A Problem in Prediction Theory.

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To Giovanni Sansone on his 70th birth day.

Summary. - *A new kind of « prediction problem » is defined and studied, first for certain special distributions and then generally. The general results are connected with conjugate trigonometric series.*

§ 1. - Introduction.

1. Let μ be a non-negative finite measure defined on the field of BOREL subsets of the half-open interval $[0, 2\pi)$. We form the HILBERT space L^2_μ of complex functions $f = f(x)$ measurable for $d\mu$ and such that

$$\| f \|^2_\mu = \int | f |^2 \, d\mu < \infty \text{ (1)}.$$

The inner product of f and g is by definition

$$(f, g)_\mu = \int f\bar{g} d\mu.$$

The trigonometric polynomials always form a dense linear subset of L^2_μ; *prediction theory* is the theory of approximation by trigonometric polynomials in the metric of that space.

We use $d\sigma = d\sigma(x)$ to denote the measure $dx/2\pi$ on $[0, 2\pi)$. Then μ has a decomposition $d\mu(x) = w(x)d\sigma(x) + d\mu_s(x)$ where $w(x)$ is a non-negative summable function on the interval and μ_s is singular with respect to LEBESGUE measure.

Let $\mathfrak{F}$, $\mathfrak{P}$, and $\mathfrak{P}_1$ be the closed linear manifolds in L^2_μ obtained, respectively, by closing the sets of trigonometric polynomials of the form

$$(1.1) \qquad F(e^{ix}) = a_1 e^{ix} + a_2 e^{2ix} + a_3 e^{3ix} + \ldots,$$

$$(1.2) \qquad P(e^{ix}) = b_1 e^{-ix} + b_2 e^{-2ix} + b_3 e^{-3ix} + \ldots,$$

$$(1.3) \qquad P_1(e^{ix}) = c_0 + c_1 e^{-ix} + c_2 e^{-2ix} + \ldots.$$

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(1) The integrals in this paper are all extended from 0 to 2π, unless stated otherwise.

We shall use F, P, and P_1 consistently to denote trigonometric polynomials given by (1.1), (1.2), and (1.3) respectively. The function identically equal to 1 will be called I.

The first prediction problem, associated with KOLMOGOROFF, SZEGÖ, and WIENER is to find the distance from I to $\mathfrak{F}$. The solution is given by the following theorem of SZEGÖ [6, 14] (²):

$$\inf \int | I + F |^2 \, d\mu = \exp\left\{ \int \log w \, d\sigma \right\}, \tag{1.4}$$

where the infimum is taken over all trigonometric polynomials F having the form (1.1). In other words, the distance from I to $\mathfrak{F}$ is $\exp\left\{ \frac{1}{2} \int \log w \, d\sigma \right\}$, whose square is the geometric mean of w. If the right-hand expression in (1.4) is zero, that is if $\int \log w \, d\sigma = -\infty$, then I lies in $\mathfrak{F}$; and indeed $\mathfrak{F}$ coincides with the whole space L^2_μ. It is curious that the distance from I to $\mathfrak{F}$ does not depend at all on μ_s.

The second prediction problem is to determine the distance from I to the smallest manifold containing $\mathfrak{F}$ and $\mathfrak{P}$. KOLMOGOROFF has shown [5, p. 83] that

$$\inf \int | I + F + P |^2 \, d\mu = \left(\int w^{-1} d\sigma \right)^{-1} \tag{1.5}$$

where F and P range over trigonometric polynomials (1.1) and (1.2), respectively. Hence the square of this distance is the harmonic mean of w, again independent of μ_s. (If the right side of (1.5) is zero, I belongs to the closure of the manifold.)

These theorems show that if w is not too «small», i. e. if the right-hand means are positive, the exponentials e^{nix} possess a certain kind of independence in L^2_μ. The purpose of this paper is to study a stronger notion of independence than the two just considered.

2. Two manifolds in a HILBERT space are said to be *at positive angle* if

$$\rho = \sup | (f, g) | < 1,$$

where f and g range over the elements of the manifolds, respectively, with norm at most 1.

The third prediction problem is to evaluate ρ for the manifolds $\mathfrak{F}$ and $\mathfrak{P}_1$ in L^2_μ. In this case

$$\rho = \sup \left| \int F(e^{ix}) \overline{P_1(e^{ix})} \, d\mu(x) \right| \tag{1.6}$$

(²) This theorem was originally proved by SZEGÖ for the case where μ is absolutely continuous, and subsequently was extended by KOLMOGOROFF.

where F and P_1 range over the trigonometric polynomials (1.1) and (1.3), respectively, subject to the restriction

$$\int |F|^2 d\mu \leqq 1, \qquad \int |P_1|^2 d\mu \leqq 1. \tag{1.7}$$

Equivalently, ρ is given by the expression

$$2 - 2\rho = \inf \int |F + P_1|^2 d\mu, \tag{1.8}$$

where now F and P_1 have norm exactly 1.

It is trivial that $\rho \leqq 1$. If for a measure μ we have $\rho < 1$, then (as one can show in an elementary way) (1.4) and (1.5) are positive, so that this condition of independence for the exponentials in L^2_μ is stronger than the conditions of independence of the first two prediction problems.

3 In § 2 and § 3 we deal with the following three special cases:

(a) $[w(x)]^{-1}$ is a positive trigonometric polynomial,

(b) $w(x)$ is a positive trigonometric polynomial,

(c) $w(x)$ is the ratio of two positive trigonometric polynomials.

The interest of these cases lies in the fact that in all three instances the determination of ρ can be reduced to an algebraic problem, whereas in general it seems to be very difficult to evaluate ρ.

There is, however, a simple necessary and sufficient condition that $\rho < 1$, in other words that $\mathfrak{F}$ and $\mathfrak{P}_1$ be at positive angle; this criterion is developed in § 4.

4. A problem in trigonometric series which has been studied previously is the following: For which measures μ does there exist a constant K such that, for every real trigonometric polynomial f with conjugate $\tilde{f}$, we have

$$\int \tilde{f}^2 d\mu \leqq K^2 \int f^2 d\mu \ ? \tag{1.9}$$

It is shown in § 5 that the answer is affirmative for μ if and only if $\mathfrak{F}$ and $\mathfrak{P}_1$ are at positive angle in L^2_μ. Therefore the preceding results of the paper give a satisfactory solution to this question. We are indebted to Professor A. Zygmund for references to other work connected with (1.9); this work is discussed at the end of § 5.

In § 6 we consider the angle between $\mathfrak{F}$ and $\mathfrak{P}$ (rather than $\mathfrak{P}_1$) in L^2_μ. If $\int w^{-1}d\sigma < \infty$, there is nothing new in the problem; without this assumption, a general criterion for $\mathfrak{F}$ and $\mathfrak{P}$ to be at positive angle exists, but the solution is less satisfactory than in the case of $\mathfrak{F}$ and $\mathfrak{P}_1$.

In § 7 we examine in detail a class of weight functions quite different from those of § 2 and § 3, including the functions

$$w_\alpha(x) = |x|^\alpha \qquad (-\pi \leqq x < \pi,\ \alpha > -1).$$

It is known that (1.9) holds for $d\mu = w_\alpha d\sigma$ if and only if $-1 < \alpha < 1$; we are able to determine the values of α such that $\mathfrak{F}$ and $\mathfrak{P}$ are at positive angle in $L^2_{w_\alpha}$. Certain more complicated functions are discussed also.

§ 2. **Distributions** (*a*) **and** (*b*).

1. In dealing with the special cases (*a*) and (*b*) defined in the Introduction, we shall make use of the following remark of HELLINGER-TOEPLITZ [9]: Let $(a_{\varkappa\lambda})$ be a finite matrix; $\varkappa, \lambda = 0, 1, \ldots, n$. We denote by ρ the maximum of

$$\left| \sum_{\varkappa=0}^{n} \sum_{\lambda=0}^{n} a_{\varkappa\lambda} x_\varkappa y_\lambda \right| \tag{2.1}$$

where $x_\varkappa$, y_λ are complex numbers and $\Sigma |x_\varkappa|^2 = \Sigma |y_\lambda|^2 = 1$. Then ρ^2 is the maximum of the Hermitian form $\Sigma\Sigma c_{\varkappa\lambda} u_\varkappa \bar{u}_\lambda$; $\varkappa, \lambda = 0, 1, \ldots, n$; $\Sigma |u_\varkappa|^2 = 1$, where

$$c_{\varkappa\lambda} = \sum_{j=0}^{n} a_{j\varkappa} \bar{a}_{j\lambda}. \tag{2.2}$$

The proof is immediate. Indeed writing $\eta_\varkappa = \sum_{\lambda=0}^{n} a_{\varkappa\lambda} y_\lambda$, the maximum of $|\sum_{\varkappa=0}^{n} \eta_\varkappa x_\varkappa|^2$ as the y_λ are fixed and the $x_\varkappa$ change, will be $\sum_{\varkappa=0}^{n} |\eta_\varkappa|^2 = \sum_{\varkappa=0}^{n}\sum_{\lambda=0}^{n} c_{\varkappa\lambda} y_\varkappa \bar{y}_\lambda$.

We shall deal first with Problem (*a*), then with Problem (*b*).

2. Problem (*a*). - According to a theorem of L. FEJÉR-F. RIESZ [6, p. 20] the positive trigonometric polynomial $[w(x)]^{-1}$ can be written as follows:

$$[w(x)]^{-1} = |h(z)|^2 = |h^*(z)|^2, \qquad z = e^{ix}, \tag{2.3}$$

where

$$h(z) = h_0 \prod_{\nu=1}^{q} (z - \alpha_\nu), \quad h_0 > 0, \quad 0 < |\alpha_\nu| < 1. \tag{2.4}$$

The rational polynomial $h(z)$ satisfying all the conditions implied by (2.3) and (2.4), is uniquely determined; $h^*(z)$ represents, as always, the reciprocal polynomial of $h(z)$, $h^*(z) = z^q \bar{h}(z^{-1}) = h_0 \prod_{\nu=1}^{q} (1 - \bar{\alpha}_\nu z)$. (Thus in the present case $D(z) = [h^*(z)]^{-1}$ is the analytic function associated with the weight function $w(x)$ in the same way as in the general case of § 4.) For the sake of simplicity we assume that *the zeros of the polynomial $h(z)$ are all distinct.*

We denote by $\{ \varphi_m(z) \}$, $m = 0, 1, 2, \ldots$, the orthonormal polynomials associated with the weight function $w(x)$ on the unit circle $z = e^{ix}$ [6, chapter 2, p. 37]:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi_\varkappa(z) \overline{\varphi_\lambda(z)} w(x) dx = \delta_{\varkappa\lambda}. \tag{2.5}$$

In the present case we have, as easily shown [loc. cit. p. 43], $\varphi_m(z) = z^{m-q} h(z)$, $m \geqq q$. We note also that [loc. cit. p. 41, (1)]

$$\sum_{m=0}^{q-1} \overline{\varphi_m(a)} \varphi_m(z) = \frac{\overline{h^*(a)} h^*(z) - \overline{h(a)} h(z)}{1 - \bar{a} z}. \tag{2.6}$$

Now

$$\rho = \sup \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{ix}) \overline{P_1(e^{ix})} \cdot w(x) dx \right| \tag{2.7}$$

where $F(z)$ and $\overline{P_1(z)}$ are rational polynomials in z, the first vanishing at $z = 0$; the following conditions are to be satisfied:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |F(z)|^2 w(x) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} |P_1(z)|^2 w(x) dx = 1, \; z = e^{ix}. \tag{2.8}$$

We write $F(z) = z f(z)$, $\overline{P_1(z)} = g(z)$ where $f(z)$ and $g(z)$ are polynomials and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(z)|^2 w(x) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(z)|^2 w(x) dx = 1, \; z = e^{ix}. \tag{2.9}$$

The expansions of $f(z)$ and $g(z)$ in terms of the orthogonal polynomials suggest the following decompositions:

$$f(z) = f_0(z) + h(z)f_1(z), \qquad g(z) = g_0(z) + h(z)g_1(z) \tag{2.10}$$

where

$$f_0(z) = \sum_{\varkappa=0}^{q-1} x_\varkappa \varphi_\varkappa(z), \qquad g_0(z) = \sum_{\lambda=0}^{q-1} y_\lambda \varphi_\lambda(z), \tag{2.11}$$

and $f_1(z)$, $g_1(z)$ are arbitrary polynomials. Indeed, $f(z)$ being given, $f_0(z)$ is identical with the uniquely determined interpolation polynomial of degree $q-1$ coinciding with $f(z)$ at the points α_ν; $\nu = 1, 2, \ldots, q$; similarly $g_0(z)$. We have, $z = e^{ix}$,

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} F(z)\overline{P_1(z)}w(x)dx &= \frac{1}{2\pi i}\int_{|z|=1} f(z)g(z)\frac{z^q dz}{h(z)h^*(z)} = \\ &= \frac{1}{2\pi i}\int_{|z|=1} f_0(z)g_0(z)\frac{z^q dz}{h(z)h^*(z)} = \sum_{\varkappa,\lambda=0}^{q-1} a_{\varkappa\lambda}x_\varkappa y_\lambda\end{aligned} \tag{2.12}$$

where

$$a_{\varkappa\lambda} = \frac{1}{2\pi i}\int_{|z|=1} \varphi_\varkappa(z)\varphi_\lambda(z)\frac{z^q dz}{h(z)h^*(z)}. \tag{2.13}$$

Also, $z = e^{ix}$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f_0(z)} \cdot h(z)f_1(z)w(x)dx = \frac{1}{2\pi i}\int_{|z|=1} \overline{f_0(z)}z^{q-1} \cdot \frac{f_1(z)\,dz}{h^*(z)} = 0$$

since $\overline{f_0(z)}z^{q-1}$ is a polynomial in z. From this we conclude that

$$\begin{aligned}1 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f_0(z)|^2 w(x)dx + \frac{1}{2\pi}\int_{-\pi}^{\pi} |h(z)f_1(z)|^2 w(x)dx \\ &\geq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f_0(z)|^2 w(x)dx = \sum_{\varkappa=0}^{q-1} |x_\varkappa|^2.\end{aligned}$$

A similar remark holds for the components of $g(z)$. Hence the quantity ρ will be the maximum of the modulus of (2.12) provided that $\Sigma|x_\varkappa|^2 = \Sigma|y_\lambda|^2 = 1$.

3. We form

$$\sum_{\varkappa,\lambda=0}^{q-1} c_{\varkappa\lambda} u_\varkappa \bar{u}_\lambda = \sum_{m=0}^{q-1} \left| \sum_{j=0}^{q-1} a_{mj} u_j \right|^2$$

where

$$\sum_{j=0}^{q-1} a_{mj} u_j = \frac{1}{2\pi i} \int_{|z|=1} \varphi_m(z) u(z) \frac{z^q dz}{h(z)h^*(z)}$$
$$= \sum_{\nu=1}^{q} \varphi_m(\alpha_\nu) u(\alpha_\nu) \frac{\alpha_\nu^q}{h'(\alpha_\nu)h^*(\alpha_\nu)};$$

$$u(z) = \sum_{j=0}^{q-1} u_j \varphi_j(z),$$

so that, using (2.6), [3]

$$\sum_{\varkappa,\lambda=0}^{q-1} c_{\varkappa\lambda} u_\varkappa \bar{u}_\lambda = \sum_{\nu,\mu=1}^{q} \frac{u(\alpha_\nu)\alpha_\nu^q}{h'(\alpha_\nu)h^*(\alpha_\nu)} \cdot \overline{\frac{u(\alpha_\mu)\alpha_\mu^q}{h'(\alpha_\mu)h^*(\alpha_\mu)}} \cdot \sum_{m=0}^{q-1} \varphi_m(\alpha_\nu)\overline{\varphi_m(\alpha_\mu)}$$
$$= \sum_{\nu,\mu=1}^{q} \frac{u(\alpha_\nu)\alpha_\nu^q}{h'(\alpha_\nu)h^*(\alpha_\nu)} \cdot \overline{\frac{u(\alpha_\mu)\alpha_\mu^q}{h'(\alpha_\mu)h^*(\alpha_\mu)}} \cdot \frac{h^*(\alpha_\nu)\overline{h^*(\alpha_\mu)}}{1-\alpha_\nu\bar{\alpha}_\mu}$$
$$= \sum_{\nu,\mu=1}^{q} \frac{u(\alpha_\nu)\alpha_\nu^q}{h'(\alpha_\nu)} \cdot \overline{\frac{u(\alpha_\mu)\alpha_\mu^q}{h'(\alpha_\mu)}} \cdot \frac{1}{1-\alpha_\nu\bar{\alpha}_\mu}.$$

Moreover

$$u(z) = \sum_{\nu=1}^{q} \frac{u(\alpha_\nu)}{h'(\alpha_\nu)} \frac{h(z)}{z-\alpha_\nu},$$

hence, $z = e^{ix}$,

$$1 = \sum_{\varkappa=0}^{q-1} |u_\varkappa|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|\frac{u(z)}{h(z)}\right|^2 dx = \sum_{\nu,\mu=1}^{q} \frac{u(\alpha_\nu)}{h'(\alpha_\nu)} \overline{\frac{u(\alpha_\mu)}{h'(\alpha_\mu)}} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{z-\alpha_\nu} \frac{1}{\bar{z}-\bar{\alpha}_\mu} dx$$
$$= \sum_{\nu,\mu=1}^{q} \frac{u(\alpha_\nu)}{h'(\alpha_\nu)} \cdot \overline{\frac{u(\alpha_\mu)}{h'(\alpha_\mu)}} \cdot \frac{1}{1-\alpha_\nu\bar{\alpha}_\mu},$$

in view of

$$\frac{1}{2\pi i} \int_{|z|=1} \frac{1}{z-\alpha_\nu} \frac{dz}{1-\bar{\alpha}_\mu z} = \frac{1}{1-\alpha_\nu\bar{\alpha}_\mu}.$$

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(3) In this and other similar formulas the bar of conjugation applies to all factors in both numerators and denominators.

Writing $\dfrac{u(\alpha_\nu)}{h'(\alpha_\nu)} = A_\nu$ we see that ρ^2 is the maximum of

$$\sum_{\nu,\mu=1}^{q} \frac{(\alpha_\nu \bar{\alpha}_\mu)^q}{1 - \alpha_\nu \bar{\alpha}_\mu} A_\nu \bar{A}_\mu$$

under the condition

$$1 = \sum_{\nu,\mu=1}^{q} \frac{A_\nu \bar{A}_\mu}{1 - \alpha_\nu \bar{\alpha}_\mu},$$

i. e. [cf. for instance 6, p. 32], ρ^2 *is the largest root of the determinantal equation*

$$\left[\frac{(\alpha_\nu \bar{\alpha}_\mu)^q - \rho^2}{1 - \alpha_\nu \bar{\alpha}_\mu}\right]_1^q = 0, \tag{2.14}$$

or, writing $\rho^2 = 1 - \gamma$, γ *is the smallest root of the determinantal equation*

$$\left[\frac{1 - (\alpha_\nu \bar{\alpha}_\mu)^q}{1 - \alpha_\nu \bar{\alpha}_\mu} - \frac{\gamma}{1 - \alpha_\nu \bar{\alpha}_\mu}\right]_1^q = 0. \tag{2.14'}$$

This reasoning needs only a slight modification if $F(z)$ runs over the polynomials containing all powers $\geqq a$, and $\overline{P_1(z)}$ over the polynomials containing all powers $\geq b$ where a and b are given non-negative integers. The equation determining $\gamma = 1 - \rho^2$ is in this case, instead of (2.14'):

$$\left[\frac{1 - (\alpha_\nu \bar{\alpha}_\mu)^{q+a+b-1}}{1 - \alpha_\nu \bar{\alpha}_\mu} - \frac{\gamma}{1 - \alpha_\nu \bar{\alpha}_\mu}\right]_1^q = 0. \tag{2.15}$$

In the case above $a = 1$, $b = 0$. For $a = b = 0$ we have $\gamma = 0$, $\rho = 1$, see (1.8), in all other cases $\rho < 1$.

4. Problem (*b*). With the previous meaning of $h(z)$ and $h^*(z)$ we have in the present case: $w(x) = |h(z)|^2 = |h^*(z)|^2$, $z = e^{ix}$; $D(z) = h^*(z)$. Writing again $F(z) = zf(z)$, $\overline{P_1(z)} = g(z)$, the functional in question will be now

$$\frac{1}{2\pi i} \int_{|z|=1} f(z) g(z) \frac{h(z) h^*(z)}{z^q}\, dz \tag{2.16}$$

and ρ is the supremum of the modulus of (2.16) under the condition

$$(2.17) \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(z)h^*(z)|^2 dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} |g(z)h^*(z)|^2 dx = 1, \qquad z = e^{ix}.$$

In the present case we shall not use orthogonal polynomials; it is convenient to set

$$(2.18) \qquad f(z)h^*(z) = f_0(z) + z^q f_1(z), \qquad g(z)h^*(z) = g_0(z) + z^q g_1(z)$$

where $f_0(z)$, $g_0(z)$ are of degree $q-1$ and $f_1(z)$, $g_1(z)$ are arbitrary. Hence (2.16) becomes

$$(2.19) \qquad \frac{1}{2\pi i}\int_{|z|=1} \frac{f_0(z)g_0(z)}{(h^*(z))^2}\,\frac{h(z)h^*(z)}{z^q}\,dz = \frac{1}{2\pi i}\int_{|z|=1} f_0(z)g_0(z)\frac{h(z)}{h^*(z)}\,\frac{dz}{z^q}.$$

Also

$$(2.20) \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f_0(z)} z^q f_1(z)\,dx = \frac{1}{2\pi i}\int_{|z|=1} \overline{f_0(z)} z^{q-1} f_1(z)\,dz = 0.$$

Thus, $z = e^{ix}$,

$$(2.21) \qquad 1 \geqq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f_0(z)|^2 dx, \qquad 1 \geqq \frac{1}{2\pi}\int_{-\pi}^{\pi} |g_0(z)|^2 dx.$$

Consequently the problem is reduced to the evaluation of the maximum of the modulus of (2.19) under the condition (2.21).

5. Writing

$$f_0(z) = \sum_{\varkappa=0}^{q-1} x_\varkappa z^\varkappa, \qquad g_0(z) = \sum_{\lambda=0}^{q-1} y_\lambda z^\lambda$$

we have

$$(2.22) \qquad \rho = \max\left|\sum_{\varkappa,\lambda=0}^{q-1} t_{\varkappa+\lambda} x_\varkappa y_\lambda\right|, \quad t_m = \frac{1}{2\pi i}\int_{|z|=1} z^{m-q}\frac{h(z)}{h^*(z)}\,dz,$$

under the condition $\Sigma\,|x_\varkappa|^2 = \Sigma\,|y_\lambda|^2 = 1$. All t_m are zero except possibly those with $m \leqq q-1$.

We introduce now the expansion

$$\frac{h(z)}{h^*(z)} = h_0 + h_1 z + h_2 z^2 + \dots \tag{2.23}$$

so that $t_m = h_{q-1-m}$. Hence

$$\sum_{\varkappa,\lambda=0}^{q-1} c_{\varkappa\lambda} u_\varkappa \bar{u}_\lambda = \sum_{m=0}^{q-1} \left| \sum_{\varkappa=0}^{q-1} t_{m+\varkappa} u_\varkappa \right|^2 \tag{2.24}$$

$$= \sum_{m=0}^{q-1} \left| \sum_{\varkappa=0}^{q-1-m} h_{q-1-m-\varkappa} u_\varkappa \right|^2,$$

$$c_{\varkappa\lambda} = \Sigma h_{q-1-m-\varkappa} \bar{h}_{q-1-m-\lambda}, \qquad 0 \leqq m \leqq \min(q-1-\varkappa,\ q-1-\lambda),$$

$$d_{\varkappa\lambda} = c_{q-1-\varkappa,\ q-1-\lambda} = \Sigma\, h_{\varkappa-m} \bar{h}_{\lambda-m}, \qquad 0 \leqq m \leqq \min(\varkappa, \lambda). \tag{2.25}$$

Consequently, ρ^2 *is the largest root of the determinantal equation*

$$[d_{\varkappa\lambda} - \rho^2 \delta_{\varkappa\lambda}]_0^{q-1} = 0. \tag{2.26}$$

Remarks: (1) The same result can be concluded from the relation (4.9) to be proved, in view of the fact that in the present case

$$e^{-i\varphi} = D^{-2} w = (h^*(z))^{-2} \,|\, h(z) \,|^2 = \frac{h(z)}{z^q h^*(z)}, \quad z = e^{ix}.$$

Hence

$$\rho = \inf \left\| \frac{h(z)}{h^*(z)} - z^q A(z) \right\|_\infty, \quad A \in \boldsymbol{H}^\infty,$$

or $\rho = \inf \| B(z) \|_\infty$ where $B \in \boldsymbol{H}^\infty$ and the expansion of $B(z)$ begins with the terms $h_0 + h_1 z + \dots + h_{q-1} z^{q-1}$. This is a well-known extremem problem [cf. 6, pp. 158-159] the solution of which is exactly that given above.

(2) As in section 3 we may assume that $F(z)$ ranges over the polynomials containing only powers $\geqq a$, $\overline{P_1(z)}$ over the polynomials containing only powers $\geqq b$ where a, b are non-negative integers. The modified result is then

$$[d'_{\varkappa\lambda} - \rho^2 \delta_{\varkappa\lambda}]_0^{q-1} = 0 \tag{2.27}$$

where

$$d'_{\varkappa\lambda} = \Sigma\ h_{\varkappa-m-\nu} \bar{h}_{\lambda-m-\nu}, \qquad \nu = a+b-1; \quad 0 \leqq m \leqq \min(\varkappa-\nu,\ \lambda-\nu). \tag{2.28}$$

(If $\nu = -1$, we must have $m \leqq q - 1$). In the case $a = b = 0$, $\nu = -1$, we have $\rho = 1$, see (1.8). If $\nu \geqq q$, we have $\rho = 0$. For $\nu = q - 1$, we have $d'_{\varkappa\lambda} = 0$ except for $d'_{q-1,q-1} = |h_0|^2$ so that $\rho = |h_0|$. For $\nu = q - 2$, ρ can be computed explicitly from a quadratic equation.

§ 3. - Distribution (c).

1. In this section we deal with Problem (c), i.e. with the case $w(x) = w_1(x)/w_2(x)$ where $w_1(x)$ and $w_2(x)$ are positive trigonometric polynomials of the precise degree p and q, respectively. We use different methods according as $p \leqq q$ or $p > q$. The results generalize those of § 2.

We write

$$(3.1) \qquad w_1(x) = |a(z)|^2, \qquad w_2(x) = |b(z)|^2, \qquad z = e^{ix},$$

where $a(z)$ and $b(z)$ are rational polynomials of the precise degree p and q, respectively, whose zeros are all in $0 < |z| < 1$, and in both polynomials the coefficient of the highest power of z is positive. For the sake of simplicity we assume again that *the zeros of the polynomial* $b(z)$, say $\beta_1, \ldots, \beta_q$, *are all distinct*. Finally we assume that $a(z)$ and $b(z)$ have no zeros in common.

2. The case $p \leqq q$. We have then

$$(3.2) \qquad \rho = \sup \left| \frac{1}{2\pi i} \int_{|z|=1} f(z) g(z) \frac{a(z) a^*(z)}{b(z) b^*(z)} z^{q-p} dz \right|,$$

where $f(z)$ and $g(z)$ are polynomials satisfying the conditions

$$(3.3) \qquad \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f(z) \frac{a^*(z)}{b(z)} \right|^2 dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| g(z) \frac{a^*(z)}{b(z)} \right|^2 dx = 1, \; z = e^{ix}.$$

In this case as a generalization of (2.10) we set

$$(3.4) \qquad f(z) a^*(z) = f_0(z) + b(z) f_1(z), \; g(z) a^*(z) = g_0(z) + b(z) g_1(z)$$

where $f_0(z)$, $g_0(z)$ are of degree $q - 1$, and $f_1(z)$, $g_1(z)$ are arbitrary. Thus

$$(3.5) \qquad \rho = \sup \left| \frac{1}{2\pi i} \int_{|z|=1} f_0(z) g_0(z) \frac{a(z) z^{q-p}}{b(z) a^*(z) b^*(z)} dz \right|.$$

Now, $z = e^{ix}$,

$$(3.6)\qquad \frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{\frac{f_0(z)}{b(z)}}\, f_1(z)dx = \frac{1}{2\pi i}\int_{|z|=1}\overline{f_0(z)}z^{q-1}\frac{f_1(z)}{b^*(z)}\,dz = 0$$

so that

$$(3.7)\qquad 1 \geqq \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{f_0(z)}{b(z)}\right|^2 dx,\quad 1 \geqq \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{g_0(z)}{b(z)}\right|^2 dx,\quad z = e^{ix}.$$

In order to compute (3.5) under the condition (3.7), we write again $f_0(z)$ and $g_0(z)$ in the form (2.11) where the system $\{\varphi_m(z)\}$ is associated with $w_2(x) = |b(z)|^2$, $z = e^{ix}$. We proceed as in § 2, sections 2 and 3:

$$a_{\varkappa\lambda} = \frac{1}{2\pi i}\int_{|z|=1}\varphi_\varkappa(z)\varphi_\lambda(z)\frac{a(z)z^{q-p}}{b(z)a^*(z)b^*(z)}\,dz,$$

$$\sum_{j=0}^{q-1} a_{mj}u_j = \frac{1}{2\pi i}\int_{|z|=1}\varphi_m(z)u(z)\frac{a(z)z^{q-p}}{b(z)a^*(z)b^*(z)}\,dz$$

$$= \sum_{\nu=1}^{q}\varphi_m(\beta_\nu)u(\beta_\nu)\frac{a(\beta_\nu)\beta_\nu^{q-p}}{b'(\beta_\nu)a^*(\beta_\nu)b^*(\beta_\nu)},$$

$$\sum_{\varkappa,\lambda=0}^{q-1} c_{\varkappa\lambda}u_\varkappa\bar{u}_\lambda = \sum_{\nu,\mu=1}^{q}\frac{u(\beta_\nu)a(\beta_\nu)\beta_\nu^{q-p}}{b'(\beta_\nu)a^*(\beta_\nu)b^*(\beta_\nu)}\cdot\overline{\frac{u(\beta_\mu)a(\beta_\mu)\beta_\mu^{q-p}}{b'(\beta_\mu)a^*(\beta_\mu)b^*(\beta_\mu)}}\cdot\frac{b^*(\beta_\nu)\overline{b^*(\beta_\mu)}}{1-\beta_\nu\bar{\beta}_\mu}$$

$$= \sum_{\nu,\mu=1}^{q}\frac{u(\beta_\nu)a(\beta_\nu)\beta_\nu^{q-p}}{b'(\beta_\nu)a^*(\beta_\nu)}\cdot\overline{\frac{u(\beta_\mu)a(\beta_\mu)\beta_\mu^{q-p}}{b'(\beta_\mu)a^*(\beta_\mu)}}\cdot\frac{1}{1-\beta_\nu\bar{\beta}_\mu}$$

$$= \sum_{\nu,\mu=1}^{q}\frac{a(\beta_\nu)\beta_\nu^{q-p}}{a^*(\beta_\nu)}\cdot\overline{\frac{a(\beta_\mu)\beta_\mu^{q-p}}{a^*(\beta_\mu)}}\cdot\frac{A_\nu A_\mu}{1-\beta_\nu\bar{\beta}_\mu}$$

where we write again

$$\frac{u(\beta_\nu)}{b'(\beta_\nu)} = A_\nu,\qquad 1 \geqq \sum_{\nu,\mu=1}^{q}\frac{A_\nu\bar{A}_\mu}{1-\alpha_\nu\bar{\alpha}_\mu}.$$

Hence, ρ^2 *is the largest root of the determinantal equation*

(3.8) $$\left[\frac{\gamma_\nu\bar{\gamma}_\mu(\beta_\nu\bar{\beta}_\mu)^{q-p}-\rho^2}{1-\beta_\nu\bar{\beta}_\mu}\right]_1^q=0,\qquad \gamma_\nu=\frac{a(\beta_\nu)}{a^*(\beta_\nu)}.$$

This reduces to (2.14) when $p=0$.

3. The case $p>q$. We set $p-q=\sigma>0$. We denote again the (distinct) zeros of $b(z)$ by $\beta_1, \beta_2, \ldots, \beta_q$, the orthonormal polynomials associated with $|b(z)|^2$, $z=e^{ix}$, by $\{\varphi_m(z)\}$.

We intend to evaluate

(3.9) $$\rho=\sup\left|\frac{1}{2\pi i}\int_{|z|=1} f(z)g(z)\frac{a(z)a^*(z)z^{-\sigma}}{b(z)b^*(z)}\,dz\right|,$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f(z)\frac{a^*(z)}{b(z)}\right|^2dx=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|g(z)\frac{a^*(z)}{b(z)}\right|^2dx=1,\ z=e^{ix},$$

$f(z)$, $g(z)$ polynomials. For every polynomial $f(z)$ we have the decomposition

(3.10) $$f(z)a^*(z)=f_0(z)+z^\sigma b(z)f_1(z)$$

where $f_0(z)$, $f_1(z)$ are polynomials, the first of degree $p-1$. Indeed, $f_0(z)$ is the unique LAGRANGE-HERMITE interpolation polynomial coinciding with the left-hand expression at the zeros of $z^\sigma b(z)$; for $z=0$ the usual convention, involving the derivatives up to the order $\sigma-1$, applies. A similar decomposition holds for $g(z)$. Now

$$\int_{-\pi}^{\pi}\overline{\frac{f_0(z)}{b(z)}}z^\sigma f_1(z)dx=\int_{|z|=1}\frac{z^{p-1}\overline{f_0(z)}}{b^*(z)}\cdot f_1(z)dz=0$$

so that we may write:

(3.11) $$\rho=\max\left|\frac{1}{2\pi i}\int_{|z|=1} f_0(z)g_0(z)\frac{a(z)z^{-\sigma}}{b(z)a^*(z)b^*(z)}\,dz\right|,$$

(3.12) $$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{f_0(z)}{z^\sigma b(z)}\right|^2dx=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{g_0(z)}{z^\sigma b(z)}\right|^2dx=1,\ z=e^{ix}.$$

We set as before

$$(3.13)\qquad a_{\varkappa\lambda}=\frac{1}{2\pi i}\int\limits_{|z|=1}\varphi_\varkappa(z)\varphi_\lambda(z)\frac{a(z)z^{-\sigma}}{b(z)a^*(z)b^*(z)}dz\,;\ \varkappa,\lambda=0,\,1,\,\ldots,\,p-1,$$

$$u(z)=\sum_{j=0}^{p-1}u_j\varphi_j(z),\qquad \sum_{j=0}^{p-1}|u_j|^2=1,$$

so that

$$\sum_{j=0}^{p-1}a_{mj}u_j=\frac{1}{2\pi i}\int\limits_{|z|=1}\varphi_m(z)u(z)\frac{a(z)z^{-\sigma}}{b(z)a^*(z)b^*(z)}dz$$

$$(3.14)\qquad =\sum_{\nu=1}^{q}\varphi_m(\beta_\nu)\frac{u(\beta_\nu)}{b'(\beta_\nu)}\left(\frac{a(z)z^{-\sigma}}{a^*(z)b^*(z)}\right)_{z=\beta_\nu}+$$

$$+\frac{1}{2\pi i}\int\limits_{|\zeta|=t}\varphi_m(\zeta)u(\zeta)\frac{a(\zeta)\zeta^{-\sigma}}{b(\zeta)a^*(\zeta)b^*(\zeta)}d\zeta$$

where t is positive and $t<\min|\beta_\nu|$.

4. To the polynomial $u(z)$ of degree $p-1$ we apply the LAGRANGE-HERMITE interpolation formula in the convenient form resulting from the fact that for $0<|z|<1$, $z\neq\beta_\nu$, $0<t<\min(|\beta_\nu|,|z|)$,

$$(3.15)\qquad 0=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{u(\zeta)d\zeta}{(\zeta-z)\zeta^\sigma b(\zeta)}=\frac{u(z)}{z^\sigma b(z)}+$$

$$+\sum_{\nu=1}^{q}\frac{u(\beta_\nu)}{(\beta_\nu-z)\beta_\nu^\sigma b'(\beta_\nu)}+\frac{1}{2\pi i}\int\limits_{|\zeta|=t}\frac{u(\zeta)d\zeta}{(\zeta-z)\zeta^\sigma b(\zeta)};$$

indeed, the left-hand integrand is regular for $|\zeta|\geqq 1$ and vanishes as ζ^{-2} for $\zeta\to\infty$ since the numerator is of degree $p-1$ and the denominator of degree $1+\sigma+q=p+1$. Thus

$$(3.16)\qquad \frac{u(z)}{z^\sigma b(z)}=\sum_{\nu=1}^{q}\frac{u(\beta_\nu)}{\beta_\nu^\sigma b'(\beta_\nu)}\frac{1}{z-\beta_\nu}+\frac{1}{2\pi i}\int\limits_{|\zeta|=t}\frac{u(\zeta)d\zeta}{(z-\zeta)\zeta^\sigma b(\zeta)}.$$

In the now familiar way we conclude that ρ^2 is the largest eigenvalue of a certain Hermitian form $P-\rho^2Q$ where P and Q are obtained just as before, with the aid of the expressions (3.14) and (3.16). As a convenient choice of the variables in the Hermitian forms we set

$$(3.17)\qquad \frac{u(\beta_\nu)}{\beta_\nu^\sigma b'(\beta_\nu)} = A_\nu, \qquad \nu = 1, 2, \ldots, q,$$

$$\frac{u(\zeta)}{\zeta^\sigma b(\zeta)} = B_0\zeta^{-\sigma} + B_1\zeta^{-\sigma+1} + \ldots + B_{\sigma-1}\zeta^{-1} + \ldots = B(\zeta) + \ldots$$

where the terms not written out contain non-negative powers of ζ. The variables $B_0, B_1, \ldots, B_{\sigma-1}$ are certain linear combinations of the derivatives of $u(\zeta)$, $\zeta = 0$, up to the order $\sigma - 1$. (We may also use the alternate notation $A_{q+\varkappa+1} = B_\varkappa$, $\varkappa = 0, 1, \ldots, \sigma - 1$.) Hence

$$(3.18)\qquad \sum_{j=0}^{p-1} a_{mj}u_j = \sum_{\nu=1}^{q} A_\nu \varphi_m(\beta_\nu)\left(\frac{a(z)}{a^*(z)b^*(z)}\right)_{z=\beta_\nu} + \\ + \frac{1}{2\pi i}\int_{|\zeta|=t} \varphi_m(\zeta)B(\zeta)\,\frac{a(\zeta)}{a^*(\zeta)b^*(\zeta)}\,d\zeta,$$

$$(3.19)\qquad \frac{u(z)}{z^\sigma b(z)} = \sum_{\nu=1}^{q}\frac{A_\nu}{z-\beta_\nu} + \frac{1}{2\pi i}\int_{|\zeta|=t}\frac{B(\zeta)}{z-\zeta}\,d\zeta, \qquad |t| < |z|.$$

5. Now, as in the above cases,

$$P = \sum_{m=0}^{p-1}\sum_{\nu,\mu=1}^{q} A_\nu\bar{A}_\mu\varphi_m(\beta_\nu)\overline{\varphi_m(\beta_\mu)}\left(\frac{a(z)}{a^*(z)b^*(z)}\right)_{z=\beta_\nu}\overline{\left(\frac{a(z)}{a^*(z)b^*(z)}\right)}_{z=\beta_\mu}$$

$$+ 2\Re\sum_{m=0}^{p-1}\sum_{\mu=1}^{q}\bar{A}_\mu\overline{\varphi_m(\beta_\mu)}\overline{\left(\frac{a(z)}{a^*(z)b^*(z)}\right)}_{z=\beta_\mu}\cdot\frac{1}{2\pi i}\int_{|\zeta|=t}\varphi_m(\zeta)B(\zeta)\,\frac{a(\zeta)}{a^*(\zeta)b^*(\zeta)}\,d\zeta$$

$$+ \sum_{m=0}^{p-1}\left|\frac{1}{2\pi i}\right|^2\int_{|\zeta_1|=t}\int_{|\zeta_2|=t}\varphi_m(\zeta_1)\overline{\varphi_m(\zeta_2)}B(\zeta_1)\overline{B(\zeta_2)}\frac{a(\zeta_1)}{a^*(\zeta_1)b^*(\zeta_1)}\,\overline{\frac{a(\zeta_2)}{a^*(\zeta_2)b^*(\zeta_2)}}\,d\zeta_1 d\bar{\zeta}_2,$$

$$1 = Q = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{u(z)}{z^\sigma b(z)}\right|^2 dx = \sum_{\nu,\mu=1}^{q}\frac{A_\nu\bar{A}_\mu}{1-\beta_\nu\bar{\beta}_\mu} +$$

$$+ 2\Re\sum_{\mu=1}^{q}\bar{A}_\mu(B_0\bar{\beta}_\mu^{\sigma-1} + B_1\bar{\beta}_\mu^{\sigma-2} + \ldots + B_{\sigma-1}) +$$

$$+ |B_0|^2 + |B_1|^2 + \ldots + |B_{\sigma-1}|^2$$

where $z = e^{ix}$. Indeed, $\nu, \mu = 1, 2, \ldots, q$; $\varkappa, \lambda = 0, 1, \ldots, \sigma - 1$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{dx}{(z-\beta_\nu)(\bar{z}-\bar{\beta}_\mu)} = \frac{1}{2\pi i}\int_{|z|=1} \frac{dz}{(z-\beta_\nu)(1-\bar{\beta}_\mu z)} = \frac{1}{1-\beta_\nu\bar{\beta}_\mu},$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{dx}{(\bar{z}-\bar{\beta}_\mu)(z-\zeta)} = \frac{1}{2\pi i}\int_{|z|=1} \frac{dz}{(1-\bar{\beta}_\mu z)(z-\zeta)} = \frac{1}{1-\bar{\beta}_\mu\zeta},$$

$$\frac{1}{2\pi i}\int_{|\zeta|=t} \frac{\zeta^{-\sigma+\varkappa}d\zeta}{1-\bar{\beta}_\mu\zeta} = \bar{\beta}_\mu^{\sigma-1-\varkappa},$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{dx}{(z-\zeta_1)(\bar{z}-\bar{\zeta}_2)} = \frac{1}{2\pi i}\int_{|z|=1} \frac{dz}{(z-\zeta_1)(1-\bar{\zeta}_2 z)} = \frac{1}{1-\zeta_1\bar{\zeta}_2},$$

$$\left|\frac{1}{2\pi i}\right|^2 \int_{|\zeta_1|=t}\int_{|\zeta_2|=t} \frac{\zeta_1^{-\sigma+\varkappa}\bar{\zeta}_2^{-\sigma+\lambda}}{1-\zeta_1\bar{\zeta}_2}\, d\zeta_1 \overline{d\zeta_2} = \delta_{\varkappa\lambda}.$$

6. In order to simplify P we note that $\varphi_p(z) = z^\sigma b(z)$, $\varphi_p^*(z) = b^*(z)$ so that (cf. (2.6))

$$\sum_{m=0}^{p-1} \varphi_m(\beta_\nu)\overline{\varphi_m(\beta_\mu)} = \frac{b^*(\beta_\nu)\overline{b^*(\beta_\mu)}}{1-\beta_\nu\bar{\beta}_\mu},$$

$$\sum_{m=0}^{p-1} \overline{\varphi_m(\beta_\mu)}\varphi_m(\zeta) = \frac{\overline{b^*(\beta_\mu)}b^*(\zeta)}{1-\bar{\beta}_\mu\zeta},$$

$$\sum_{m=0}^{p-1} \varphi_m(\zeta_1)\overline{\varphi_m(\zeta_2)} = \frac{b^*(\zeta_1)\overline{b^*(\zeta_2)} - (\zeta_1\bar{\zeta}_2)^\sigma b(\zeta_1)\overline{b(\zeta_2)}}{1-\zeta_1\bar{\zeta}_2}.$$

Hence

$$P = \sum_{\nu,\mu=1}^{q} \frac{A_\nu\bar{A}_\mu}{1-\beta_\nu\bar{\beta}_\mu}\,\frac{a(\beta_\nu)}{a^*(\beta_\nu)}\,\frac{\overline{a(\beta_\mu)}}{\overline{a^*(\beta_\mu)}}$$

$$+ 2\Re \sum_{\mu=1}^{q} \bar{A}_\mu \frac{\overline{a(\beta_\mu)}}{\overline{a^*(\beta_\mu)}} \cdot \frac{1}{2\pi i}\int_{|\zeta|=t} B(\zeta)\frac{a(\zeta)}{a^*(\zeta)}\,\frac{d\zeta}{1-\bar{\beta}_\mu\zeta}$$

$$+ \left|\frac{1}{2\pi i}\right|^2 \int_{|\zeta_1|=t}\int_{|\zeta_2|=t} B(\zeta_1)\overline{B(\zeta_2)}\,\frac{a(\zeta_1)}{a^*(\zeta_1)}\,\frac{\overline{a(\zeta_2)}}{\overline{a^*(\zeta_2)}}\,\frac{d\zeta_1\overline{d\zeta_2}}{1-\zeta_1\bar{\zeta}_2}$$

since $\zeta^\sigma B(\zeta)$ is regular at $\zeta = 0$. Thus ρ^2 *is the largest root of the determinantal equation*

$$\begin{vmatrix} P_{11} - \rho^2 Q_{11} & P_{12} - \rho^2 Q_{12} \\ P_{21} - \rho^2 Q_{21} & P_{22} - \rho^2 Q_{22} \end{vmatrix}_1^p = 0. \tag{3.20}$$

Here the q-by-q matrix $(P_{11} - \rho^2 Q_{11})$ is exactly the same as the matrix corresponding to (3.8), $p = q$. Further we write

$$\frac{a(\zeta)}{a^*(\zeta)} = a_0 + a_1\zeta + a_2\zeta^2 + \dots \tag{3.21}$$

so that the σ-by-q matrix $(P_{21} - \rho^2 Q_{21}) = (P_{12} - \rho^2 Q_{12})^*$ contains the elements

$$\bar{\gamma}_\mu(a_0\bar{\beta}_\mu^{\sigma-1-\varkappa} + a_1\bar{\beta}_\mu^{\sigma-2-\varkappa} + \dots + a_{\sigma-1-\varkappa}) - \rho^2\bar{\beta}_\mu^{\sigma-1-\varkappa}, \tag{3.22}$$

where γ has the same meaning as in (3.8). Reversing the order of the rows, see (2.25), we have the elements

$$\bar{\gamma}_\mu(a_0\bar{\beta}_\mu^\varkappa + a_1\bar{\beta}_\mu^{\varkappa-1} + \dots + a_\varkappa) - \rho^2\bar{\beta}_\mu^\varkappa, \quad \varkappa = 0, 1, \dots, \sigma - 1; \; \mu = 1, 2, \dots, q. \tag{3.23}$$

Finally, the σ-by-σ matrix $(P_{22} - \rho^2 Q_{22})$ has the elements $c_{\varkappa\lambda} - \rho^2\delta_{\varkappa\lambda}$ where

$$\begin{gathered} c_{\varkappa\lambda} = \Sigma\, a_{\sigma-1-m-\varkappa}\bar{a}_{\sigma-1-m-\lambda}, \quad 0 \leqq m \leqq \min(\sigma - 1 - \varkappa,\ \sigma - 1 - \lambda), \\ c_{\sigma-1-\varkappa,\,\sigma-1-\lambda} = \Sigma\, a_{\varkappa-m}\bar{a}_{\lambda-m}, \quad 0 \leqq m \leqq \min(\varkappa, \lambda); \\ \varkappa, \lambda = 0, 1, \dots, \sigma - 1. \end{gathered} \tag{3.24}$$

This is in agreement with the result (2.25), (2.26) of Problem (*b*) as can be seen by writing $q = 0$, $p = \sigma$.

§ 4. Criterion for positive angle between $\mathfrak{F}$ and $\mathfrak{P}_1$.

1. **Theorem 1.** - *The manifolds* $\mathfrak{F}$ *and* $\mathfrak{P}_1$ *are at positive angle in* L^2_μ *if and only if* μ *is absolutely continuous* (so that $d\mu = w\,d\sigma$) *and*

$$w = e^{u+\tilde{v}} \tag{4.1}$$

where u *is a bounded real function, and* $\tilde{v}$ *is the conjugate of a real function* v *which is bounded and satisfies the condition* $\| v \|_\infty < \pi/2$.

In order to proceed directly to the main difficulty, we shall assume first that μ is absolutely continuous. Afterwards we show that $\mathfrak{F}$ and $\mathfrak{P}_1$ are at zero angle if μ has any singular part. For that we shall need a fact stated as Lemma 2, whose proof is however conveniently postponed until § 5.

It is enough to consider summable functions w for which $\log w$ is summable; for if $\int \log w d\sigma = -\infty$, the answer to the first prediction problem shows that $\mathfrak{F}$ and $\mathfrak{P}_1$ both coincide with L^2_w and so are at zero angle. We form the FOURIER series

$$\log w(x) \sim \sum_{-\infty}^{\infty} d_n e^{nix}$$

and define the analytic functions

$$(4.2)\qquad \begin{aligned} D(z) &= \exp(d_0/2 + d_1 z + d_2 z^2 + \dots), \\ H(z) &= [D(z)]^2. \end{aligned}$$

Both functions are analytic for $|z| < 1$. Obviously D is of class $\boldsymbol{H}^2$, H is of class $\boldsymbol{H}^1$, and their radial limits satisfy

$$(4.3)\qquad |D(e^{ix})|^2 = |H(e^{ix})| = w(x) \qquad \text{almost everywhere.}$$

Moreover D and H are *outer functions* [3, 10]:

$$(4.4)\qquad \int \log |D(e^{ix})| d\sigma(x) = \log D(0),$$

with a similar equation for H. Later we shall make use of the fact that (4.3) and (4.4) characterize D and H.

We require this fundamental fact about outer functions [3]: *the linear set of functions $\overline{P}_1(e^{ix})D(e^{ix})$, where P_1 ranges over all trigonometric polynomials* (1.3), *is dense in $\boldsymbol{H}^2$*. More specifically, *the set of all such functions for which*

$$\int |\overline{P}_1 D|^2 d\sigma \leqq 1$$

is dense in the unit sphere of $\boldsymbol{H}^2$.

2. We are considering the quantity ρ given by (1.6) and (1.7). If we define φ by $w = D^2 e^{-i\varphi}$, then (1.6) takes the form

$$(4.5)\qquad \rho = \sup \left| \int (FD)(\overline{P}_1 D) e^{-i\varphi} d\sigma \right|.$$

As we have asserted above, the set of $P_1 D$ appearing in (4.5) is dense in the unit sphere of $\boldsymbol{H}^2$. Let $\boldsymbol{H}_0^2$ be the subspace of $\boldsymbol{H}^2$ consisting of those functions with mean value zero. Then similarly the functions FD in (4.5) range over a dense subset of the unit sphere of $\boldsymbol{H}_0^2$. Now it is well-known [15, p. 275] that the set of products fg where f and g belong to the unit sphere of $\boldsymbol{H}^2$ exactly covers the unit sphere of $\boldsymbol{H}^1$. Therefore the products FP_1D^2 of (4.5) range over a dense subset of the unit sphere of $\boldsymbol{H}_0^1$ (the subspace of $\boldsymbol{H}^1$ consisting of the functions with mean value zero). In place of the quadratic functional (4.5) we now have the linear expression

$$\rho = \sup \left| \int Fe^{-i\varphi} d\sigma \right| \tag{4.6}$$

where F ranges over the functions with FOURIER series (1.1) such that $\int |F| d\sigma \leqq 1$; and the supremum is the same if F is restricted to be a trigonometric polynomial.

$\boldsymbol{H}_0^1$ is a closed subspace of $\boldsymbol{L}$, the BANACH space of complex functions summable on the unit circle. Evidently (4.6) expresses ρ as the norm of the linear functional in $\boldsymbol{H}_0^1$ defined on F by

$$\int Fe^{-i\varphi} d\sigma.$$

Of course this integral gives a linear functional in all of $\boldsymbol{L}$ which has norm 1, but its norm restricted to $\boldsymbol{H}_0^1$ may be smaller. Indeed, if A is any bounded function such that

$$\int FA\, d\sigma = 0 \qquad (\text{all } F \text{ in } \boldsymbol{H}_0^1), \tag{4.7}$$

then clearly $\rho \leq \| e^{-i\varphi} - A \|_\infty$. The HAHN-BANACH extension theorem implies that the norm of the functional in $\boldsymbol{H}_0^1$ is precisely the infimum of such numbers. Now A satisfies (4.7) if and only if it has FOURIER series

$$A(e^{ix}) \backsim \alpha_0 + \alpha_1 e^{ix} + \alpha_2 e^{2ix} + \ldots ; \tag{4.8}$$

that is to say, just if A belongs to $\boldsymbol{H}^\infty$. Thus we deduce from (4.6) the dual relation

$$\rho = \inf \| e^{-i\varphi} - A \|_\infty \qquad (A \in \boldsymbol{H}^\infty). \tag{4.9}$$

This passage to the dual of the given problem was used by NEHARI [11] in order to find the bounds of certain bilinear forms. In a study as yet unpublished, MARVIN ROSENBLUM has extended NEHARI' s work to several variables, using the duality principle more explicitly than NEHARI. Our application of BEURLING' s Theorem seems to be new in this context, and so is the analysis of (4.9) which follows.

3. **Lemma 1.** - *In order that $\rho < 1$ it is necessary and sufficient that there exist an $\varepsilon > 0$ and an element A of $\boldsymbol{H}^\infty$ such that almost everywhere*

$$(a) \qquad | A(e^{ix}) | \geqq \varepsilon \qquad \textit{and}$$

$$(b) \qquad | -\varphi(x) - \arg A(e^{ix}) | \leqq \frac{\pi}{2} - \varepsilon \qquad (\textit{mod } 2\pi).$$

If $\rho < 1$, we take for A any function in $\boldsymbol{H}^\infty$ such that $\| e^{-i\varphi} - A \|_\infty < 1$. It is clear that (a) must hold, and then (b) is geometrically obvious, perhaps with a smaller value of ε. Conversely, if A satisfies (a) and (b), then one can verify that $\| e^{-i\varphi} - \lambda A \|_\infty < 1$ for sufficiently small positive values of λ, and so $\rho < 1$.

4. By definition, $H(e^{ix}) = w(x)e^{i\varphi(x)}$. Therefore the conditions of the lemma can be expressed as follows:

$$(4.10) \qquad \begin{aligned} &| A(e^{ix}) | \geqq \varepsilon, \\ &| \arg (A(e^{ix})H(e^{ix})) | \leqq \frac{\pi}{2} - \varepsilon \qquad (\textit{mod } 2\pi). \end{aligned}$$

The second inequality states that $A(e^{ix})H(e^{ix})$, the boundary function of $\zeta = A(z)H(z)$ belonging to $\boldsymbol{H}^1$, assumes its values in the sector $S_\varepsilon : | \arg \zeta | \leqq \leqq \frac{\pi}{2} - \varepsilon$. But $A(z)H(z)$ is the POISSON integral of its boundary values, and so $A(z)H(z)$ lies in S_ε also for $| z | < 1$. It follows that $A(z)H(z)$ cannot vanish for $| z | < 1$ (since otherwise its values would cover a neighborhood of 0), and $\log (A(z)H(z))$ is single-valued and analytic in that circle. We can choose the argument of $A(0)H(0)$ so that the second inequality of (4.10) holds without the qualification *modulo* 2π.

For $r < 1$ and $z = re^{ix}$ we have

$$\log (A(z)H(z)) = \log | A(z)H(z) | + i \arg (A(z)H(z)).$$

Thus $\arg (A(re^{ix})H(re^{ix}))$ is a function of x which, for fixed $r < 1$, is conjugate to $\log | A(re^{ix})H(re^{ix}) |$. We have to conclude that the same relation holds for $r = 1$. This is indeed so because AH is an outer function [10, p. 469], but it can be shown directly as follows. Since $\arg (AH)$ is bounded in the unit circle, it tends to its boundary function in the metric of $\boldsymbol{L}^2$ (this is trivial); therefore the same is true for $\log | AH |$, and it follows that the boundary functions are conjugate.

So we have exhibited $\log |A(e^{ix})H(e^{ix})|$ as the conjugate of $-\arg(A(e^{ix})H(e^{ix}))$, a function with bound $\frac{\pi}{2}-\varepsilon$.

Therefore

$$w(x)=\frac{1}{|A(e^{ix})|}\cdot |A(e^{ix})H(e^{ix})| = e^{u+\tilde{v}}$$

is a representation for w of the desired form (4.1). We have inferred this representation from the existence of an element A of $\boldsymbol{H}^\infty$ satisfying conditions (a) and (b) of Lemma 1; there is such an element if $\rho<1$.

5. Conversely, suppose w has the form (4.1). Multiplying w by a factor bounded from zero and from infinity does not change the property $\rho<1$, and so we may assume $u=0$. We set

$$K(e^{ix}) = e^{\tilde{v}(x)-iv(x)};$$

then $w(x)=|K(e^{ix})|$. It is not difficult to see that K belongs to $\boldsymbol{H}^1$ and that $D(z)=\lambda[K(z)]^{1/2}$ (where λ is chosen so that $\lambda[K(0)]^{1/2}>0$) is exactly the analytic function associated with w by (4.2). It suffices to show that (4.10) holds for $H=\lambda^2K$ if A is chosen appropriately in $\boldsymbol{H}^\infty$. Obviously $A(e^{ix})\equiv\lambda^{-2}$ makes (4.10) true. Therefore $\rho<1$, and this completes the proof for absolutely continuous measures μ.

6. We still have to show that if $\rho<1$, then μ is absolutely continuous. Actually a little more is true: μ is absolutely continuous merely under the assumption that $\mathfrak{F}$ and $\mathfrak{P}$ are at positive angle (which is true *a fortiori* if $\mathfrak{F}$ and $\mathfrak{P}_1$ are at positive angle).

Lemma 2. - *$\mathfrak{F}$ and $\mathfrak{P}_1$ are at positive angle if and only if there is a constant K such that* (1.9) *holds for every real trigonometric polynomial f. $\mathfrak{F}$ and $\mathfrak{P}$ are at positive angle if and only if* (1.9) *is merely required to hold for those real trigonometric polynomials f having mean value zero.*

The proof will be given in § 5. Assuming the lemma, we show that μ is absolutely continuous if $\mathfrak{F}$ and $\mathfrak{P}$ are at positive angle. Let E be any closed subset of the circle $|z|=1$ with measure zero. We choose an arbitrary point e^{ix_0} not in E. According to a theorem of RUDIN [12], there is a function $G(z)$ analytic for $|z|<1$, continuous for $|z|\leq 1$, and equal to 1 on E but 0 at e^{ix_0}. Furthermore, G can be chosen so that $|G(e^{ix})|<1$ for every point e^{ix} where $G(e^{ix})\neq 1$. (The set where $G(e^{ix})=1$ may be larger than E, but it must have measure zero since G is not constant.) If $G(0)=a$, then $|a|<1$ by the maximum principle.

Now we define the sequence of functions

$$H_n(z) = [G(z)]^n - a^n \qquad (n = 1,\ 2, \dots).$$

Each H_n is analytic in $|z| < 1$, continuous for $|z| \leqq 1$, and vanishes at 0. If $H_n(e^{ix}) = \tilde{f}_n(e^{ix}) - if_n(e^{ix})$ with f_n, $\tilde{f}_n$ real, then f_n, $\tilde{f}_n$ are conjugate functions with mean values zero. We are assuming that $\mathfrak{F}$ and $\mathfrak{P}$ are at positive angle, so that (1.9) holds for trigonometric polynomials having mean value zero, and consequently also for uniform limits of such functions. Therefore we can find a constant K such that

$$\int \tilde{f}_n^2 d\mu \leqq K \int f_n^2 d\sigma \qquad (n = 1,\ 2, \dots). \tag{4.11}$$

From the construction of G it is clear that the functions H_n are uniformly bounded, and that for each point e^{ix}, $H_n(e^{ix})$ tends either to 1 or to 0; the first alternative holds if e^{ix} is in E. Therefore $f_n(e^{ix})$ tends to zero everywhere and boundedly, and $\tilde{f}_n(e^{ix})$ tends to 1 at least on E. Consequently the right side of (4.11) tends to zero, but the left side has upper limit at least $\mu(E)$. Hence $\mu(E) = 0$ for every closed set E with LEBESGUE measure zero, and this proves that μ is absolutely continuous.

§ 5. - Conjugate functions.

1. Let

$$\Sigma\ a_n e^{nix} \tag{5.1}$$

be an arbitrary trigonometric series. The series *conjugate* to (5.1) is by definition

$$\Sigma\ (-i\varepsilon_n) a_n e^{nix}, \qquad \varepsilon_n = sgn\ n, \tag{5.2}$$

and if (5.1) and (5.2) represent functions f and $\tilde{f}$ in some sense, then $\tilde{f}$ is said to be conjugate to f. The convention that $sgn\ 0 = 0$ implies that every conjugate function has mean value zero. If f belongs to $\boldsymbol{L}^2$ it is trivial that $\tilde{f}$ exists in $\boldsymbol{L}^2$ and

$$\int |\tilde{f}|^2 d\sigma \leqq \int |f|^2 d\sigma \tag{5.3}$$

with equality if and only if $a_0 = 0$. A well-known theorem of M. RIESZ states that (5.3) holds with exponent p for $1 < p < \infty$, provided a suitable constant factor is introduced on the right side. Indeed, it is known that the

BANACH space L^p can be replaced by an ORLICZ space of quite general type. It does not seem to be known, however, in what spaces L^p_μ the operation of passing to the conjugate is continuous, even with $p = 2$. Our contribution to this question is

Theorem 2. - *There is a constant K such that $\| \tilde{f} \| \leqq K \| f \|$ for every real trigonometric polynomial f, in the norm of L^2_μ, if and only if $\mathfrak{F}$ and $\mathfrak{P}_1$ are at positive angle in L^2_μ.*

A necessary and sufficient condition is therefore given by Theorem 1. A summary of previous work on the problem, mostly with p arbitrary, is given at the end of this section.

2. Theorem 2 is simply the statement of Lemma 2 for the manifolds $\mathfrak{F}$ and $\mathfrak{P}_1$. We proceed now to the proof of both parts of the lemma.

An operator T is defined on the set of all trigonometric polynomials by the formula

$$T \, \Sigma \, a_n e^{nix} = \sum_{n>0} a_n e^{nix}.$$

Using the characterization of positive angle given by (1.8), it is easy to verify that $\mathfrak{F}$ and $\mathfrak{P}_1$ are at positive angle in L^2_μ if and only if T is bounded in the norm of that space. Similarly, $\mathfrak{F}$ and $\mathfrak{P}$ are at positive angle just if T is bounded on the smaller set of trigonometric polynomials such that $a_0 = 0$.

Consider first the angle between $\mathfrak{F}$ and $\mathfrak{P}$. If T is bounded then

$$\| F \| \leqq K \| \mathfrak{R} F \|$$

for each trigonometric polynomial F of the form (1.1), where K is independent of F and the norm refers to the space L^2_μ.

Since trivially

$$\| \mathfrak{I} F \| \leqq \| F \|,$$

we have

$$\| \mathfrak{I} F \| \leqq K \| \mathfrak{R} F \|. \tag{5.4}$$

Setting $f = \mathfrak{R} F$, which is the general form of a real trigonometric polynomial having mean value zero, we have exactly (1.9).

Conversely, if (5.4) is true for each F of the form (1.1), then

$$\| F \|^2 = \| \mathfrak{R} F \|^2 + \| \mathfrak{I} F \|^2 \leqq (K^2 + 1) \| \mathfrak{R} F \|^2,$$

so that T is bounded at least on the set of real trigonometric polynomials having mean value zero. But if $f = f_1 + if_2$ (where f_1 and f_2 are real trigonometric polynomials with mean value zero), then

$$\| T(f_1 + if_2) \| \leqq \| Tf_1 \| + \| Tf_2 \| \leqq B(\| f_1 \| + \| f_2 \|) \leqq \sqrt{2} B \| f_1 + if_2 \|.$$

This completes the proof for the manifolds $\mathfrak{F}$ and $\mathfrak{P}$.

Now suppose T is bounded on the set of *all* trigonometric polynomials. Then

$$T_1 \, \Sigma \, a_n e^{nix} = \sum_{n<0} a_n e^{nix}$$

defines a second operator which is also bounded (because passing to the complex conjugate leaves the norm of a function unchanged), and consequently

$$\| \sum_{n \neq 0} a_n e^{nix} \| \leqq C \| \, \Sigma \, a_n e^{nix} \|. \tag{5.5}$$

Suppose the trigonometric polynomial on the right side of (5.5) is real. Then, by what has just been proved,

$$\| \, \Sigma \, (-i\varepsilon_n) a_n e^{nix} \| \leqq K \| \sum_{n \neq 0} a_n e^{nix} \|. \tag{5.6}$$

Combining (5.5) and (5.6) we find, with $f(x) = \Sigma a_n e^{nix}$,

$$\| \tilde{f} \| < KC \| f \|. \tag{5.7}$$

Finally suppose that (5.7) holds for all real (and hence, with a new constant, also complex) trigonometric polynomials f. We have shown that T is bounded on the set of complex trigonometric polynomials having mean value zero, and we only have to remove this restriction on the domain of T. Apply the conjugacy operation to f twice; we obtain $-f + a_0$. Therefore a_0 is a bounded functional of f, and so

$$\| Tf \| = \| T(f - a_0) \| \leqq D \| f - a_0 \| \leqq E \| f \|.$$

This completes the proof of Lemma 2, and so also of Theorem 2. The proof of Theorem 1 is also finally complete at this point.

3. Still another form of Theorem 1 may be mentioned. The functions $\{ e^{nix} \}$ are said to be a *basis* in L^2_μ if every element f of the space has a unique expansion

$$f(x) \sim \sum_{-\infty}^{\infty} a_n(f) e^{nix},$$

convergent in norm [2, p. 110]. It is easy to show that *the exponentials form a basis in* L^2_μ *if and only if the manifolds* $\mathfrak{F}$ *and* $\mathfrak{P}_1$ *are at positive angle.* The question which spaces L^2_μ have the exponentials for basis is therefore answered by Theorem 1.

4. The history of these problems is rather involved. HARDY and LITTLEWOOD [7] showed that the conjugacy operation is bounded in L^2_w if the weight function w has the form

$$w(x) = |x|^\alpha \qquad (-\pi < x < \pi), \tag{5.8}$$

with $-1 < \alpha < 1$. (Actually this is a special case of their theorem, which applies to L^p_w with $p > 1$.) BABENKO [1] rediscovered their result, and applied it to the basis problem. N. BARI had asked essentially whether the exponentials could be a basis in L^2_w if w were unbounded from zero or from infinity, and BABENKO showed that the weight functions (5.8) provide an affirmative answer.

Recently GAPOŠKIN [4] proved a theorem which, for $p = 2$, states that the conjugacy operation is bounded if

$$|\tilde{w}| \leqq kw \tag{5.9}$$

for some constant k. (It is easy to apply this result to the functions (5.8).) In correspondence with us, GAPOŠKIN has observed that his condition (5.9) is the same as our (4.1), in the following sense: *w has a representation* (4.1) *if and only if there is a bounded function u_1 such that*

$$|\tilde{w}_1| < kw_1,$$

where

$$w_1 = e^{-u_1}w.$$

The paper of GAPOŠKIN [4] contains the best results known for $p \neq 2$, but no condition has been found which is necessary and sufficient for the conjugacy operation to be bounded in L^p_w if $p \neq 2$.

From (1.6) and (4.5) it is clear that the problem being studied can be stated as a problem about the bounds of certain bilinear forms. As we have mentioned, our methods are related to those used by NEHARI [11] to study bilinear forms, and to those of ROSENBLUM in work still unpublished.

§ 6. - The angle between $\mathfrak{F}$ and $\mathfrak{P}$.

1. If $\mathfrak{F}$ and $\mathfrak{P}_1$ are at positive angle, it is true *a fortiori* that $\mathfrak{F}$ and $\mathfrak{P}$ are at positive angle (since $\mathfrak{P}$ is contained in $\mathfrak{P}_1$), but the converse is not true. We have proved that μ must be absolutely continuous if $\mathfrak{F}$ and $\mathfrak{P}$ are at positive angle in L^2_μ and so we consider only such measures.

Theorem 3. - *If $\int w^{-1}d\sigma < \infty$, then $\mathfrak{F}$ and $\mathfrak{P}$ are at positive angle in L^2_w if and only if $\mathfrak{F}$ and $\mathfrak{P}_1$ are at positive angle.*

Suppose that w^{-1} is summable and that $\mathfrak{F}$ and $\mathfrak{P}$ are at positive angle in L^2_w. From the solution (1.5) of the second prediction problem we know that there is a constant A such that

$$|a| \leqq A \, \| a + F + P \|$$

for all constants a and trigonometric polynomials F and P of the form (1.1) and (1.2) respectively. Moreover

$$\| F \| \leqq B \, \| F + P \|$$

because $\mathfrak{F}$ and $\mathfrak{P}$ are at positive angle. Therefore

$$\| F \| \leqq B \| F + P \| \leqq B[\, \| a + F + P \| + |a| \, \| I \| \,] \leqq C \, \| a + F + P \| .$$

This shows that $\mathfrak{F}$ and $\mathfrak{P}_1$ are at positive angle.

2. If w^{-1} is not summable the situation may be very complicated. The argument used to prove Theorem 1 can be repeated up to a certain point, and gives the following result.

Theorem 4. - *$\mathfrak{F}$ and $\mathfrak{P}$ are at positive angle in L^2_w if and only if w has the form*

$$w = e^u \, |K| \tag{6.1}$$

where u is a bounded real function, and K belongs to H^1 and satisfies

$$|\arg K(e^{ix}) - x| \leqq \frac{\pi}{2} - \varepsilon \qquad (mod\ 2\pi) \tag{6.2}$$

for some $\varepsilon > 0$.

The function K is the same function as AH in (4.10); but we cannot wave aside the qualification *mod* 2π as previously, and it may not be possible to choose K as an outer function. In spite of its similarity to (4.10), (6.2) really gives much less information.

If $K(z)$ has a zero inside the unit circle, then K can be replaced by a new function K_1 having the same modulus on the boundary and satisfying $|\arg K_1(e^{ix})| \leqq \frac{\pi}{2} - \varepsilon$, so that actually w has the form of Theorem 1. In the representation (6.1) it suffices therefore to consider functions K without zeros in the circle, provided w^{-1} is not summable.

The proof of Theorem 4 requires no new idea, and we omit it. A more satisfactory solution of the problem would depend on the deeper properties of inner and outer functions. An impressive but still incomplete treatment of that subject is given in [10].

§ 7. - Some particular weight functions.

1. As we have mentioned, the validity of (1.9) for the weight functions $w(x) = |x|^\alpha$ $(-\pi \leqq x < \pi)$ has been studied before. For $-1 < \alpha < 1$ it is easy to deduce that $\mathfrak{F}$ and $\mathfrak{P}_1$ are at positive angle from Theorem 1; for α outside this interval either w or w^{-1} is not summable, and so $\mathfrak{F}$ and $\mathfrak{P}_1$ cannot be at positive angle. This is the extent of what was known about these weight functions. It is not obvious, however, for which values of α the manifolds $\mathfrak{F}$ and $\mathfrak{P}$ are at positive angle, and we consider that question now. Instead of $|x|^\alpha$ we shall deal with the periodic functions

$$w_\alpha(x) = |1 + e^{ix}|^\alpha.$$

Theorem 5. - *$\mathfrak{F}$ and $\mathfrak{P}$ are at positive angle in $L^2_{w_\alpha}$ if $-1 < \alpha < 1$, and if $1 < \alpha < 3$, but not for other values of α.*

Let $K(z) = (1 + z)^\alpha$, a single-valued analytic function in the unit circle, taking the value 1 at the origin. For $\alpha > -1$, K belongs to H^1 and is an outer function. Its radial limits are

$$K(e^{ix}) = (1 + e^{ix})^\alpha$$

where $\arg(1 + e^{ix})^\alpha$ is defined to be zero for $x = 0$, continuous for $-\pi < x < \pi$, with jump of $\pi\alpha$ at $x = \pm\pi$. For $-1 < \alpha < 1$, K takes the place of H in (4.10) with $A \equiv 1$, showing that $\mathfrak{F}$ and $\mathfrak{P}_1$ are at positive angle, and therefore also $\mathfrak{F}$ and $\mathfrak{P}$. For $1 < \alpha < 3$, K satisfies (6.2) because

$$\arg(1 + e^{ix})^\alpha = \alpha x/2 \qquad (-\pi < x < \pi).$$

Hence $\mathfrak{F}$ and $\mathfrak{P}$ are at positive angle, although $\mathfrak{F}$ and $\mathfrak{P}_1$ are not. The more interesting part of the theorem is the negative part which remains.

2. From (4.5) and the remarks which followed, it is easy to see that $\mathfrak{F}$ and $\mathfrak{P}$ are at positive angle in L^2_w if and only if

$$\tau = \sup \left| \int FGe^{-i\varphi} d\sigma \right| < 1, \tag{7.1}$$

where F and G range over the trigonometric polynomials of the form (1.1) with

$$\int |F|^2 d\sigma = \int |G|^2 d\sigma = 1$$

and φ is the argument of the function H associated with w_α by (4.2). Now

$$K(e^{ix}) = (1 + e^{ix})^\alpha = we^{i\frac{\alpha}{2}x} \qquad (-\pi < x < \pi)$$

is an outer function, and must therefore be the same as H. Hence in this case

$$\varphi(x) = \frac{\alpha}{2} x \qquad (-\pi < x < \pi)$$

and (7.1) becomes

$$\tau = \sup \left| \int_{-\pi}^{\pi} FGe^{-i\frac{\alpha}{2}x} d\sigma \right|. \tag{7.2}$$

The limits of integration are relevant because the integrand is not periodic.

Suppose that

$$F(e^{ix}) = \sum_{n>0} a_n e^{nix}, \qquad G(e^{ix}) = \sum_{n>0} b_n e^{nix}.$$

Using the formula

$$\int_{-\pi}^{\pi} e^{\lambda ix} d\sigma(x) = \frac{\sin \pi\lambda}{\pi\lambda} \qquad (\lambda \neq 0)$$

we have from (7.2)

$$\tau = \sup \left| \sum_{m,n>0} a_m b_n \frac{\sin \pi\left(m + n - \frac{\alpha}{2}\right)}{\pi\left(m + n - \frac{\alpha}{2}\right)} \right|$$

under the restriction

$$\Sigma \mid a_n \mid^2 = \Sigma \mid b_n \mid^2 = 1.$$

Replacing a_m by $(-1)^m a_m$, b_n by $(-1)^n b_n$, we find the simpler expression

$$\tau = \sup \left| \frac{\sin \pi\lambda}{\pi} \sum_{m,n>0} \frac{a_m b_n}{m+n+\lambda} \right| \qquad \left(\lambda = -\frac{\alpha}{2}\right). \tag{7.3}$$

It is well-known that the supremum is not diminished if we take $b_n = a_n$. We denote by $B(\lambda)$ the bound of the bilinear form

$$\sum_{m,n>0} \frac{a_m a_n}{m+n+\lambda}; \qquad \Sigma \mid a_n \mid^2 = 1 \tag{7.4}$$

for each real λ such that $\lambda + 1$ is not a negative integer.

3. It is well-known [13] that $B(\lambda) \leqq |\pi \operatorname{cosec} \pi\lambda|$ for all λ. (This merely says that τ, defined by (7.2), does not exceed 1, which is obvious here.) Moreover $B(0) = \pi$, a fact which is known [8, p. 226] but more difficult to prove. In computing $B(\lambda)$ from (7.4) it is clearly enough to consider non-negative a_n, so that $B(\lambda)$ is non-increasing, for $\lambda > -2$. Hence

$$\pi \leqq B(\lambda) \leqq |\pi \operatorname{cosec} \pi\lambda| \qquad (-2 < \lambda \leqq 0).$$

It follows that $B(-3/2) = \pi$, and therefore

$$B(\lambda) = \pi \qquad (-3/2 \leqq \lambda \leqq 0).$$

From (7.3) we find

$$\tau = \left| \sin \frac{\pi\alpha}{2} \right| \qquad (0 \leqq \alpha \leqq 3),$$

and in particular

$$\tau = 1 \text{ for } \alpha = 1,\ 3.$$

Suppose now $\tau < 1$ for some $\alpha > 3$. Then rewrite (7.2) with $G(e^{ix}) = e^{ix} G_1(e^{ix})$:

$$\tau = \sup \left| \int_{-\pi}^{\pi} F G_1 e^{ix} e^{-i\frac{\alpha}{2}x} \, d\sigma \right|, \tag{7.5}$$

where F ranges over the unit sphere of $\boldsymbol{H}_0^2$ as before, but G_1 over the unit sphere of $\boldsymbol{H}^2$. By (4.5), the supremum in (7.5) is the number ρ for the weight function w_β with $\beta = \alpha - 2$. But w_β^{-1} is not summable if $\beta \geqq 1$, and so it is impossible that $\rho < 1$. This contradiction means that $\tau = 1$ for $\alpha > 3$. This completes the proof of the theorem.

It is curious that $\mathfrak{F}$ and $\mathfrak{P}$ are at positive angle for values of α close to 1, but at zero angle if $\alpha = 1$.

4. We do not know whether the local properties alone of w determine whether $\mathfrak{F}$ and $\mathfrak{P}_1$ are at positive angle. Under additional hypotheses, however, we can answer the question, and thereby deal with certain new weight functions.

Theorem 6. – *Let w be a weight function with associated functions H and*

$$\varphi(e^{ix}) = \lim_{r \uparrow 1} \arg H(re^{ix}). \tag{7.6}$$

Suppose that φ is continuous except for jumps at a finite number of points. In order for $\mathfrak{F}$ and $\mathfrak{P}_1$ to be at positive angle in L^2_w, it is necessary and sufficient that each jump have magnitude smaller than π.

Let φ have jumps whose magnitudes are smaller than π. There is a continuous, periodic function Ψ_1 such that for some $\varepsilon > 0$

$$|\varphi - \Psi_1| \leq \frac{\pi}{2} - \frac{\varepsilon}{2}. \tag{7.7}$$

Indeed, define Ψ_1 to be equal to φ except near the jumps of φ; near those points interpolate Ψ_1 so that (7.7) holds. The periodicity of Ψ_1 follows of itself.

Now we approximate Ψ_1 with error less than $\varepsilon/2$ by a continuously differentiable function Ψ. Then the conjugate function $\tilde{\Psi}$ is continuous and periodic. and

$$A(e^{ix}) = e^{\tilde{\Psi}(x) - i\Psi(x)}$$

is the boundary function of $A(z)$, analytic for $|z| < 1$ and continuous in $|z| \leq 1$. We have

$$(a) \qquad |A(e^{ix})| \geqq \inf e^{\tilde{\Psi}} > 0,$$

$$(b) \qquad |-\varphi(x) - \arg A(e^{ix})| = |-\varphi(x) + \Psi(x)| \leqq \frac{\pi}{2} - \varepsilon.$$

By Lemma 1, $\mathfrak{F}$ and $\mathfrak{P}_1$ are at positive angle.

Conversely, suppose that $\mathfrak{F}$ and $\mathfrak{P}_1$ are at positive angle in L^2_w, and that φ has only simple discontinuities. For any x_0 and real α the function

$$(e^{ix_0} - z)^{\alpha}$$

is analytic in the circle, and has boundary function whose argument is linear except for a jump of magnitude $\pi\alpha$ at e^{ix_0}. Therefore we can choose points x_j and weights α_j $(j = 1, 2, \ldots, n)$ so that

$$K(z) = H(z) \prod_1^n (e^{ix_j} - z)^{\alpha_j} \tag{7.8}$$

has a boundary function $K(e^{ix})$ whose argument is continuous everywhere. Moreover arg $K(e^{ix})$ is the conjugate of log $|K(e^{ix})|$ (except for an additive constant). This fact is easy to verify by considering each factor in (7.8) separately, and referring to the definition of H given by (4.2). The continuity of arg $K(e^{ix})$ implies that both K and K^{-1} belong to L^q for every finite q [15, p. 254]. But if w has the form (4.1), as we are assuming, then w belongs to L^p for some $p > 1$, so that the product $H(e^{ix})K^{-1}(e^{ix})$ is at least summable. We conclude from (7.8) that each α_j is smaller than 1. Now if w has the form (4.1), the same is true of w^{-1}, and the associated outer function is H^{-1}, whose argument has the same jumps as those of H but of opposite sign. Therefore each α_j is larger than -1, and Theorem 6 is proved.

Corollary. - *$\mathfrak{F}$ and $\mathfrak{P}_1$ are at positive angle in L^2_w, where*

$$w(x) = \prod_1^n |e^{ix_j} - e^{ix}|^{\alpha_j}$$

and the points e^{ix_j} are distinct on the circle, if and only if

$$|\alpha_j| < 1 \qquad (j = 1, \ldots, n).$$

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