_{v^*}(v) \cdot [c_{z^*}(z) - c_{z^*}(0)] \cdot v = 0$$

or all values of v and z.

LEMMA 5. $c(z) = c(0) + Cz$

where C is a constant $n \times n$ matrix.

PROOF *(first part)*. From

$$\sum_i \sum_j \frac{\partial h(v)}{\partial v_i} \frac{\partial e_i(z)}{\partial z_j} v_j = 0$$

(Lemma 4) we obtain

$$\begin{cases} \sum_i \sum_j \frac{\partial e_i(z)}{\partial z_j} \int_{R^n} \frac{\partial h(v)}{\partial v_i} v_j \, dv = 0 \,, \\ \sum_i \sum_j \frac{\partial e_i(z)}{\partial z_j} \int_{R^n} \frac{\partial h(v)}{\partial v_i} v_j v_k v_l \, dv = 0 \,, \qquad 1 \leq k,\ l \leq n. \end{cases}$$

These last integrals exist because of

$$\frac{\partial}{\partial v_i} [h(v) v_j v_k v_l] = \frac{\partial h(v)}{\partial v_i} v_j v_k v_l + h(v) [\delta_{ij} v_k v_l + \delta_{ik} v_j v_l + \delta_{il} v_j v_k]$$

(where $\{\delta_{ij}\}$ is the unit matrix) and the additional assumption of Theorem I. Thus, by partial integrations,

$$\begin{cases} \sum_i \frac{\partial e_i(z)}{\partial z_i} = 0 \,, \\ \sum_i \sum_j \frac{\partial e_i(z)}{\partial z_j} (\delta_{ij} \overline{v_k v_l} + \delta_{ik} \overline{v_j v_l} + \delta_{il} \overline{v_j v_k}) = 0 \end{cases}$$

where

$$\overline{v_i v_j} = \int_{R^n} v_i v_j h(v) \, dv \qquad \text{(Def.)},$$

and

$$\sum_j \left(\overline{v_j v_l} \frac{\partial e_k(z)}{\partial z_j} + \overline{v_j v_k} \frac{\partial e_l(z)}{\partial z_j} \right) = 0 \,.$$

Taking the principal axes of the matrix $\{\overline{v_i v_j}\}$ as the axes of coordinates, we obtain

$$\overline{v_l^2} \frac{\partial e_k(z)}{\partial z_l} + \overline{v_k^2} \frac{\partial e_l(z)}{\partial z_k} = 0 \,.$$

Now we shall apply the following:

Lemma 6. Let $\varepsilon_i(\zeta)$, $i = 1, 2, \ldots, n$, be n functions of $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n)$ which are defined and have continuous derivatives in (a non-empty open subset of) R^n and for which

$$(*) \qquad \frac{\partial \varepsilon_i(\zeta)}{\partial \zeta_j} + \frac{\partial \varepsilon_j(\zeta)}{\partial \zeta_i} = 0, \; i, \; j = 1, \; 2, \ldots, \; n,$$

for all vectors ζ. Then,

$$\varepsilon_i(\zeta) = \sum_{j=1}^{n} \varepsilon_{ij} \zeta_j + \alpha_i, \qquad i = 1, \ldots, n,$$

where $\{\varepsilon_{ij}\}$ is a skew-symmetric constant matrix and $\{\alpha_i\}$ is a constant vector.

Proof of lemma 6. Let $j = i$: thus,

$$\frac{\partial \varepsilon_i(\zeta)}{\partial \zeta_i} = 0,$$

and $\varepsilon_i(\zeta)$ and $\partial \varepsilon_i(\zeta) / \partial \zeta_j$ are independent of ζ_i.

Therefore, the derivatives $\frac{\partial}{\partial \zeta_i} \frac{\partial \varepsilon_i(\zeta)}{\partial \zeta_j}$ exist and vanish identically. This implies, because of (*), that the second-order derivatives

$$\frac{\partial}{\partial \zeta_i} \frac{\partial \varepsilon_j(\zeta)}{\partial \zeta_i} = - \frac{\partial}{\partial \zeta_i} \frac{\partial \varepsilon_i(\zeta)}{\partial \zeta_j}$$

exist and vanish also. Hence, the functions $\varepsilon_j(\zeta)$ are linear (or constant) in each of the variables $\zeta_1, \zeta_2, \ldots, \zeta_n$ and all their second-order derivatives $\frac{\partial^2 \varepsilon_j(\zeta)}{\partial \zeta_i \partial \zeta_k}$ exist and are continuous. From

$$d\varepsilon_j(\zeta) = \sum_i \frac{\partial \varepsilon_j(\zeta)}{\partial \zeta_i} d\zeta_i = - \sum_i \frac{\partial \varepsilon_i(\zeta)}{\partial \zeta_j} d\zeta_i$$

it now follows that

$$\frac{\partial}{\partial \zeta_k} \frac{\partial \varepsilon_i(\zeta)}{\partial \zeta_j} = \frac{\partial}{\partial \zeta_i} \frac{\partial \varepsilon_k(\zeta)}{\partial \zeta_j}.$$

On the other hand, the hypothesis (*) implies that

$$\frac{\partial}{\partial \zeta_k} \frac{\partial \varepsilon_i(\zeta)}{\partial \zeta_j} = - \frac{\partial}{\partial \zeta_i} \frac{\partial \varepsilon_k(\zeta)}{\partial \zeta_j}:$$

hence,

$$\frac{\partial^2 \varepsilon_i(\zeta)}{\partial\zeta_j \partial\zeta_k} = 0\,;$$

i.e., all second-order derivatives vanish identically, and

$$\varepsilon_i(\zeta) = \sum_j \varepsilon_{ij}\,\zeta_j + \text{const}$$

where $\varepsilon_{ij} = \text{const}$. Substituting the right-hand sides in (*) shows that $\{\varepsilon_{ij}\}$ is skew-symmetric.

PROOF OF LEMMA 5 *(continued)*. By Lemma 6 and the last equation before Lemma 6,

$$e_i(z) = \Sigma\, e_{ij}\, z_j + \text{const}$$

and $e_{ij} = \text{const}$. Hence,

$$c_{z^*}(z) - c_{z^*}(0) = e_{z^*}(z) = \{e_{ij}\} = \text{const}.$$

Letting $z = 0$, we obtain

$$e_{z^*}(z) = 0$$

and

$$c(z) = c(0) + Cz$$

where $C = c_{z^*}(0)$.

PROOF OF THEOREM I. (iii) By Lemma 2,

$$\bar{u} = c(x - \bar{u}t)\,;$$

hence, by Lemma 5,

$$\bar{u} = (J + Ct)^{-1}\, Cx + (J + Ct)^{-1}\, c(0)\,.$$

$$\begin{aligned} \text{(i)}\quad f(u, x \mid t) &= \rho_0 \cdot h[u - c(x - ut)] \quad \text{(Lemma 4)} \\ &= \rho_0 \cdot h[(J + Ct)\,u - Cx - c\,(0)] \quad \text{(Lemma 5)} \\ = g(v \mid t) &= \rho_0 \cdot [(J + Ct)\,\bar{u} - Cx - c(0) + (J + Ct)\,v] \\ &= \rho_0 \cdot h[(J + Ct)\,v] \quad \text{(iii)}. \end{aligned}$$

$$\text{(ii)} \quad \rho(x \mid t) = \int_{R^n} g(v \mid t)\, dv = \rho_0 \cdot \{\det(J + Ct)\}^{-1} \quad \text{(i)}.$$

$$\text{(iv)} \quad \rho\,\Theta_{ij} = \int_{R^n} v_i v_j f(\bar{u} + v, x \mid t)\, dv$$

$$= \int_{R^n} v_i v_j \rho_0 h[(J + Ct) v]\, dv \quad \text{by (i)}.$$

Introducing the integration variable

$$w = (J + Ct) v$$

yields, after a short computation,

$$\Theta(x, t) = (J + Ct)^{-1} \cdot \Theta(0, 0) \cdot (J + Ct)^{-1*},$$

and this completes the proof of Theorem I.

COROLLARY. The statements of Theorem I hold if its additional assumption is replaced by the following one: the frequency function $g(v \mid t)$ of the residual velocities v has, at $v = 0$, $t = 0$, second-order derivatives with respect to the components of v and their matrix, $g_{vv^*}(0 \mid 0)$, is non-singular, i.e.,

$$\det g_{vv^*}(0 \mid 0) \neq 0.$$

PROOF. The additional assumption of Theorem I was used only in the proof of Lemma 5. Thus it is sufficient to show that this lemma holds under the new assumption also.

By Lemma 4,

$$h_{v^*}(v)\, e_{z^*}(z)\, v = 0.$$

Hence, by TAYLOR'S Theorem,

$$h_{v^*}(0)\, e_{z^*}(z)\, v + v^* h_{vv^*}(0)\, e_{z^*}(z)\, v + o(|v|^2) = 0 \text{ if } |v| \to 0.$$

This implies that

$$h_{vv^*}(0)\, e_{z^*}(z) + e^*_z(z)\, h_{vv^*}(0) = 0.$$

Now take the principal axes of $h_{vv^*}(0)$ as the axes of coordinates and let

$$h_{vv^*}(0) = \mathrm{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n).$$

Then,

$$\lambda_i \frac{\partial e_i(z)}{\partial z_j} + \lambda_j \frac{\partial e_j(z)}{\partial z_i} = 0$$

and, by Lemma 6,

$$e_i(z) = \Sigma\, e_{ij} z_j + \text{const}.$$

The remainder of the proof is identical with that of Lemma 5. Hence the Corollary.

The following reverse of Theorem I shows the consistency of the Axioms I - IV. It is proved by straight-forward verifications.

THEOREM II. Let $h(v)$ be any normalized frequency function of the n-dimensional random variable v defined on R^n, and $h(v)$ have continuous derivatives and vanishing moments of first order; C be an arbitrary constant $n \times n$ matrix, T' be a neighbourhood of the time-zero, $t = 0$, such that $(J + Ct)^{-1}$ exists, $c(0)$ be an arbitrary constant n-vector, and ρ_0 be an arbitrary positive constant scalar. Then the function

$$f(u, x \mid t) = \rho_0 \cdot h[(J + Ct)u - Cx - c(0)],$$

together with the corresponding density $\rho(x \mid t)$ and potential $V(x, t)$, satisfies the Axioms I - IV of § 2, except possibly those concerning the existence of second-order moments of $f(u, x \mid t)$.

§ 4. - Comments.

1. - Theorems I and II together yield a complete survey over the set of universes which satisfy Axioms I - IV and the additional hypotheses of Theorem I or its Corollary. These universes are, in a natural fashion, classified by their initial frequency functions, $h(v)$, of the residual velocities v and their initial « velocity matrices » C. BÒTH are entirely independent of each other. The constants $c(0)$ are inessential: they are removed by choosing suitable systems of coordinates. The velocity matrix C need not be symmetric, i. e., there may be a « rigid-body rotation » of the universe with respect

to any « inertial system of coordinates » (i. e., a system for which NEWTON'S equation of motion hold). Such an (apparent) rotation is by no means at variance with the uniform (« Galileian ») motions of the mass-elements of the universe: the mean velocity $\bar{u}(x, t)$ is defined as a mean value of the velocities of mass-elements at any given place and time, rather than as such a velocity itself: there need not be any mass-element at the point x and the time t which has the velocity $\bar{u}(x, t)$. In the literature it is sometimes assumed that this « rotation » can be eliminated by choosing a new system of coordinates which « rotates with the universe » (cf., e. g., [1]). But such a new system is not an inertial system and, therefore, inadmissible. A permanent « pure rotation », however, is indeed not possible: the velocity matrix $(J + Ct)^{-1} C$ is skew-symmetric if and only if C is skew-symmetric and $t = 0$. « Pure dilatation », on the other hand, is permanent. This implies that rotation (though not « pure rotation ») is permanent also.

The directions of « purely radial motions » are permanent. I. e., the vectors x and $\bar{u}(x, t)$ are (for $c(0) = 0$) collinear if and only if the vectors x and $\bar{u}(x, 0)$ ase collinear. For,

$$Cx = \gamma x$$

for some vector x and some scalar constant γ implies that

$$(J + Ct)^{-1} x = (1 + \gamma t)^{-1} x$$

and, since

$$C(J + Ct)^{-1} = (J + Ct)^{-1} C ,$$

$$(J + Ct)^{-1} Cx = (1 + \gamma t)^{-1} \gamma x ,$$

as has been asserted.

2. - In the hydrodynamical cosmologies it is assumed that the density and pressure are independent of the position vector x (« microscopic homogeneity postulate ») and the relative mean velocity $\bar{u}(x + \Delta x , t) - \bar{u}(x, t)$, is also independent of x (« macroscopic homogeneity postulate »). In the stellar dynamical approach of §§ 2-3 macroscopic homogeneity is a consequence of microscopic homogeneity. The reverse, however, need not be true: this is demonstrated by a frequency function of the form

$$h_1[(J + C_1 t) u - C_1 x] + h_2[(J + C_2 t) u - C_2 x]$$

for suitable fnnctions h_1, h_2 and matrices C_1, C_2. « Microscopic isotropy » (i. e., $g(v \mid t)$ is spherically symmetrical) and « macroscopic isotropy » (i. e., the relative mean velocities, $\bar{u}(x + \Delta x, t) - \bar{u}(x, t)$, and the relative position vectors, Δx, are collinear) are not related to each other.

3. - Let the field of mean velocities be « isotropic in r dimensions » $(0 \leq r \leq n)$, i.e.,

$$C = k \cdot \begin{pmatrix} J_r & 0 \\ 0 & 0 \end{pmatrix}$$

where J_r denotes a $r \times r$ unit matrix, the other elements of C vanish and $k \neq 0$ is a scalar constant (the « Hubble constant »). Let p and ϑ be the scalar pressure and temperature, defined by

$$\begin{cases} p = \dfrac{1}{3} \text{ trace } P, \\ \vartheta = \dfrac{1}{3} \text{ trace } \Theta \end{cases}$$

(cf. [7]), and p_0 and ϑ_0 be their values at the time $t = 0$. Then,

$$\begin{cases} \rho = \rho_0 \cdot (1 + kt)^{-r}, \\ \vartheta = \vartheta_0 \cdot (1 + kt)^{-2}, \\ p = p_0 \cdot (1 + kt)^{-r+2}. \end{cases}$$

Hence, if $r > 0$,

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{1+2/r};$$

i.e., the material of the universe behaves like a polytropic gas of the index $r/2$. In particular, if $n = r = 3$,

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{5/3}.$$

This is the equation which describes adiabatic, or isentropic, changes of state of a monatomic ideal gas. The number r (which corresponds to the number of degrees of freedom in the theory of gases), however, is determined by the matrix C rather than by the function $h(v)$ — i.e., by the properties of the macroscopic motions rather than those of the microscopic motions of the material. In this respect the thermodynamics of the universe appears significantly different from that of an ideal gas.

4. - In § 2 the integral $\iint_M f(u, x \mid t)\, du\, dx$ was interpreted as the mass contained in the set M at the time t. The following, different, interpretation may appear preferable. Suppose that (i) the material of the universe consists of a countable number of particles, (ii) the number actually contained in the set M at the time t is a POISSON random variable with the mean value $\mu = \iint_M f(u, x \mid t)\, du\, dx$, and (iii), Axioms I - IV hold with this interpretation of $f(u, x \mid t)$ also. Thus the particle structure of the real universe is taken into account. For example, a set M with a corresponding POISSON mean μ should, with the probability $1 - \varepsilon$, contain $\mu \pm \xi_\varepsilon \sqrt{\mu}$ particles where ε is any real number between 0 and 1 and ξ_ε denotes the corresponding quantile of the standardized POISSON distribution. (Cf., e.g., [3]).

REFERENCES

[1] H. BONDI, *Cosmology*. (Cambridge U. P. 1960).

[2] R. COURANT, and D. HILBERT, *Methods of Mathematical Physics*, Vol. II. (New York, 1962).

[3] H. CRAMÉR, *Mathematical Melhods of Statistics*. (Princeton U. P.).

[4] O. HECKMANN, and E. SCHÜCKING, *Newtonsche und Einsteinsche Kosmologie*. (Handbuch der Physik, Vol. LIII. Springer-Verlag).

[5] R. KURTH, *Über den Zusammenhang zwischen dynamischer Kosmologie und Stellardynamik*. Z. f Astrophysik **26**, p. 272 (1949).

[6] — — *Introduction to the Mechanics of Stellar Systems*. (Pergamon Press 1957).

[7] — — *Axiomatics of Classical Statistical Mechanics*. (Pergamon Press 1960).

[8] D. LAYZER, *On the Significance of Newtonian Cosmology*. Astrom. J. **59**, p. 268, (1954).

[9] W. H. MCCREA, *Newtonian Cosmology*. Nature **175**, p. 466, 1955).