Problems and methods in partial differential equations (*).

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A Giovanni Sansone nel suo 70^mo compleanno.

Summary. - *Finite part and logarithmic part of some divergent integrals with applications to the* CAUCHY *problem.*

This report represents a part of a group of lectures given at Duke University during the academic year 1955-56. It introduces the theory of the finite part and the logarithmic part of some divergent integrals and applies it to the study of the CAUCHY problem for simple partial differential equations, namely, the wave equation, the damped wave equation and the singular equation of EULER-POISSON-DARBOUX.

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I. Derivatives of some improper integrals.

1. - **Differentiation under the integral sign. Generalisation of the Leibnitz rule.** In its classical form, LEIBNITZ'S rule is as follows. Let

(i) $$a(t), \qquad b(t) \in C^1,$$

(ii) $$f(t, x) \text{ and } \frac{\partial f(t, x)}{\partial t} \in C;$$

then

$$\frac{d}{dt}\int_a^b f(t, x)dx = \int_a^b \frac{\partial f(t, x)}{\partial t}\, dx + f(t, b)\,\frac{db}{dt} - f(t, a)\,\frac{da}{dt}.$$

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In this rule, it is essential to suppose that $f(t, x)$ is bounded in the closed interval $[a, b]$. Therefore, this rule cannot be applied to the integrals (even though they are differentiable)

1. $$\int_0^t \frac{dx}{\sqrt{t-x}} = -2\sqrt{t-x}\Big|_0^t = 2\sqrt{t},$$

2. $$\int_0^t \frac{dx}{\sqrt{x(t-x)}} = \pi,$$

3. $$\int_0^t \lg(t-x)dx = -(t-x)\lg(t-x)\Big|_0^t + (t-x)\Big|_0^t$$
$$= t\lg t - t,$$

the integrands being infinite for $x = t$ (and in 2 also for $x = 0$).

If the LEIBNITZ rule in applied, one would indeed obtain a difference of two terms, each of which is infinite, and which difference has no sense. For instance, from 1

$$\frac{1}{\sqrt{t-x}}\Big|_{x=t} - \frac{1}{2}\int_0^t \frac{dx}{(t-x)^{3/2}},$$

and from 3

$$\lg(t-x)\Big|_{x=t} + \int_0^t \frac{dx}{t-x}.$$

Nevertheless, it is possible to compute the derivatives from the right hand side: for istance, for 1 and 3, one obtains respectively $\frac{1}{\sqrt{t}}$ and $\lg t$.

To correct this situation, it is necessary to generalise LEIBNITZ'S rule. We shall first study two typical examples.

2. Consider the integral

$$F(t) = \int_0^t \frac{A(x)}{\sqrt{t-x}}\, dx, \tag{1}$$

where $A(x)$, $A_x(x)$ are continuous in the closed interval $[0, t]$. Setting $x = yt$, one has

$$F(t) = \int_0^1 \sqrt{t}\, A(yt)\, \frac{dy}{\sqrt{1-y}},$$

which is uniformly convergent and therefore continuous. Its derivative with respect to t is

$$\frac{dF}{dt} = \int_0^1 \frac{\partial}{\partial t} [\sqrt{t}\, A(yt)]\, \frac{dy}{\sqrt{1-y}}, \tag{2}$$

owing to the uniform convergence of this last integral. Consequently

$$\frac{dF}{dt} = \frac{1}{2\sqrt{t}} \int_0^1 A(x)\, \frac{dy}{\sqrt{1-y}} + \frac{1}{\sqrt{t}} \int_0^1 x A_x(x)\, \frac{dy}{\sqrt{1-y}}$$

$$= \frac{1}{2t} \int_0^t A(x)\, \frac{dx}{\sqrt{t-x}} + \frac{1}{t} \int_0^t x A_x(x)\, \frac{dx}{\sqrt{t-x}}.$$

But,

$$\frac{xA_x}{\sqrt{t-x}} = \frac{tA_x}{\sqrt{t-x}} - \sqrt{t-x}\, A_x$$

$$= \frac{tA_x}{\sqrt{t-x}} - \frac{\partial}{\partial x} [\sqrt{t-x}\, A] - \frac{A}{2\sqrt{t-x}}.$$

Hence,

$$\frac{dF}{dt} = \frac{A(0)}{\sqrt{t}} + \int_0^t A_x\, \frac{dx}{\sqrt{t-x}}. \tag{3}$$

To gain a better view, set

$$F_\varepsilon(t) = \int_0^{1-\varepsilon} \sqrt{t}\, A(yt)\, \frac{dy}{\sqrt{1-y}} = \int_0^{t(1-\varepsilon)} A(x)\, \frac{dx}{\sqrt{t-x}},$$

where $0 < \varepsilon < 1$. We have

$$F(t) = \lim_{\varepsilon \to 0} F_\varepsilon(t), \qquad \frac{dF}{dt} = \frac{d}{dt} \lim_{\varepsilon \to 0} F_\varepsilon(t). \tag{4}$$

But, from (2)

$$(5)\qquad \frac{dF}{dt}=\lim_{\varepsilon\to 0}\int_0^{1-\varepsilon}\frac{\partial}{\partial t}[\sqrt{t}\,A(yt)]\,\frac{dy}{\sqrt{1-y}}=\lim_{\varepsilon\to 0}\frac{dF_\varepsilon(t)}{dt}.$$

A comparison of (4) and (5) yields

$$\frac{dF}{dt}=\frac{d}{dt}\lim_{\varepsilon\to 0}F_\varepsilon(t)=\lim_{\varepsilon\to 0}\frac{dF_\varepsilon(t)}{dt}$$

so that $\frac{d}{dt}$ and $\lim_{\varepsilon\to 0}$ commute.

Therefore, to compute $\frac{dF(t)}{dt}$, replace $F(t)$ by $F_\varepsilon(t)$ and compute the derivative with respect to t using Leibnitz's rule [1] and then let ε tend to zero. This gives

$$(6)\qquad \frac{dF_\varepsilon}{dt}=\int_0^{t(1-\varepsilon)}A(x)\frac{\partial}{\partial t}\frac{1}{\sqrt{t-x}}\,dx+\frac{1-\varepsilon}{\sqrt{t\varepsilon}}\,A[t(1-\varepsilon)].$$

Integrating by parts, noting that $\frac{\partial}{\partial t}=-\frac{\partial}{\partial x}$, one finds

$$\frac{dF_\varepsilon}{dt}=-\frac{A(x)}{\sqrt{t-x}}\Bigg|_0^{t(1-\varepsilon)}+\int_0^{t(1-\varepsilon)}A_x\frac{dx}{\sqrt{t-x}}+\frac{1-\varepsilon}{\sqrt{t\varepsilon}}\,A[t(1-\varepsilon)]$$

$$=\int_0^{t(1-\varepsilon)}A_x\frac{dx}{\sqrt{t-x}}+\frac{A(0)}{\sqrt{t}}+\frac{1-\varepsilon}{\sqrt{t\varepsilon}}\,A[t(1-\varepsilon)]-\frac{A[t(1-\varepsilon)]}{\sqrt{t\varepsilon}}.$$

The last two terms become infinite when $\varepsilon\to 0$; but their difference tends to zero. Therefore, when $\varepsilon\to 0$, formula (3) is obtained.

Remark I. - Note that one can also write [cf. (6)]

$$\frac{dF_\varepsilon}{dt}=-A(t)\int_0^{t(1-\varepsilon)}\frac{\partial}{\partial x}\frac{1}{\sqrt{t-x}}\,dx-\frac{1}{2}\int_0^{t(1-\varepsilon)}\frac{A(t)-A(x)}{(t-x)^{3/2}}\,dx+\frac{1-\varepsilon}{\sqrt{t\varepsilon}}\,A[t(1-\varepsilon)]$$

$$=\frac{A(t)}{\sqrt{t}}-\frac{1}{2}\int_0^{t(1-\varepsilon)}\frac{A(t)-A(x)}{(t-x)^{3/2}}\,dx+\frac{1-\varepsilon}{\sqrt{t\varepsilon}}\,A[t(1-\varepsilon)]-\frac{A(t)}{\sqrt{t\varepsilon}}.$$

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[1] Note that the integrand is continuous in the closed interval $[0,\,t(1-\varepsilon)]$.

As a result,

$$\frac{dF}{dt} = \frac{A(t)}{\sqrt{t}} - \frac{1}{2}\int_0^t \frac{A(t) - A(x)}{(t-x)^{3/2}}\, dx .$$

REMARK II. - Suppose $A(t, x)$ $A_t(t, x)$, $A_x(t, x)$ are continuous in the closed rectangle $x \leq t \leq b$, $0 \leq x \leq t$ and set

$$F(t) = \int_0^t \frac{A(t, x)}{\sqrt{t-x}}\, dx .$$

Using the above method, one finds

$$\frac{dF(t)}{dt} = \frac{A(t, 0)}{\sqrt{t}} + \int_0^t (A_t + A_x) \frac{dx}{\sqrt{t-x}} .$$

3. Similar computations can be carried out for the integral

$$\mathfrak{F}(t) = \int_0^t A(x) \lg (t-x) dx,$$

where A, A_x are continuous on the closed interval $[0, t]$. Setting $x = yt$ and $g(t, y) = tA(y, t) \lg t(1-y)$, it follows that

$$\frac{d\mathfrak{F}}{dt} = \int_0 \frac{\partial}{\partial t} g(t, y) dy$$

$$= \frac{1}{t}\int_0^t [A(x) + A(x) \lg (t-x) + xA_x \lg (t-x)]\, dx .$$

But,

$$\frac{\partial}{\partial x}[xA(x) \lg (t-x)] = (A + xA_x) \lg (t-x) - \frac{xA}{t-x}$$

$$= (A + xA_x) \lg (t-x) + \left(1 - \frac{t}{t-x}\right) A .$$

Hence, integrating from 0 to $t-\varepsilon$, $t>\varepsilon>0$, letting $\varepsilon\to 0$, and noting that $\lim\limits_{\varepsilon\to 0}\varepsilon\lg\varepsilon=0$,

$$\int_0^t [A + A\lg(t-x) + xA_x\lg(t-x)]\,dx$$
$$= \lim_{\varepsilon\to 0}\left[tA(t-\varepsilon)\lg\varepsilon + \int_0^{t-\varepsilon}\frac{A(x)}{t-x}\,dx\right]$$

and therefore

$$\frac{d\mathfrak{F}}{dt} = \lim_{\varepsilon\to 0}\left[A(t-\varepsilon)\lg\varepsilon + \int_0^{t-\varepsilon}\frac{A(x)}{t-x}\,dx\right]. \tag{7}$$

Further,

$$\frac{A(x)}{t-x} = -A(x)\frac{\partial}{\partial x}\lg(t-x)$$

and, integrating by parts from 0 to $t-\varepsilon$,

$$\int_0^{t-\varepsilon}\frac{A(x)}{t-x}\,dx = -A(t-\varepsilon)\lg\varepsilon + A(0)\lg t + \int_0^{t-\varepsilon}A_x\lg(t-x)\,dx.$$

Thus, from (7),

$$\frac{d\mathfrak{F}}{dt} = A(0)\lg t + \int_0^t A_x\lg(t-x)\,dx.$$

We may also write [cf. (7)]

$$A(x) = A(t) + (t-x)B(t,\ x)$$

and find

$$\frac{d\mathfrak{F}}{dt} = A(t)\lg t + \int_0^t B(t,\ x)dx = A(t)\lg t + \int_0^t\frac{A(x)-A(t)}{t-x}\,dx. \tag{8}$$

If $0<\varepsilon<1$ and

$$\mathfrak{F}_\varepsilon(t) = \int_0^{1-\varepsilon}g(t,\ y)dy = \int_0^{t(1-\varepsilon)}A(x)\lg(t-x)dx,$$

then

$$\mathfrak{F}(t) = \lim_{\varepsilon \to 0} \mathfrak{F}_\varepsilon(t),$$

$$\frac{d\mathfrak{F}}{dt} = \frac{d}{dt} \lim_{\varepsilon \to 0} \mathfrak{F}_\varepsilon(t) = \lim_{\varepsilon \to 0} \int_0^{1-\varepsilon} \frac{\partial}{\partial t} g(t, y)\, dy$$

$$= \lim_{\varepsilon \to 0} \frac{d\mathfrak{F}_\varepsilon(t)}{dt} = \lim_{\varepsilon \to 0} \frac{d}{dt} \int_0^{t(1-\varepsilon)} A(x) \lg (t - x)\, dx.$$

An application of LEIBNITZ's rule to the last integral gives

(9) $$\frac{d}{dt} \mathfrak{F}_\varepsilon(t) = (1 - \varepsilon) A[t(1 - \varepsilon)] \lg t\varepsilon + \int_0^{t(1-\varepsilon)} A(x) \frac{\partial}{\partial t} \lg (t - x)\, dx.$$

But,

$$\int_0^{t(1-\varepsilon)} A(x) \frac{\partial}{\partial t} \lg (t - x) dx = - \int_0^{t(1-\varepsilon)} A(x) \frac{\partial}{\partial x} \lg (t - x)\, dx$$

$$= - A(x) \lg (t - x) \Big|_0^{t(1-\varepsilon)} + \int_0^{t(1-\varepsilon)} A_x \lg (t - x)\, dx$$

(10) $$= A(0) \lg t - A[t(1 - \varepsilon)] \lg t\varepsilon + \int_0^{t(1-\varepsilon)} A_x \lg (t - x)\, dx.$$

Combining (9) and (10), one has

$$\frac{d}{dt} \mathfrak{F}_\varepsilon(t) = A(0) \lg t + \int_0^{t(1-\varepsilon)} A_x \lg (t - x)\, dx$$

$$+ (1 - \varepsilon) A[t(1 - \varepsilon)] \lg t\varepsilon - A[t(1 - \varepsilon)] \lg t\varepsilon.$$

Here again, each of the last two terms becomes infinite when $\varepsilon \to 0$ while their sum tends to zero.

II. Finite part and logarithmic part of some divergent integrals.

1. Simple integrals. - Let $A(x)$ be a real function of the real single variable x satisfying

(i) $A(x) \in C$ on an interval I;

(ii) $A(x) \in C^k$ in a neighborhood U of $t \in I$.

For $x \in U$,

$$A(x) = \sum_{h=0}^{k-1} (-1)^h \frac{A_h(t)}{h!} (t-x)^h + B_k(t, x), \tag{1}$$

where $A_h(t)$ is the *h-th* derivative of $A(x)$ at $x = t$, $A_0(t) = A(t)$, and

$$B_k(t, x) = \frac{(-1)^{k-1}}{(k-1)!} \int_t^x A_k(y)(y-x)^{k-1} dy.$$

Note that $B_k(t, x) = O([t-x]^k)$. Let the closed interval $[a, t]$ be interior to I and consider

$$I_s(t) = \int_a^t A(x)(t-x)^s dx, \qquad s \text{ real}. \tag{2}$$

$I_s(t)$ exists for $s > -1$; for values of $s \leq -1$, it may or may not be convergent. It is the purpose here to define some useful expressions connected with $I_s(t)$. First, the finite part (abbreviated pf) of $I_s(t)$, symbolically pf $I_s(t)$ is defined for $a \in U$, and then in terms of this definition, the more general definition covering the case $a \notin U$ is given. This is followed by a definition of pl $I_s(t)$, the logarithmic part (abbreviated pl) of $I_s(t)$.

To abbreviate, write

$$P(y;\, n,\, q+1) = \sum_{h=0}^{n} (-1)^h \frac{A_h(t) y^{h-q}}{h!(h-q)}, \quad n \geq 0 \tag{3}$$

so that

$$P_y(y;\, n,\, q+1) \equiv \frac{\partial}{\partial y} P(y;\, n,\, q+1) = \sum_{h=0}^{n} (-1)^h \frac{A_h(t)}{h!} y^{h-q-1}.$$

Throughout, μ is a real number such that $0 < \mu < 1$ and k is zero or a positive integer.

If $a \in U$, the finite part of $I_s(t)$ is defined as follows:

$$\text{pf}\, I_s(t) = I_s(t), \qquad s > -1. \tag{4}$$

$$\text{pf}\, I_{-k-\mu}(t) = P(t-a;\ k-1,\ k+\mu) + \int_a^t \frac{B_k(t,\ x)}{(t-x)^{k+\mu}}\, dx. \tag{5}$$

$$\text{pf}\, I_{-k}(t) = P(t-a;\ k-2,\ k) + (-1)^{k-1} \frac{A_{k-1}(t)}{(k-1)!} \lg (t-a) \tag{6}$$

$$+ \int_a^t \frac{B_k(t,\ x)}{(t-x)^k}\, dx, \quad [P(t-a;\ -1,\ q+1) = 0].$$

If it is not known that $a \in U$, then let a' be any point of U such that $a' < t$ and define

$$\text{pf} \int_a^t A(x)\,(t-x)^s dx = \int_a^{a'} + \text{pf} \int_{a'}^t . \tag{7}$$

For $s > -1$, it is trivially true that $\text{pf}\, I_s(t)$ is independent of the choice of a'. To show that this is also true for $s \leq -1$, let $a'' < t$, $a'' \in U$. Then from (1) and (3),

$$A(x)\,(t-x)^{-k-\mu} = P_y(t-x;\ k-1,\ k+\mu) + \frac{B_k(t,\ x)}{(t-x)^{k+\mu}}, \tag{8}$$

$$\text{pf} \int_a^t A(x)\,(t-x)^{-k-\mu}\, dx = \int_a^{a''} + \text{pf} \int_{a''}^t$$

$$= \left(\int_a^{a'} + \int_{a'}^{a''} \right) A(x)\,(t-x)^{-k-\mu}\, dx + P(t-a'';\ k-1,\ k+\mu)$$

$$+ \left(\int_{a''}^{a'} + \int_{a'}^{t} \right) \frac{B_k(t,\ x)}{(t-x)^{k+\mu}}\, dx .$$

But, with the aid of (8),

$$\int_{a'}^{a''} A(x)\,(t-x)^{-k-\mu}\, dx = -P(t-a'';\ k-1,\ k+\mu) + P(t-a';\ k-1,\ k+\mu)$$

$$+ \int_{a'}^{a''} \frac{B_k(t,\ x)}{(t-x)^{k+\mu}}\, dx.$$

Hence,

$$\int_a^{a''} + \operatorname{pf}\int_{a''}^{t} = \int_a^{a'} + \operatorname{pf}\int_{a'}^{t} .$$

In exactly the same way it can be shown that

$$\operatorname{pf}\int_a^t A(x)\,(t-x)^{-k}\,dx = \int_a^{a'} + \operatorname{pf}\int_{a'}^{t} = \int_a^{a''} + \operatorname{pf}\int_{a''}^{t} .$$

The logarithmic part of $I_s(t)$ is defined by

(9) $$\operatorname{pl} I_s(t) = \begin{cases} (-1)^k \dfrac{A_{k-1}(t)}{(k-1)\,!}, & s = -k \leq -1 \\ 0, & \text{otherwise} \end{cases}$$

where k is a positive integer.

Note that $\operatorname{pl} I_s(t)$ is the negative of the coefficient of $\lg(t-a)$ in $\operatorname{pf} I_s(t)$; $\operatorname{pl} I_s(t)$ is zero when there is no $\lg(t-a)$ term in $\operatorname{pf} I_s(t)$. Also, $\operatorname{pl} I_s(t)$ depends only on the values of $A_{k-1}(x)$ *at the point* $x = t$ and consequently only on the values of $A(x)$ in a neighborhood of $x = t$. This fact parallels the so called HUYGEN'S principle (HUYGEN'S minor premise following HADAMARD'S classification) in the theory of CAUCHY'S problem for partial differential equations of the hyperbolic type.

Contrary to the purely local character of $\operatorname{pl} I_{-k}(t)$, it is clear that $\operatorname{pf} I_s(t)$ is a global operator depending on all the values of $A(x)$ on $[a, t]$. It is important to note too that, as symbols, $\operatorname{pf}\int_a^t$ and $\operatorname{pl}\int_a^t$ are each to be regard d as a « whole », as a single symbol.

The above definitions are easily generalized by replacing $A(x)$ by $A(t, x)$ depending on t, it being understood that $A_{k-1}(x)$ is replaced by

$$A_{k-1}(t, x) = \frac{\partial^{k-1}}{\partial x^{k-1}} A(t, x) \Big|_{x=t} .$$

2. The problem of computing the derivates of improper integrals and the method of singularities used extensively in the theory of partial differential equations lead one to associate with

$$\int_a^t A(x)\,(t-x)^s\,dx, \ (-1 < s < 0),$$

the integral

$$\int_a^{t-\varepsilon} A(x)\,(t-x)^s\,dx,$$

$\varepsilon > 0$ and small. In fact, this amounts to cutting off the singularity at $x = t$.

Using (1) and performing the integration, one finds

(10) $$\operatorname{pf} I_s(t) = \lim_{\varepsilon=0} \int_a^{t-\varepsilon} A(x)\,(t-x)^s\,dx, \quad s > -1,$$

(11) $$\operatorname{pf} I_{-k-\mu}(t) = \lim_{\varepsilon=0} \left[\int_a^{t-\varepsilon} A(x)\,(t-x)^{-k-\mu}\,dx + P(\varepsilon\,;\,k-1,\,k+\mu) \right],$$

(12) $$\operatorname{pl} I_{-k}(t) \lg \varepsilon + \operatorname{pf} I_{-k}(t)$$

$$= P(\varepsilon\,;\,k-2,\,k) + \int_a^{t-\varepsilon} A(x)\,(t-x)^{-k}\,dx + \int_{t-\varepsilon}^{t} B_k(t,\,x)\,(t-x)^{-k}\,dx$$

which clearly indicate the meaning of (4) ... (9).

Considered in itself, the integral given by (2) may, and usually does, diverge for $s \leq -1$. Nevertheless, $\operatorname{pf} I_s(t)$ and $\operatorname{pl} I_s(t)$ exist independently of ε and are finite.

3. In the preceding, the singularity of the integrand was at t, the upper limit of the integral $I_s(t)$. It is sometimes necessary to consider integrals

$$\mathfrak{I}_s(a) = \int_a^{t} A(x)\,(x-a)^s\,dx, \quad (t > a),$$

whose integrands have a singularity at the lower limit.

It is clear that the above definitions may be extended in a natural way, to cover this case. Indeed with reference to (10), (11), (12), define $[t < b\,;\,[t,\,b] \in I]$

$$\operatorname{pf} \mathfrak{I}_s(t) = \lim_{\varepsilon=0} \int_{t+\varepsilon}^{b} A(x)\,(x-t)^s\,dx, \quad s > -1,$$

$$\operatorname{pf} \mathfrak{I}_{-k-\mu}(t) = \lim_{\varepsilon=0} \left[\int_{t+\varepsilon}^{b} A(x)\,(x-t)^{-k-\mu}\,dx + \mathfrak{P}(\varepsilon\,;\; k-1,\; k+\mu) \right],$$

$$\operatorname{pl} \mathfrak{I}_{-k}(t) \lg \varepsilon + \operatorname{pf} \mathfrak{I}_{-k}(t) =$$

$$= \mathfrak{P}(\varepsilon\,;\; k-2,\, k) + \int_{t+\varepsilon}^{b} A(x)\,(x-t)^{-k}\,dx + \int_{t}^{t+\varepsilon} B_k(t,\; x)\,(x-t)^{-k}\,dx$$

where

$$\mathfrak{S}(\varepsilon;\ n,\ q+1)=\sum_{h=0}^{n}\frac{A_h(t)}{h\,!}\,\frac{\varepsilon^{h-q}}{h-q},\quad n\geq 0,$$

$$\mathfrak{S}(\varepsilon;\ -1,\ q+1)=0.$$

4. In connection with pf $I_s(t)$ and pl $I_s(t)$, it is sometimes useful to consider the integral

$$J(\alpha)=\int_a^t A(x)\,(t-x)^\alpha\,dx$$

of the complex variable α. For (²) $\Re\alpha>-1$, $J(\alpha)$ exists and is a holomorphic function of α. Using (1),

$$J(\alpha)=P(t-a;\ r-1,\ -\alpha)+\int_a^t B_r(t,\ x)\,(t-x)^\alpha\,dx\,;$$

the right hand side of the equality gives the analytic continuation $F(\alpha)$ of $J(\alpha)$ for $\Re\alpha\leq-1$. It shows that $F(\alpha)$ is meromorphic in the half plane $\Re\alpha>-(k+1)$ and has simple poles at $\alpha=-1,\ -2,\ldots,\ -k$; it follows that pl $I_s(t)$ *is equal to the negative of the residue of* $F(\alpha)$ *at* $\alpha=s$ and that

$$\text{pf}\,I_{-r-\mu}(t)=F(-r-\mu),$$

$$\text{pf}\,I_{-r}(t)=\lim_{\alpha=-r}\left[F(\alpha)+\frac{\text{pl}\,I_{-r}(t)}{\alpha+r}\right]+(-1)^{r-1}\frac{A_{r-1}(t)}{(r-1)\,!}\lg(t-a).$$

Moreover, it is a simple matter to modify the above so that one has to consider only holomorphic functions. To this end, note that the gamma function $\Gamma(\alpha)$ is meromorphic in the α — plane and has simple poles at $\alpha=-n$ $(n=0,\ 1,\ldots)$ with residue $\dfrac{(-1)^n}{n\,!}$ at $\alpha=-n$. Therefore, the function $f(\alpha)=$ $=\dfrac{F(\alpha)}{\Gamma(\alpha+1)}$ is holomorphic for $\Re\alpha>-(k+1)$ and

$$\text{(13)}\qquad \text{pl}\,I_{-k}(t)=\frac{(-1)^k}{(k-1)\,!}\,[f(\alpha)]_{\alpha=-k},$$

$$\text{(14)}\qquad \text{pf}\,I_{-k-\mu}(t)=\Gamma(-k-\mu+1)\,f(-k-\mu).$$

(²) When α is a complex number, we write as usual $\alpha=\Re\alpha+i\Im\alpha$.

In a similar way, we may obtain a comparable expression for pf $I_{-k}(t)$; however, it is more complicated and not as easy to handle as the other expressions.

These results show the connection between the RIEMANN-LIOUVILLE integral

$$f(\alpha - 1) = \frac{1}{\Gamma(\alpha)} \int_a^t A(x)(t-x)^{\alpha-1} dx$$

and pf, pl of some divergent integrals.

It must be noted that the RIEMANN-LIOUVILLE integral introduces a factor $\Gamma(-k-\mu+1)$ different from zero if $\mu \neq 0$ and which does not appear in the final result. Moreover, the definitions for pf and pl are easily extended to include some multiple integrals and as a matter of fact prove easier to handle when actual computation is required. This will become clearer later on.

5. pf $I_s(t)$ and pl $I_s(t)$ exibit simple properties which shall be very useful later on. These properties are given below with proofs for those which are not immediately evident. Throughout, it is assumed, unless otherwise stated, that $A(x) \in C^k$ on $[a, t]$. This is no restriction in view of the fact that (7) does not depend on a'. Furthermore, since pf $I_s(t) = I_s(t)$ for $s > -1$, in which case the known theory can be applied, the properties below will be concerned with the case $s \leq -1$. Note that the case pl $I_s(t) = 0$, $s \neq 0, -1, -2, \dots,$ is considered trivial and discarded.

PROPERTY 1. - pf $\int_a^t$ and pl $\int_a^t$ are linear operators with respect to $A(x)$.

DEFINITION. - Suppose that $f_j(x) \in C^k$, $j = 1, 2, \dots, n$ on $[a, t]$. Then by definition

$$\text{pf} \int_a^t \sum_{k=1}^n \frac{f_k(x)}{(t-x)^{r_k}} dx = \sum_{k=1}^n \text{pf} \int_a^t \frac{f_k(x)}{(t-x)^{r_k}} dx.$$

PROPERTY 2. - If c is such that $a \leq c < t$, then

$$\text{pf} \int_a^t = \int_a^c + \text{pf} \int_c^t \quad \text{and} \quad \text{pl} \int_a^t = \text{pl} \int_c^t .$$

PROPERTY 3. - *Change of variable* · pf $I_{-r-\mu}(t)$ is invariant under a change $x = x(y)$ of the variable of integration provided $x(y)$ has enough derivatives and $\dfrac{dx}{dy} \neq 0, \infty$.

This condition due to HADAMARD is sufficient, not necessary.

EXAMPLE. - To find the changes of variable leaving

$$\operatorname{pf}\int_0^t x^\alpha dx = \frac{t^{\alpha+1}}{\alpha+1}, \quad t > 0, \ \alpha \neq -1$$

invariant, set $x = y^\mu$, $\mu \neq 0$. Then

$$\operatorname{pf}\int_0^t x^\alpha dx = \operatorname{pf}\int_0^{t^{1/\mu}} y^{(\alpha+1)\mu-1}\, dy = \frac{t^{\alpha+1}}{\alpha+1}.$$

It is clear that we may put $x = y^\mu f(y)$, $\mu \neq 0$ provided $f(y)$ is regular and $f(0) \neq 0$.

pl $I_s(t)$ is also invariant under a suitably chosen change of variable carefully performed. As an example, consider $\int_0^1 \frac{dx}{x}$ and set $x = y/a$, $a > 0$; one has

$$-\lg \varepsilon = \int_\varepsilon^1 \frac{dx}{x} = \int_{a\varepsilon}^a \frac{dy}{y} = \lg a - \lg a\varepsilon = -\lg \varepsilon,$$

from which it is clear that the coefficient of $\lg \varepsilon$, [cf. (12)], remains the same.

PROPERTY 4. - **Integration by parts.** If $A(x) \in C^{k+1}$ on $[a, t]$, then

$$\operatorname{pf}\int_a^t A(x)(t-x)^s\, dx$$

$$= \frac{1}{s+1}\left[A(a)(t-a)^{s+1} + \operatorname{pf}\int_a^t A'(x)(t-x)^{s+1}\, dx\right], \ s \neq -1,$$

$$= A(a)\lg(t-a) + \int_a^t A'(x)\lg(t-x)\, dx, \quad s = -1.$$

PROOF. - If $s = -k-\mu$, use (11); if $s = -k$, use (12).

PROPERTY 5. - Suppose that the functions $A(t, x)$ and $g(x)$ are both $\in C^k$ and that $g(x)$ is given as a series

$$g(x) = \sum_{j=0}^{\infty} g_j(x),$$

uniformly convergent on $[a, t]$, with the property that $\sum_{j=0}^{\infty} g_j^{(i)}(x)$, $(i = 1, \dots, k)$, also converge uniformly on $[a, t]$. Then

$$\mathrm{pf}\int_a^t \frac{A(t, x)g(x)}{(t-x)^{k+\mu}}\,dx = \sum_{j=0}^{\infty} \mathrm{pf}\int_a^t \frac{A(t, x)g_j(x)}{(t-x)^{k+\mu}}\,dx.$$

PROOF. - From (5),

$$\mathrm{pf}\int_a^t \frac{A(t, x)g_j(x)}{(t-x)^{k+\mu}}\,dx$$

$$= \sum_{h=0}^{k-1} (-1)^h \frac{[A(t, x)g_j(x)]_{x=t}^{(h)}(t-a)^{h-k-\mu+1}}{h!\,(h-k-\mu+1)}$$

$$+ \int_a^t \frac{B_{k,j}(t, x)}{(t-x)^{k+\mu}}\,dx$$

holds for $j = 0, 1, \dots$ and also for $g_j(x) = g(x)$.

Summing on j and then making use of the uniform convergence hypothesis, one obtains the desired result.

Note that

$$B_k(t, x) = \frac{(-1)^{k-1}}{(k-1)!}\int_t^x A_k(y)(y-x)^{k-1}dy$$

so that we can safely commute integration and summation.

PROPERTY 6. - *A bound for* $\mathrm{pf}\, I_s(t)$. Let M be chosen so that $|A_h(x)| \leq M$ $(h = 1, \dots, k)$ on $[a, t]$. Then from (5),

$$\mathrm{pf}\, I_{-k-\mu}(t) \leq |P(t-a;\, k-1,\, k+\mu)| + \int_a^t \frac{|B_k(t, x)|}{(t-x)^{k+\mu}}\,dx$$

$$\leq \sum_{h=0}^{k-1} \frac{|A_h(t)|\,(t-a)^{h-k-\mu+1}}{h!\,(k+\mu-h-1)} + \int_a^t \frac{M(t-x)^k}{k!\,(t-x)^{k+\mu}}\,dx$$

$$\leq M\left[\sum_{h=0}^{k-1} \frac{(t-a)^{h-k-\mu+1}}{k!\,(k+\mu-h-1)} + \frac{(t-a)^{1-\mu}}{k!\,(1-\mu)}\right]$$

and from (6),

$$\text{pf}\, I_{-k}(t) \leq M\left[\sum_{h=0}^{k-2} \frac{(t-a)^{h-k-1}}{k!\,(k-h+1)} + \frac{|\lg(t-a)|}{(k-1)!} + \frac{t-a}{k!}\right].$$

Thus, a bound for $\text{pf}\, I_s(t)$ is known if M and $t-a$ are known.

6. **Property 7. Differentiation.** - If $A(x) \in C^{k+1}$ on $[a, t]$, $k = 0, 1, \ldots$, then

(15)
$$\frac{d}{dt}\text{pf}\int_a^t \frac{A(x)}{(t-x)^{k+\mu}}\,dx = \text{pf}\int_a^t \frac{d}{dt}\frac{A(x)}{(t-x)^{k+\mu}}\,dx,$$

(16)
$$\frac{d}{dt}\text{pf}\int_a^t \frac{A(x)}{(t-x)^k}\,dx = \text{pf}\int_a^t \frac{d}{dt}\frac{A(x)}{(t-x)^k}\,dx + (-1)^k \frac{A_k(t)}{k!},$$

(17)
$$\frac{d}{dt}\text{pl}\int_a^t \frac{A(x)}{(t-x)^k}\,dx = \text{pl}\int_a^t \frac{d}{dt}\frac{A(x)}{(t-x)^k}\,dx.$$

Note that (16) and (17) are trivially true for $k = 0$.

That (17) follows easily is seen in (9).

To establish (15), first a proof by direct calculation is given, then a proof making sole use of repeated integration by parts, followed by a proof by induction again using integration by parts as the core.

7. **Proof of (15) by direct calculation.** - Write

(18)
$$B_k(t, x) = (-1)^k \frac{A_k(t)}{k!}(t-x)^k + B_{k+1}(t, x),$$

where

$$B_{k+1}(t, x) = \frac{(-1)^k}{k!}\int_t^x A_{k+1}(y)(y-x)^k\,dy$$

so that

$$\frac{\partial}{\partial t} B_{k+1}(t, x) = -\frac{(-1)^k}{k!} A_{k+1}(t)(t-x)^k. \tag{19}$$

Further, an easy calculation shows that

$$\frac{\partial}{\partial t} P(t-a;\ s,\ q+1) \tag{20}$$

$$= -(q+1)P(t-a;\ s+1,\ q+2) + \frac{(-1)^s A_{s+1}(t)(t-a)^{s-q}}{(s+1)!}.$$

Indeed,

$$\frac{\partial}{\partial t} P(t-a;\ s,\ q+1) = \frac{\partial}{\partial t} \sum_{h=0}^{s} (-1)^h \frac{A_h(t)(t-a)^{h-q}}{h!\,(h-q)}$$

$$= \sum_{h=0}^{s} (-1)^h \frac{A_{h+1}(t)(t-a)^{h-q}}{h!\,(h-q)} + \sum_{h=0}^{s} (-1)^h \frac{A_h(t)(t-a)^{h-q-1}}{h!}$$

$$= \sum_{h=1}^{s+1} (-1)^{h-1} \frac{A_h(t)(t-a)^{h-q-1}}{(h-1)!\,(h-q-1)} + \sum_{h=0}^{s} (-1)^h \frac{A_h(t)(t-a)^{h-q-1}}{h!}$$

$$= -(q+1) \sum_{h=0}^{s} (-1)^h \frac{A_h(t)(t-a)^{h-q-1}}{h!\,(h-q-1)} + \frac{(-1)^s A_{s+1}(t)(t-a)^{s-q}}{s!\,(s-q)}$$

$$= -(q+1)P(t-a;\ s+1,\ q+2) + \frac{(-1)^s A_{s+1}(t)(t-a)^{s-q}}{(s+1)!}.$$

Because of complications arising from the use of the symbol $P(t-a;\ k-1,\ k+\mu)$ the case $k=0$ will be treated first as a special case; then the case $k \geq 1$ will be considered.

a. In case $k=0$,

$$\frac{d}{dt}\int_a^t \frac{A(x)}{(t-x)^\mu}\,dx = \frac{d}{dt}\int_a^t \frac{A(t)+A(x)-A(t)}{(t-x)^\mu}\,dx$$

$$= \frac{d}{dt}\left[\frac{A(t)\,(t-a)^{1-\mu}}{1-\mu} + \int_a^t \frac{A(x)-A(t)}{(t-x)^\mu}\,dx\right.$$

$$= \frac{A(t)}{(t-a)^\mu} + \frac{A'(t)\,(t-a)^{1-\mu}}{1-\mu} + \frac{d}{dt}\int_a^t \frac{A(x)-A(t)}{(t-x)^\mu}\,dx.$$

But, since

$$\frac{A(x) - A(t)}{(t-x)^{\mu}} = \frac{\int_t^x A'(y)dy}{(t-x)^{\mu}} = O([t-a]^{1-\mu}),$$

one has

$$\frac{d}{dt}\int_a^t \frac{A(x) - A(t)}{(t-x)^{\mu}}\, dx = \int_a^t \frac{d}{dt}\, \frac{A(x) - A(t)}{(t-x)^{\mu}}\, dx$$

$$= -\mu \int_a^t \frac{A(x) - A(t)}{(t-x)^{1+\mu}}\, dx - \frac{A'(t)\,(t-a)^{1-\mu}}{1-\mu}.$$

Hence,

$$\frac{d}{dt}\int_a^t \frac{A(x)}{(t-x)^{\mu}}\, dx = -\mu\left[\frac{A(t)\,(t-a)^{-\mu}}{-\mu} + \int_a^t \frac{A(x) - A(t)}{(t-x)^{1+\mu}}\, dx\right]$$

$$= \mathrm{pf}\int_a^t \frac{d}{dt}\, \frac{A(x)}{(t-x)^{\mu}}\, dx.$$

b. When $k \geq 1$, consider

$$\frac{d}{dt}\,\mathrm{pf}\int_a^t \frac{A(x)}{(t-x)^{k+\mu}}\, dx \tag{21}$$

$$= \frac{d}{dt}\, P(t-a\,;\; k-1,\; k+\mu) + \frac{d}{dt}\int_a^t \frac{B_k(t,\; x)}{(t-x)^{k+\mu}}\, dx$$

$$= -(k+\mu)P(t-a;\; k,\; k+\mu+1) + \frac{(-1)^{k-1}A_k(t)\,(t-a)^{-\mu}}{k\,!}$$

$$+ \frac{d}{dt}\int_a^t \frac{B_k(t,\; x)}{(t-x)^{k+\mu}}\, dx.$$

But, using (18) and noting that

$$\frac{d}{dt}\int_a^t \frac{B_{k+1}(t,\; x)}{(t-x)^{k+\mu}}\, dx = \int_a^t \frac{d}{dt}\, \frac{B_{k+1}(t,\; x)}{(t-x)^{k+\mu}}\, dx,$$

one finds

$$\frac{d}{dt}\int_a^t \frac{B_k(t,\ x)}{(t-x)^{k+\mu}}\,dx$$

$$= \frac{d}{dt}\left[(-1)^k \frac{A_k(t)}{k\,!}\,\frac{(t-a)^{1-\mu}}{1-\mu} + \int_a^t \frac{B_{k+1}(t,\ x)}{(t-x)^{k+\mu}}\,dx\right]$$

$$= \frac{(-1)^k A_{k+1}(t)\,(t-a)^{1-\mu}}{k\,!(1-\mu)} + (-1)^k\,\frac{A_k(t)\,(t-a)^{-\mu}}{k\,!}$$

$$+ \int_a^t \frac{\frac{d}{dt}\,B_{k+1}(t,\ x)}{(t-x)^{k+\mu}}\,dx - (k+\mu)\int_a^t \frac{B_{k+1}(t,\ x)}{(t-x)^{k+\mu+1}}\,dx$$

which, upon 'using (19), gives

(22)
$$\frac{d}{dt}\int_a^t \frac{B_k(t,\ x)}{(t-x)^{k+\mu}}\,dx$$

$$= (-1)^k\,\frac{A_{k+1}(t)\,(t-a)^{1-\mu}}{k\,!\ (1-\mu)} + (-1)^k\,\frac{A_k(t)\,(t-a)^{-\mu}}{k\,!}$$

$$+ (-1)^{k+1}\,\frac{A_{k+1}(t)}{k\,!}\,\frac{(t-a)^{1-\mu}}{1-\mu} - (k+\mu)\int_a^t \frac{B_{k+1}(t,\ x)}{(t-x)^{k+\mu+1}}\,dx$$

$$= (-1)^k\,\frac{A_k(t)\,(t-a)^{-\mu}}{k\,!} - (k+\mu)\int_a^t \frac{B_{k+1}(t,\ x)}{(t-x)^{k+\mu+1}}\,dx.$$

Combining the results of (21) and (22), we have

$$\frac{d}{dt}\,\mathrm{pf}\int_a^t \frac{A(x)}{(t-x)^{k+\mu}}\,dx$$

$$= -(k+\mu)\left[P(t-a\,;\ k,\ k+\mu+1) + \int_a^t \frac{B_{k+1}(t,\ x\ )}{(t-x)^{k+\mu+1}}\,dx\right]$$

$$= -(k+\mu)\,\mathrm{pf}\int_a^t \frac{A(x)}{(t-x)^{k+\mu+1}}\,dx = \mathrm{pf}\int_a^t \frac{d}{dt}\,\frac{A(x)}{(t-x)^{k+\mu}}\,dx.$$

8. **Proof by repeated integration by parts.** Integrating by parts $k+1$ times, we have

$$\frac{d}{dt}\,\mathrm{pf}\int_a^t \frac{A(x)}{(t-x)^{k+\mu}}\,dx$$

$$= \frac{d}{dt}\left[\sum_{h=1}^{k+1} \frac{A_{h-1}(a)\,(t-a)^{h-k-\mu}}{(1-k-\mu)\dots(h-k-\mu)}\right.$$

$$\left. + \frac{1}{(1-k-\mu)\dots(1-\mu)}\int_a^t A_{k+1}(x)\,(t-x)^{1-\mu}\,dx\right.$$

$$= \sum_{h=1}^{k+1} \frac{A_{h-1}(a)\,(t-a)^{h-k-\mu-1}}{(1-k-\mu)\dots(h-k-\mu-1)}$$

$$+ \frac{1}{(1-k-\mu)\dots(-\mu)}\int_a^t A_{k+1}(x)\,(t-x)^{-\mu}\,dx$$

$$= -(k+\mu)\,\mathrm{pf}\int_a^t \frac{A(x)}{(t-x)^{k+\mu+1}}\,dx$$

$$= \mathrm{pf}\int_a^t \frac{d}{dt}\,\frac{A(x)}{(t-x)^{k+\mu}}\,dx\,.$$

9. **Proof by induction using integration by parts.** (15) holds for $k=0$. For

$$\frac{d}{dt}\int_a^t \frac{A(x)}{(t-x)^{\mu}}\,dx$$

$$= \frac{d}{dt}\left[A(a)\,\frac{(t-a)^{1-\mu}}{1-\mu} + \frac{1}{1-\mu}\int_a^t A'(x)\,(t-x)^{1-\mu}\,dx\right]$$

$$= A(a)\,(t-a)^{-\mu} + \int_a^t \frac{A'(x)}{(t-x)^{\mu}}\,d\bar{x}$$

$$= -\mu\,\mathrm{pf}\int_a^t \frac{A(x)}{(t-x)^{1+\mu}}\,dx = \mathrm{pf}\int_a^t \frac{d}{dt}\,\frac{A(x)}{(t-x)^{\mu}}\,dx\,.$$

If (15) holds for $k-1\geq 0$,

$$\frac{d}{dt}\,\mathrm{pf}\int_a^t \frac{A(x)}{(t-x)^{k+\mu}}\,dx$$

$$=\frac{d}{dt}\left[A(a)\,\frac{(t-a)^{1-k-\mu}}{1-k-\mu}+\frac{1}{1-k-\mu}\,\mathrm{pf}\int_a^t A'(x)(t-x)^{1-k-\mu}\,dx\right]$$

$$=A(a)\,(t-a)^{-k-\mu}+\mathrm{pf}\int_a^t \frac{A'(x)}{(t-x)^{k+\mu}}\,dx$$

$$=-(k+\mu)\,\mathrm{pf}\int_a^t \frac{A(x)}{(t-x)^{k+\mu+1}}\,dx$$

$$=\mathrm{pf}\int_a^t \frac{d}{dt}\,\frac{A(x)}{(t-x)^{k+\mu}}\,dx,$$

and it holds for k.

10. In very much the same way, the three proofs used to show that (15) holds for $k\geq 0$ can be used to show that (16) holds for $k\geq 1$, $k=0$ regarded as trivial. Only the proof by direct calculation is given below. The special case $k=1$ arises, again due to notational difficulties. The special case is treated first.

If $k=1$ and $A(x)\in C^2$, then

$$\frac{d}{dt}\,\mathrm{pf}\int_a^t \frac{A(x)}{t-x}\,dx$$

$$=\frac{d}{dt}\left[A(t)\lg(t-a)-\int_a^t \frac{A'(t)(t-x)+\int_t^x A''(y)(y-x)\,dy}{t-x}\,dx\right]$$

$$=\frac{A(t)}{t-a}+A'(t)\lg(t-a)-A'(t)-A''(t)\,(t-a)$$

$$-\frac{d}{dt}\int_a^t \frac{\int_t^x A''(y)\,(y-x)\,dy}{t-x}\,dx.$$

Because

$$\int_t^x A''(y)\,(y-x)\,dy=O([t-x]^2),$$

it follows that

$$\frac{d}{dt}\int_a^t \frac{\int_t^x A''(y)(y-x)\,dy}{t-x}\,d\bar{x} = \int_a^x \frac{d}{dt}[\ldots]\,dx$$

$$= - A''(t)(t-a) - \int_a^t \frac{\int_t^x A''(y)(y-x)\,dy}{(t-x)^2}\,dx.$$

Thus,

$$\frac{d}{dt}\,\mathrm{pf}\int_a^t \frac{A(x)}{t-x}\,dx$$

$$= \frac{A(t)}{t-a} + A'(t)\lg(t-a) + \int_a^t \frac{\int_t^x A''(y)(y-x)\,dy}{(t-x)^2}\,dx - A'(t)$$

$$= -\,\mathrm{pf}\int_a^t \frac{A(x)}{(t-x)^2}\,dx - A'(t)$$

$$= \mathrm{pf}\int_a^t \frac{d}{dt}\frac{A(x)}{t-x}\,dx - A'(t).$$

Now suppose $A(x) \in C^{k+1}$ on $[a, t]$, $k \geq 2$. Then

$$\frac{d}{dt}\,\mathrm{pf}\int_a^t \frac{A(x)}{(t-x)^k}\,dx$$

$$= \frac{d}{dt}\Big[P(t-a;\ k-2,\ k) + (-1)^{k-1}\frac{A_{k-1}(t)}{(k-1)!}\lg(t-a)$$

$$+ \int_a^t \frac{B_k(t,x)}{(t-x)^k}\,dx\Big]$$

$$= -\,kP(t-a;\ k-1,\ k+1) + (-1)^{k-1}\frac{A_k(t)}{(k-1)!}\lg(t-a) +$$

$$+ \frac{d}{dt}\int_a^t \Big[(-1)^k\frac{A_k(t)}{k!}(t-x)^k + \frac{(-1)^k}{k!}\int_t^x A_{k+1}(y)(y-x)^k dy\Big]\frac{dx}{(t-x)^k}$$

$$= -\,kP(t-a;\ k-1,\ k+1) + (-1)^{k-1}\frac{A_k(t)}{(k-1)!}\lg(t-a) + (-1)^k\frac{A_k(t)}{k!}$$

$$+ (-1)^k \frac{A_{k+1}(t)(t-a)}{k!} + \int_a^t \frac{d}{dt} \frac{(-1)^k}{k!} \frac{\int_t^x A_{k+1}(y)(y-x)^k dy}{(t-x)^k} dx$$

$$\left(\text{because} \int_t^x = O[(t-x)^{k+1}]\right)$$

$$= -k \left[P(t-a); k-1, k+1) + (-1)^k \frac{A_k(t)}{k!} \lg(t-a) + \int_a^t \frac{B_{k+1}(t, x)}{(t-x)^{k+1}} dx \right]$$

$$+ (-1)^k \frac{A_k(t)}{k!}$$

which is the right hand side of (16).

11. Examples of differentiation. If $A(x) \in C^1$, then

$$\frac{d}{dt} \int_a^t \frac{A(x)}{\sqrt{t-x}} dx = -\frac{1}{2} \operatorname{pf} \int_a^t \frac{A(x)}{(t-x)^{3/2}} dx,$$

as indicated long ago by R. d'ADHÉMAR.

In general, if $A(x) \in C^{k+1}$, $(0 < \mu < 1$ understood)

$$\frac{d^k}{dt^k} \int_a^t \frac{A(x)}{(t-x)^\mu} dx = (-1)^k \frac{\Gamma(\mu+k)}{\Gamma(\mu)} \operatorname{pf} \int_a^t \frac{A(x)}{(t-x)^{k+\mu}} dx$$

because

$$\frac{d^k}{dt^k} (t-x)^{-\mu} = (-1)^k \mu(\mu+1) \dots (\mu+k-1)(t-x)^{-k-\mu}$$

$$= (-1)^k \frac{\Gamma(\mu+1)}{\Gamma(\mu)} \cdot \frac{1}{(t-x)^{k+\mu}}.$$

From I, 3, form. 8, and from the definition of pf $I_{-k}(t)$, [II, 1, form. 6], one deduces

$$\frac{d}{dt} \int_a^t A(x) lg(t-x) dx = A(t) lg(t-a) + \int_a^t \frac{A(x) - A(t)}{t-x} dx$$

$$= \operatorname{pf} \int_a^t \frac{A(x)}{t-x} dx = \operatorname{pf} \int_a^t A(x) \frac{d}{dt} \lg(t-x) dx.$$

12, Cauchy's principal value of an integral. There is a close relation between the CAUCHY principal value (abbreviated vp.) of a simple integral and the finite parts associated with that integral. As we hope to develop this point of view on another occasion, it will serve our purpose to consider here the most simple case.

Let $A(x)$ be a function having the properties described earlier (see II, 1). We have

$$\text{pf}\int_a^t \frac{A(x)}{t-x}\,dx = A(t)\lg(t-a) + \int_a^t \frac{A(x)-A(t)}{t-x}\,dx,$$

$$\text{pf}\int_t^b \frac{A(x)}{x-t}\,dx = A(t)\lg(b-t) + \int_t^b \frac{A(x)-A(t)}{x-t}\,dx,$$

$$\text{vp}\int_a^b \frac{A(x)}{x-t}\,dx = \lim_{\varepsilon=0}\left[\int_a^{t-\varepsilon} + \int_{t+\varepsilon}^b\right]\frac{A(x)}{x-t}\,dx$$

$$= \int_a^b \frac{A(x)-A(t)}{x-t}\,dt + A(t)\lg\frac{b-t}{t-a}$$

$$= \text{pf}\int_t^b \frac{A(x)}{x-t}\,dx - \text{pf}\int_a^t \frac{A(x)}{t-x}\,dx.$$

It is not difficult to prove that $\text{vp}\int$ enjoys most of the properties of an ordinary integral. In particular, using II, 6, (16), one has

$$\frac{d}{dt}\,\text{vp}\int_a^b \frac{A(x)}{x-t}\,dx = \text{pf}\int_t^b \frac{d}{dt}\frac{A(x)}{x-t}\,dx - A_1(t)$$

$$- \text{pf}\int_a^t \frac{d}{dt}\frac{A(x)}{t-x}\,dx + A_1(t).$$

Integrating by parts, one finds ($A(x)\in C^2$ on $[a, b]$)

$$\text{pf}\int_t^b \frac{d}{dt}\frac{A(x)}{x-t}\,dx = -\,\text{pf}\int_t^b A(x)\frac{d}{dt}\frac{1}{x-t}\,dx$$

$$= \text{pf}\int_t^b A_x\frac{dx}{x-t} - \frac{A(b)}{b-t},$$

$$\text{pf}\int \frac{d}{dt}\frac{A(x)}{t-x}\,dx = \text{pf}\int_a^t A_x\frac{dx}{t-x} + \frac{A(a)}{t-a}$$

and finally

$$\frac{d}{dt}\,\mathrm{vp}\int_a^b \frac{A(x)}{x-t}\,dx = \mathrm{pf}\int_t^b A_x \frac{dx}{x-t} - \frac{A(b)}{b-t}$$

$$- \mathrm{pf}\int_a^t A_x \frac{dx}{x-t} - \frac{A(a)}{t-a}$$

$$= \mathrm{vp}\int_a^b A_x \frac{dx}{x-t} - \frac{A(b)}{b-t} + \frac{A(a)}{a-t}.$$

13. In order to facilitate a study of TRICOMI's equation and its generalization, it is desirable to introduce the finite part of the usually divergent integral

$$\mathfrak{F}_{-k-\mu}(t) = \int_a^t A(x)\,(t-x)^{-k-\mu}\,\lg(t-x)\,dx,$$

where $0 < \mu < 1$, k is a positive integer, and $A(x) \in C^k$ on $[a,\ t]$.

Let us write

$$Q(y;\ k-1,\ q+1) = \sum_{h=0}^{k-1} (-1)^h \frac{A_h(t)}{h\,!}\,\frac{y^{h-q}}{h-q}\left[\lg y - \frac{1}{h-q}\right]$$

and define the finite part of $\mathfrak{F}_{-k-\mu}(t)$ as

$$\mathrm{pf}\,\mathfrak{F}_{-k-\mu}(t) = Q(t-a;\ k-1,\ k+\mu) + \int_a^t B_k(t,\ x)\,\frac{\lg(t-x)}{(t-x)^{k+\mu}}\,dx \qquad (k \geq 1).$$

Should it be only known that $A(x) \in C$ on $[a,\ t]$ and $A(x) \in C^k$ in a neighborhood U of t then let $a' \in U$, $a' < t$ and define

$$\mathrm{pf}\,\mathfrak{F}_{-k-\mu}(t) = \int_a^{a'} + \mathrm{pf}\int_{a'}^t.$$

As for $\mathrm{pf}\,I_s(t)$, it follows that $\mathfrak{F}_{-k-\mu}(t)$ is unique for all choices of a'.

In cases k is a non-positive integer and $A(x) \in C$ on $[a,\ t]$, $\mathfrak{F}_{-k-\mu}(t)$ exists and we define

$$\mathrm{pf}\,\mathfrak{F}_{-k-\mu}(t) = \mathfrak{F}_{-k-\mu}(t).$$

Clearly, defining $Q(y;\ k-1,\ q+1)=0$ for $k=0,\ -1,\ -2, \ldots$ (in which case $A(x) \in C$),

$$\lim_{\varepsilon \to 0}\left[\int_a^{t-\varepsilon} A(x)\frac{\lg(t-x)}{(t-x)^{k+\mu}}\,dx - Q(\varepsilon;\ k-1,\ k+\mu)\right] = \mathrm{pf}\ \mathfrak{F}_{-k-\mu}(t) \tag{23}$$

for any integer k.

$\mathrm{pf}\ \mathfrak{F}_{-k-\mu}(t)$ enjoys the same properties that $\mathrm{pf}\ I_s(t)$ was shown to have, namely properties 1-7. While most of these properties follow rather easily, integration by parts may require clarification and, as the task is not a trivial one, it may also be worthwhile to establish the rule for differentiation. These are treated below for $k \geq 0$.

14. Integration by parts of $\mathfrak{F}_{-k-\mu}(t)$. If $A(x) \in C^{k+1}$, $k \geq 0$, then

$$\mathrm{pf}\int_a^t A(x)\,\frac{\lg(t-x)}{(t-x)^{k+\mu}}\,dx$$

$$= A(a)\,\frac{(t-a)^{1-k-\mu}}{1-k-\mu}\left[\lg(t-a) - \frac{1}{1-k-\mu}\right] + \mathrm{pf}\int_a^t A'(x)\,\frac{\lg(t-x)\,dx}{(t-x)^{k+\mu-1}}$$

PROOF. - Use (23).

15. **Differentiation with respect to** t **of** $\mathfrak{F}_{-k-\mu}(t)$. If $k=0, 1, \ldots$ and $A(x) \in C^{k+1}$, then

$$\frac{d}{dt}\,\mathrm{pf}\int_a^t A(x)\,\frac{\lg(t-x)}{(t-x)^{k+\mu}}\,dx = \mathrm{pf}\int_a^t A(x)\,\frac{d}{dt}\,\frac{\lg(t-x)}{(t-x)^{k+\mu}}\,dx, \tag{24}$$

where, by agreement,

$$\mathrm{pf}\int_a^t A(x)\,\frac{d}{dt}\,\frac{\lg(t-x)}{(t-x)^{k+\mu}}\,dx = \mathrm{pf}\int_a^t A(x)\left[\frac{1-(k+\mu)\lg(t-x)}{(t-x)^{k+\mu+1}}\right]dx$$

$$= \mathrm{pf}\int_a^t \frac{A(x)}{(t-x)^{k+\mu+1}}\,dx - (k+\mu)\,\mathrm{pf}\int_a^t A(x)\,\frac{\lg(t-x)}{(t-x)^{k+\mu+1}}\,dx.$$

As in the case of $\mathrm{pf}\ I_s(t)$, three different proofs may be given. Only the proof by direct calculation will be given below.

First, as a result of some elementary computations,

$$(25)\qquad \frac{d}{dt}Q(t-a;\,k-1,\,q+1) = -(q+1)\,Q(t-a;\,k,\,q+2) +$$

$$+\sum_{h=0}^{k}(-1)^h\frac{A_h(t)}{h\,!}\frac{(t-a)^{h-q-1}}{h-q-1} + (-1)^{k+1}\frac{A_k(t)}{k\,!}(t-a)^{k-q-1}\lg(t-a)$$

$$= -(q+1)\,Q(t-a;\,k,\,q+2) + P(t-a;\,k,\,q+2)$$

$$+(-1)^{k+1}\frac{A_k(t)}{h\,!}(t-a)^{k-q-1}\lg(t-a),\ k\geq 1.$$

Formula (24), $k \geq 1$, is equivalent to

$$(26)\qquad \frac{d}{dt}Q(t-a;\,k-1,\,k+\mu) + \frac{d}{dt}\int_a^t B_k(t,\,x)\frac{\lg(t-x)}{(t-x)^{k+\mu}}dx$$

$$= -(k+\mu)\,\mathrm{pf}\,\mathcal{F}_{-k-\mu-1}(t) + \mathrm{pf}\int_a^t\frac{A(x)}{(t-x)^{k+\mu+1}}dx,$$

as may be seen using for instance (23).

Because

$$B_k(t,\,x) = (-1)^k\frac{A_k(t)}{k\,!}(t-x)^k + B_{k+1}(t,\,x),$$

$$\frac{d}{dt}B_{k+1}(t,\,x) = (-1)^{k+1}\frac{A_{k+1}(t)}{k\,!}(t-x)^k,$$

it is clear that (³)

$$\int_a^t B_k(t,\,x)\frac{\lg(t-x)}{(t-x)^{k+\mu}}dx = \int_a^t B_{k+1}(t,\,x)\frac{\lg(t-x)}{(t-x)^{k+\mu}}dx$$

$$+(-1)^k\frac{A_k(t)}{k\,!}\frac{(t-a)^{1-\mu}}{1-\mu}\left[\lg(t-a)-\frac{1}{1-\mu}\right],$$

$$\int_a^t\frac{d}{dt}[B_{k+1}(t,\,x)]\frac{\lg(t-x)}{(t-x)^{k+\mu}}dx$$

$$= (-1)^{k+1}\frac{A_{k+1}(t)}{k\,!}\frac{(t-a)^{1-\mu}}{1-\mu}\left[\lg(t-a)-\frac{1}{1-\mu}\right].$$

(³) $\int(t-x)^{-\mu}\lg(t-x)dx = -\frac{(t-x)^{1-\mu}}{1-\mu}\left[\lg(t-x)-\frac{1}{1-\mu}\right]+C.$

Consequently,

$$\text{(27)} \qquad \frac{d}{dt}\int_a^t B_k(t,\ x)\frac{\lg(t-x)}{(t-x)^{k+\mu}}\,dx$$

$$=(-1)^k\frac{A_k(t)}{k\,!}\,\frac{\lg(t-a)}{(t-a)^{\mu}}+\int_a^t B_{k+1}(t,\ x)\frac{d}{dt}\frac{\lg(t-x)}{(t-x)^{k+\mu}}\,dx\,.$$

Finally, using (25) and (27), the left hand member of (26) becomes

$$-(k+\mu)\,Q(t-a\,;\,k,\,k+\mu+1)+P(t-a\,;\,k,\,k+\mu+1)$$

$$+\int_a^t B_{k+1}(t,\ x)\frac{d}{dt}\,\frac{\lg(t-x)}{(t-x)^{k+\mu}}\,dx$$

$$=-(k+\mu)\left[Q(t-a\,;\,k,\,k+\mu+1)+\int_a^t B_{k+1}(t,\ \ x)\frac{\lg(t-x)}{(t-x)^{k+\mu+1}}\,dx\right]$$

$$+\,P(t-a\,;\,k,\,k+\mu+1)+\int_a^t\frac{B_{k+1}(t,\ x)}{(t-x)^{k+\mu+1}}\,dx$$

$$=-(k+\mu)\,\mathrm{pf}\,\mathfrak{F}_{-k-\mu-1}(t)+\mathrm{pf}\int_a^t\frac{A(x)}{(t-x)^{k+\mu+1}}\,dx\,.$$

The above may easily be generalized by appropriately replacing $A(x)$ by $A(t,\ x)$ depending on t and x.

16. **Examples.** i. Set

$$\Gamma(\alpha)=\int_0^\infty e^{-x}x^{\alpha-1}dx,\qquad \Re\alpha>0.$$

It is well known that $\Gamma(\alpha)$ may be continued analytically for $\Re\alpha\leq 0$; it is a meromorphic function in the α — plane and has simple poles at $\alpha=-n$, $(n=0,\ 1,\ldots)$ with the residue $\dfrac{(-1)^n}{n\,!}$.

If α is real and $\neq 0,\ -1,\ldots,$ then

$$\Gamma(\alpha)=\mathrm{pf}\int_0^1 e^{-x}x^{\alpha-1}dx+\int_1^\infty e^{-x}x^{\alpha-1}dx\,;$$

if n is a non-positive integer, then

$$\mathrm{pl}\int_0^\infty e^{-x}x^{\alpha-1}dx = \mathrm{pl}\int_0^1 + \int_1^\infty = -\frac{(-1)^n}{n!}.$$

Integrating by parts, one finds (α real and $\neq 0, -1, \ldots$)

$$\Gamma(\alpha) = \mathrm{pf}\left\{\frac{e^{-x}x^{-\alpha}}{\alpha}\Big|_0^\infty\right\} + \frac{1}{\alpha}\,\mathrm{pf}\int_0^\infty e^{-x}x^{\alpha}dx$$

$$= \frac{1}{\alpha}\,\Gamma(\alpha+1),$$

if by definition, we set

$$\mathrm{pf}\left\{\frac{e^{-x}x^{-\alpha}}{\alpha}\Big|_0^\infty\right\} = 0.$$

Moreover, we have ($\alpha = z + n$)

$$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin \pi\alpha} = \frac{(-1)^n\pi}{\sin \pi z},$$

n a positive integer or zero; therefore,

$$\Gamma(\alpha) = \frac{(-1)^n}{n!\,(z+n)} + (-1)^n\frac{\Gamma'(1+n)}{\Gamma^2(1+n)} + \ldots. \tag{28}$$

Further, $\Gamma'(1) = -C$ and

$$\left[\frac{d}{d\alpha}\lg\Gamma[\alpha+1)\right]_{\alpha=m} = \frac{\Gamma'(m+1)}{\Gamma(m+1)} = -C + 1 + \frac{1}{2} + \ldots + \frac{1}{m},$$

where m is a positive integer and C the EULER constant.

ii. It is also useful to recall some properties of the Beta function $B(r, s)$ First, $B(r, s)$ is defined by

$$B(r, s) = \int_0^1 x^{r-1}(1-x)^{s-1}dx = 2\int_0^1 x^{2r-1}(1-x^2)^{s-1}dx \tag{29}$$

for r, s positive and by

$$B(r, s) = \frac{\Gamma(r)\,\Gamma(s)}{\Gamma(r+s)} \tag{30}$$

for r, s complex and $r + s \neq 0, -1, \ldots$.

Let us now consider the integral

$$F(\alpha) = \int_0^1 x^{2r-1}(1-x^2)^{\alpha}\,dx, \; r > 0,$$

regarded as a function of the complex variable α, $\Re\alpha > 0$. Since

$$F(\alpha) \equiv \frac{1}{2} B(r, \; \alpha + 1), \; \Re\alpha > 0, \; r > 0,$$

and $B(r, \alpha + 1)$ is an analytic function of α, it follows that $B(r, \alpha + 1)$ is the analytic continuation of $F(\alpha)$. Thus, making use of (23) and (12), if $r > 0$, m a non-negative integer and $0 < \mu < 1$,

$$\mathrm{pf} \int_0^1 x^{2r-1}(1-x^2)^{-m-\mu}dx = \frac{1}{2} B(r, \; -m-\mu+1)$$

which extends the range of the relations (29) and (30) to some divergent integrals.

To calculate $\mathrm{pl}\int_0^1 x^{2r-1}(1-x^2)^{-m}dx$, m a positive integer and $r > 0$, observe that this expression is equal to the negative of the residue of the analytic continuation (in α) of

$$F(\alpha) = \int_0^1 x^{2r-1}(1-x^2)^{\alpha}dx$$

at $\alpha = -m$, i. e., the negative of the residue of

$$\frac{1}{2} B(r, \; \alpha+1) = \frac{1}{2}\frac{\Gamma(r)\Gamma(\alpha+1)}{\Gamma(r+\alpha+1)}$$

at $\alpha = -m$ which is taken to be zero when $r - m + 1 = 0, \; -1, \ldots$. But, it is also equal to the value of

$$\frac{(-1)^m B(r, \; \alpha+1)}{2\Gamma(m)\Gamma(\alpha+1)}$$

at $\alpha = -m$.

Therefore,

$$\mathrm{pl}\int_0^1 x^{2r-1}(1-x^2)^{-m}dx = \frac{(-1)^m\Gamma(r)}{2\Gamma(m)\Gamma(r+1-m)}$$

$$(= 0 \text{ if } r+1-m = 0, \; -1, \ldots).$$

In particular, if p and m are positive integers,

$$\text{pl}\int_0^1 x^{2p+2}(1-x^2)^{-m}dx = \frac{(-1)^m\Gamma\left(p+\frac{3}{2}\right)}{2\Gamma(m)\Gamma\left(p+\frac{5}{2}-m\right)}.$$

It is also easy to evaluate, in terms of the gamma function,

$$A = \text{pf}\int_0^1 x^{2r-1}(1-x^2)^{-m}dx,$$

$r > 0$, m a positive integer. Indeed, making use of (12)

$$A = \lim_{\alpha=-m}\left[\frac{1}{2}B(r,\ \alpha+1) + \frac{1}{\alpha+m}\,\text{pl}\int_0^1 x^{2r-1}(1-x^2)^{-m}dx\right]$$

$$= \frac{1}{2}\frac{\Gamma(r)}{\Gamma(m)}\lim_{\alpha=-m}\left[\frac{\Gamma(\alpha+1)\,\Gamma(m)}{\Gamma(r+\alpha+1)} + \frac{(-1)^m}{\Gamma(r+1-m)}\cdot\frac{1}{\alpha+m}\right],$$

from which upon applying (28), it follows that

$$A = \frac{(-1)^{m-1}}{2}\frac{\Gamma'(m)\Gamma(r)}{\Gamma^3(m)}.$$

iii. Let us denote by k a positive integer, by r a positive number, and by J_k the BESSEL function of order k, we have

$$J_k(z) = \sum_{h=0}^{\infty}\frac{(-1)^h}{h\,!\,\Gamma(h+k+1)}\left(\frac{z}{2}\right)^{2h+k}.$$

Because

$$\text{pf}\int_0^1 \frac{\rho^{2(h+k)+1}}{(1-\rho^2)^{k+\frac{1}{2}}}\,d\rho$$

$$= \frac{1}{2}B\left(h+k+1,\ -k+\frac{1}{2}\right) = \frac{(-1)^k\pi\Gamma(h+k+1)}{2\Gamma\left(h+\frac{3}{2}\right)\Gamma\left(h+\frac{1}{2}\right)}$$

and

$$\sin z = \sqrt{\frac{\pi z}{2}}\,J_{\frac{1}{2}}(z),$$

il follows that

$$F_k = \mathrm{pf}\int_0^1 \frac{\rho^{k+1} J_k(r\rho)}{(1-\rho^2)^{k+\frac{1}{2}}}\, d\rho$$

$$= \sum_{h=0}^{\infty} \frac{(-1)^h}{h!\,\Gamma(h+k+1)} \left(\frac{r}{2}\right)^{2h+k} \mathrm{pf}\int_0^1 \frac{\rho^{2(h+k)+1}}{(1-\rho^2)^{k+\frac{1}{2}}}\, d\rho$$

$$= \sum_{h=0}^{\infty} \frac{(-1)^h}{h!\,\Gamma(h+k+1)} \left(\frac{r}{2}\right)^{2h+k} \cdot \frac{1}{2} \frac{\Gamma(h+k+1)\Gamma\left(-k+\frac{1}{2}\right)}{\Gamma\left(h+\frac{3}{2}\right)}$$

$$= \frac{\pi}{2} \sum_{h=0}^{\infty} \frac{(-1)^{h+k}}{h!\,\Gamma\left(h+\frac{3}{2}\right)\Gamma\left(k+\frac{1}{2}\right)} \left(\frac{r}{2}\right)^{2h+k}$$

$$= \frac{(-1)^k \pi r^{k-\frac{1}{2}}}{2^{k+\frac{1}{2}}\,\Gamma\left(k+\frac{1}{2}\right)} J_{\frac{1}{2}}(r) = \frac{(-1)^k \sqrt{\pi}}{2^k \Gamma\left(k+\frac{1}{2}\right)} r^{k-1} \sin r.$$

iv. To compute

$$L_{k,s} = \mathrm{pl}\int_0^1 \frac{\rho^{k+\frac{3}{2}} J_{k+\frac{1}{2}}(r\rho)}{(1-\rho^2)^{k+1-s}}\, d\rho,$$

where k is a positive integer such that $0 \leq s < k$, and $r > 0$, use the expansion form of $J_{k+\frac{1}{2}}(r\rho)$ and the fact that

$$\mathrm{pl}\int_0^1 \rho^{2(p+k+1)}(1-\rho^2)^{-k-1+s} d\rho$$

$$= \frac{(-1)^{k+1+s}\Gamma\left(p+k+\frac{3}{2}\right)}{2\Gamma(k+1-s)\Gamma\left(p+s+\frac{3}{2}\right)} \qquad \text{(see above)}$$

and obtain

$$L_{k,s} = \frac{(-1)^{k+s+1} r^{k-s}}{2^{k-s+1}(k-s)!} J_{s+\frac{1}{2}}(r), \qquad s = 0, 1, \ldots, k.$$

17. **Multiple integrals.** The above definitions can be extended to some multiple divergent integrals by the usual reduction of multiple integrals to simple ones.

Before considering to some details this generalisation, we shall consider several particular cases which will be useful in what follows.

Let $x = (x_1, \ldots, x_p)$ denote a point in p-dimensional euclidian space. Let $k > 0$ be an integer and, as is customary, $dx = dx_1 \ldots dx_p$.

Consider the divergent integral $\int\limits_{|x|\leq 1} (1 - |x|^2)^{-k-\frac{1}{2}} dx$. To motivate a meaning for $\mathrm{pf}\int\limits_{|x|\leq 1} (1 - |x|^2)^{-k-\frac{1}{2}} dx$, apply to the integral $\int\limits_{|x|\leq 1-\varepsilon} (1 - |x|^2)^{-k-\frac{1}{2}} dx$, the spherical coordinate transformation

$$
\begin{aligned}
&x_1 = r \cos \theta_1 , \\
&x_2 = r \sin \theta_1 \cos \theta_2 , \\
(31) \qquad & \ldots\ldots\ldots\ldots \\
&x_{p-1} = r \sin \theta_1 \ldots \sin \theta_{p-2} \cos \theta_{p-1} , \\
&x_p = r \sin \theta_1 \ldots \sin \theta_{p-2} \sin \theta_{p-1} ,
\end{aligned}
$$

$$0 \leq \theta_i \leq \pi, \qquad i = 1, \ldots, p-2, \qquad 0 \leq \theta_{p-1} \leq 2\pi,$$

for which

$$
\begin{aligned}
dx &= \frac{\partial(x_1, \ldots, x_p)}{\partial(r, \theta_1, \ldots, \theta_{p-1})} dr\, d\theta_1 \ldots d\theta_{p-1} \\
&= r^{p-1} \sin^{p-2} \theta_1 \sin^{p-3} \theta_2 \ldots \sin \theta_{p-2}\, dr\, d\theta_1 \ldots d\theta_{p-1} \\
&= r^{p-1}\, dr\, d\omega_p .
\end{aligned}
$$

This gives

$$
\begin{aligned}
\int\limits_{|x|\leq 1-\varepsilon} (1 - |x|^2)^{-k-\frac{1}{2}} dx &= \int\limits_{\omega_p} d\omega_p \int\limits_0^{1-\varepsilon} r^{p-1}(1 - r^2)^{-k-\frac{1}{2}} dr \\
&= \omega_p \int\limits_0^{1-\varepsilon} r^{p-1}(1 - r^2)^{-k-\frac{1}{2}} dr,
\end{aligned}
$$

where $\omega_p = \dfrac{2\pi^{\frac{p}{2}}}{\Gamma\left(\frac{p}{2}\right)}$ represents the surface area of the p-dimensional unit sphere.

It thus becomes natural to define

$$\operatorname{pf}\int\limits_{|x|\leq 1}(1-|x|^2)^{-k-\frac{1}{2}}\,dx=\omega_p\operatorname{pf}\int\limits_0^1 r^{p-1}(1-r^2)^{-k-\frac{1}{2}}\,dr$$

$$=\frac{\omega_p}{2}B\left(\frac{p}{2},\ -k+\frac{1}{2}\right)=\frac{\pi^{p/2}\Gamma\left(-k+\frac{1}{2}\right)}{\Gamma\left(\frac{p}{2}-k+\frac{1}{2}\right)},\qquad p \text{ even.}$$

And because

$$\Gamma\left(-k+\frac{1}{2}\right)=(-1)^k\frac{\pi}{\Gamma\left(k+\frac{1}{2}\right)},$$

we also have

$$\operatorname{pf}\int\limits_{|x|\leq 1}(1-|x|^2)^{-k-\frac{1}{2}}\,dx=\frac{(-1)^k\pi^{\frac{p}{2}+1}}{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(\frac{p}{2}-k+\frac{1}{2}\right)},\qquad p \text{ even.}$$

In case k is a non-positive integer, take

$$\operatorname{pf}\int\limits_{|x|\leq 1}(1-|x|^2)^{-k-\frac{1}{2}}\,dx=\int\limits_{|x|\leq 1}(1-|x|^2)^{-k-\frac{1}{2}}\,dx,\qquad p \text{ even.}$$

Whenever $k>0$ is an integer and p is odd, define

$$\operatorname{pl}\int\limits_{|x|\leq 1}(1-|x|^2)^{-k}dx=\omega_p\operatorname{pl}\int\limits_0^1 r^{p-1}(1-r^2)^{-k}dr$$

$$=\frac{\omega_p(-1)^k\Gamma\left(\frac{p}{2}\right)}{2\Gamma(k)\Gamma\left(\frac{p}{2}+1-k\right)}=\frac{(-1)^k\pi^{\frac{p}{2}}}{\Gamma(k)\Gamma\left(\frac{p}{2}-k+1\right)}.$$

18. At this point, it is very useful to introduce the following definition.

A function $g(x)$ of one or more independent variables is said to be ***regular*** if it is continuous together with its derivates up to a certain order s. This order will naturally depend upon the nature of the question. Although it is often easy to indicate this order, we shall usually refrain from doing so in order to avoid, being mainly concerned with method, the somewhat tedious precaution it would require.

For the remainder of the chapter, certain agreements as to notation are made. Let $x = (x_1, \ldots, x_p)$ denote a point in the p-dimensional euclidian space E_p. We define

$$r = r_{xy} = |x - y| = \sqrt{(x_1 - y_1)^2 + \ldots + (x_p - y_p)^2}$$

and understand that

$$g(x + r\alpha) \equiv g(x_1 + r\alpha_1, \ldots, x_p + r\alpha_p),$$

where

$$\begin{aligned}
&\alpha_1 = \cos\theta_1,\\
&\alpha_2 = \sin\theta_1 \cos\theta_2,\\
(32)\qquad &\ldots\ldots\ldots\ldots\\
&\alpha_{p-1} = \sin\theta_1 \ldots \sin\theta_{p-2} \cos\theta_{p-1},\\
&\alpha_p = \sin\theta_1 \ldots \sin\theta_{p-2} \sin\theta_{p-1},
\end{aligned}$$

$$0 \le \theta_i \le \pi, \qquad i = 1, \ldots, p-2, \qquad 0 \le \theta_{p-1} \le 2\pi,$$

$$d\alpha = \sin^{p-2}\theta_1 \sin^{p-3}\theta_2 \ldots \sin\theta_{p-2}\, d\theta_1 \ldots d\theta_{p-1}.$$

Further, $\Omega_{x,r}$ is used to denote the hypersphere of center x and radius r, Ω_r its surface area and $d\Omega_r$ its element of area; ω_p is used to denote the p-dimensional unit sphere and $d\omega_p$ its surface element. ω_p is also used to denote the area of the unit sphere in E_p so that $\omega_p = \dfrac{2\pi^{p/2}}{\Gamma\left(\frac{p}{2}\right)}$.

Note that $\Omega_r = r^{p-1}\omega_p$ and that $d\Omega_r = r^{p-1}d\omega_p$.

Now let $(y = x + r\alpha)$

$$\begin{aligned}
\bar{g}(x;\, r) = \bar{g}_p(x;\, r) &= \frac{1}{\Omega_r}\int_{\Omega_{x,r}} g(y)\,d\Omega_r\\
&= \frac{1}{\omega_p}\int_{\omega_p} g(x + r\alpha)\,d\omega_p
\end{aligned}$$

be the mean value of the regular function $g(y)$ on $\Omega_{x,r}$ in E_p.

It follows that $(0 < \varepsilon < t)$

$$\int_{|x-y|\le t-\varepsilon} g(y)(t^2 - r_{xy}^2)^{-\frac{p-1}{2}}\,dy = \omega_p \int_0^{t-\varepsilon} \bar{g}(x;\, r)\,\frac{r^{p-1}}{(t^2 - r^2)^{(p-1)/2}}\,dr.$$

As a result and in agreement with the definitions given above (cf. II), let us define

$$\text{(33)}\qquad \begin{matrix}\text{pf}\\ \text{pl}\end{matrix}\int\limits_{|x-y|\leq t} g(y)(t^2 - r_{xy}^2)^{-\frac{p-1}{2}}dy = \omega_p \begin{matrix}\text{pf}\\ \text{pl}\end{matrix}\int\limits_0^t \bar{g}(x;\ r)\frac{r^{p-1}}{(t^2 - r^2)^{(p-1)/2}}dr$$

depending on the value of p.

In particular,

$$\text{(34)}\qquad \text{pl}\int\limits_{|x-y|\leq t} g(y)(t^2 - r_{xy}^2)^{-1}dy = \omega_p\ \text{pl}\int\limits_0^t \bar{g}(x;\ r)r^{p-1}(t^2 - r^2)^{-1}dr$$

$$= \omega_p\ \text{pl}\int\limits_0^t \frac{\bar{g}(x;\ r)r^{p-1}}{t+r}\cdot\frac{dr}{t-r} = -\frac{\omega_p}{2}t^{p-2}\bar{g}(x;\ t).$$

Thus,

$$\frac{\partial}{\partial t}\int\limits_{|x-y|\leq t} g(y)dy = \omega_p\frac{\partial}{\partial t}\int\limits_0^t \bar{g}(x;\ r)r^{p-1}dr = \omega_p t^{p-1}\bar{g}(x;\ t)$$

$$= -2t\ \text{pl}\int\limits_{|x-y|\leq t} g(y)(t^2 - r^2)^{-1}dy.$$

Note also that

$$\mathfrak{J}(\alpha) = \int\limits_{|x-y|\leq t} g(y)(t^2 - r_{xy}^2)^{\alpha}dy = \omega_p\int\limits_0^t \bar{g}(x;\ r)r^{p-1}(t^2 - r^2)^{\alpha}dr$$

exists for $\Re\alpha \geq -1$ and is a holomorphic function of the complexv ariable α.

In accordance with II, 4, it is easy to find a relation between the analytic continuation of $\mathfrak{J}(\alpha)$ for $\Re\alpha \leq -1$ (when it exists) and

$$\begin{matrix}\text{pf}\\ \text{pl}\end{matrix}\int\limits_{|x-y|\leq t} g(y)\,(t^2 - r_{xy}^2)^{-\frac{p-1}{2}}dy.$$

This will be left to the reader.

19. Let $g(y)$ be a regular (as defined earlier) function of y, $\bar{g}(x;\ r)$ the mean value of $g(y)$ on the hypersphere $\Omega_{x,r}$, k a positive integer or zero,

μ a real number such that $0 < \mu < 1$. In conformity with the case of one simple integral, the following definitions are obvious.

(36) $$\mathrm{pf} \int_{|x-y| \leq t} g(y)\,(t^2 - r_{xy}^2)^{-k-\mu}\, dy = \omega_p \, \mathrm{pf} \int_0^t \bar{g}(x;\, r)\, \frac{r^{p-1}}{(t^2 - r^2)^{k+\mu}}\, dr,$$

(37) $$\mathrm{pl} \int_{|x-y| \leq t} g(y)\,(t^2 - r_{xy}^2)^{-k}\, dy = \omega_p \, \mathrm{pl} \int_0^t \bar{g}(x;\, r)\, \frac{r^{p-1}}{(t^2 - r^2)^{k}}\, dr.$$

These definitions are easily generalized by replacing $g(y)$ by a regular function $g(t, y)$ depending on t and y.

It is not difficult to prove that the operators pf and pl just defined enjoy most of the properties proved for pf and pl of a simple integral. We shall restrict ourselves to the consideration of the derivates of (36) and (37) with respect to t and $x_1, \ldots, x_p$.

20. **Differentiation with respect to t.** From the definitions (36), (37) and from the properties of pf and pl of a simple integral, one deduces

(38) $$\frac{\partial}{\partial t} \mathrm{pf} \int_{|x-y| \leq t} g(y)(t^2 - r_{xy}^2)^{-k-\mu} dy = \omega_p \, \mathrm{pf} \int_0^t \bar{g}(x;\, r)\, \frac{\partial}{\partial t} \frac{r^{p-1}}{(t^2 - r^2)^{k+\mu}}\, dr$$
$$= \mathrm{pf} \int_{|x-y| \leq t} \frac{\partial}{\partial t} \frac{g(y)}{(t^2 - r_{xy}^2)^{k+\mu}}\, dy.$$

Similarly,

(39) $$\frac{\partial}{\partial t} \mathrm{pl} \int_{|x-y| \leq t} g(y)(t^2 - r_{xy}^2)^{-k} dy = \mathrm{pl} \int_{|x-y| \leq t} \frac{\partial}{\partial t} \frac{g(y)}{(t^2 - r_{xy}^2)^{k}}\, dy.$$

21. **Differentiation with respect to x_i.** Under the conditions stated above, we have

(40) $$\frac{\partial}{\partial x_i} \mathrm{pf} \int_{|x-y| \leq t} g(y)(t^2 - r_{xy}^2)^{-k-\mu} dy = \mathrm{pf} \int_{|x-y| \leq t} g(y)\, \frac{\partial}{\partial x_i} (t^2 - r_{xy}^2)^{-k-\mu} dy,$$

(41) $$\frac{\partial}{\partial x_i} \mathrm{pl} \int_{|x-y| \leq t} g(y)(t^2 - r_{xy}^2)^{-k} dy = \mathrm{pl} \int_{|x-y| \leq t} g(y)\, \frac{\partial}{\partial x_i} (t^2 - r_{xy}^2)^{-k} dy.$$

PROOF. - First we note that with reference to

$$\bar{g}(x;\ r) = \frac{1}{\omega_p}\int_{\omega_p} g(x + r\alpha)d\omega_p\,,$$

one has

$$\frac{\partial}{\partial x_i}\cdot\frac{\partial^k \bar{g}(x;\ r)}{\partial r^k} = \frac{\partial^k}{\partial r^k}\cdot\frac{\partial}{\partial x_i}\bar{g}(x;\ r),$$

$k > 0$ an integer.

Consider

$$\frac{\partial}{\partial x_i}\,\mathrm{pf}\int_{|x-y|\leq t} g(y)(t^2 - r_{xy}^2)^{-k-\mu}dy = \frac{\partial}{\partial x_i}\,\mathrm{pf}\int_0^t \bar{g}(x;\ r)\frac{r^{p-1}}{(t^2 - r^2)^{k+\mu}}\,dr.$$

Because x_i is a parameter, it is clear from the definition of pf of a simple integral (see II, 1, (5)) that

$$\frac{\partial}{\partial x_1}\,\mathrm{pf}\int_0^t \bar{g}(x;\ r)\frac{r^{p-1}}{(t^2 - r^2)^{k+\mu}}\,dr = \mathrm{pf}\int_0^t \frac{\partial}{\partial x_i}\bar{g}(x;\ r)\frac{r^{p-1}}{(t^2 - r^2)^{k+\mu}}\,dr.$$

Set $x + h = (x_1 + h,\ x_2,\ \ldots,\ x_p)$; we have

$$\frac{\partial}{\partial x_1}\bar{g}(x;\ r) = \lim_{h=0}\frac{\bar{g}(x + h;\ r)\quad \bar{g}(x;\ r)}{h}.$$

$$\bar{g}(x + h;\ r) = \frac{1}{\Omega_r}\int_{|x+h-z|=r} g(z)d\Omega_r = \frac{1}{\Omega_r}\int_{|x-y|=r} g(h + y)d\Omega_r\,;$$

consequently,

$$\frac{\partial}{\partial x_1}\bar{g}(x;\ r) = \frac{1}{\Omega_r}\lim_{h=0}\int_{|x-y|=r}\frac{g(h + y) - g(y)}{h}\,d\Omega_r$$

$$= \frac{1}{\Omega_r}\int_{|x-y|=r}\frac{\partial g(y)}{\partial y_1}\,d\Omega_r$$

and

$$\frac{\partial}{\partial x_i}\,\mathrm{pf}\int_0^t \bar{g}(x;\ r)\frac{r^{p-1}dr}{(t^2 - r^2)^{k+\mu}}\,dr = \mathrm{pf}\int_0^t \frac{1}{\Omega_r}\int_{|x-y|\ r}\frac{\partial g(y)}{\partial y_i}\,d\Omega_r\cdot\frac{r^{p-1}dr}{(t^2 - r^2)^{k+\mu}}\,.$$

According to II, 2, (11), consider $(0 < \varepsilon < 1)$

$$\int_0^{t(1-\varepsilon)}\frac{1}{\Omega_r}\int_{|x-y|=r}\frac{\partial g(y)}{\partial y_i}\,d\Omega_r\cdot\frac{r^{p-1}}{(t^2 - r^2)^{k+\mu}}\,dr = \int_{|x-y|\leq t(1-\varepsilon)}\frac{\partial g(y)}{\partial y_i}\,\frac{dy}{(t^2 - r_{xy}^2)^{k+\mu}}\,.$$

Because

$$\frac{\partial g(y)}{\partial y_i}\cdot\frac{1}{(t^2-r_{xy}^2)^{k+\mu}}=\frac{\partial}{\partial y_i}\left[\frac{g(y)}{(t^2-r_{xy}^2)^{k+\mu}}\right]-g(y)\cdot\frac{\partial}{\partial y_i}(t^2-r_{xy}^2)^{-k-\mu}$$

$$=\frac{\partial}{\partial y_i}\left[\frac{g(y)}{(t^2-r_{xy}^2)^{k+\mu}}\right]+g(y)\frac{\partial}{\partial x_i}(t^2-r_{xy}^2)^{-k-\mu}$$

and on applying the divergence theorem, one finds

$$\int\limits_{|x-y|\le t(1-\varepsilon)}\frac{\partial g(y)}{\partial y_i}\cdot\frac{dy}{(t^2-r_{xy}^2)^{k+\mu}}=\int\limits_{|x-y|\le t(1-\varepsilon)}g(y)\frac{\partial}{\partial x_i}(t^2-r_{xy}^2)^{-k-\mu}dy$$

$$-\int\limits_{|x-y|=t(1-\varepsilon)}g(y)(t^2-r_{xy}^2)^{-k-\mu}\pi_i\,d\Omega_{t(1-\varepsilon)}$$

where $(\pi_1, \ldots, \pi_p)$ are the direction cosines of the inward normal to $\Omega_{t(1-\varepsilon)}$.

But

$$\frac{\partial}{\partial x_i}(t^2-r_{xy}^2)^{-k-\mu}=2(k+\mu)\frac{x_i-y_i}{(t^2-r_{xy}^2)^{k+\mu+1}}$$

so that

$$\int\limits_{|x-y|\le t(1-\varepsilon)}g(y)\frac{\partial}{\partial x_i}(t^2-r_{xy}^2)^{-k-\mu}dy$$

$$=\int_0^{t(1-\varepsilon)}\frac{2(k+\mu)}{\Omega_r}\int\limits_{\Omega_r}g(y)(x_i-y_i)d\Omega_r\cdot\frac{r^{p-1}}{(t^2-r^2)^{k+\mu+1}}dr.$$

Combining these results and noting that $t^2-r_{xy}^2=t^2\varepsilon(2-\varepsilon)$ on $|x-y|=$ $=t(1-\varepsilon)$, one obtains

$$\int\limits_{|x-y|\le t(1-\varepsilon)}\frac{\partial g(y)}{\partial y_i}\cdot\frac{dy}{(t^2-r_{xy}^2)^{k+\mu}}$$

$$=\int_0^{t(1-\varepsilon)}\frac{2(k+\mu)}{\Omega_r}\int\limits_{\Omega_r}g(y)(x_i-y_i)d\Omega_r\cdot\frac{r^{p-1}}{(t^2-r^2)^{k+\mu+1}}dr$$

$$-\,t^{-2(k+\mu)}\varepsilon^{-k-\mu}(2-\varepsilon)^{-k-\mu}\int\limits_{|x-y|=t(1-\varepsilon)}g(y)\pi_i\,d\Omega_{t(1-\varepsilon)}.$$

This last term does not contribute to the pf because $\varepsilon^{-k-\mu}$ becomes infinite when $\varepsilon \to 0$; it remains

$$\frac{\partial}{\partial x_i}\,\mathrm{pf}\int_0^t \bar{g}(x;\ r)\frac{r^{p-1}}{(t^2-r^2)^{k+\mu}}\,dr$$

$$= \mathrm{pf}\int_0^t \frac{2(k+\mu)}{\Omega_r}\int_{\Omega_r} g(y)(x_i - y_i)d\Omega_r \cdot \frac{r^{p-1}}{(t^2-r^2)^{k+\mu+1}}\,dr$$

$$= \mathrm{pf}\int_{|x-y|\leq t} g(y)\cdot\frac{2(k+\mu)(x_i-y_i)dy}{(t^2-r_{xy}^2)^{k+\mu+1}} = \mathrm{pf}\int_{|x-y|\leq t} g(y)\frac{\partial}{\partial x_i}\frac{1}{(t^2-r_{xy}^2)^{k+\mu}}\,dy$$

which proves (40).

22. The proof of formula (41) being along the same line, we only sketch the proof.

Because x_i is a parameter, we have again

$$\frac{\partial}{\partial x_i}\,\mathrm{pl}\int_{|x-x|\leq t} g(y)(t^2-r_{xy}^2)^{-k}dy = \frac{\partial}{\partial x_i}\,\mathrm{pl}\int_0^t \bar{g}(x;\ r)\frac{r^{p-1}}{(t^2-r^2)^k}\,dr$$

$$= \mathrm{pl}\int_0^t \frac{\partial}{\partial x_i}\bar{g}(x;\ r)\cdot\frac{r^{p-1}}{(t^2-r^2)^k}\,dr$$

$$= \mathrm{pl}\int_0^t \frac{1}{\Omega_r}\int_{|x-y|=r}\frac{\partial g(y)}{\partial y_i}\,d\Omega_r\cdot\frac{r^{p-1}}{(t^2-r^2)^k}\,dr.$$

Now, consider $(0 < \varepsilon < 1)$

$$G_\varepsilon = \int_0^{t(1-\varepsilon)}\frac{1}{\Omega_r}\int_{|x-y|=r}\frac{\partial g(y)}{\partial y_i}\,d\Omega_r\cdot\frac{r^{p-1}}{(t^2-r^2)^k}\,dr;$$

we have

$$G_\varepsilon = \int_{|x-y|\leq t(1-\varepsilon)}\frac{\partial g(y)}{\partial y_i}\cdot\frac{dy}{(t^2-r_{xy}^2)^k}$$

$$= \int_{|x-y|\leq t(1-\varepsilon)} g(y)\frac{\partial}{\partial x_i}(t^2-r_{xy}^2)^{-k}dy - \int_{|x-y|=t(1-\varepsilon)} g(y)(t^2-r_{xy}^2)^{-k}\pi_i\,d\Omega_{t(1-\varepsilon)}$$

$$= \int_{|x-y|\leq t(1-\varepsilon)} g(y)\frac{\partial}{\partial x_i}(t^2-r_{xy}^2)^{-k}dy - \frac{1}{t^{2k}\varepsilon^k(2-\varepsilon)^k}\int_{|x-y|=t(1-\varepsilon)} g(y)\pi_i\,d\Omega_{t(1-\varepsilon)}.$$

The second term on the right hand side does not contribute to the pl; for, it is obvious that that expression does not imply any term in $\lg \varepsilon$. Finally, we have

$$\frac{\partial}{\partial x_i} \text{pl} \int\limits_{|x-y|\leq t} g(y)(t^2 - r_{xy}^2)^{-k} dy = \text{pl} \int\limits_{|x-y|\leq t} g(y) \frac{\partial}{\partial x_i} (t^2 - r_{xy}^2)^{-k} dy$$

i. e. formula (41).

III. **The Cauchy problem for the wave equation.**

1. In the following, we consider real quantities $x_1, \ldots, x_p, t, u$ and abbreviate

$$x = (x_1, \ldots, x_p), \qquad p \geq 1$$

$$u(x, t) = u(x_1, \ldots, x_p, t),$$

$$r^2 = |x - y|^2, \qquad r > 0,$$

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \ldots + \frac{\partial^2 u}{\partial x_p^2}.$$

The CAUCHY problem is: to find a solution of

$$Lu \equiv u_{tt} - \Delta u = 0, \tag{1}$$

$$u(x, 0) = f(x), \qquad u_t(x, 0) = g(x) \tag{2}$$

where $f(x)$ and $g(x)$ are regular functions (regular as defined earlier).

Assume that $f(x)$ is identically zero and observe that $v = (t^2 - r^2)^{-\frac{p-1}{2}}$ is a solution of (1). By making use of the properties of pf, pl and their evaluations, the solution of the CAUCHY problem (1), (2) with $f(x) \equiv 0$ is obtained in terms of v and g. This in turn leads to the solution of the general case, $f(x)$ not necessarily identically zero.

Two cases are considered according as p, the number of space variables, is even or odd.

2. If $p = 2k + 2$ is even, the solution to the problem $(f \equiv 0)$ is

$$u_g(x, t) = A_p \,\text{pf} \int\limits_{|x-y|\leq t} g(y)(t^2 - r^2)^{-\frac{p-1}{2}} dy \tag{3}$$

where

$$A_p = \frac{(-1)^{\frac{p-2}{2}}}{2\pi^{(p+1)/2}}\Gamma\left(\frac{p-1}{2}\right).$$

That $u_g(x, t)$ satisfies (1) is clear since

$$Lu_g = A_p \operatorname{pf}\int\limits_{|x-y|\leq t} g(y) L(t^2 - r^2)^{-\frac{p-1}{2}}\, dy$$

and $L(t^2 - r^2)^{-\frac{p-1}{2}} = 0$.

To verify the initial condition (2), set $y = x + t\alpha$ where $\alpha = (\alpha_1, \ldots, \alpha_p)$ and $d\alpha = d\alpha_1 \ldots d\alpha_p$ are given by (32) of chapter II; one has

$$\begin{aligned} u_g(x, t) &= A_p t \operatorname{pf}\int\limits_{|\alpha|\leq 1} g(x + t\alpha)(1 - |\alpha|^2)^{-\frac{p-1}{2}}\, d\alpha \\ &= A_p t g(x) \operatorname{pf}\int\limits_{|\alpha|\leq 1} (1 - |\alpha|^2)^{-\frac{p-1}{2}}\, d\alpha + O(t^2) \\ &= t g(x) + O(t^2) \end{aligned}$$

because, when $p = 2k + 2$ is even,

$$\begin{aligned} \operatorname{pf}\int\limits_{|\alpha|\leq 1} (1 - |\alpha|^2)^{-\frac{p-1}{2}}\, d\alpha &= \operatorname{pf}\int\limits_{|\alpha|\leq 1} (1 - |\alpha|^2)^{-k-\frac{1}{2}}\, d\alpha \\ &= \frac{(-1)^k \pi^{\frac{p}{2}+1}}{\Gamma\left(k + \frac{1}{2}\right)\Gamma\left(\frac{p}{2} - k + \frac{1}{2}\right)} = \frac{(-1)^{\frac{p-2}{2}} \pi^{\frac{p}{2}+1}}{\Gamma\left(\frac{p-1}{2}\right)\Gamma\left(\frac{3}{2}\right)} \\ &= \frac{2(-1)^{\frac{p-2}{2}} \pi^{\frac{p+1}{2}}}{\Gamma\left(\frac{p-1}{2}\right)} = A_p^{-1}. \end{aligned}$$

Note that

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}.$$

3. If $p = 2k + 3$ is odd, the solution to the problem $(f \equiv 0)$ is

$$u_g(x, t) = B_p \operatorname{pl}\int\limits_{|x-y|\leq t} g(y)(t^2 - r^2)^{-\frac{p-1}{2}}\, dy \tag{4}$$

where

$$B_p = \frac{(-1)^{\frac{p-1}{2}}}{2\pi^{(p-1)/2}}\Gamma\left(\frac{p-1}{2}\right).$$

As above, $Lu_g = 0$ and

$$u_g(x,\ t) = B_p t \ \mathrm{pl} \int\limits_{|\alpha| \leq 1} g(x + t\alpha)(1 - |\alpha|^2)^{-\frac{p-1}{2}} d\alpha$$

$$= tg(x) + O(t^2)$$

in view of

$$\mathrm{pl} \int\limits_{|\alpha| \leq 1} (1 - |\alpha|^2)^{-\frac{p-1}{2}} d\alpha = \mathrm{pl} \int\limits_{|\alpha| \leq 1} (1 - |\alpha|^2)^{-k-1} d\alpha$$

$$= \frac{(-1)^{k+1} \pi^{\frac{p}{2}}}{\Gamma(k+1)\Gamma\left(\frac{p}{2} - k\right)} = \frac{(-1)^{\frac{p-1}{2}} \pi^{\frac{p}{2}}}{\Gamma\left(\frac{p-1}{2}\right)\Gamma\left(\frac{3}{2}\right)} = B_p^{-1}.$$

4. Now assume that $f(x) \not\equiv 0$ and $g(x) \equiv 0$ i. e., the initial conditions (2) are

$$u(x,\ 0) = f(x), \qquad u_t(x,\ 0) = 0.$$

It is easy to verify that the solution of the new CAUCHY problem is $\frac{\partial}{\partial t} u_f(x,\ t)$ where $u_f(x,\ t)$, is obtained from (3) and (4) by replacing g by f. This result confirms STOKE'S rule: «the terms of the solution of the CAUCHY problem depending on the initial configuration are obtained from those depending on the initial velocity by replacing the function giving the velocity by that giving the displacement and then differentiating with respect to time».

Finally, combining the above results, the solution of the original CAUCHY problem (1), (2) is seen to be

$$u(x,\ t) = u_g(x,\ t) + \frac{\partial}{\partial t} u_f(x,\ t).$$

5. The properties of the finite part and logarithmic part of divergent integrals enable one to derive with ease the *known* solutions of the CAUCHY problem under consideration.

Indeed, by using the mean value $\bar{g}(r)$ of $g(x)$ on the hypersphere of center x and radius r, one has

$$\begin{matrix}\mathrm{pf} \\ \mathrm{pl}\end{matrix} \int\limits_{|x-y| \leq t} g(y)(t^2 - r^2)^{-\frac{p-1}{2}} dy = \omega_p \begin{matrix}\mathrm{pf} \\ \mathrm{pl}\end{matrix} \int\limits_0^t \bar{g}(r) \frac{r^{p-1}}{(t^2 - r^2)^{(p-1)/2}} dr.$$

Two cases arise according as p is even or odd.

(i) If $p = 2k + 2$ is even,

$$u(x, t) = U_k = C_k \operatorname{pf} \int_0^t \frac{r^{2k+1}}{(t^2 - r^2)^{k+\frac{1}{2}}} \bar{g}(r) dr,$$

where

$$C_k = \frac{(-1)^k}{\sqrt{\pi}} \frac{\Gamma\left(k + \frac{1}{2}\right)}{\Gamma(k+1)}.$$

Consider U_k as a function of k. For $k = 0$, one obtains

$$U_0 = \int_0^t \frac{r}{\sqrt{t^2 - r^2}} \bar{g}(r) dr ;$$

differentiating U_{k-1} with respect to t and writing $t^2 = (t^2 - r^2) + r^2$ yields

(5) $$t \frac{\partial U_{k-1}}{\partial t} = -(2k-1) U_{k-1} + 2k U_k,$$

which determines U_k starting from U_0.

(ii) If $p = 2k + 3$ is odd.

$$u(x, t) = V_k = D_k \operatorname{pl} \int_0^t \frac{r^{2k+2}}{(t^2 - r^2)^{k+1}} \bar{g}(r) dr$$

where

$$D_k = (-1)^{k+1} \sqrt{\pi} \frac{\Gamma(k+1)}{\Gamma\left(k + \frac{3}{2}\right)}.$$

Again, consider V_k as a function of k. For $k = 0$, one has

$$V_0 = -2 \operatorname{pl} \int_0^t \frac{r^2 \bar{g}(r)}{t + r} \cdot \frac{dr}{t - r} = t\bar{g}(t).$$

Repeating the method used for (i) gives the recursion formula

(6) $$t \frac{\partial V_{k-1}}{\partial t} = -2k V_{k-1} + (2k+1) V_k,$$

which determines V_k starting from V_0.

6. Recursion formulas (5) and (6) may be reduced to one equation by a transformation of the unknown function. Indeed, setting

$$U_k = \frac{t}{k!} w_k \tag{7}$$

relation (5) becomes

$$w_k = k w_{k-1} + \frac{t}{2} \frac{\partial w_{k-1}}{\partial t} \tag{8}$$

with

$$w_0 = t^{-1} U_0 .$$

By setting

$$V_k = \frac{2^{2k} k!}{(2k+1)!} w_k \tag{9}$$

in (6), we again obtain (8) with $w_0 = V_0$. Let us now write $t^2 = s$ and hence $\frac{1}{2t}\frac{d}{dt} = \frac{d}{ds}$; equation (8) becomes

$$w_k = k w_{k-1} + s \frac{\partial w_{k-1}}{\partial s} , \tag{10}$$

which admits the solution

$$w_k = \frac{\partial^k}{\partial s^k} (s^k w_0) \tag{11}$$

because if $a = s^{p-1} w_0$, one has, using Leibnitz' s rule,

$$w_p = \frac{d^p}{ds^p}(sa) = s a^{(p)} + p a^{(p-1)} \tag{12}$$

$$= s \frac{d w_{p-1}}{ds} + p w_{p-1} .$$

Thus, it follows that

$$w_k = \frac{1}{2^{2k} \cdot k!} \cdot \frac{\partial^{2k+1}}{\partial t^{2k+1}} \int_0^t (t^2 - \alpha^2)^k w_0(\alpha) d\alpha \tag{13}$$

must satisfy (8). Indeed,

$$w_k = \frac{2k}{2^{2k}\cdot k!}\frac{\partial^{2k}}{\partial t^{2k}}\, t\int_0^t (t^2-\alpha^2)^{k-1} w_0(\alpha)d\alpha$$

which upon applying (12), with $a = \int_0^t \ldots$, becomes

$$w_k = \frac{2k}{2^{2k}\cdot k!}\left[t\frac{\partial^{2k}}{\partial t^{2k}}\int_0 \ldots + 2k\frac{\partial^{2k-1}}{\partial t^{2k-1}}\int \ldots\right]$$

$$= \frac{t}{2}\frac{\partial w_{k-1}}{\partial t} + k w_{k-1}.$$

Consequently, when $p = 2k+3$, (9) is applied to (13) with $w_0(\alpha) = V_0(\alpha) = \alpha g(\alpha)$ yielding the well known solution

$$u(x;t) = \frac{1}{(p-2)!}\frac{\partial^{p-2}}{\partial t^{p-2}}\int_0^t (t^2-\alpha^2)^{\frac{p-3}{2}}\alpha\bar{g}(\alpha)d\alpha. \tag{14}$$

In a like manner it can be shown that

$$w_k = \frac{k!}{(2k)!}\cdot\frac{1}{t}\cdot\frac{\partial^{2k}}{\partial t^{2k}}\int_0^t (t^2-\alpha^2)^{k-\frac{1}{2}}\alpha\bar{g}(\alpha)d\alpha \tag{15}$$

is also a solution of (8) when $w_0 = t^{-1}U_0$. Indeed, rewrite (8) thus

$$tw_k = \left(k-\frac{1}{2}\right)tw_{k-1} + \frac{t}{2}\frac{\partial}{\partial t}(tw_{k-1})$$

and note that

$$tw_k = \frac{k!}{(2k)!}\frac{\partial^{2k}}{\partial t^{2k}}\int^t (t^2-\alpha^2)^{k-\frac{1}{2}}\alpha\bar{g}(\alpha)d\alpha$$

and that

$$\frac{\partial^{2k}}{\partial t^{2k}}\int_0^t (t^2-\alpha^2)^{k-\frac{1}{2}}\alpha\bar{g}(\alpha)d\alpha$$

$$= (2k-1)\frac{\partial^{2k-1}}{\partial t^{2k-1}}t\int_0^t (t^2-\alpha^2)^{k-\frac{3}{2}}\alpha\bar{g}(\alpha)d\alpha$$

$$= (2k-1)\left[t\frac{\partial^{2k-1}}{\partial t^{2k-1}}\int_0^t \ldots + (2k-1)\frac{\partial^{2k-2}}{\partial t^{2k-2}}\int_0^t \ldots\right]$$

$$= 2(2k-1)\left[\frac{t}{2}\frac{\partial}{\partial t}\frac{\partial^{2k-2}}{\partial t^{2k-2}}\int_0^t \ldots + \left(k-\frac{1}{2}\right)\frac{\partial^{2k-2}}{\partial t^{2k-2}}\int \ldots\right].$$

Finally, when $p=2k+2$, the use of (7) in (15) with $w_0=t^{-1}U_0$ gives again (14).

7. In order to evaluate (13), set

$$(t^2-\alpha^2)^k = (t-\alpha)^k(t-\alpha+2\alpha)^k$$

$$= \sum_{i=0}^{k} C_k^i(2\alpha)^i(t-\alpha)^{2k-i}$$

so that

$$w_k = \frac{1}{2^{2k}k!}\sum_{i=0}^{k} 2^i C_k^i \frac{\partial^{2k+1}}{\partial t^{2k+1}}\int_0^t (t-\alpha)^{2k-i}\alpha^i w_0(\alpha)d\alpha$$

$$= \frac{1}{2^{2k}k!}\sum_{i=0}^{k} 2^i C_k^i \frac{\partial^{2k+1}}{\partial t^{2k+1}} t^{2k-i} * [t^i w_0(t)]$$

(16) $$= \frac{1}{2^{2k}k!}\sum_{i=0}^{k} 2^i C_k^i (2k-i)!\frac{\partial^i}{\partial t^i}[t^i w_0(t)]$$

because

$$\frac{d^i}{dt^i}[t^{i-1} * \varphi] = (i-1)!\,\varphi(t).$$

From LEIBNITZ's formula, one has

$$\frac{\partial^i}{\partial t^i}[t^i w_0(t)] = \sum_{j=0}^{i} C_i^j \frac{i\,!}{(i-j)\,!}\, t^{i-j} \frac{\partial^{i-j} w_0(t)}{\partial t^{i-j}}. \tag{17}$$

Thus combining (16) and (17)

$$w_k = \sum_{i=0}^{k} b_i t^i w_0^{(i)},$$

the coefficients b_i being constants.

For a recursion relation among the b_i's, consider the polynomial

$$P_k(t) = \sum_{i=0}^{k} b_i t^i$$

and write symbolically

$$w_k = P_k[tw_0],$$

understanding that the power w_0^r is to be replaced by the derivative $w_0^{(r)}$ and $w_0^{(0)}$ by w_0. Then, upon replacing w_0 by e^t, one has

$$P_k(te^t) = \frac{1}{2^{2k}} \sum_{i=0}^{k} \frac{2^i(2k-i)\,!}{i\cdot(k-i)\,!} \frac{d^i}{dt^i}(t^i e^t) \qquad \text{[cf. (16)]}$$

and

$$w_k(t) = e^t P_k(t)\,;$$

but

$$\frac{d}{dt}[e^t P_k(t)] = [P_k(t) + P'_k(t)]e^t$$

so that, combining with

$$w_k = kw_{k-1} + \frac{t}{2}\frac{d}{dt} w_{k-1} \qquad \text{[cf. (8)]}$$

$P_k(t)$ satisfies the recursion formula

$$2P_k = (2k + t)P_{k-1} + tP'_{k-1}, \qquad P_0 = 1.$$

Therefore

$$2P_1(t) = 2 + t,$$

$$2^2 P_2(t) = 8 + 7t + t^2$$

to which correspond the solutions of (8), namely,

$$2w_1 = 2w_0 + tw'_0\,,$$
$$4w_2 = 8w_0 + 7tw'_0 + t^2w''_0.$$

The corresponding solutions of the wave equation are

1. If $w_0 = t\bar{g}(t)$, then for

$$p = 3 \qquad u = t\bar{g}$$

$$p = 5 \qquad u = t\left(\bar{g} + \frac{t}{3}\bar{g}'\right)$$

$$p = 7 \qquad u = t\left(\bar{g} + \frac{3}{5}t\bar{g}' + \frac{1}{15}t^2\bar{g}''\right)$$

. . . .

2. If $w_0 = \frac{1}{t}U_0 = \frac{1}{t}\int_0^t \frac{r}{\sqrt{t^2 - r^2}}\,\bar{g}(r)dr$, then for

$$p = 2 \qquad u = U_0$$

$$p = 4 \qquad u = \frac{1}{2}(U_0 + tU'_0)$$

$$p = 6 \qquad u = \frac{1}{8}(3U_0 + 5tU'_0 + t^2U''_0)$$

. . . .

8. **The method of descent.** The solution of the CAUCHY problem for p even involves a pf, for p odd, a pl. It is possible to obtain the solution of the CAUCHY problem for $p-1$ from the solution for p by using the method of descent. This method helps to explain the transition from pf to pl.

Assume p is even and that $g \equiv g(x) \equiv g(x_1, \ldots, x_{p-1})$ is independent of x_p. Consider the CAUCHY problem

$$u_{tt} - \Delta u = 0, \qquad \Delta = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_p^2}\,,$$

$$u(x,\,0) = 0, \qquad u_t(x,\,0) = g(x) = g(x_1, \ldots, x_{p-1}) \quad .$$

which has for its solution

$$u(x, t) = (-1)^{\frac{p-2}{2}} \frac{\Gamma\left(\frac{p-1}{2}\right)}{2\pi^{\frac{p+1}{2}}} \operatorname{pf} \int\limits_{|x-y| \leq t} g(y)(t^2 - r^2)^{-\frac{p-1}{2}} dy,$$

where $r^2 = |x - y|^2 = (x_1 - y_1)^2 + \ldots + (x_p - y_p)^2$. Note that

$$\operatorname{pf} \int\limits_{|x-y| \leq t} = \omega_p \operatorname{pf} \int_0^t \bar{g}_p(r) \frac{r^{p-1} dr}{(t^2 - r^2)^{\frac{p-1}{2}}}, \qquad \omega_p = \frac{2\pi^{p/2}}{\Gamma(p/2)}$$

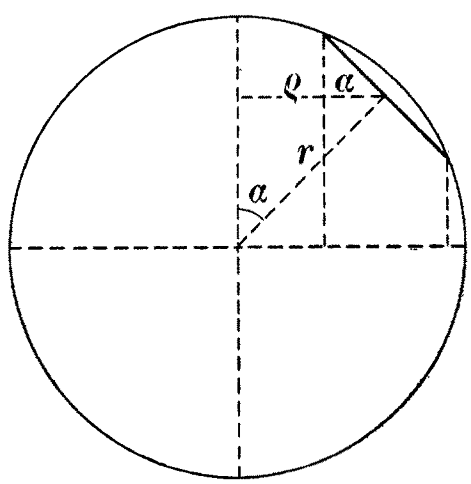
and remember that g does not depend on x_p. Setting $\rho^2 = r^2 - (x_p - y_p)^2$, there exists a relation between $\bar{g}_p(r)$ and $\bar{g}_{p-1}(\rho)$. For, if $d\Omega_{p,r}$ is used to denote the surface element of the sphere in p-dimensional space with radius r and center at x,

$$d\Omega_{p,r} = d\Omega_{p-1,\rho} \cdot \frac{d\rho}{\cos\alpha},$$

$$r \cos\alpha = \sqrt{r^2 - \rho^2}$$

so that

$$r^{p-1} d\omega_p = \frac{r\rho^{p-2}}{\sqrt{r^2 - \rho^2}} d\omega_{p-1} d\rho,$$



where $d\omega_p$ is the surface element for the unit sphere in p-dimensional space with center at x, i. e., $d\omega_p = d\Omega_{p,1}$. Further,

$$\omega_p \bar{g}_p(r) = 2\omega_{p-1} \frac{1}{r^{p-2}} \int_0^r \frac{\rho^{p-2}}{\sqrt{r^2 - \rho^2}} \bar{g}_{p-1}(\rho) d\rho.$$

Therefore, the solution of the CAUCHY problem under consideration is

$$u(x,\ t)=(-1)^{\frac{p-2}{2}}\frac{2}{\pi}\,\mathrm{pf}\int_0^t\frac{rdr}{(t^2-r^2)^{\frac{p-1}{2}}}\int_0^r\frac{\rho^{p-2}}{\sqrt{r^2-\rho^2}}\bar{g}_{p-1}(\rho)d\rho.$$

Now, with the above in mind, consider the function

$$I(\alpha)=\int_0^t(t^2-r^2)^\alpha rdr\int_0^r\frac{\rho^{p-2}}{\sqrt{r^2-\rho^2}}\bar{g}_{p-1}(\rho)d\rho$$

so that

$$u(x,\ t)=(-1)^{\frac{p-2}{2}}\frac{2}{\pi}\,\mathrm{pf}\,I\left(-\frac{p-1}{2}\right).$$

Assume $\Re\alpha>0$. Then, using DIRICHLET's formula for changing the order of integration

$$I(\alpha)=\int_0^t\rho^{p-2}\bar{g}_{p-1}(\rho)d\rho\int_\rho^t\frac{(t^2-r^2)^\alpha rdr}{\sqrt{r^2-\rho^2}}.$$

In the last integral on the right, set

$$t^2-r^2=(t^2-\rho^2)x.$$

This gives

$$\begin{aligned}\int^t &=\frac{1}{2}(t^2-\rho^2)^{\alpha+\frac{1}{2}}\int_0^1 x^\alpha(1-x)^{-\frac{1}{2}}dx\\ &=\frac{1}{2}(t^2-\rho^2)^{\alpha+\frac{1}{2}}B\left(\alpha+1;\ \frac{1}{2}\right)\\ &=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(\alpha+1)}{2\Gamma\left(\alpha+\frac{3}{2}\right)}(t^2-\rho^2)^{\alpha+\frac{1}{2}}\end{aligned}$$

and in turn

$$I(\alpha)=\frac{\sqrt{\pi}}{2}\frac{\Gamma(\alpha+1)}{\Gamma\left(\alpha+\frac{3}{2}\right)}\int_0^t\rho^{p-2}(t^2-\rho^2)^{\alpha+\frac{1}{2}}\bar{g}_{p-1}(\rho)d\rho.$$

Since p was assumed even, $p-1$ is odd Suppose $p-1 \geq 3$ so that $p \geq 4$. Note that for $\alpha = -\frac{p-1}{2}$, $\alpha + 1 = -\frac{p-3}{2}$ is not a negative integer while $\alpha + \frac{3}{2} = -\frac{p-4}{2}$ is. Because $\Gamma(z)$ has a simple pole at $-\frac{p-4}{2}$ with the residue

$$\frac{(-1)^{\frac{p-4}{2}}}{\Gamma\left(\frac{p-2}{2}\right)}$$

one has

$$\frac{\Gamma(\alpha+1)}{\Gamma\left(\alpha+\frac{3}{2}\right)} = (-1)^{\frac{p-4}{2}} \Gamma\left(\frac{p-2}{2}\right) \Gamma\left(-\frac{p-3}{2}\right)\left(\alpha + \frac{p-1}{2}\right)$$
$$+ O\left(\alpha + \frac{p-1}{2}\right).$$

On the other hand, consider

$$\int_0^t (t^2 - \rho^2)^{\alpha + \frac{1}{2}} \rho^{p-2} \bar{g}_{p-1}(\rho) d\rho$$

and note that for $\alpha = -\frac{p-1}{2}$, $\alpha + \frac{1}{2} = -\frac{p-2}{2}$ is a negative integer; therefore, the analytic continuation of the integral has a simple pole at $\alpha + 1 = -\frac{p-2}{2}$ with the residue

$$-\,\mathrm{pl} \int_0^t \frac{\rho^{p-2}}{(t^2-\rho^2)^{\frac{p-2}{2}}} g_{p-1}(\rho) d\rho = -\frac{1}{\omega_{p-1}} \mathrm{pl} \int_{|x-y| \leq t} gv dy,$$

where $v = (t^2 - \rho^2)^{-(p-2)/2}$ and the pl on the right is performed in $p-1$ dimensional space. Thus, its analytic continuation is equal to

$$\left(\alpha + \frac{p-1}{2}\right)^{-1} \left(-\frac{1}{\omega_{p-1}} \mathrm{pl} \int_{|x-y| \leq t} gv dy\right) + O\left(\alpha + \frac{p-1}{2}\right).$$

Summing up, one has

$$u(x,\, t) = -(-1)^{\frac{p-2}{2}} \frac{2}{\pi} \frac{\sqrt{\pi}}{2} (-1)^{\frac{p-4}{2}} \Gamma\left(\frac{p-2}{2}\right) \Gamma\left(-\frac{p-3}{2}\right) \frac{1}{\omega_{p-1}} \mathrm{pl} \int_{|x-y| \leq t} gv dy.$$

Because

$$\omega_{p-1} = \frac{2\pi^{\frac{p-1}{2}}}{\Gamma\left(\frac{p-1}{2}\right)},$$

$$\Gamma\left(-\frac{p-3}{2}\right) = -\frac{p-1}{2}\Gamma\left(-\frac{p-1}{2}\right),$$

$$\Gamma\left(\frac{p-1}{2}\right)\Gamma\left(-\frac{p-1}{2}\right) = (-1)^{p/2}\frac{2\pi}{p-1},$$

it follows that

$$u(x,\ t) = -\frac{(-1)^{\frac{p+2}{2}}\Gamma\left(\frac{p-2}{2}\right)}{2\pi^{(p-2)/2}}\ \mathrm{pl}\int_{|x-y|\leq t} gv dy$$

which is formula (4) with p replaced by $p-1$. The same method can be applied to pass from a pl to a pf, i. e., to pass from the solution for an odd number of space variables to the solution for an even number of space variables.

The result just obtained shows why « la méthode de descente se montre donc, en fin de compte, beaucoup moins artificielle qu'elle ne le semblait au premier abord et apparaît comme liée à la nature des choses ».

IV. The Cauchy problem for the damped wave equation.

1. The method used to solve the CAUCHY problem for the wave equation may also be applied to find the solution of the CAUCHY problem for the damped wave equation. In its simplest form, the problem is to determine $u(x,\ t)$ satisfying

$$Lu \equiv L_{x,t}u \equiv u_{tt} - \Delta u - u = 0, \tag{1}$$

$$u(x,\ 0) = 0, \qquad u_t(x,\ 0) = g(x) \tag{2}$$

where $g(x)$ is a regular function (regular as defined earlier).

First of all, it is necessary to find an elementary solution $v(x,\ t)$ of (1), i. e., a solution having a singularity along the characteristic cone $t^2 - r^2 = 0$ [compare with the corresponding solution v of the wave equation]. To do this, write

$$\gamma = \sqrt{t^2 - r^2}, \qquad (t > r)$$

and assume that the required solution $v(x, t)$ of (1) depends only on γ. Let $v(x, t) = v(\gamma)$; because [4]

$$\gamma_t = \frac{t}{\gamma}, \qquad \gamma_{tt} = \frac{1}{\gamma} - \frac{t^2}{\gamma^3},$$

$$\gamma_r = -\frac{r}{\gamma}, \qquad \gamma_{rr} = -\frac{1}{\gamma} + \frac{r^2}{\gamma^3},$$

$$v_t = \frac{t}{\gamma} v', \qquad \gamma_{tt} = \frac{t^2}{\gamma^2} v'' + \left(\frac{1}{\gamma} - \frac{t^2}{\gamma^3}\right) v',$$

$$v_r = -\frac{r}{\gamma} v', \qquad v_{rr} = \frac{r^2}{\gamma^2} v'' - \left(\frac{1}{\gamma} + \frac{r^2}{\gamma^3}\right) v',$$

$$\Delta v = v_{rr} + \frac{p-1}{r} v_r,$$

$$v_{tt} - \Delta v = v'' + \frac{pv'}{\gamma},$$

it is clear that $v(\gamma)$ must be a solution of

$$\mathcal{E}_p(v) \equiv v_{\gamma\gamma} + \frac{p}{\gamma} v_\gamma - v = 0. \tag{3}$$

To find the solution of (3), set $v = \frac{1}{\gamma} w'$, where w is an unknown function of γ; one has

$$v' = \frac{1}{\gamma} w'' - \frac{1}{\gamma^2} w', \qquad v'' = \frac{1}{\gamma} w''' - \frac{2}{\gamma^2} w'' + \frac{2}{\gamma^3} w',$$

$$\mathcal{E}_p(v) = \frac{1}{\gamma} w''' + \frac{p-2}{\gamma^2} w'' - \frac{p-2}{\gamma^3} w' - \frac{1}{\gamma} w'$$
$$= \frac{1}{\gamma} \frac{d}{d\gamma} \mathcal{E}_{p-2}(w).$$

Consequently, if w is a solution of (3) for a particular value p_0 of p, i. e. if $\mathcal{E}_{p_0}(w) = 0$, then $v = \frac{1}{\gamma} w'$ is a solution of $\mathcal{E}_{p_0+2}(v) = 0$. As a result, to deter-

(4) Since $v = v(\gamma)$, a function of γ alone, we use $v', v'', \ldots$ instead of $v_\gamma, v_{\gamma\gamma}, \ldots$ and similarly for w.

mine the solutions of (3), we have only to consider two cases, namely $p = 0$ or 1.

1. $p = 2k + 2$ is *even*. It is well known that the equation $\mathcal{E}_0(v) = v'' - v = 0$ has two independent solutions, namely $\operatorname{ch}\gamma$ and $\operatorname{sh}\gamma$. Therefore, as a consequence of the above, a pair of independent solutions of (3) is

$$\left(\frac{1}{\gamma}\frac{d}{d\gamma}\right)^{k+1}\operatorname{ch}\gamma, \qquad \left(\frac{1}{\gamma}\frac{d}{d\gamma}\right)^{k+1}\operatorname{sh}\gamma.$$

Because the required solution $v(\gamma)$ must be singular for $\gamma = 0$, we have

$$v(x,\ t) = v(\gamma) = \left(\frac{1}{\gamma}\frac{d}{d\gamma}\right)^{k+1}\operatorname{sh}\gamma$$

$$= \left(\frac{1}{\gamma}\frac{d}{d\gamma}\right)^{k}\frac{\operatorname{ch}\gamma}{\gamma} = 2^k\left[\frac{d}{d(\gamma^2)}\right]^k\frac{\operatorname{ch}\gamma}{\gamma}.$$

From the known formulas,

$$ch\gamma = \cos(i\gamma),$$

$$\left(\frac{1}{z}\frac{d}{dz}\right)^k\frac{\cos z}{z} = \sqrt{\frac{\pi}{2}}\,\frac{J_{-k-\frac{1}{2}}(z)}{z^{k+\frac{1}{2}}}$$

where k is a positive integer or zero and $J_k(z)$ the BESSEL function of order k, it follows that

$$v(x,\ t) = (-1)^k\sqrt{\frac{\pi}{2}}\,i^{-k+\frac{1}{2}}\,\frac{J_{-k-\frac{1}{2}}(i\gamma)}{\gamma^{k+\frac{1}{2}}}.$$

For $p = 2$, we note that

$$v(x,\ t) = \frac{1}{\gamma}\operatorname{ch}\gamma.$$

2. $p = 2k + 3$ is *odd*. The equation

$$\mathcal{E}_1(u) = u_{\gamma\gamma} + \frac{1}{\gamma}u_\gamma - u = 0$$

has two independent solutions, namely

$$u_1 = I_0(\gamma) = J_0(i\gamma),$$

$$u_2 = J_0(i\gamma)\lg\gamma + \text{holomorphic function of } \gamma^2 \ (^5).$$

---

(5) This expression of u_2 is easily obtained by setting $u = J_0(i\gamma)\lg\gamma + w$ and $\gamma^2 = \Gamma$ n the equation $\mathcal{E}_1(u) = 0$.

For convenience, set $\Gamma = \gamma^2$. Because u_2 is singular for $\gamma = 0$, the required solution of (3) is

$$v(x,\ t) = 2^{k+1}\left[\frac{d}{d\,\gamma^2)}\right]^{k+1} u_2$$

(4) $$= 2^k \frac{d^{k+1}}{d\Gamma^{k+1}} J_0(i\sqrt{\Gamma}) \lg \Gamma + \text{holomorphic function of } \Gamma.$$

It will be convenient to write this solution in another way.

Note that if a and b are regular functions of $\Gamma = \gamma^2$, one has from Leibnitz's formula

$$\frac{d^k ab}{d\Gamma^k} = \sum_{j=0}^{k} C_j^k \frac{d^j a}{d\Gamma^j} \cdot \frac{d^{k-j} b}{d\Gamma^{k-j}}$$

and because

$$\frac{d^j \lg \Gamma}{d\Gamma^j} = \frac{d^{j-1}\Gamma^{-1}}{d\Gamma^{j-1}} = (-1)^{j-1}(j-1)!\,\Gamma^{-j},$$

$$\frac{d^j J_0(i\gamma)}{(d\gamma^2)^j} = \left(\frac{1}{2i}\right)^j \frac{J_j(i\gamma)}{\gamma^j},$$

solution (4) becomes

$$v(x,\ t) = 2^k \frac{d^{k+1}}{d\Gamma^{k+1}} [J_0(i\sqrt{\Gamma}) \lg \Gamma] + \text{holomorphic function of } \Gamma$$

$$= 2^k (k+1)!\,(-1)^k \sum_{j=0}^{k} \frac{\left(\frac{i}{2}\right)^j}{j!\,(k+1-j)} \cdot \frac{J_j(i\gamma)}{\gamma^{2(k+1)-j}}$$

$$+ \frac{(-i)^{k+1}}{2} \cdot \frac{J_{k+1}(i\gamma)}{\gamma^{k+1}} \lg \gamma^2 + \text{holomorphic function of } \gamma^2.$$

2. As a summary and for convenience, the elementary solutions $v(x,\ t)$ which will be used to solve the Cauchy problem (1), (2) are given once again.

a. When $p = 1$,

$$v(x,\ t) = J_0(i\gamma) \lg \gamma + \varphi_1(\gamma^2)$$

where $\varphi_1(\gamma^2)$ is a holomorphic function of γ^2.

b. When $p = 2$,

$$v(x,\ t) = \gamma^{-1} \operatorname{ch} \gamma.$$

c. When $p = 2k + 2$,

$$v(x,\ t) = (-1)^k \sqrt{\frac{\pi}{2}}\, i^{-k+\frac{1}{2}} J_{-k-\frac{1}{2}}(i\gamma)\gamma^{-k-\frac{1}{2}}.$$

d. When $p = 2k + 3$,

$$v(x,\ t) = v_1(\gamma) \lg \gamma^2 + v_2(\gamma) + \varphi_2(\gamma^2),$$

where

$$v_1(\gamma) = \frac{(-i)^{k+1}}{2} \gamma^{-k-1} J_{k+1}(i\gamma)$$

$$= 2^{-k-2} \sum_{h=0}^{\infty} \frac{\gamma^{2h}}{h\,!\,\Gamma(h+k+2) \cdot 2^{2h}},$$

$$v_2(\gamma) = 2^k(k+1)\,!\,(-1)^k \sum_{j=0}^{k} \frac{i^j}{2^j \cdot j\,!\,(k+1-j)} \frac{J_j(i\gamma)}{\gamma^{2(k+1)-j}};$$

$\varphi_2(\gamma^2)$ is a holomorphic function of γ^2.

For future reference, note that

(5) $$Lv_1 = 0,$$

(6) $$L(v_2 + \varphi_2) = -2\left(\frac{2}{\gamma}\frac{dv_1}{d\gamma} + \frac{p-1}{\gamma^2} v_1\right).$$

Indeed, set $w = v_2 + \varphi_2$; then

$$v = 2v_1 \lg \gamma + w,$$

$$v_\gamma = 2\frac{\partial v_1}{\partial \gamma} \lg \gamma + \frac{2v_1}{\gamma} + \frac{\partial w}{\partial \gamma},$$

$$v_{\gamma\gamma} = 2\frac{\partial^2 v_1}{\partial \gamma^2} \lg \gamma + \frac{4}{\gamma}\frac{\partial v_1}{\partial \gamma} - \frac{2v_1}{\gamma^2} + \frac{\partial^2 w}{\partial \gamma^2},$$

$$Lv = 2Lv_1 \cdot \lg \gamma + Lw + \frac{4}{\gamma}\frac{\partial v_1}{\partial \gamma} + \frac{2(p-1)}{\gamma^2} v_1 = 0.$$

Taking into account the singularities of v_1 and w at $\gamma = 0$, one finds

$$Lv_1 = 0,$$

$$Lw + \frac{4}{\gamma}\frac{dv_1}{d\gamma} + \frac{2(p-1)}{\gamma^2} v_1 = 0.$$

3. The solution of the CAUCHY problem (1), (2) is

a. When $p = 1$,

$$u(x\,;\ t) = \frac{1}{2}\int_{x-t}^{x+t} g(y) J_0[i\sqrt{t^2 - |x-y|^2}]dy.$$

b. When $p = 2$,

$$u(x\,;\ t) = \int_{|x-y|\leq t} g(y) \frac{\operatorname{ch}\sqrt{t^2 - |x-y|^2}}{\sqrt{t^2 - |x-y|^2}}\, dy.$$

c. When $p = 2k + 2$,

(7) $$(2\pi)^{p/2} u(x\,;\ t) = \operatorname{pf}\int_{|x-y|\leq t} g(y) v(x-y\,;\ t) dy.$$

d. When $p = 2k + 3$,

(8) $$(2\pi)^{\frac{p-1}{2}} u(x\,;\ t) = \int_{|x-y|\leq t} g(y) v_1(x-y\,;\ t)dy - \operatorname{pl}\int_{|x-y|\leq t} g(y) v_2(x-y\,;\ t) dy.$$

In cases *a* and *b*, the synthesis of the solution is easy and left to the reader.

Cases *c* and *d* are considered below.

4. $p = 2k + 2$ is *even*. That $u(x\,;\ t)$ given by (7) satisfies equation (1) is clear since differentiation under the integral sign is permitted and $L_{x,\,t} v(x-y\,;\ t) = 0$.

To verify the initial conditions, set $y = x + t\alpha$, where $\alpha = (\alpha_1, \ldots, \alpha_p)$; one has

$$(2\pi)^{p/2} u(x\,;\ t) = t^p \operatorname{pf}\int_{|\alpha|\leq 1} g(x + t\alpha) v(t\alpha\,;\ t) d\alpha.$$

When $t \to 0$, the most important term of $v(t\alpha\,;\ t)$ is

$$\frac{2^k\sqrt{\pi}}{\Gamma\left(-k+\frac{1}{2}\right)}\, t^{-2k-1}(1 - |\alpha|^2)^{-k-\frac{1}{2}};$$

consequently,

$$(2\pi)^{p/2}u(x\,;\,t) = \frac{2^k\sqrt{\pi}}{\Gamma\left(-k+\frac{1}{2}\right)}\, g(x)t \operatorname{pf}\int\limits_{|\alpha|\leq 1} \frac{d\alpha}{(1-|\alpha|^2)^{k+\frac{1}{2}}} + O(t^2)$$

$$= \frac{2^k\sqrt{\pi}}{\Gamma\left(-k+\frac{1}{2}\right)} \cdot \frac{\pi^{p/2}\Gamma\left(-k+\frac{1}{2}\right)}{\Gamma\left(\frac{p}{2}-k+\frac{1}{2}\right)}\, tg(x) + O(t^2)$$

$$= (2\pi)^{p/2}tg(x) + O(t^2).$$

5. $p = 2k + 3$ is *odd*. Let A and B be respectively the first and the second terms of the right hand side of (8); the solution of the CAUCHY problem (1) (2) is

$$(2\pi)^{\frac{p-1}{2}}u(x\,;\,t) = A - B. \tag{9}$$

We have to verify that $u(x\,;\,t)$ given by (9) is a solution of $L_{x,t}u = 0$ First, consider the term B. Differentiation under the integral sign gives

$$L_{x,t}B = \operatorname{pl}\int\limits_{|x-y|\leq t} g(y)L_{x,t}v_2(x-y\,;\,t)dy$$

$$= -\,2\operatorname{pl}\int\limits_{|x-y|\leq t}\left(\frac{2}{\gamma}\frac{dv_1}{d\gamma} + \frac{p-1}{\gamma^2}\,v_1\right)g(y)dy$$

where (6) is used.

Because regular terms contribute zero to the pl, it follows that

$$L_{x,t}B = -\,2(p-1)v_1(0)\operatorname{pl}\int\limits_{|x-y|\leq t} g(y)\,\frac{dy}{t^2-r^2}$$

where

$$v_1(0) = \frac{1}{2^{k+2}\Gamma(k+2)}\,.$$

Now, by using $\bar{g}(t)$ the mean value of $g(x)$ on the hypersphere of center x and radius t, one finds

$$\operatorname{pl}\int\limits_{|x-y|\leq t} g(y)\,\frac{dy}{t^2-r^2} = \frac{1}{2t}\operatorname{pl}\int\limits_{|x-y|\leq t} g(y)\,\frac{dy}{t-r}$$

$$= -\,\frac{\omega_p}{2}\,t^{p-2}\bar{g}(t) = -\,\frac{1}{2t}\int\limits_{|x-y|=t} g(y)d\Omega_t$$

and therefore

$$L_{x,t}B = \frac{p-1}{2^{k+2}\Gamma(k+2)} \cdot \frac{1}{t}\int\limits_{|x-y|=t} g(y)d\Omega_t .$$

Now, consider

$$A = \int\limits_{|x-y|\leq t} g(y)v_1(x-y\,;\,t)dy \tag{10}$$

and take into account that $v_1(x-y\,;\,t)$ is a regular function of $\gamma^2 = t^2 - r^2$. Differentiation with respect to t gives

$$A_t = \int\limits_{|x-y|\leq t} g(y)\frac{\partial v_1(x-y\,;\,t)}{\partial t}dy + \int\limits_{|x-y|=t} g(y)v_1(x-y\,;\,t)d\Omega_t \tag{11}$$

$$= \int\limits_{|x-y|\leq t} g(y)\frac{\partial v_1(x-y\,;\,t)}{\partial t}dy - \mathrm{pl}\int\limits_{|x-y|\leq t} g(y)v_1(x-y\,;\,t)\frac{dy}{t-r},$$

$$A_{tt} = \int\limits_{|x-y|\leq t} g(y)\frac{\partial^2 v_1(x-y\,;\,t)}{\partial t^2}dy - 2\,\mathrm{pl}\int\limits_{|x-y|\leq t} g(y)\frac{\partial v_1(x-y\,;\,t)}{\partial t}\frac{dy}{t-r}$$

$$+ \mathrm{pl}\int\limits_{|x-y|\leq t} g(y)v_1(x-y\,;\,t)\frac{dy}{(t-r)^2}.$$

To determine A_{x_1}, set $x+h = (x_1+h, x_2, \ldots, x_p)$ and write

$$A_{x_1} = \lim_{h=0}\frac{A(x+h)-A(x)}{h}$$

where

$$A(x+h) = \int\limits_{|x+h-z|\leq t} g(z)v_1(x+h-z\,;\,t)dz.$$

Observe that $z = y+h$ is a one to one transformation of $|x+h-z| \leq t$ on $|x-y| \leq t$, $\frac{\partial(z)}{\partial(y)} = 1$; consequently,

$$A(x+h) = \int_{|x-y|\leq t} g(y+h) v_1(x-y\,;\,t)\,dy$$

so that

$$A_{x_1} = \lim_{h=0} \int_{|x-y|\leq t} \frac{1}{h}[g(y+h) - g(y)] v_1(x-y\,;\,t)\,dy$$

$$= \int_{|x-y|\leq t} \frac{\partial g(y)}{\partial y_1} v_1(x-y\,;\,t)\,dy.$$

Because

$$\frac{\partial g}{\partial y_1} v_1 = \frac{\partial (g v_1)}{\partial y_1} - g \frac{\partial v_1}{\partial y_1} = \frac{\partial (g v_1)}{\partial y_1} + g \frac{\partial v_1}{\partial x_1}$$

and in view of the divergence theorem, one finds

$$A_{x_1} = \int_{|x-y|\leq t} g(y) \frac{\partial v_1}{\partial x_1} dv - \int_{|x-y|=t} g(y) v_1(x-y\,;\,t) \frac{\partial r_{xy}}{\partial x_1} d\Omega_t$$

$$= \int_{|x-y|\leq t} g(y) \frac{\partial v_1}{\partial x_1} dy + \mathrm{pl} \int_{|x-y|\leq t} g(y) v_1(x-y\,;\,t) \frac{\partial r_{xy}}{\partial x_1} \frac{dy}{t-r}$$

$$A_{x_1 x_1} = \int_{|x-y|\leq t} g(y) \frac{\partial^2 v_1}{\partial x_1^2} dy + 2\,\mathrm{pl} \int_{|x-y|\leq t} g(y) \frac{\partial v_1}{\partial x_1} \frac{\partial r_{xy}}{\partial x_1} \cdot \frac{dy}{t-r}$$

$$+ \mathrm{pl} \int_{|x-y|\leq t} g(y) v_1(x-y\,;\,t) \frac{\partial^2 r_{xy}}{\partial x_1^2} \frac{dy}{t-r} + \mathrm{pl} \int_{|x-y|\leq t} g(y) v_1(x-y\,;\,t) \left(\frac{\partial r_{xy}}{\partial x_1}\right)^2 \frac{dy}{(t-r)^2}.$$

Noting that

$$\sum_{i=1}^{p} \left(\frac{\partial r}{\partial x_i}\right)^2 = 1, \qquad \Delta r = \frac{p-1}{r},$$

$$\frac{\partial v_1}{\partial x_i} = \frac{\partial v_1}{\partial r} \cdot \frac{\partial r}{\partial x_i}, \qquad \sum_{i=1}^{p} \frac{\partial v_1}{\partial x_i} \frac{\partial r}{\partial x_i} = \frac{\partial v_1}{\partial r},$$

we obtain

$$\Delta A = \int_{|x-y|\leq t} g(y) \Delta v_1 dy + 2\,\mathrm{pl} \int_{|x-y|\leq t} g(y) \frac{\partial v_1}{\partial r} \frac{dy}{t-r}$$

$$+ (p-1)\,\mathrm{pl} \int_{|x-y|\leq t} g(y) \frac{v_1}{r} \cdot \frac{dy}{t-r} + \mathrm{pl} \int_{|x-y|\leq t} g(y) v_1 \frac{dy}{(t-r)^2}.$$

Finally, combining the above results

$$L_{x,t}A = \int\limits_{|x-y|\leq t} g(y)L_{x,t}v_1 dy$$

$$-\operatorname{pl}\int\limits_{|x-y|\leq t}\left[2\left(\frac{\partial v_1}{\partial t}+\frac{\partial v_1}{\partial r}\right)+\frac{p-1}{r}v_1\right]g(y)\frac{dy}{t-r}$$

$$=\int\limits_{|x-y|\leq t} g(y)L_{x,t}v_1 dy + \int\limits_{|x-y|=t}\left[2\left(\frac{\partial v_1}{\partial t}+\frac{\partial v_1}{\partial r}\right)+\frac{p-1}{t}v_1\right]g(y)d\Omega_t$$

$$=\frac{p-1}{t}\int\limits_{|x-y|=t} v_1 g(y)d\Omega_t = \frac{p-1}{t}v_1(0)\int\limits_{|x-y|=t} g(y)dy = L_{x,t}B$$

because $L_{x,t}v_1 = 0$ in $|x-y|\leq t$ and $\dfrac{\partial v_1}{\partial t}+\dfrac{\partial v_1}{\partial r}=\dfrac{1}{\gamma}\dfrac{dv_1}{d\gamma}\,(t-r)=0$ on $|x-y|=t$.

Therefore, $L_{x,t}u = 0$.

6. Let us now verify the initial conditions. First, note that when t tends to zero, A and A_t [cf. 10, 11] tend also to zero (set as usual $y = x + t\alpha$), so that it remains to consider the term B.

From the definition of $\operatorname{pl}\int$, it is clear that the terms of v_2 giving a logarithmic part different from zero are

$$w(x) = 2^k(k+1)!\,(-1)^k\sum_{j=0}^{k}\sum_{l=0}^{k-j}\frac{(-1)^j\gamma^{-2(k+1-j-l)}}{2^{2(l+j)}j!\,l!\,(l+j)!\,(k+1-j)!}.$$

To compute the double sum (call it S), set $l+j=s$; s varying from 0 to k and j from 0 to s; one has

$$S=\sum_{s=0}^{k}\frac{1}{2^{2s}s!\,\gamma^{2(k+1-s)}}\sum_{j=0}^{s}\frac{(-1)^j}{j!\,(s-j)!\,(k+1-j)}.$$

From the formula

$$\frac{1}{z(z-1)\ldots(z-s)}=\sum_{j=0}^{s}\frac{(-1)^{s-j}}{j!\,(s-j)!\,(z-j)},$$

one deduces (set $z = k + 1$),

$$\frac{(-1)^s}{(k+1)k \dots (k+1-s)} = \sum_{j=0}^{s} \frac{(-1)^j}{j!\,(s-j)!\,(k+1-j)}$$

$$= \frac{(-1)^s (k-s)!}{(k+1)!}$$

and consequently

$$S = \sum_{s=0}^{k} \frac{(-1)^s (k-s)!}{(k+1)!\, 2^{2s} s!\, \gamma^{2(k+1-s)}},$$

so that

$$w(x) = \sum_{s=0}^{k} \frac{(-1)^{s+k} 2^{k-2s} (k-s)!}{s!\, \gamma^{2(k+1-s)}}.$$

Therefore

$$B = (-1)^k 2^k k!\, tg(x) \operatorname{pl} \int_{|\alpha| \leq 1} (1 - |\alpha|^2)^{-k-1} d\alpha + O(t^2)$$

$$= -(2\pi)^{\frac{p-1}{2}}\, tg(x) + O(t^2)$$

from which it follows easily that $u(x, 0) = 0$, $u_t(x, 0) = g(x)$.

V. The Cauchy Problem for the equation of Euler-Poisson-Darboux.

1. The original form of the TRICOMI equation is

$$(1) \qquad y^q u_{xx} + u_{yy} = 0,$$

where x, y are two independent variables and $q > 0$, an odd integer. When $y > 0$, equation (1) is of elliptic type; when $y < 0$, it is of hyperbolic type. Therefore, when the variable y crosses the line $y = 0$ (called the parabolic line of the equation), equation (1) changes its type.

More generally, consider

$$(2) \qquad Lu \equiv y^{-q} u_{yy} + \Delta u.$$

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \dots + \frac{\partial^2 u}{\partial x_p^2},$$

where $q > 0$ is an odd integer.

In order to simplify equation (2), a change of the independent variable is made. Two cases arise according as $y > 0$ or $y < 0$. Set

$$t = (1 - k) \mid y \mid^{\frac{1}{1-k}}, \qquad q = \frac{2k}{1-k}.$$

Because

$$u_y = \mid y \mid^{\frac{k}{1-k}} u_t \cdot \frac{d \mid y \mid}{dy},$$

$$\mid y \mid^{-q} u_{yy} = \frac{k}{t} u_t = u_{tt}$$

and

$$\frac{d \mid y \mid}{dy} = 1, \qquad y > 0,$$

$$= -1, \qquad y < 0,$$

one finds

$$Lu = u_{tt} + \frac{k}{t} u_t + \Delta u, \qquad y > 0,$$

$$= - u_{tt} - \frac{k}{t} u_t + \Delta u, \qquad y < 0.$$

Note that $\frac{1}{3} \leq k < 1$.

2. Consider the so called EULER-POISSON-DARBOUX equation (abbreviated E-P-D equation)

$$L_k u = u_{tt} + \frac{k}{t} u_t - \Delta u = 0,$$

where k is any real number $-\infty < k < \infty$.

The CAUCHY problem dealt with below reads:

to find a solution $u(x; t)$ of

$$L_k u = 0 \tag{3}$$

satisfying

$$u(x; 0) = f(x), \qquad u_t(x; 0) = 0, \tag{4}$$

where $f(x)$ is a regular function.

When $k=0$, equation (1) reduces to the wave equation for which the CAUCHY problem has been solved.

When $k \neq 0$, the coefficient kt^{-1} varies and becomes infinite on the hyperplane $t=0$. Consequently, the existence and uniqueness theorems for the regular CAUCHY problem (i. e. for partial differential equation whose coefficients are regular for $t=0$) cannot be applied to the current problem (3), (4). In fact, (3), (4) is a *singular* CAUCHY problem.

3. To indicate the dependence of u on the parameter k, write $u^{(k)}$ instead of u. The solutions of equation $L_k u^{(k)}=0$ exhibit the following recursion formulas

$$u_t^{(k)}(x\,;\ t) = t u^{(k+2)}(x\,;\ t), \tag{5}$$

$$u^{(k)}(x\,;\ t) = t^{1-k} u^{(2-k)}(x\,;\ t). \tag{6}$$

These formulas were used by various authors, among them EULER and DARBOUX; however their usefulness in the theory of the EULER-POISSON-DARBOUX equation was fully recognized and emphasized by A. WEINSTEIN. Their application can be extended to more general equations.

i. To prove (5), set $w_t = tv$ where w, as a function of t, $\in C^3$. By elementary computations, one has

$$\frac{1}{t}\frac{\partial}{\partial t}\left(\frac{\partial^2 w}{\partial t^2} + \frac{k}{t}\frac{\partial w}{\partial t}\right)$$

$$= \frac{1}{t}\left[\frac{\partial^2}{\partial t^2}(tv) + \frac{k}{t}\frac{\partial}{\partial t}(tv) - \frac{k}{t}v\right] = v_{tt} + \frac{k+2}{t}v_t$$

and

$$\frac{1}{t}\frac{\partial}{\partial t}L_k w = L_{k+2} v.$$

If $w = u^{(k)}$, then $v = u^{(k+2)}$ and $\dfrac{\partial u^{(k)}}{\partial t} = t u^{(k+2)}$.

ii. To prove (6), set

$$A = t^{1-k}\left(v_{tt} + \frac{2-k}{t} v_t\right)$$

and note that

$$\frac{\partial}{\partial t}(t^{1-k}v) = t^{1-k}v_t + (1-k)t^{-k}v,$$

$$\frac{\partial^2}{\partial t^2}(t^{1-k}v) = t^{1-k}v_{tt} + 2(1-k)t^{-k}v_t - k(1-k)t^{-k-1}v\,;$$

we have

$$A = \frac{\partial^2}{\partial t^2}(t^{1-k}v) + \frac{k}{t}t^{1-k}v_t + k(1-k)t^{-k-1}v$$

$$= \frac{\partial^2}{\partial t^2}(t^{1-k}v) + \frac{k}{t}\frac{\partial}{\partial t}(t^{1-k}v)$$

and therefore

$$L_k(t^{1-k}v) = t^{1-k}L_{2-k}v\,.$$

If $v \equiv u^{(2-k)}$, then $u^{(k)} = t^{1-k}u^{(2-k)}$.

4. In order to solve the CAUCHY problem (3), (4), it is necessary to determine a solution of (3) which corresponds to the elementary solution of the wave equation. To do this, observe that

$$(7)\qquad v^{(k)} = (t^2 - r^2)^{-\frac{k+p-1}{2}}, \qquad r = |\,x - y\,|$$

is a solution of (3). For, any solution of (3) depending on t and r only is a solution of

$$(8)\qquad L_k u \equiv u_{tt} + \frac{k}{t}u_t - u_{rr} - \frac{p-1}{r}u_r = 0\,;$$

and, conversely, any solution of (8) is also a solution of (3). Thus,

$$v_t^{(k)} = -(k+p-1)\,t(t^2-r^2)^{-\frac{k+p+1}{2}},$$

$$v_{tt}^{(k)} = (k+p-1)(k+p+1)\,t^2(t^2-r^2)^{-\frac{k+p+1}{2}-1} - (k+p-1)(t^2-r^2)^{-\frac{k+p+1}{2}},$$

$$v_r^{(k)} = (k+p-1)\,r(t^2-r^2)^{-\frac{k+p+1}{2}},$$

$$v_{rr}^{(k)} = (k+p-1)(k+p+1)\,r^2(t^2-r^2)^{-\frac{k+p+1}{2}-1} + (k+p-1)(t^2-r^2)^{-\frac{k+p+1}{2}}$$

and consequently, $L_k v^{(k)} = 0$.

From (6), it follows that

$$u^{(k)} = t^{1-k} v^{(2-k)} = t^{1-k}(t^2 - r^2)^{\frac{k-p-1}{2}} \tag{9}$$

is again a solution of (3).

Now, consider [as usual $r = r_{xy} = |x - y|$]

$$U^{(k)}(x;\ t) = t^{1-k} \int_{|x-y| \le t} f(y)\,(t^2 - r_{xy}^2)^{\frac{k-p-1}{2}}\, dy; \tag{10}$$

if $\Re k$ is large enough, $U^{(k)}(x;\ t)$ is clearly a solution of (3); for, under these circumstances, differentiation under the integral sign is permitted.

To verify the initial conditions, set as usual $y = x + t\alpha$; one has

$$U^{(k)}(x;\ t) = \int_{|\alpha| \le 1} f(x + t\alpha)\,(1 - |\alpha|^2)^{\frac{k-p-1}{2}}\, d\alpha$$

$$= f(x) \int_{|\alpha| \le 1} (1 - |\alpha|^2)^{\frac{k-p-1}{2}}\, d\alpha + O(t)$$

$$= A_k f(x) + O(t)$$

where

$$A_k = \frac{\pi^{\frac{p}{2}} \Gamma\left(\frac{k-p+1}{2}\right)}{\Gamma\left(\frac{k+1}{2}\right)}$$

since

$$\int_{|\alpha| \le 1} (1 - |\alpha|^2)^{\frac{k-p-1}{2}}\, d\alpha = \omega_p \int_0^1 (1 - r^2)^{\frac{k-p-1}{2}}\, r^{p-1} dr$$

$$= \frac{\omega_p}{2} B\left(\frac{p}{2}, \frac{k-p+1}{2}\right) = \frac{1}{2}\,\frac{2\pi^{\frac{p}{2}}}{\Gamma\left(\frac{p}{2}\right)} \cdot \frac{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{k-p+1}{2}\right)}{\Gamma\left(\frac{k+1}{2}\right)}$$

$$= \frac{\pi^{\frac{p}{2}}\, \Gamma\left(\frac{k-p+1}{2}\right)}{\Gamma\left(\frac{k+1}{2}\right)}.$$

To verify $\frac{\partial}{\partial t} U_k(x; 0) = 0$, set

$$w = \frac{\partial}{\partial t} t^{1-k}(t^2 - r^2)^{\frac{k-p-1}{2}} = t^{-k}(t^2 - r^2)^{\frac{k-p-1}{2}-1}[(k-1)r^2 - pt^2];$$

on differentiating under the integral sign, one obtains

$$\frac{\partial U^{(k)}}{\partial t} = \int\limits_{|x-y| \leq t} f(y) \frac{\partial w}{\partial t}\, dy.$$

In view of

$$f(x + t\alpha) = f(x) + t \sum_{i=l}^{p} \frac{\partial f(x)}{\partial x_i} \alpha_i + O(t^2),$$

$$\int\limits_{|x-y| \leq t} w dy = t^{-1} \int\limits_{|\alpha| \leq 1} (1 - |\alpha|^2)^{\frac{k-p-1}{2}-1} [(k-1)|\alpha|^2 - p] d\alpha$$

$$= t^{-1}\omega_p \int_0^1 [(k-1)r^2 - p](1 - r^2)^{\frac{k-p-1}{2}-1} r^{p-1} dr$$

$$= t^{-1} \frac{\omega_p}{2} \left[(k-1) B\left(\frac{p}{2} + 1, \frac{k-p-1}{2}\right) - pB\left(\frac{p}{2}, \frac{k-p-1}{2}\right)\right]$$

$$= 0 \qquad \text{(by elementary computations)};$$

$$\int\limits_{|\alpha| \leq 1} \alpha_i[(k-1)|\alpha|^2 - p](1 - |\alpha|^2)^{\frac{k-p-1}{2}-1} d\alpha = 0$$

(because of symmetry), we have $\frac{\partial U^{(k)}}{\partial t} = O(t)$ and $\frac{\partial}{\partial t} U^{(k)}(x; 0) = 0.$

Therefore, if $\Re k$ is large enough, the solution of the Cauchy problem (3), (4) is

$$u^{(k)}(x; t) = A_k^{-1} t^{1-k} \int\limits_{|x-y| \leq t} f(y)(t^2 - r_{xy}^2)^{\frac{k-p-1}{2}} dy \tag{11}$$

$$A_k = \frac{\pi^{\frac{p}{2}} \Gamma\left(\frac{k-p+1}{2}\right)}{\Gamma\left(\frac{k+1}{2}\right)}. \tag{12}$$

5. If k in (11) be permitted to range over the complex domain, the function $u^{(k)}(x; t)$ regarded as function of the complex variable k is an analytic function of k for $\Re k$ sufficiently large and hence may be analytically continued. Consider the analytic continuation of $u^{(k)}(x; t)$.

Because $\Gamma(z)$ has simple poles at $z = -n$, $(n = 0, 1, \ldots)$, it is clear that A_k is zero when $k = -2q - 1$, $q > 0$ an integer, and becomes infinite when $\frac{1}{2}(k - q - 1)$ is a negative integer.

Therefore three cases arise according as $k = -2q - 1$, $q \geq 0$ an integer, and as $\frac{1}{2}(k - p - 1)$ is or is not a negative integer.

i. $k = -2q - 1$, $q \geq 0$ an *integer*. We shall consider later on these exceptional values of k.

ii. $\frac{1}{2}(k - p - 1)$ *is not a negative integer*. Consider

$$u^{(k)}(x;\, t) = A_k^{-}\, t^{1-k}\, \mathrm{pf} \int\limits_{|x-y| \leq t} f(y)(t^2 - r_{xy}^2)^{\frac{k-p-1}{2}}\, dy. \tag{13}$$

On differentiating under the pf$\int$ sign and in view of preceding results, it is obvious that $u^{(k)}(x;\, t)$ given by (13) is a solution of the CAUCHY problem (3), (4).

iii. $\frac{1}{2}(k - p - 1) = -m$, $(m = 0, 1, \ldots)$ *is a negative integer*. Clearly, $k \leq p + 1$. Consider

$$u^{(k)}(x;\, t) = B_k^{-1}\, t^{1-k}\, \mathrm{pl} \int\limits_{|x-y| \leq t} f(y)(t^2 - r_{xy}^2)^{\frac{k-p-1}{2}}\, dy \tag{14}$$

where

$$B_k = \frac{(-1)^m\, \pi^{p/2}}{\Gamma\left(\frac{p-k+1}{2}\right)\Gamma\left(\frac{k+1}{2}\right)}. \tag{15}$$

[Note that (see II, § 17)

$$\mathrm{pl} \int\limits_{|\alpha| \leq 1} (1 - |\alpha|^2)^{-h} d\alpha = \frac{(-1)^h\, \pi^{p/2}}{\Gamma(h)\, \Gamma\left(\frac{p}{2} - h + 1\right)}$$

$h \geq 0$ an integer].

On differentiating under the pl$\int$ sign, one finds that $u^{(k)}(x;\, t)$ given by (14) satisfies (3). As above, $u^{(k)}(x;\, t)$ satisfies (4); the formal proof is left to the reader.

Again, note that $B_k = 0$ for the exceptional values of k (see i).

6. **Remarks.** *i.* On differentiating under the integral sign, it is easy to verify that

$$u^{(k)}(x\,;\,t) =$$

$$= 2^{-n} A_{k+2n}^{-1} \frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k+1}{2}+n\right)} t^{1-k} \left(\frac{\partial}{t\partial t}\right)^n \int_{|x-y|\le t} f(y)(t^2 - r_{xy}^2)^{\frac{k+2n-p-1}{2}} dy$$

where $n \geq 0$, an integer, is such that $k + 2n > p - 1$. That formula was obtained by A. Weinstein as a consequence of the recursion formulas (5), (6).

ii) Consider $u^{(k)}(x\,;\,t)$ as a function of p; call it $u_p^{(k)}$. The solution (13) satisfies the recursion formula

$$t\frac{\partial u_p^{(k)}}{\partial t} + p u_p^{(k)} = p u_{p+2}^{(k)}$$

which determines $u_p^{(k)}$ starting from $u_1^{(k)}$ and $u_2^{(k)}$.

7. **The exceptional values** $k = -2q - 1$, $q \geq 0$ an integer. For these exceptional values of k, the solution of equation (3) was considered by A. Weinstein (loc. cit.) who emphasized the role of polyharmonic initial values. For arbitrary initial values, the solution of (3), (4) was first considered by E. K. Blum.

In the following, we shall consider the same problem from another viewpoint and use a method suitable for further generalization. When $k = -2q - 1$, $q \geq 0$, an integer, $u^{(k)}(x\,;\,t)$ given by (13) or (14) becomes infinite and is not a solution of the Cauchy problem (3), (4).

To examine this situation, we use the recursion formulas

$$u_t^{(k)} = t u^{(k+2)}, \tag{5}$$

$$u^{(k)} = t^{1-k} u^{(2-k)}. \tag{6}$$

From (6), it follows

$$u^{(-2q-1)} = t^{2q+2} u^{(2q+3)}. \tag{16}$$

Now set $s = t^2$; then (5) yields

$$u^{(k+2)} = \frac{1}{t}\frac{\partial}{\partial t} u^{(k)} = 2\frac{\partial}{\partial s} u^{(k)}. \tag{17}$$

Set $k = 2q + 1$; repeated applications of formula (17) give

$$u^{(2q+3)} = 2\frac{\partial}{\partial s} u^{(2q+1)} = 2^{q+1} \frac{\partial^{q+1} u^{(1)}}{\partial s^{q+1}}$$

and combining with (16),

$$u^{(-2q-1)} = 2^{q+1} s^{q+1} \frac{\partial^{q+1} u^{(1)}}{\partial s^{q+1}}. \tag{18}$$

Suppose that $u^{(-2q-1)}$ given by (18) is a solution of the CAUCHY problem (3), (4) for an arbitrary initial function $f(x)$; then $s^{q+1} \frac{\partial^{q+1} u^{(1)}}{\partial s^{q+1}}$ remains finite and not identically zero as $s \to 0$ (or $t \to 0$). Therefore at $s = 0$, $\frac{\partial^{q+1} u^{(1)}}{\partial s^{q+1}}$ must have a singularity of type s^{-q-1}, i. e. at $s = 0$, $u^{(1)}$ must have a singularity of type $\log s$; for, if $\frac{\partial^{q+1} u^{(1)}}{\partial s^{q+1}}$ is finite and not identically zero when $s = 0$, we have $u^{(-2q-1)}(x;\,0) = 0$.

A solution of that sort is given below when p is *odd*.

When p is *even*, an analogous method or the method of descent (see III) may be used; the formal proof is left to the reader.

8. Because $(t^2 - r_{xy}^2)^{-\frac{k+p-1}{2}}$ and $t^{1-k}\;(t^2 - r_{xy}^2)^{\frac{k-p-1}{2}}$ are solutions of $L_k u = 0$, it is clear that

$$U^{(k)}(x;\; t) = \mathrm{pf} \int\limits_{|x-y| \leq t} f(y)(t^2 - r^2)^{-\frac{k+p-1}{2}} dy,$$

$$U^{(k)*}(x;\; t) = t^{1-k}\, \mathrm{pf} \int\limits_{|x-y| \leq t} f(y)(t^2 - r^2)^{\frac{k-p-1}{2}} dy$$

are solutions of $L_k u = 0$. For, differentiation under the $\mathrm{pf}\int$ sign is permitted.

Set $k = 1 + 2\varepsilon$, $\varepsilon > 0$ and small; then

$$u_1(x;\; t) = U^{(1+2\varepsilon)} = \mathrm{pf} \int\limits_{|x-y| \leq t} f(y)(t^2 - r^2)^{-\frac{p}{2} - \varepsilon} dy,$$

$$u_2(x;\; t) = U^{(1+2\varepsilon)*}(x,\; t) = t^{-2}\, \mathrm{pf} \int\limits_{|x-y| \leq t} f(y)(t^2 - r^2)^{-\frac{p}{2} + \varepsilon} dy$$

are solutions of $L_{1+2\varepsilon} u = 0$.

Again, $\frac{1}{2\varepsilon}(u_1 - u_2)$ is a solution of $L_{1+2\varepsilon}u = 0$; on setting

$$F(\varepsilon) = t^\varepsilon(t^2 - r^2)^{-\varepsilon},$$

one finds

$$\frac{1}{2\varepsilon}(u_1 - u_2) = \mathrm{pf}\int\limits_{|x-y|\leq t} f(y)t^{-\varepsilon}\left[\frac{F(\varepsilon) - F(-\varepsilon)}{2\varepsilon}\right](t^2 - r^2)^{-\frac{p}{2}}dy.$$

Now let ε tend to 0 under the $\mathrm{pf}\int$ sign; one gets

$$u^{(1)}(x;\ t) = \mathrm{pf}\int\limits_{|x-y|\leq t} f(y)[\lg t - \lg(t^2 - r^2)](t^2 - r^2)^{-\frac{p}{2}}dy.$$

It is hoped that this limiting expression, a formal solution of $L_1 u = 0$, is also an actual solution of $L_1 u = 0$ having at $t = 0$ the singularity described above. That such is the case is proven below.

9. To prove that $u^{(1)}(x;\ t)$ satisfies $L_1 u = 0$, it is necessary, since differentiation under the $\mathrm{pf}\int$ sign is permissible, only to show that

$$v = [\lg t - \lg(t^2 - r^2)](t^2 - r^2)^{-\frac{p}{2}}$$

is a solution of $L_1 u = 0$.

Set

$$t^2 - r^2 = \rho^2,$$

$$a = \rho^{-p}\lg t, \qquad b = 2\rho^{-p}\lg\rho,$$

$$v = a - b.$$

Further, note that

$$\frac{1}{t}\frac{\partial u}{\partial t} = \frac{1}{\rho}\frac{\partial u}{\partial \rho},$$

$$Du \equiv u_{tt} - \Delta u = \frac{\partial^2 u}{\partial \rho^2} + \frac{p}{\rho}\frac{\partial u}{\partial \rho},$$

$$L_1 u = Du + \frac{1}{t}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial \rho^2} + \frac{p+1}{\rho}\frac{\partial u}{\partial \rho},$$

$$L_1 b = -2p\rho^{-p-2}.$$

To compute $L_1 a$, set $c = \rho^{-p}$, $a = c\lg t$ and note that $L_1 c = 0$; we find

$$L_1 a = (L_1 c)\cdot\lg t + \frac{2}{\rho}\frac{\partial c}{\partial \rho} = -2p\rho^{-p-2} = L_1 b$$

i. e.

$$L_1 v = L_1(a - b) = 0.$$

10. To verify the initial conditions, set as usual $y = x + t\alpha$; one has

$$u^{(1)} = \lg t \ \mathrm{pf} \int\limits_{|\alpha| \leq 1} f(x + t\alpha)(1 - |\alpha|^2)^{-\frac{p}{2}} d\alpha$$

$$- \mathrm{pf} \int\limits_{|\alpha| \leq 1} f(x + t\alpha)(1 - |\alpha|^2)^{-\frac{p}{2}} \lg(1 - |\alpha|^2) d\alpha.$$

The second term on the right hand side is equal to

$$Af(x) + O(t)$$

where A is a constant.

Further,

$$\mathrm{pf} \int\limits_{|\alpha| \leq 1} f(x + t\alpha)(1 - |\alpha|^2)^{-\frac{p}{2}} d\alpha = f(x) \int\limits_{|\alpha| \leq 1} (1 - |\alpha|\,)^{-\frac{p}{2}} d\alpha + O(t)$$

$$= \frac{(-1)^{\frac{p-1}{2}} \pi^{\frac{p}{2}+1}}{\Gamma\left(\dfrac{p}{2}\right)} f(x) + O(t) = (-1)^{\frac{p-1}{2}} \frac{\pi}{2} \omega_p f(x) + O(t).$$

Combining these results and setting $s = t^2$, we obtain

$$u^{(1)}(x\,;\ t) = A(s) \lg s + B(s),$$

where $A(s)$ and $B(s)$ are regular functions of s, $A(0) \neq 0$.

Finally, we have

$$u^{(-2q-1)}(x\,;\ t) = cA(s) + o(s)$$

where c is a constant, not zero and $\lim\limits_{t=0} o(s) = 0$.

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