Analytic expressions for bounded solutions of non-linear ordinary differential equations with an irregular type singular point.

Dedicated to Professor Masuo Hukuhara on his sixty-third birthday

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Summary. - *We shall discuss how to construct analytic expressions for bounded solutions of non-linear ordinary differential equations of the form* (A) *which tend to* 0 *as* x *approaches the origin along the positive real axis.*

§ 1. - Introduction.

In this paper we consider two systems of non-linear ordinary differential equations of the form

$$\text{(A)} \qquad x^{\sigma+1}y' = f(x,\ y,\ z), \qquad xz' = g(x,\ y,\ z) \qquad \left(' = \frac{d}{dx}\right),$$

where we assume that:

1) *σ is a positive integer.*

2) *x is a complex independent variable, y and z are m-and n-column vectors with elements $\{y_j\}$ and $\{z_k\}$ respectively.*

3) *$f(x, y, z)$ and $g(x, y, z)$ are respectively m-and n-column vectors of components $\{f_j\}$ and $\{g_k\}$ which are holomorphic and bounded functions of (x, y, z) for*

$$|x| < \xi, \qquad \|y\| < d, \qquad \|z\| < d \qquad \left(\|y\| = \max_{j=1}^{m} |y_j|\right)$$

and we have $f(0, 0, 0) = 0$, $g(0, 0, 0) = 0$.

4) *An $m \times m$ matrix $\mathfrak{A} = f_y(0, 0, 0)$ with elements $\{(\partial f_j/\partial y_h)_{x=y=z=0}\}$ is non-singular and has Jordan's canonical form with upper triangular form.*

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Since $\mathfrak{A}$ is a non-singular matrix, we can assume without loss of generality that:

5) *The following relations hold:*

$$f_z(0) \equiv f_z(0,\ 0,\ 0) = 0,\ f_x(0) \equiv f_x(0,\ 0,\ 0) = 0, \tag{1.1}$$

$$g_y(0) \equiv g_y(0,\ 0,\ 0) = 0, \tag{1.2}$$

where

$$f_z(0) = \{(\partial f_j/\partial z_h)_{x=y=z=0}\}, \qquad g_y(0) = \{(\partial g_k/\partial y_h)_{x=y=z=0}\}.$$

Indeed, if (1.1) is not true, it is sufficient to make a linear transformation with constant coefficients of the form

$$y = \widehat{y} - \mathfrak{A}^{-1}f_z(0)\widehat{z} - x\mathfrak{A}^{-1}f_x(0), \qquad z = \widehat{z},$$

where $\mathfrak{A}^{-1}$ is the inverse matrix of $\mathfrak{A}$. If (1.2) is false, it is sufficient to make the change of variables

$$y = \tilde{y}, \qquad z = -x^{\sigma}g_y(0)\mathfrak{A}^{-1}\tilde{y} + \tilde{z}.$$

The last transformation does not disturb any components of the vector $g_x(0,\ 0,\ 0)$ because $f_x(0) = 0$.

We assume moreover that:

6) ***All the eigenvalues*** μ_k ***of an*** $n \times n$ ***matrix*** $\mathfrak{B} \equiv g_z(0,\ 0,\ 0)$ ***with elements*** $\{(\partial g_k/\partial z_h)_{x=y=z=0}\}$ ***have positive real parts and*** $\mathfrak{B}$ ***has Jordan's canonical form:***

$$Re\,\mu_k > 0 \qquad (k = 1,\ \dots,\ n). \tag{1.3}$$

The purpose of this paper is to solve a problem on constructing analytic expressions for bounded solutions of equations (A) that tend to 0 as x approaches the origin along the positive real axis under further additional assumptions (see Sections 3 and 4). The motivation of this study was the problem for the case of $m = n = 1$ that Professor M. HUKUHARA proposed in connection with the study of the boundary layer differential equation.

Previously the author developed, in his papers [3, 4], a general theory to construct analytic expressions for bounded solutions for differential equations of the form

$$x\mathbf{1}_m(x^{\tau})y' = A(x)y + x^{\|\tau\|}f(x,\ y), \qquad f(0,\ 0) = 0,$$

where $1_m(x^\tau)$ is an $m \times m$ diagonal matrix with elements $\{x^{\tau_j}\}$ with non-negative integers τ_j, $\|\tau\|$ denotes $\max_{j=1}^{m} (\tau_j)$, $A(x)$ is an $m \times m$ diagonal matrix whose components are polynomials of x of degree at most $\|\tau\| - 1$, $f(x, y)$ is an m-column vector function such that $f_y(0, 0)$ has JORDAN'S canonical form.

However, as will be shown below, our previous theory is ont always useful for our purpose of this paper. Therefore there is need for an improvement, if it is possible, on the method to construct analytic expressions for bounded solutions. We have succeeded in improving the method for a special case of equations (A).

1°. EXAMPLE. - We shall illustrate a comparison between the previous method and the improved method, which we are going to develop in this paper, by the following example:

$$(A_1) \qquad x^{\sigma+1}y' = f(x, y, z), \; xz' = \mu z \qquad (\sigma > 0, \; \mu \neq 0),$$

where y and z are both scalars, $f(x, y, z)$ is a holomorphic scalar function of (x, y, z) at $(0, 0, 0)$ and vanishes there.

Assume that

$$f_y(0, 0, 0) \equiv \nu \neq 0, \qquad f(x, 0, z) \not\equiv 0.$$

Obviously the equations (A_1) have a form similar to equations (A).

It is expected that, if $\nu > 0$ and $\mu > 0$, equations (A_1) have a general solution which tends to 0 as x approaches the origin *along the positive real axis.* However, our previous theory does not give any information about the existence of such a solution. The reason is that, when we construct an analytic expression for a particular solution of equations (A_1) which tends to 0 as x approaches the origin through some sector, say $\underline{\Theta} < \arg x < \bar{\Theta}$, the opening angle of this sector is too small to contain the positive real axis. Indeed, by applying directly our previous theory to equations (A_1), we can get the following result:

Equations (A_1) *have a particular solution of the form* $\{\Phi(x, x^\mu C''), x^\mu C''\}$, *whenever the values of* x *and* $x^\mu C'$ *satisfy inequalities of the form*

$$(1.4) \qquad 0 < |x| < \xi', \; \underline{\Theta} < \arg x < \bar{\Theta}, \; |x^\mu C''| < \delta',$$

where the angles $\underline{\Theta}$ *and* $\bar{\Theta}$ *are given by either*

$$(1.5) \qquad \underline{\Theta} = \frac{1}{\sigma}(\arg \nu - \arg \mu) + \varepsilon'', \; \bar{\Theta} = \frac{1}{\sigma}(\arg \nu - \arg \mu + 2\pi) - \varepsilon''$$

or

$$(1.6)\qquad \underline{\Theta} = \frac{1}{\sigma}(\arg\nu - \arg\mu - 2\pi) + \varepsilon'',\ \bar{\Theta} = \frac{1}{\sigma}(\arg\nu - \arg\mu) - \varepsilon''$$

for a sufficiently small positive constant ε''. $\Phi(x, z)$ *is expanded to a uniformly convergent power series of* z *for*

$$(1.7)\qquad 0 < |x| < \xi',\ \underline{\Theta} < \arg x < \bar{\Theta},\ |z| < \delta'$$

whose coefficients are functions admitting asymptotic expansions in powers of x *as* x *tends to* 0 *through the sector* $\underline{\Theta} < \arg x < \bar{\Theta}$. C'' *is an arbitrary constant.*

If $\nu > 0$ and $\mu > 0$, the sectors $\underline{\Theta} < \arg x < \bar{\Theta}$ with (1.5) and (1.6) can never contain the positive real axis. To explain how to determine the angles $\underline{\Theta}$ and $\bar{\Theta}$, we shall state a lemma which, in our previous theory, played a fundamental role in constructing the solution $\{\Phi(x, x^{\mu}C''), x^{\mu}C''\}$. The lemma can be stated as follows:

LEMMA Γ. - *Let* $\Lambda(x) \equiv -\nu/\sigma x^{\sigma}$. *We can determine a function* $\omega(\varphi)$, *which is strictly positive valued, bounded and continuous for* $\underline{\Theta} \leqq \varphi \leqq \bar{\Theta}$, *in such a way that:*

For any point (x_1, z^1) *in a domain of the form*

$$(1.8)\qquad 0 < |x| < \xi''\omega(\arg x),\ \underline{\Theta} < \arg x < \bar{\Theta},\ |z| < \delta'',$$

there exists a curve Γ_{x_1}, *joining the point* x_1 *with the origin, which satisfies the following two conditions:*

i) *The curve* Γ_{x_1} *is entirely contained in the domain* (1.8) *except for the origin.*

ii) *As* x *moves on the curve* Γ_{x_1}, *we have an inequality of the form*

$$(1.9)\qquad \frac{d}{ds}e^{-Re\Lambda(x)} \geqq |x|^{-\sigma-1}e^{-Re\Lambda(x)}|\nu|\varepsilon$$

and, moreover, an inequality of the form

$$(1.10)\qquad \frac{d}{ds}|x^{\mu}C''| \geqq |x|^{-1}|x^{\mu}C''|\,|\mu|\varepsilon \qquad (C'' = x_1^{-\mu}z^1).$$

Here s *is the arc length of the curve* Γ_{x_1} *measured from the origin to the variable point* x *and* $\varepsilon > 0$ *is a sufficiently small constant.*

In other words, the existence of a function $\omega(\varphi)$ and a curve Γ_{x_1} with the above specified properties is the condition to determine the sector $\underline{\Theta} < \arg x < \bar{\Theta}$. Obviously a domain of the form (1.8) is equivalent to a domain of the form (1.7).

However the above lemma can not always be extended to the higher dimensional case of equations (A_1) such as

$$(A_2)\qquad x^{\sigma+1}y' = f(x, y, z),\qquad xz' = 1_n(\mu)z,$$

where y is an m-column vector, z is an n-column vector, $f(x, y, z)$ is an m-column vector whose components are holomorphic functions of (x, y, z) at $(0, 0, 0)$ and satisfies $f(0, 0, 0) = 0$ and $\det f_y(0, 0, 0) \neq 0$, $1_n(\mu)$ is an $n \times n$ diagonal matrix with elements $\{\mu_k\}$ such that $Re\,\mu_k > 0$. Therefore, we studied *a sufficient condition,* which the author called «Hypothèse $B_{k'}$» in Section 40 in [4], in order that we can construct a function with the properties similar to $\omega(\varphi)$ for a non-empty sector $\underline{\Theta}' < \arg x < \bar{\Theta}'$.

2°. IMPROVEMENT ON THE METHOD. - Since $x = 0$ is an irregular type singular point of equations (A_1), an inequality of the form (1.9) plays an essential role for constructing bounded solutions of equations (A_1). As can be easily seen, an inequality of the form (1.10) shows that a solution $Z(x, x_1, z^1) \equiv x^{\mu}(x_1^{-\mu} z^1)$ of the equation $xz' = \mu z$ satisfying initial conditions $z = z^1$ at $x = x_1$ is in absolute value a monotone increasing function of s for $x \in \Gamma_{x_1}$. However, as we have already remarked (See «Théorème 5» in Section 44 in [4]), the inequality (1.10) is not always necessary, but it seems to be sufficient that there exist an angular domain Δ in the x-plane and a simply connected bounded domain $\mathfrak{D}$ in the vicinity of $z = 0$ in the z-plane such that:

We have always $\{x, Z(x, x_1, z^1)\} \in \Delta \times \mathfrak{D}$ when x moves on a curve Γ_{x_1} (except for the origin) on which an inequality of the form (1.9) *is satisfied, no matter how we choose the initial values x_1 and z^1 in $\Delta \times \mathfrak{D}$. Moreover, the boundary of $\mathfrak{D}$ may depend on* arg x *for $x \in \Delta$ provided that $\mathfrak{D}$ contains a circle $|z| = \rho$ with small radius ρ independent of x.*

To have $Z(x, x_1, z^1) \in \mathfrak{D}$ for $x \in \Gamma_{x_1}$ it is necessary that, as x moves on Γ_{x_1}, the function $|Z(x, x_1, z^1)/z^1|$, considered as a function of (x, x_1, z^1), is uniformly bounded for $x \in \Gamma_{x_1}$ no matter how we choose x_1 and z^1 in $\Delta \times \mathfrak{D}$.

Thus the author [6] has succeded in improving the method to construct analytic expressions for some bounded solutions of equations (A_2). By applying the reasonings in [6] to equations (A_1), an improved result can be stated as follows:

Equations (A_1) *have a particular solution of the form* $\{\Phi^0(x, x^{\mu}C''), x^{\mu}C''\}$, *whenever x and $x^{\mu}C''$ satisfy inequalities of the form*

$$(1.4)^0 \qquad 0 < |x| < \xi', \quad \underline{\Theta}^0 < \arg x < \bar{\Theta}^0, \quad |x^{\mu}C''| < \delta',$$

where $\underline{\Theta}^0$ and $\bar{\Theta}^0$ are given by either

$$(1.5)^0 \qquad \underline{\Theta}^0 = \frac{1}{\sigma}\left(\arg \nu - \frac{5\pi}{2}\right) + \varepsilon'', \quad \bar{\Theta}^0 = \frac{1}{\sigma}\left(\arg \nu + \frac{\pi}{2}\right) - \varepsilon''$$

or

$$(1.6)^0 \qquad \underline{\Theta}^0 = \frac{1}{\sigma}\left(\arg \nu - \frac{\pi}{2}\right) + \varepsilon'', \quad \bar{\Theta}^0 = \frac{1}{\sigma}\left(\arg \nu + \frac{5\pi}{2}\right) - \varepsilon''.$$

$\Phi^0(x, z)$ is a holomorphic and bounded function of (x, z) for

$$(1.7)^0 \qquad 0 < |x| < \xi', \ \underline{\Theta}^0 < \arg x < \bar{\Theta}^0, \ |z| < \delta'$$

and admits there a uniformly convergent expansion in powers of z with coefficients asymptotically developable in power of x as x tends to the origin through the sector $\underline{\Theta}^0 < \arg x < \bar{\Theta}^0$.

Obviously the sectors $\underline{\Theta}^0 < \arg x < \bar{\Theta}^0$ with $(1.5)^0$ and $(1.6)^0$ contain both the positive real axis whatever the value of arg ν is. Therefore the improved theory can construct a solution of equations (A_1) which tends to 0 with the order of a certain positive power of x as x approaches the origin along the positive real axis.

The condition which is imposed upon the sector $\underline{\Theta}^0 < \arg x < \bar{\Theta}^0$ will be clarified by the lemma below.

LEMMA Γ^0. - *We can determine functions $\omega^0(\varphi)$ and $\chi^0(\varphi)$, which are strictly positive valued, bounded and continuous for $\underline{\Theta}^0 \leqq \varphi \leqq \bar{\Theta}^0$ in such a way that:*

For any point (x_1, z^1) in a domain of the form

$$(1.8)^0 \qquad 0 < |x| < \xi''\omega^0(\arg x), \ |z| < \delta''\chi^0(\arg x), \ \underline{\Theta}^0 < \arg x < \bar{\Theta}^0,$$

there exists a curve $\Gamma^0_{x_1}$, similar to the curve Γ_{x_1}, on which the inequality (1.9) *is satisfied and an inequality of the form*

$$(1.10)^0 \qquad |x^\mu C''| < \delta''\chi^0(\arg x), \qquad \underline{\Theta}^0 < \arg x < \bar{\Theta}^0$$

with $C'' = z^1 x_1^{-\mu}$ also is satisfied.

Clearly a domain of the form $(1\ 8)^0$ is equivalent to a domain of the form $(1.7)^0$.

3°. OUTLINE OF CONTENTS. - Chapter I will be devoted to the statement of our main results with further additional assumptions. We shall explain briefly the reasons why we have introduced those assumptions.

Unfortunately the improved method is not still useful for equations such as

$$(A_3) \qquad x^{\sigma+1}y' = f(x, y, z), \qquad xz' = \mu z + x^\mu,$$

where μ *is a positive integer* and $f(x, y, z)$ is a scalar function holomorphic in (x, y, z) at (0, 0, 0) and vanishing there. As is well known, the second equation of (A_3) arises in the theory of BRIOT-BOUQUET type singular points as one of the reduced (simplified) equations. In this case a general solution of the second equation of (A_3) has the form $x^\mu(C'' + \log x)$ and depends actually on log x. It seems for me to be doubtful that we can determine a function $\omega(\varphi)$ with the above specified property in such a way that $|Z(x, x_1, z^1)/z^1|$ is uniformly bounded for x on Γ_{x_1} and for any x_1 unless an inequality of the form (1.10) is satisfied on Γ_{x_1}. Of course the range of x_1 is restricted within an angular domain.

By virtue of this reason, we consider first the case when equations (A) have a particular formal solution of the form

$$(1.11)\qquad y \sim \sum_{l,q} x^l(\mathbf{1}_n(x^\mu)C'')^q P_{lq}, \quad z \sim \sum_{l,q} x^l(\mathbf{1}_n(x^\mu)C'')^q Q_{lq}.$$

Here P_{lq} and Q_{lq} are m- and n-column constant vectors respectively, $\mathbf{1}_n(x^\mu)$ is an $n\times n$ diagonal matrix with element $\{x^{\mu_k}\}$, C'' is an n-column constant vector with elements $\{C''_k\}$, q is an n-row vector with elements $\{q_k\}$ with non-negative integers q_k and

$$(\mathbf{1}_n(x^\mu)C'')^q \equiv (x^{\mu_1}C''_1)^{q_1} \dots (x^{\mu_n}C''_n)^{q_n}.$$

In order for equations (A) to have a formal solution of the form (1.11), we introduce Assumptions I and II in Section 3. Concerning an analytic meaning of this formal solution, we have Theorem 1 in Section 3. This theorem asserts that:

If we formally rearrange the formal solution (1.11) *in the form of a single power series of* $\mathbf{1}_n(x^\mu)C''$, *the resulting formal solution is uniformly convergent with coefficients admitting asymptotic expansions in powers of* x.

We write this uniformly convergent solution as $\{\Phi(x, \mathbf{1}_n(x^\mu)C''), \Psi(x, \mathbf{1}_n(x^\mu)C'')\}$ and apply a transformation of the form

$$(\mathrm{T}_1)\qquad y = \tilde{y} + \Phi(x, \tilde{z}), \qquad z = \Psi(x, \tilde{z})$$

to equations (A). Then it will be verified that the equations satisfied by $(\tilde{y}, \tilde{z})$ are written as

$$(\mathrm{B}_1)\qquad x^{\sigma+1}\tilde{y}' = \tilde{F}(x, \tilde{y}, \tilde{z}), \qquad x\tilde{z}' = \tilde{G}(x, \tilde{y}, \tilde{z}),$$

where $\tilde{F}(x, \tilde{y}, \tilde{z})$ and $\tilde{G}(x, \tilde{y}, \tilde{z})$ are respectively m- and n-column vectors whose components are expressed by uniformly convergent power series of $\tilde{y}$ and $\tilde{z}$ with coefficients admitting asymptotic expansions in powers of x. Since $\tilde{y} = 0$ and $\tilde{z} = \mathbf{1}_n(x^\mu)C''$ are a particular solution of equations (B_1), we have

$$\tilde{F}(x, 0, \tilde{z}) \equiv 0, \; \tilde{G}(x, 0, \tilde{z}) = \mathbf{1}_n(\mu)\tilde{z}.$$

Let us apply next a transformation of the form

$$(\mathrm{T}_2) \qquad \tilde{y} = Y + A(x, Z)Y, \qquad \tilde{z} = Z + x^{\sigma}B(x, Z)Y$$

to equations (B_1), where $A(x, Z)$ and $B(x, Z)$ are respectively $m \times m$ and $n \times m$ matrices whose components are expressed by uniformly convergent power series of Z with coefficients admitting asymptotic expansions in powers of x. Let

$$(\mathrm{B}_2) \qquad x^{\sigma+1}Y' = F(x, Y, Z), \qquad xZ' = G(x, Y, Z)$$

be the equations derived from equations (B_1) by applying the transformation T_2). We will try to simplify the matrices $F_Y(x, 0, Z)$ and $G_Y(x, 0, Z)$. From the formal point of view, it is easy to see that $G_Y(x, 0, Z)$ can be reduced to the zero matrix while $F_Y(x, 0, Z)$ involves still a power series of Z even in the case of $m = 1$. In order for the matrix $F_Y(x, 0, Z)$ to have a very simple form, we introduce Assumption IV in Section 4. Then we have Theorem 2 in Section 4 which asserts that:

We can choose the matrices $A(x, Z)$ and $B(x, Z)$ with the above specified properties in such a way that $G_Y(x, 0, Z)$ is reduced to the zero matrix and $F_Y(x, 0, Z)$ is reduced to a diagonal matrix, say $1_m(\lambda(x, Z))$, whose diagonal components are polynomials of x of degree at most σ with coefficients admitting uniformly convergent expansions in powers of Z.

This case seems for me to be an only case when our simplified equations can be integrated by quadratures. However some troubles will arise in the attempt at the proof of uniform convergence of power series of Z appearing in the components of the diagonal matrix $1_m(\lambda(x, Z))$. In order to overcome those troubles, *we must construct* first of all *the solution* $\{\Phi(x, 1_n(x^{\mu})C''),$ $\Psi(x, 1_n(x^{\mu})C'')\}$ *in a particular way.* This is the reason why the construction of this solution which is going to be developed in Chapter III needs slightly lengthy reasonings.

Finally, in order to construct a general solution of equations (B_2), we consider a formal transformation of the form

$$(\mathrm{T}_3) \qquad Y \sim u + \sum_{|p|=2}^{\infty} u^p A_p(x, v), \qquad Z \sim v + x^{\sigma} \sum_{|p|=2}^{\infty} u^p B_p(x, v),$$

where $A_p(x, v)$ and $B_p(x, v)$ are respectively m- and n-column vectors which are expressed by uniformly convergent power series of v with coefficients admitting asymptotic expansions in powers of x. Here p is an m-row vector with elements $\{p_j\}$ with non-negative integers p_j, and $|p| = p_1 + \dots + p_m$. In order for the equations on $\{u, v\}$ to have the simplest form, we introduce

Assumptions III and V in Section 4. Then we have Theorem 3 in Section 4 which asserts that:

We can choose the vector functions $A_p(x, v)$ *and* $B_p(x, v)$ *with the above specified properties in such a way that the power series* (T_3) *are uniformly convergent and the simplified equations take the form*

$$x^{\sigma+1}u' = \mathbf{1}_m(\lambda(x, v))u, \qquad xv' = \mathbf{1}_n(\mu)v. \tag{B}$$

In Section 5, equations (B) will be integrated by quadratures.

In Chapter II we shall establish two fundamental existence theorems (Theorem A in Section 6 and Theorem B in Section 11) which will play an important role in the proof of Theorems 1, 2 and 3. From the proof of Theorem B one can know our basic ideas about how to study an analytic meaning of formal solutions. Theorems 1, 2 and 3 will be proved in Chapters III, IV and V respectively.

CHAPTER I.

Assumptions and main results.

§ 2. - Notation and Definitions.

1°. NOTATION. - $\mathbf{1}_m$ is the $m \times m$ unit-matrix, e_j is an m-dimensional row unit-vector whose j^{th} component is equal to 1.

For an m-column vector y with element $\{y_j\}$, the expression $\mathbf{1}_m(y)$ denotes an $m \times m$ diagonal matrix with diagonal elements $\{y_j\}$.

If u is an m-column vector with elements $\{u_j\}$, $[u]$ denotes an m-column vector with elements $\{|u_j|\}$. In particular, if all the components u_j are non-negative real numbers, the m-column vector $[u]$ coincides with the m-column vector u.

For m-column vectors u and $\tilde{u}$ with elements $\{u_j\}$ and $\{\tilde{u}_j\}$ respectively, a vectorial inequality of the form $[u] \leqq [\tilde{u}]$ means that we have $|u_j| \leqq |\tilde{u}_j|$ for each index j.

The components of an m- and n-row vectors $p = (p_1, \ldots, p_m)$ and $q = (q_1, \ldots, q_n)$ are all non-negative integers and

$$|p| = p_1 + p_2 + \cdots + p_m. \tag{2.1}$$

For an m-column vector y with elements $\{y_j\}$, the symbol $p.y$ is the inner product given by $\sum_{j=1}^{m} p_j y_j$ and the symbol y^p stands for the scalar expression

$$y^p = y_1^{p_1} y_2^{p_2} \ldots y_m^{p_m}, \tag{2.2}$$

For an m-column vector y with elements $\{y_j\}$ and an n-column vector function $f(x, y)$ with elements $\{f_j(x, y)\}$, the symbol $f_y(x, y)$ denotes an $n \times m$ matrix given by

$$f_y(x, y) = \left(\frac{\partial}{\partial y_1} f(x, y), \ldots, \frac{\partial}{\partial y_m} f(x, y)\right). \tag{2.3}$$

The norm of an m-vector y with elements $\{y_j\}$ is

$$\|y\| = \max_{j=1}^{m} |y_j|. \tag{2.4}$$

To simplify the description, we use the following symbols for a scalar w and for an m-row vector y with elements $\{y_j\}$,

$$w^y = (w^{y_1}, \ldots, w^{y_m}), \tag{2.5}$$

$$\exp y = (\exp y_1, \ldots, \exp y_m) \text{ or } e^y = (e^{y_1}, \ldots, e^{y_m}), \tag{2.6}$$

$$Re\ y = (Re\ y_1, \ldots, Re\ y_m), \qquad Im\ y = (Im\ y_1, \ldots, Im\ y_m) \tag{2.7}$$

with $y = Re\ y + \sqrt{-1}\ Im\ y$. If y is a column vector, w^y, e^y, $Re\ y$ and $Im\ y$ are all column vectors.

2°. DEFINITIONS. - A function $f(x)$, which is holomorphic and bounded in x for

$$0 < |x| < \xi, \qquad \underline{\Theta} < \arg x < \bar{\Theta}$$

and admits an asymptotic expansion in powers of x as x tends to 0 through the sectors $\underline{\Theta} < \arg x < \bar{\Theta}$, is said to *belong to class* $\mathcal{C}(\underline{\Theta}, \bar{\Theta}; \xi)$.

The symbol $f[x; y, z]$ denotes a polynomial of x of degree σ. If the coefficients of this polynomial are holomorphic vector functions of (y, z) for $\|y\| < \delta$, $\|z\| < \delta$, we shall say that $f[x; y, z]$ *has Property-σ with respect to x for* $\|y\| < \delta$, $\|z\| < \delta$.

A vector $f(x, y, z)$, which is a holomorphic function of (x, y, z) for

$$0 < |x| < \xi, \qquad \underline{\Theta} < \arg x < \bar{\Theta}, \qquad \|y\| < \delta,\ \|z\| < \delta, \tag{D}$$

is called to *have Property-$\mathfrak{A}$ with respect to y and z in* (D), if the components of $f(x, y, z)$ admit uniformly convergent expansions in powers of y and z for (D) and if the coefficients of this expansion belong te class $\mathcal{C}$ $(\underline{\Theta}, \bar{\Theta}; \xi)$.

Let there be given a finite number of monomials of x^{-1} of the same degree, say σ:

$$\Omega_j(x) = -\frac{\gamma_j}{\sigma x^\sigma} \qquad (j = 1, 2, \dots, M).$$

Then sectors of the form

$$\frac{1}{\sigma}\left(\arg \gamma_j - \frac{\pi}{2} + 2\pi h\right) < \arg x < \frac{1}{\sigma}\left(\arg \gamma_j + \frac{\pi}{2} + 2\pi h\right) \tag{2.8}$$

and

$$\frac{1}{\sigma}\left(\arg \gamma_j + \frac{\pi}{2} + 2\pi h'\right) < \arg x < \frac{1}{\sigma}\left(\arg \gamma_j + \frac{3\pi}{2} + 2\pi h'\right) \tag{2.9}$$

are said to *be a maximal negative region of* $\Omega_j(x)$ and *a maximal positive region of* $\Omega_j(x)$ respectively, where h and h are any integers. The maximal negative (or positive) region has the meaning such that, if x approaches the origin through any subsector of the sector (2.8) (or the sector (2.9)), the function exp $(Re\ \Omega_j(x))$ tends to 0 (or infinity) exponentially.

We shall say that a sector $\Theta < \arg x < \bar{\Theta}$ *has Property-$\mathfrak{S}$ with respect to monomials* $\{\Omega_1(x), \dots, \Omega_M(x)\}$ if this sector does not contain any maximal negative region of $\Omega_j(x)$ for each index j and if there exists in this sector a direction for each index j such that, as x approaches the origin along this direction, exp $(Re\ \Omega_j(x))$ tends to infinity exponentially.

REMARKS 1°. - In 1942 Professor MASUO HUKUHARA [1] introduced the notion of Property-$\mathfrak{S}$ in order to complete the theory of asymptotic expansions of solutions of a system of linear ordinary differential equations with an irregular singular point which was founded first by H. POINCARÉ and was studied by J. TRJITZINSKY, J. MALMQUIST. HUKUHARA'S condition for a sector $\Theta < \arg x < \bar{\Theta}$ to have Property-$\mathfrak{S}$ with respect to monomials $\{\Omega_1(x), \dots, \Omega_M(x)\}$ is weaker than ours. Namely M. HUKUHARA assumed only that the sector $\Theta < \arg x < \bar{\Theta}$ does not contain any maximal negative region of $\Omega_j(x)$ for each index j. Moreover, in his case, $\Omega_j(x)$ may have *distinct* degrees.

2°. - Assume that $\Omega_j(x)$ have the same degree, say σ. Then, for a preassigned angle θ_0, there exists always a sector $\Theta < \arg x < \bar{\Theta}$ which has Property-$\mathfrak{S}$ with respect to the monomials $\{\Omega_1(x), \dots, \Omega_M(x)\}$ and does contain the direction $\arg x = \theta_0$.

In the case of $\theta_0 = 0$, we choose $\arg \gamma_j$ so that $-\pi < \arg \gamma_j \leqq \pi$ $(j = 1, 2, \dots, M)$ and define θ', θ'' by $\theta' = \min\{\arg \gamma_j;\ \arg \gamma_j > 0\}$, $\theta'' = \max\{\arg \gamma_j;\ \arg \gamma_j < 0\}$.

Then, as can be easily verified, sectors of the form

$$-\frac{\pi}{2\sigma}+\varepsilon''<\arg x<\frac{1}{\sigma}\left(\theta'+\frac{\pi}{2}\right)-\varepsilon'',$$

$$\frac{1}{\sigma}\left(\theta''-\frac{\pi}{2}\right)+\varepsilon''<\arg x<\frac{\pi}{2\sigma}-\varepsilon''$$

have both Property-$\mathcal{S}$ with respect to $\{\Omega_1(x), \dots, \Omega_M(x)\}$ and do contain the positive real axis, where $\varepsilon''>0$ is sufficiently small.

§ 3. - Assumptions I, II and Theorem 1.

In order to construct analytic expressions for bounded solutions which tend to 0 with the order of a certain positive power of x as x approaches the origin *along the positive real axis*, we introduce the following assumptions:

ASSUMPTION I. - *$\mathcal{B}=g_z(0, 0, 0)$ is a diagonal matrix with elements $\{\mu_k\}$.* We denote this diagonal matrix by $\mathbf{1}_n(\mu)$, where μ is an n-column vector with elements $\{\mu_k\}$.

ASSUMPTION II - i) *The k^{th} component of the n-vector $g_x(0)\equiv g_x(0, 0, 0)$ is zero if $\mu_k=1$.* ii) *For all the arrangements (l, q) of $1+n$ non-negative integers l, $\{q_k\}$ such that $l+|q|\geqq 2$, we have*

$$\mu_k \neq l+q\cdot\mu \qquad \textit{for each index } k. \tag{3.1}$$

By virtue of the first portion of Assumption II we can assume without loss of generality that:

$$g_x(0)=0.$$

Indeed, if the k^{th} component of $g_x(0)$, say β_k, is different from zero, we have $\mu_k\neq 1$. We make then a linear transformation of the form

$$y=\tilde{y};\ z_h=\tilde{z}_h(h\neq k),\ z_k=-x(\mu_k-1)^{-1}\beta_k+\tilde{z}_k,$$

which reduces β_k to 0 without disturbing any other components of $g_x(0)$.

For equations (A_1) and (A_2) that appeared in Introduction, Assumption I is automatically satisfied and Assumption II is unnecessary.

Let ν_j be the eigenvalues of the matrix $\mathcal{A}\equiv f_y(0, 0, 0)$ and put

$$\Lambda_j(x)=-\frac{\nu_j}{\sigma x^\sigma} \qquad (j=1, 2, \dots, m).$$

Then we have the following theorem:

THEOREM 1. - *Assume that, besides Assumptions* 1) $\sim$ 6) *in Section* 1, *Assumptions* I *and* II *are satisfied. Let* $\underline{\Theta}_1 < \arg x < \bar{\Theta}_1$ *be a sector with Property-$\mathcal{S}$ with respect to* $\{\Lambda_1(x), \ldots, \Lambda_m(x)\}$ *and containing the positive real axis.*

Then equations (A) *have a particular solution of the form*

$$(S_1) \qquad y = \Phi(x,\ \mathbf{1}_n(x^{\mu})C''), \qquad z = \Psi(x,\ \mathbf{1}_n(x^{\mu})C'')$$

whenever

$$(3.2) \qquad 0 < |x| < \xi_1'',\ \underline{\Theta}_1 < \arg x < \bar{\Theta}_1,\ \|\mathbf{1}_n(x^{\mu})C''\| < \delta_1''$$

for suitably chosen positive constants ξ_1'' *and* δ_1''.

Here $\Phi(x, v)$ *and* $\Psi(x, v)$ *are respectively m- and n-column vectors with a unique representation of the form*

$$(3.3) \qquad \Phi(x,\ v) = \varphi[x;\ v] + x^{\sigma+1}\Phi^0(x,\ v),\ \Psi(x,\ v) = \psi[x;\ v] + x^{\sigma+1}\Psi^0(x,\ v),$$

where $\varphi[x;\ v]$ *and* $\psi[x;\ v]$ *have Property-σ with respect to* x *for* $\|v\| < \delta_1''$, *while* $\Phi^0(x, v)$ *and* $\Psi^0(x, v)$ *have Property-$\mathfrak{A}$ with respect to* v *in*

$$(3.4) \qquad 0 < |x| < \xi_1'',\ \underline{\Theta}_1 < \arg x < \bar{\Theta}_1, \qquad \|v\| < \delta_1''.$$

In particular, we have

$$(3.5) \qquad \frac{\partial}{\partial v}\psi[x;\ v]\Big|_{x=v=0} = \mathbf{1}_n.$$

This theorem will be proved in Chapter III.

The solution $\{\Phi(x,\ \mathbf{1}_n(x^{\mu})C''),\ \Psi(x,\ \mathbf{1}_n(x^{\mu})C'')\}$ tends to 0 with the order of a certain power of x as x approaches the origin along the positive real axis. By the definitions of Property-σ and Property-$\mathfrak{A}$, it is immediately seen that the vector functions $\Phi(x, v)$ and $\Psi(x, v)$ admit uniformly convergent expansions in powers of v for (3.4) with coefficients belonging to class $\mathcal{C}(\underline{\Theta}_1, \bar{\Theta}_1; \xi_1'')$.

§ 4. - Assumptions III, IV, V and Theorems 2, 3.

1°. PRELIMINARY TRANSFORMATION. - By using the vector functions $\Phi(x, v)$ and $\Psi(x, v)$ appearing in Theorem 1, we apply a transformation of the form

$$(T_1) \qquad y = \tilde{y} + \Phi(x,\ \tilde{z}), \qquad z = \Psi(x,\ \tilde{z})$$

to equations (A). This transformation is non-singular by virtue of (3.5). Observe that the equations satisfied by $\{\tilde{y}, \tilde{z}\}$ have $\tilde{y}=0$, $\tilde{z}=1_n(x^{\mu})C''$ as a particular solution.

Hence the transformed equations can be written as

$$(\mathrm{B}_1)\qquad \begin{cases} x^{\sigma+1}\tilde{y}' = \mathcal{A}\tilde{y} + C(x, \tilde{z})\tilde{y} + \sum_{|p|=2}^{\infty} \tilde{y}^p F_p(x, \tilde{z}), \\ x\tilde{z}' = 1_n(\mu)\tilde{z} + D(x, \tilde{z})\tilde{y} + \sum_{|p|=2}^{\infty} \tilde{y}^p G_p(x, \tilde{z}), \end{cases}$$

where the power series in the right-hand members are uniformly convergent for

$$(4.1)\qquad 0<|x|<\xi_1,\ \underline{\Theta}_1<\arg x<\bar{\Theta}_1,\ \|\tilde{y}\|<d_1,\ \|\tilde{z}\|<d_1$$

for suitably chosen positive constants ξ_1 and d_1. Here $C(x, \tilde{z})$, $D(x, \tilde{z})$, $F_p(x, \tilde{z})$ and $G_p(x, \tilde{z})$ are respectively $m\times m$, $n\times m$, $m\times 1$ and $n\times 1$ matrices whose components have Property-$\mathfrak{A}$ with respect to $\tilde{z}$ in

$$(4.2)\qquad 0<|x|<\xi_1,\ \underline{\Theta}_1<\arg x<\Theta_1,\ \|\tilde{z}\|<d_1$$

and, moreover, we have

$$(4.3)\qquad C(0, 0)=0, \qquad D(0, 0)=0.$$

To simplify the description, we used here the symbol $C(0, 0)$ in place of $\lim_{x\to 0} C(x, 0)$. We shall use this symbol hereafter throughout this paper.

Since

$$C(x, \tilde{z}) = f_y(x, \Phi(x, \tilde{z}), \Psi(x, \tilde{z})) - \mathcal{A}$$

and $f_y(x, y, z)$ is holomorphic at $(0, 0, 0)$, the relations (3.3) imply that the matrix $C(x, \tilde{z})$ has a unique representation of the form

$$(4.4)\qquad C(x, \tilde{z}) = c[x;\tilde{z}] + x^{\sigma+1}C^0(x, \tilde{z}).$$

Here $c[x;\tilde{z}]$ has Property-σ with respect to x for $\|\tilde{z}\|<d_1$ while $C^0(x, \tilde{z})$ has Property-$\mathfrak{A}$ with respect to $\tilde{z}$ in (4.2).

2°. ASSUMPTIONS. - In order to construct an analytic expression for a bounded general solution of equations (B_1) which tends to 0 as x approaches the origin along the positive real axis, we introduce the following assumptions:

ASSUMPTION III. - *The inequalities*

$$Re\,\Lambda_1(x) \leqq Re\,\Lambda_2(x) \leqq \dots \leqq Re\,\Lambda_m(x) < 0 \tag{4.5}$$

hold for x on the positive real axis, where

$$\Lambda_j(x) = -\nu_j/\sigma x^\sigma \qquad (j = 1, 2, \dots, m). \tag{4.6}$$

ASSUMPTION IV. - *The eigenvalues $\{\nu_j\}$ of the matrix $\mathfrak{A} \equiv f_y(0, 0, 0)$ are mutually distinct.*

Since $\mathfrak{A}$ is supposed to be JORDAN'S canonical form, this assumption implies that $\mathfrak{A}$ is a diagonal matrix with elements $\{\nu_j\}$. We denote $\mathfrak{A}$ by $\mathbf{1}_m(\nu)$, where ν is an m-column vector with elements $\{\nu_j\}$.

ASSUMPTION V. - *For any m-row vector p with elements $\{p_j\}$ with non-negative integers p_j such that $|p| \geqq 2$, we have*

$$\nu_j \neq p\cdot\nu \qquad \textit{for each index } j. \tag{4.7}$$

3°. STATEMENT OF THEOREMS. - In order to construct a formal solution of equations (B_1), we try to simplify, according to our usual method, equations (B_1) by applying a formal transformation and we expect for the simplified equations to be integrated by quadratures. However, in the present case, the simplified equations *do contain still power series* with respect to some dependent variables. If these power series would have a positive radius of convergence, we might have no trouble for an analytic integration of the simplified equations by quadratures.

Fortunately, this is the case for our simplified equations. But, to have such simplified equations, we have to construct the formal transformation in a particular way.

Let $\underline{\Theta}_2 < \arg x < \overline{\Theta}_2$ be the common part of the sector $\underline{\Theta}_1 < \arg x < \overline{\Theta}_1$ appearing in Theorem 1 and of a sector with Property-$\mathfrak{S}$ with respect to monomials of the form

$$\{\Lambda_j(x) - \Lambda_k(x),\ j \neq k;\ -\Lambda_j(x)\}. \tag{4.8}$$

We can assume without loss of generality that *the sector $\underline{\Theta}_2 < \arg x < \overline{\Theta}_2$ does contain the positive real axis.*

We shall prove then the following theorem:

THEOREM 2. - *Under Assumption* IV, *there exists a transformation of the form*

$$\tilde{y} = Y + A(x, Z)Y,\ \tilde{z} = Z + x^\sigma B(x, Z)Y \tag{T$_2$}$$

such that equations (B_1) *are transformed into equations of the form*

$$(B_2)\qquad \begin{cases} x^{\sigma+1}Y' = \mathbf{1}_m(\lambda[x;\ Z])Y + \sum_{|p|=2}^{\infty} Y^p F_p(x,\ Z), \\ xZ' = \mathbf{1}_n(\mu)Z + \sum_{|p|=2}^{\infty} Y^p G_p(x,\ Z), \end{cases}$$

where the power series of the right-hand members are uniformly convergent for

$$(4.9)\qquad 0 < |x| < \xi_2'',\ \underline{\Theta}_2 < \arg\ x < \bar{\Theta}_2,\ \|Y\| < d_2'',\ \|Z\| < d_2''.$$

ξ_2'' *and* d_2'' *are suitably chosen positive constants.*

Here $\lambda[x;\ Z]$ *is an m-column vector function with Property-*σ *with respect to* x *for* $\|Z\| < d_2''$ *and satisfies* $\lambda[0;\ 0] = \nu$. *Furthermore* $\frac{\partial^\sigma}{\partial x^\sigma}\lambda[0;\ Z] \equiv \frac{\partial^\sigma}{\partial x^\sigma}\lambda[x;\ Z]\Big|_{x=0}$ *is a constant vector.*

$A(x,\ Z)$, $B(x,\ Z)$, $F_p(x,\ Z)$ *and* $G_p(x,\ Z)$ *are respectively* $m \times m$, $n \times m$, $m \times 1$ *and* $n \times 1$ *matrices whose components have Property-*$\mathfrak{A}$ *with respect to* Z *in*

$$(4.10)\qquad 0 < |x| < \xi_2'',\ \underline{\Theta}_2 < \arg\ x < \bar{\Theta}_2,\ \|Z\| < d_2''$$

and, in particular, we have

$$(4.11)\qquad A(0,\ 0) = 0,\ B(0,\ 0) = 0.$$

This theorem will be proved in Chapter IV.

We consider next monomials of the form

$$(4.12)\qquad \{\Lambda_j(x) - p\cdot\Lambda(x),\ -p\cdot\Lambda(x);\ 2 \leqq |p| \leqq M'\}.$$

where $\Lambda(x)$ is an m-column vector with elements $\{\Lambda_j(x)\}$ and M' is supposed to be sufficiently large. Since all these monomials have the same degree with respect to x^{-1}, it is easy to verify that, in the sector $\underline{\Theta}_2 < \arg\ x < \bar{\Theta}_2$, *there exists a subsector* $\underline{\Theta}_3 < \arg x < \bar{\Theta}_3$ *which has Property-*$\mathfrak{S}$ *with respect to the monomials* (4.12) *for each index* j *and does contain the positive real axis.* Then we see by virtue of Assumption III that the sector $\underline{\Theta}_3 < \arg\ x < \bar{\Theta}_3$ has Property-$\mathfrak{S}$ with respect to all the monomials

$$(4.13)\qquad \{\Lambda_j(x) - p\cdot\Lambda(x),\ -p\cdot\Lambda(x);\ 2 \leqq |p|\}.$$

Then we shall prove the following theorem:

THEOREM 3. – *Let there be given equations of the form* (B_2) *satisfying the conditions mentioned before. Assume that Assumptions* III *and* V *are satisfied. Then equations* (B_2) *possess a general solution of the form*

$$(S_2) \qquad Y = \Phi(x,\ U(x),\ V(x)),\ Z = \Psi(x,\ U(x),\ V(x))$$

whenever x, $U(x)$, $V(x)$ *satisfy inequalities of the form*

$$(4.14) \qquad 0 < |x| < \xi_3'',\ \underline{\Theta}_3 < \arg x < \bar{\Theta}_3,\ \|U(x)\| < \delta_3'',\ \|V(x)\| < \delta_3'',$$

where ξ_3'' *and* δ_3'' *are suitably chosen positive constants.*

Here $\{U(x),\ V(x)\}$ *is a general solution of equations of the form*

$$(B) \qquad x^{\sigma+1}u' = \mathbf{1}_m(\lambda[x;\ v])u,\ xv' = \mathbf{1}_n(\mu)v$$

and is obtained by quadratures. $\Phi(x, u, v)$ *and* $\Psi(x, u, v)$ *are respectively m- and n-column vector functions which have Property-*$\mathfrak{A}$ *with respect to u and v in*

$$(4.15) \qquad 0 < |x| < \xi_3',\ \underline{\Theta}_3 < \arg x < \bar{\Theta}_3,\ \|u\| < \delta_3'',\ \|v\| < \delta_3''.$$

The proof of this theorem will be given in Chapter V. The integration of the equations (B) will be studied in the next section.

§ 5. – Bounded General Solutions for Equations (A).

1°. INTEGRATION OF (B) BY QUADRATURES. – The second equation of (B) can be immediately integrated and we have a general solution $V(x) = \mathbf{1}_n(x^\mu)C''$. Let τ be any integer: $0 \leqq \tau \leqq \sigma$. A simple calculation shows that

$$\int \frac{V(x)^q}{x^{\sigma+1-\tau}}\,dx = \begin{cases} (q\cdot\mu - \sigma + \tau)^{-1}V(x)^q x^{\tau-\sigma} & (q\cdot\mu \neq \sigma - \tau), \\ (C'')^q \log x & (q\cdot\mu = \sigma - \tau). \end{cases}$$

It is to be noticed that, for each integer τ, there exists a finite number of the vectors $\{q\}$ satisfying the equation $q\cdot\mu = \sigma - \tau$.

On the other hand, the j^{th} component of the first equation of (B) has the form

$$x^{\sigma+1}u_j' = \lambda_j[x;\ V(x)]u_j,$$

where $\lambda_j[x;\ v]$ is a scalar function with Property-σ with respect to x for $\|v\| < d_2''$ and admits an expansion of the form

$$\lambda_j[x;\ v] = \lambda_j^0(v) + x\lambda_j^1(v) + \dots + x^{\sigma-1}\lambda_j^{\sigma-1}(v) + x^\sigma\lambda_j^\sigma$$

with coefficients holomorphic in v for $\|v\| < d_2'$. By an elementary calculation we have a general solution of the form

$$u_j = e^{\Lambda_j[x;\, V(x)]} x^{\lambda_j(C'')} C_j',$$

where $\Lambda_j[x;\, v]$ has Property-σ with respect to x^{-1} for $\|v\| < d_2'$ and satisfies

$$\Lambda_j[x;\, v] = \Lambda_j^{\cdot}(x)\ (1 + O(|x| + \|v\|)),$$

and $\lambda_j(C'')$ is a polynomial of C'' satisfying $\lambda_j(0) = \lambda_j^{\sigma}$.

We denote by $\Lambda[x;\, v]$, $\lambda(C'')$ and C' m-column vectors with elements $\{\Lambda_j[x;\, v]\}$, $\{\lambda_j(C'')\}$ and $\{C_j'\}$ respectively.

Then we have *a general solution of equations* (B) *which is written as*

$$U(x) = \mathbf{1}_m(e^{\Lambda[x;\, V(x)]})\,\mathbf{1}_m(x^{\lambda(C'')})C', \quad V(x) = \mathbf{1}_n(x^{\mu})C'', \tag{5.1}$$

where C' and C'' are m- and n-column constant vectors respectively.

2°. GENERAL SOLUTIONS OF EQUATIONS (A). – We assume that, besides Assumptions 1) ~ 6) in Section 1, Assumptions I ~ V in Sections 3, 4 are satisfied. If we combine the transformation (T_1) with the transformation (T_2), we have a transformation from (y, z) to (Y, Z), say (T). Substituting (S_2) for (Y, Z) into the transformation (T), we get an analytic expression for a bounded general solution of equations (A). Thus, owing to Theorems 1, 2 and 3 we have at once the following theorem:

THEOREM 4. – *Assume that, besides Assumptions* 1) ~ 6) *in Section* 1, *Assumptions* I ~ V *are satisfied.*

Then equations (A) *have a general solution of the form*

$$y = \mathfrak{Y}(x,\ U(x),\ V(x)), \qquad z = \mathfrak{Z}(x,\ U(x),\ V(x)) \tag{S}$$

whenever the values of x, $U(x)$, $V(x)$ stay in a domain of the form

$$0 < |x| < \xi^0,\ \underline{\Theta}_3 < \arg\ x < \bar{\Theta}_3,\ \|u\| < \delta^0,\ \|v\| < \delta^0. \tag{5.2}$$

Here $\{U(x),\ V(x)\}$ is a general solution of equations (B) *and has the representation* (5.1). *$\mathfrak{Y}(x, u, v)$ and $\mathfrak{Z}(x, u, v)$ are respectively m- and n-column vectors whose components are functions with Property-$\mathfrak{A}$ with respect to u and v in the domain* (5.2). *ξ^0 and δ^0 are positive constants.*

By virtue of Assumption III, the solution (S) tends to 0 as x apporaches the origin along the positive real axis. If we take $C' = 0$, this solution is reduced to the solution (S_1) appearing in Theorem 1. If we take $C'' = 0$, the solution (S) represents a particular solution which tends to 0 exponentially as x approaches the origin along the positive real axis.

3°. CONCLUDING REMARK. – In order for simplified equations (in the present case equations (B)) to be integrated by quadratures, we introduced, besides Assumption III, Assumptions IV and V. Without the last two assumptions, we can construct analytic expressions for bounded solutions of equations (A).

However, in this case, the equation corresponding to the first equation of (B) has a very complicated form. Though its right-hand member is a polynomial of u, we can no longer solve it by quadratures.

CHAPTER II.

Fundamental existence theorems.

I. - First Existence Theorem.

§ 6. Statement of Theorem A. – Let there be given two systems of $\alpha+\beta$ non-linear ordinary differential equations of the form

$$x^{\sigma+1}\mathcal{Y}' = \mathcal{A}(x, \mathcal{Y}, \mathcal{Z}), \qquad x\mathcal{Z}' = \mathcal{B}(x, \mathcal{Y}, \mathcal{Z}). \tag{6.1}$$

Here we suppose that

i) *σ is a positive integer, $\mathcal{Y}$ and $\mathcal{Z}$ are α- and β-column vectors with elements $\{\mathcal{Y}_j\}$ and $\{\mathcal{Z}_k\}$ respectively.*

ii) *$\mathcal{A}(x, \mathcal{Y}, \mathcal{Z})$ and $\mathcal{B}(x, \mathcal{Y}, \mathcal{Z})$ are respectively α- and β-column vectors whose components have Property-$\mathcal{U}$ with respect to $\mathcal{Y}$ and $\mathcal{Z}$ in a domain of the form*

$$0 < |x| < \xi, \ \underline{\Theta} < \arg x < \bar{\Theta}, \ \|\mathcal{Y}\| < d, \ \|\mathcal{Z}\| < d, \tag{6.2}$$

ξ and d being positive constants.

iii) *We have*

$$\mathcal{A}_{\mathcal{Y}}(0, 0, 0) = \mathbf{1}_\alpha(\gamma) + D, \ \det \mathbf{1}_\alpha(\gamma) \neq 0, \ \mathcal{A}_{\mathcal{Z}}(0, 0, 0) = 0, \tag{6.3}$$

where γ is an α-column vector with elements $\{\gamma_j\}$ and D is an $\alpha\times\alpha$ nil-potent matrix with upper triangular form.

iv) *Equations* (6.1) *possess a formal solution of the form*

$$\mathcal{Y} \sim \sum_{l=0}^{\infty} x^l f_l, \qquad \mathcal{Z} \sim \sum_{l=0}^{\infty} x^l g_l, \tag{6.4}$$

where f_l and g_l are α- and β-column constant vectors respectively and, in particular,

$$\|f_0\| < d, \qquad \|g_0\| < d.$$

Let

$$\Omega_j(x) = -\frac{\gamma_j}{\sigma x^\sigma} \qquad (j = 1, 2, \dots, \alpha). \tag{6.5}$$

THEOREM A (FIRST EXISTENCE THEOREM). – *Assume that, in the sector* $\underline{\Theta} < \arg x < \bar{\Theta}$, *there exists a subsector* $\underline{\Theta}^* < \arg x < \bar{\Theta}^*$ *which has Property-$\mathfrak{S}$ with respect to* $\{\Omega_1(x), \dots, \Omega_\alpha(x)\}$.

Then equations (6.1) *have a unique solution* $\{\mathfrak{Y}(x), \mathfrak{W}(x)\}$ *which is holomorphic and bounded in x for*

$$0 < |x| < \xi_0, \qquad \underline{\Theta}^* < \arg x < \bar{\Theta}^* \tag{6.6}$$

and admits asymptotic expansions of the form (6.4) *as x tends to* 0 *through* (6.6).

This theorem has been already proved by M. IWANO [5] by using the method of M. HUKUHARA [1]. However, in order to explain our usual method for giving an analytic meaning to formal solutions, we want to reproduce the proof of this theorem.

REMARK. – Let $\underline{\Theta}' < \arg x < \bar{\Theta}'$ be any subsector contained in the sector $\underline{\Theta}^* < \arg x < \bar{\Theta}^*$. Then it is known that there exists at least one solution which is asymptotically developable to the formal solution (6.4) as x tends to 0 through

$$0 < |x| < \xi_0, \ \underline{\Theta}' < \arg x < \bar{\Theta}'. \tag{6.6$'$}$$

However, such a solution is not uniquely determined unless the sector $\underline{\Theta}' < \arg x < \bar{\Theta}'$ has Property-$\mathfrak{S}$ with respect to $\{\Omega_1(x), \dots, \Omega_\alpha(x)\}$.

§ 7. **Determination of $\omega^*(\varphi)$.** – In order to prove Theorem A, it is necessary to replace a domain of the form (6.6) by a domain the form

$$0 < |x| < \xi_0' \omega^*(\arg x), \qquad \underline{\Theta}^* < \arg x < \Theta^*,$$

which is equivalent to (6.6), where $\omega^*(\varphi)$ is a strictly positive valued, bounded and continuous function of φ for $\underline{\Theta}^* \leqq \varphi \leqq \bar{\Theta}^*$.

Therefore we begin with the determination of the function $\omega^*(\varphi)$.

By assumption, if $Re\ \Omega_j(x)$ is non-positive for

$$\underline{\Theta}^* < \arg x < \bar{\Theta}^*, \tag{7.1}$$

there exists at least one direction $\arg x = \theta_j$ in (7.1) such that we have $Re\, \Omega_j(x) = 0$ for $\arg x = \theta_j$. Such directions are called *singular directions of* $\Omega(x)$ and given by

$$(7.2) \qquad \frac{1}{\sigma}\left(\arg \gamma_j + \frac{\pi}{2} + 2\pi h'\right), \qquad (7.2)' \qquad \frac{1}{\sigma}\left(\arg \gamma_j - \frac{\pi}{2} + 2\pi h''\right),$$

where h' and h'' are any integers. Singular directions of the form (7.2) are called *ascending singular directions of* $\Omega_j(x)$ and those of the form (7.2)′ *descending singular directions*. It is to be noticed that, when we consider $Re\, \Omega_j(x)$ as a function of $\arg x = \theta$, $Re\, \Omega_j(x)$ is a monotone increasing (or decreasing) function of $\arg x$ in a small neighborhood of each singular direction of the form (7.2) (or the form (7.2)′).

For the indices j such that $Re\, \Omega_j(x)$ change their sign in (7.1), we choose the arguments of the complex constants γ_j so that at least either one of two singular directions

$$(7.2)^+ \qquad \theta_{j+} = \frac{1}{\sigma}\left(\arg \gamma_j + \frac{\pi}{2}\right), \qquad (7.2)^- \qquad \theta_{j-} = \frac{1}{\sigma}\left(\arg \gamma_j + \frac{3\pi}{2}\right)$$

is contained in (7.1). We classify the set $J = \{1, 2, \ldots, \alpha\}$ of indices j into J_0, J_1, J_2, J_3 where

$$J_0 = \{j;\ Re\, \Omega_j(x) > 0 \text{ for } \underline{\Theta}^* \leqq \arg x \leqq \bar{\Theta}^*\},$$

$$J_1 = \{j;\ \underline{\Theta}^* < \theta_{j+} < \theta_{j-} < \bar{\Theta}^*\},$$

$$J_2 = \{j;\ \underline{\Theta}^* < \theta_{j+} < \bar{\Theta}^* < \theta_{j-}\},$$

$$J_3 = \{j;\ \theta_{j+} < \underline{\Theta}^* < \theta_{j-} < \bar{\Theta}^*\}.$$

For $j \in J_2$, we define θ_{j-} by $(7.2)^-$ and for $j \in J_3$ we define θ_{j+} by $(7.2)^+$. Some of these four sets may be empty. It is easy to verify that either J_0 or J_1 must be empty. To simplify the discussion, we assume that the set J_0 is empty.

Since the sector (7.1) has Property-$\mathfrak{S}$ with respect to $\{\Omega_1(x), \ldots, \Omega_\alpha(x)\}$, the angles $\underline{\Theta}^*$ and $\bar{\Theta}^*$ must satisfy inequalities of the form

$$(7.3) \qquad \max_{j=1}^{\alpha}\ \theta_{j+} - \left(\frac{\pi}{\sigma} + 6\varepsilon\right) \leqq \underline{\Theta}^* < \bar{\Theta}^* \leqq \min_{j=1}^{\alpha}\ \theta_{j-} + \left(\frac{\pi}{\sigma} - 6\varepsilon\right)$$

for $\varepsilon > 0$ sufficiently small.

We put

$$(7.4)\qquad \Theta^*_{\varkappa+} = \max_{j\in J_\varkappa} \{\theta_{j+}\}, \qquad \Theta^*_{\varkappa-} = \min_{j\in J_\varkappa} \{\theta_{j-}\} \qquad (\varkappa = 1,\ 2,\ 3)$$

and define a function $L^*(\varphi)$ by

$$(7.5)\qquad L^*(\varphi) = \begin{cases} \sigma(\varphi - \Theta^*_{3-} + 4\varepsilon), & \Theta^*_{3-} + \dfrac{\pi}{2\sigma} - 4\varepsilon \leqq \varphi \leqq \bar{\Theta}^*, \\[2mm] \dfrac{\pi}{2}, & \Theta^*_{2+} - \dfrac{\pi}{2\sigma} + 4\varepsilon \leqq \varphi \leqq \Theta^*_{3-} + \dfrac{\pi}{2\sigma} - 4\varepsilon, \\[2mm] \sigma(\varphi - \Theta^*_{2+} - 4\varepsilon) + \pi, & \underline{\Theta}^* \leqq \varphi \leqq \Theta^*_{2+} - \dfrac{\pi}{2\sigma} + 4\varepsilon. \end{cases}$$

Noticing that

$$\Theta^*_{2+} = \max_j\ \theta_{j+}, \qquad \Theta^*_{3-} = \min_j\ \theta_{j-} \qquad (j \in J_1 \cup J_2 \cup J_3),$$

we see by (7.3) that the function $L^*(\varphi)$ satisfies

$$(7.6)\qquad 2\sigma\varepsilon \leqq L^*(\varphi) \leqq \pi - 2\sigma\varepsilon \qquad \text{for} \qquad \underline{\Theta}^* \leqq \varphi \leqq \bar{\Theta}^*.$$

The function $\omega^*(\varphi)$ is to be defined as

$$(7.7)\qquad \omega^*(\varphi) = \exp \int_{\theta_0}^{\varphi} \cot\, L^*(\tau)d\tau,$$

where θ_0 is an arbitrary angle in (7.1). Obviously the function $\omega^*(\varphi)$ thus defined satisfies the above specified conditions.

§ 8. - Fundamental Lemma for the Proof of Theorem A.

1°. STATEMENT OF LEMMA. - Before going into an essential part of the proof of Theorem A, we must prove a lemma.

LEMMA A. - *Let x_1 be an arbitrary point in a domain of the form*

$$(8.1)\qquad 0 < |x| < \xi_N\,\omega^*(\arg x), \qquad \underline{\Theta}^* < \arg x < \bar{\Theta}^*,$$

where ξ_N is a certain positive constant. Then there exists an α-vector path $\Gamma^*_{x_1}$ with elements $\{\Gamma^*_{jx_1}\}$ such that·

i) *Each curve $\Gamma^*_{jx_1}$ joins the point x_1 with the origin and is contained in the domain* (8.1) *except for the origin.*

ii) *As x moves on the curve $\Gamma^*_{jx_1}$, we have*

$$\frac{d}{ds_j} e^{-Re\Omega_j(x)} \geqq |x|^{-\sigma-1} e^{-Re\Omega_j(x)} \|\gamma\|' \sin 2\sigma\varepsilon \tag{8.2.1}$$

and, if $\|\gamma\|' \sin 2\sigma\varepsilon \geqq 2N(\xi_N \max_{\varphi} \omega^*(\varphi))^N$, *we have moreover*

$$\frac{d}{ds_j}\left(|x|^N e^{-Re\Omega_j(x)}\right) \geqq \frac{\|\gamma\|' \sin 2\sigma\varepsilon}{2} |x|^{N-\sigma-1} e^{-Re\Omega_j(x)} \tag{8.2.2}$$

with $\|\gamma\|' = \min_{j=1}^{\alpha} |\gamma_j|$. *Here s_j is the arc length of the curve Γ_{jx_1} measured from the origin to the variable point x on this curve.*

This lemma will be proved in the next section.

2°. DEFINITION OF THE PATH VECTOR $\Gamma^*_{x_1}$. - In order to define the path vector $\Gamma^*_{x_1}$, we shall define first an α-column vector function $l(\varphi)$ with elements $\{l_j(\varphi)\}$ as follows.

If $j \in J_1$,

$$l_j(\varphi) = \begin{cases} \sigma(\varphi - \theta_{j-} + 4\varepsilon), & \theta_{j-} - 2\varepsilon \leqq \varphi \leqq \bar{\Theta}^*, \\ \dfrac{\pi}{2} & \theta_{j+} + 2\varepsilon < \varphi < \theta_{j-} - 2\varepsilon, \\ \sigma(\varphi - \theta_{j+} - 4\varepsilon) + \pi, & \underline{\Theta}^* \leqq \varphi \leqq \theta_{j+} + 2\varepsilon. \end{cases} \tag{8.3.1}$$

If $j \in J_2$,

$$l_j(\varphi) = \begin{cases} \dfrac{\pi}{2}, & \theta_{j+} + 2\varepsilon \leqq \varphi \leqq \bar{\Theta}^*, \\ \sigma(\varphi - \theta_{j+} - 4\varepsilon) + \pi, & \underline{\Theta}^* \leqq \varphi \leqq \theta_{j+} + 2\varepsilon. \end{cases} \tag{8.3.2}$$

If $j \in J_3$,

$$l_j(\varphi) = \begin{cases} \sigma(\varphi - \theta_{j-} + 4\varepsilon), & \theta_{j-} - 2\varepsilon \leqq \varphi \leqq \bar{\Theta}^*, \\ \dfrac{\pi}{2}, & \underline{\Theta}^* \leqq \varphi \leqq \theta_{j-} - 2\varepsilon. \end{cases} \tag{8.3.3}$$

REMARK. - In the case when the set J_0 is non-empty, we take $l_j(\varphi) = \pi/2$ for $\underline{\Theta}^* \leqq \varphi \leqq \bar{\Theta}^*$.

Observing that $\underline{\Theta}^*$ and $\bar{\Theta}^*$ satisfy the inequalities (7.3), it is easily seen that

$$2\sigma\varepsilon \leqq l_j(\varphi) \leqq \pi - 2\sigma\varepsilon \qquad \text{for } \underline{\Theta}^* \leqq \varphi \leqq \bar{\Theta}^*. \tag{8.4}$$

Moreover the definitions of Θ^*_{x+} and Θ^*_{x-} imply that

$$(8.5)\qquad \begin{cases} l_j(\varphi) \leqq L^*(\varphi), & \theta_{j-} - 2\varepsilon \leqq \varphi \leqq \bar{\Theta}^* \quad (j \in J_1,\ J_3), \\ l_j(\varphi) \geqq L^*(\varphi), & \underline{\Theta}^* \leqq \varphi \leqq \theta_{j+} + 2\varepsilon \quad (j \in J_1,\ J_2). \end{cases}$$

Hence we have, by a simple consideration,

$$(8.6)\qquad \int_\theta^\varphi \cot\, l_j(\tau) d\tau \leqq \int_\theta^\varphi \cot\, L^*(\tau) d\tau$$

for $\theta \leqq \varphi \leqq \theta_{j+} + 2\varepsilon (j \in J_1,\ J_2)$ and for $\theta_{j-} - 2\varepsilon \leqq \varphi \leqq \theta\ (j \in J_1,\ J_3)$.

Let $x_1 = re^{i\theta}$ and let $(\rho,\ \varphi)$ be the polar coordinate of the variable point x on the curve $\Gamma^*_{jx_1}$. Then the curves $\Gamma^*_{jx_1}$ are defined as follows:

If $\theta < \theta_{j+} + 2\varepsilon$ or $\theta_{j-} - 2\varepsilon < \theta$, the curve $\Gamma^*_{jx_1}$ consists of a curvilinear part Γ'_j:

$$(8.7)\qquad \rho = r \exp \int_\theta^\varphi \cot\, l_j(\tau) d\tau$$

$$\theta \leqq \varphi \leqq \theta_{j+} + 2\varepsilon \text{ or } \theta_{j-} - 2\varepsilon \leqq \varphi \leqq \theta$$

and of a rectilinear part $\Gamma^{\prime\prime}$:

$$(8.8)\qquad 0 \leqq \varphi \leqq r \exp \int_\theta^\varphi \cot\, l_j(\tau) d\tau,\ \varphi = \theta_{j+} + 2\varepsilon \text{ or } \theta_{j-} - 2\varepsilon.$$

If $\theta_{j+} + 2\varepsilon \leqq \theta \leqq \theta_{j-} - 2\varepsilon$, the curve $\Gamma^*_{jx_1}$ consists of a rectilinear part Γ''_j only:

$$(8.9)\qquad 0 \leqq \rho \leqq r,\ \varphi = \theta.$$

§ 9. **Proof of Lemma A.** - By virtue of the inequality (8.6), we see by inspection that the curves $\Gamma^*_{jx_1}$ defined by ((8.7), (8 8)) or (8.9) are contained entirely in the interior of the domain (8.1) except for the origin.

This proves the first portion of Lemma A.

We shall prove first the inequality (8.2.1). On the curvilinear part Γ'_j, ρ is a function of φ given by (8.7). A simple calculation shows that we have

$$(9.1)\qquad \frac{dx}{ds_j} = -e^{(l_j(\varphi)+\varphi)i} \text{ or } +e^{(l_j(\varphi)+\varphi)i},\ i = \sqrt{-1},$$

according as $\theta \leqq \varphi \leqq \theta_{j+} + 2\varepsilon$ or $\theta_- - 2\varepsilon \leqq \varphi \leqq$.

Hence we have the equality

$$\frac{d}{ds_j}(-Re\ \Omega_j(x)) = \pm\rho^{-\sigma-1}|\gamma_j|\ \cos\ (l_j(\varphi) - \sigma\varphi + \arg\ \gamma_j) \tag{9.2}$$

where we must take the positive sign or the negative sign according as

$$\theta \leqq \varphi \leqq \theta_{j+} + 2\varepsilon \text{ or } \theta_{j-} - 2\varepsilon \leqq \varphi \leqq \theta.$$

On the other hand the definitions of the functions $l_j(\varphi)$ and of the angles θ_{j+}, θ_{j-} imply that we have the relations

$$l_j(\varphi) + \arg\ \gamma_j - \sigma\varphi = \frac{\pi}{2} - 4\sigma\varepsilon \text{ or } -\frac{3\pi}{2} + 4\sigma\varepsilon$$

according as φ is in the interval $\theta \leqq \varphi \leqq \theta_{j+} + 2\varepsilon$ or $\theta_{j-} - 2\varepsilon \leqq \varphi \leqq \theta$. It follows from these relations that we have

$$\cos\ (l_j(\varphi) - \sigma\varphi + \arg\ \gamma_j) = \sin\ 4\sigma\varepsilon > \sin\ 2\sigma\varepsilon.$$

This proves the inequality (8.2.1) for x on Γ'_j.

On the rectilinear part Γ''_j, we have $s = \rho = |x|$ and $\theta_{j+} + 2\varepsilon \leqq \varphi \leqq \theta_{j-} - 2\varepsilon$. It is readily seen that

$$Re\ (-\Omega_j(x)) = \frac{|\gamma_j|}{\sigma\rho^{\sigma}}\cos\ (\arg\ \gamma_j - \sigma\varphi),\ |\arg\ \gamma_j - \sigma\varphi + \pi| \leqq \frac{\pi}{2} - 2\sigma\varepsilon.$$

Hence $\frac{d}{ds_j} Re\ (-\Omega_j(x))$ is a monotone increasing function of ρ and we have the inequality (8.2.1) for x on Γ''_j. Thus the inequality (8.2.1) has been proved.

In order to prove (8.2.2), we observe by a simple consideration that

$$|x|^{-1}\frac{d|x|}{ds_j} = \frac{d}{ds_j}\log\ |x| = Re\left(x^{-1}\frac{dx}{ds_j}\right) \geqq -|x|^{-1}.$$

The inequality (8.2.1) implies that

$$\frac{d}{ds_j}(|x|^N e^{-Re\Omega_j(x)}) \geqq |x|^{N-\sigma-1}e^{-Re\Omega_j(x)}(\|\gamma\|'\ \sin\ 2\sigma\varepsilon - N|x|^{\sigma}).$$

Hence if we choose ξ_N small enough to have

$$2(\|\gamma\|'\ \sin\ 2\sigma\varepsilon - N|x|^{\sigma}) \geqq \|\gamma\|'\ \sin\ 2\sigma\varepsilon \qquad \text{for } |x| \leqq \xi_N\omega^*(\arg\ x),$$

the inequality (8.2.2) holds for x on $\Gamma^*_{jx_1}$. This completes the proof of Lemma A.

§ 10. Proof of Theorem A. - According to our usual method, we apply successively two transformations of the form

$$(10.1)\qquad \begin{cases} \mathfrak{Y} = \sum_{l=0}^{N-1} x^l f_l + \eta, \ \mathfrak{Z} = \sum_{l=0}^{N-1} x^l g_l + \zeta, \\ \eta = \mathbf{1}_\alpha(e^{\Omega(x)})P, \ \zeta = Q \end{cases}$$

to equations (6.1), where $\mathbf{1}_\alpha(e^{\Omega(x)})$ is an $\alpha \times \alpha$ diagonal matrix with elements $\{e^{\Omega_j(x)}\}$. By a direct calculation we see that the equations satisfied by P and Q can be written as

$$(10.2)\qquad \begin{cases} x^{\sigma+1}P' = \mathbf{1}_\alpha(e^{-\Omega(x)})\mathfrak{A}(x, \mathbf{1}_\alpha(e^{\Omega(x)})P, Q), \\ xQ' = \mathfrak{B}(x, \mathbf{1}_\alpha(e^{\Omega(x)})P, Q), \end{cases}$$

where $\mathfrak{A}(x, \eta, \zeta)$ and $\mathfrak{B}(x, \eta, \zeta)$ are respectively α- and β-column vector functions which have Property-$\mathfrak{U}$ with respect to η and ζ in

$$(10.3)\qquad 0 < |x| < \xi', \ \underline{\Theta} < \arg x < \bar{\Theta}, \ \|\eta\| < d', \ \|\zeta\| < d'.$$

It is easy to verify that

$$(10.4)\qquad \begin{cases} \|\mathfrak{A}(x, \eta, \zeta)\| \leqq H'(\|\eta\| + \|\zeta\|) + B_N|x|^N, \\ \|\mathfrak{B}(x, \eta, \zeta)\| \leqq H''(\|\eta\| + \|\zeta\|) + B_N|x|^N \end{cases}$$

for (10.3). Moreover $\mathfrak{A}$ and $\mathfrak{B}$ satisfy there LIPSCHITZ'S conditions with respect to (η, ζ) with LIPSCHITZ'S constants H' and H'' respectively. Namely we have

$$\|\mathfrak{A}(x, \eta^1, \zeta^1) - \mathfrak{A}(x, \eta^2, \zeta^2)\| \leqq H'(\|\eta^1 - \eta^2\| + \|\zeta^1 - \zeta^2\|)$$

and $\mathfrak{B}$ satisfies an analogous inequality. Here H' and H'' are positive constants independent of N. By virtue of Assumption iii) in Section 6, we can assume without loss of generality that H' satisfies

$$(10.5)\qquad 8H' < \|\gamma\|' \sin 2\sigma\varepsilon \qquad (\|\gamma\|' = \min_{j=1}^{\alpha} |\gamma_j|)$$

for a preassigned positive number ε. Indeed, this inequality is accomplishes by applying, if it is necessary, a suitable linear transformation with constant coefficients. And we take N so large that

$$(10.6)\qquad 4H'' < N.$$

By repeating the arguments which will be developed in Section 14 word by word, we can solve the following problem:

PROBLEM A. - *If we have* (10.5) *and* (10.6), *there exists a unique solution of equations* (10.2) *satisfying the conditions*

$$(10.7)_N \qquad [P] = O(|x|^N)[e^{-Re\,\Omega(x)}], \qquad \|Q\| = O(|x|^N),$$

where $[P]$ *denotes an* α*-column vector with elements* $\{|P_j|\}$ *and* $[e^{-Re\,\Omega(x)}]$ *is an* α*-column vector with elements* $\{e^{-Re\,\Omega_j(x)}\}$.

Using the solution of this problem, we can prove Theorem A by an easy application of the reasonings which will be given in 2° of Section 13.

II. - Second Existence Theorem.

§ 11. Statement of Theorem B. - In this part we consider again equations of the form (6.1) for the case when their right-hand members depend, besides x, $\mathfrak{Y}$, $\mathfrak{Z}$, on an arbitrary function of the form $V(x) \equiv 1_n(x^{\mu})C''$. Equations of this type will play an essential role for the proof of Theorems 1, 2 and 3.

We use the same notation as before.

Let there be given two systems of $\alpha + \beta$ non-linear ordinary differential equations of the form

$$(11.1) \qquad x^{\sigma+1}\mathfrak{Y}' = \mathfrak{A}(x, V(x); \mathfrak{Y}, \mathfrak{Z}), \quad x\mathfrak{Z}' = \mathfrak{B}(x, V(x); \mathfrak{Y}, \mathfrak{Z}).$$

Here we suppose that

i) $\mathfrak{A}(x, v; \mathfrak{Y}, \mathfrak{Z})$ *and* $\mathfrak{B}(x, v; \mathfrak{Y}, \mathfrak{Z})$ *are respectively* α*- and* β*-column vector functions which admit uniformly convergent expansions in powers of* $\mathfrak{Y}$ *and* $\mathfrak{Z}$ *in a domain of the form*

$$(11.2) \qquad 0 < |x| < \xi, \ \underline{\Theta} < \arg x < \bar{\Theta}, \ \|v\| < \delta, \ \|\mathfrak{Y}\| < d, \ \|\mathfrak{Z}\| < d,$$

whose coefficients are functions with Property-$\mathfrak{A}$ *with respect to* v *in*

$$(11.3) \qquad 0 < |x| < \xi, \ \underline{\Theta} < \arg x < \bar{\Theta}, \ \|v\| < \delta.$$

ii) *We have*

$$(11.4) \qquad \mathfrak{A}_{\mathfrak{Y}}(0, 0; 0, 0) = 1_\alpha(\gamma) + D, \ \mathfrak{A}_{\mathfrak{Z}}(0, 0; 0, 0) = 0, \ \det 1_\alpha(\gamma) \neq 0.$$

iii) *Equations* (11.1) *have a formal solution of the form*

$$(11.5) \qquad \mathfrak{Y} \sim \sum_{|q|=0}^{\infty} V(x)^q f_q(x), \qquad \mathfrak{Z} \sim \sum_{|q|=0}^{\infty} V(x)^q g_q(x),$$

where $f_q(x)$ and $g_q(x)$ are respectively α- and β-column vector functions which belong to class $\mathfrak{C}(\underline{\Theta}, \bar{\Theta}; \xi)$ and, in particular,

$$\|f_0(x)\| < d, \qquad \|g_0(x)\| < d.$$

THEOREM B (SECOND EXISTENCE THEOREM). - *Assume that, in the sector $\underline{\Theta} < \arg x < \bar{\Theta}$, there exists a subsector $\underline{\Theta}^* < \arg x < \bar{\Theta}^*$ which has Property-$\mathfrak{S}$ with respect to $\{\Omega_1(x), \ldots, \Omega_\alpha(x)\}$.*

Then equations (11.1) *have a solution of the form*

$$\mathfrak{Y} = \mathfrak{V}(x, V(x)), \qquad \mathfrak{Z} = \mathfrak{W}(x, V(x)), \tag{11.6}$$

whenever x and $V(x)$ are in a domain of the form

$$0 < |x| < \xi^0, \ \underline{\Theta}^* < \arg x < \bar{\Theta}^*, \ \|v\| < \delta^0. \tag{11.7}$$

This solution admits uniformly convergent expansions of the form (11.5), *so that $\mathfrak{V}(x, v)$ and $\mathfrak{W}(x, v)$ are respectively α- and β-column vector functions with Property-$\mathfrak{U}$ with respect to v in the domain* (11.7).

The proof of this theorem will be given in Section 13.

For the proof of Theorem B as well as that of Theorem A, a domain of the form (11.7) must be replaced by *an equivalent domain* of the form

$$0 < |x| < \xi^0 \omega^*(\arg x), \ [v] < \delta^0[\chi^*(\arg x)], \ \underline{\Theta}^* < \arg x < \bar{\Theta}^*. \tag{11.8}$$

Here $\omega^*(\varphi)$ is a scalar function and $\chi^*(\varphi)$ is an n-column vector function with elements $\{\chi_k^*(\varphi)\}$:

$$\begin{cases} \omega^*(\varphi) = \exp \displaystyle\int_{\theta_0}^{\varphi} \cot L^*(\tau) d\tau, \\ \chi_k^*(\varphi) = \exp \left\{ (Re\ \mu_k) \displaystyle\int_{\theta_0}^{\varphi} \cot L^*(\tau) d\tau + (Im\ \mu_k)(\theta_0 - \varphi) \right\}, \end{cases} \tag{11.9}$$

where $L^*(\tau)$ is given by the formula (7.5) and θ_0 is a fixed angle satisfying $\underline{\Theta}^* \leqq \theta_0 \leqq \bar{\Theta}^*$.

§ 12. - Fundamental Lemma for the Proof of Theorem B.

1°. LEMMA B. - We must prove a lemma, analogous to Lemma A in Section 8 which will play a fundamental role in the proof of Theorem B.

Let $\omega^*(\varphi)$ and $\chi^*(\varphi)$ be the functions given by (11.9). Then the lemma can be stated as follows:

LEMMA B. - *Let x_1 and v^1 be arbitrary points in a domain of the form*

$$0 < |x| < \xi_N \omega^*(\arg x),\ [v] < \delta_N[\chi^*(\arg x)],\ \underline{\Theta}^* < \arg x < \bar{\Theta}^*. \tag{12.1}$$

Choose C'' so that $V(x_1) = v^1$, namely let $C'' = \mathbf{1}_n(x_1^{-\mu})\, v^1$.

Then there exists an α-vector path $\Gamma^*_{x_1}$ with element $\{\Gamma^*_{jx_1}\}$ such that:

1) *The curves $\Gamma^*_{jx_1}$ join the point x_1 with the origin and are contained in the domain*

$$0 < |x| < \xi_N \omega^*(\arg x), \qquad \underline{\Theta}^* < \arg x < \bar{\Theta}^* \tag{12.2}$$

except for the origin.

2) *As x moves on the curve $\Gamma^*_{jx_1}$ for each index j, we have*

$$[V(x)] < \delta_N[\chi^*(\arg x)], \qquad \underline{\Theta}^* < \arg x < \bar{\Theta}^*, \tag{12.3}$$

$$\frac{d}{ds_j}\| V(x)\| \geqq -|x|^{-1}\|\mu\|\| V(x)\|, \qquad (\|\mu\| = \max|\mu_k|) \tag{12.4}$$

and, if $\|\gamma\|' \sin 2\sigma\varepsilon \geqq 2N(\xi_N \max \omega^*(\varphi))^\sigma$, moreover

$$\frac{d}{ds_j}(\| V(x)\|^N e^{-Re\Omega_j(x)}) \geqq \frac{\|\gamma\|' \sin 2\sigma\varepsilon}{2}|x|^{-\sigma-1}\| V(x)\|^N e^{-Re\Omega_j(x)}. \tag{12.5}$$

Here s_j is the arc length of the curve $\Gamma^*_{jx_1}$ measured from the origin to the variable point x on this curve and $\|\gamma\|' = \min|\gamma_j|$.

2°. PROOF OF LEMMA B. - We define the curves $\Gamma^*_{jx_1}$ in the exactly same way as in the proof of Lemma A in Section 8. Then Assertion 1) is evidently satisfied. Therefore, in order to prove Lemma B, we have only to prove the inequalities (12.3), (12.4) and (12.5).

By definition the vectorial inequality (12.3) is equivalent to n inequalities

$$|V_k(x)| < \delta_N \exp\left\{(Re\ \mu_k)\int_{\theta_0}^{\arg x} \cot L^*(\tau)d\tau + (Im\mu_k)(\theta_0 - \arg x)\right\} \tag{12.6.k}$$

as x is on the curve $\Gamma^*_{jx_1}$. Observe that the curve $\Gamma^*_{jx_1}$ consists of two parts Γ'_j and Γ''_j in general and we have $V_k(x) = v^1_k(x/x_1)^{\mu_k}$ and, consequently,

$$|V_k(x)| = |v^1_k|\left|\frac{x}{x_1}\right|^{Re\mu_k}\exp\left\{(Im\mu_k)\arg\left(\frac{x_1}{x}\right)\right\}. \tag{12.7.k}$$

On the curvilinear part Γ_j', $\rho=|x|$ is a function of φ given by (8.7). Hence we have

$$|V_k(x)|=|v_k^1|\exp\left\{(Re\mu_k)\int_{\arg x_1}^{\arg x}\cot l_j(\tau)d\tau+(Im\mu_k)'\arg x_1-\arg x)\right\}$$

and, by (8.6),

$$|V_k(x)|\leqq|v_k^1|\exp\left\{(Re\mu_k)\int_{\arg x_1}^{\arg x}\cot L^*(\tau)d\tau+(Im\mu_k)(\arg x_1-\arg x)\right\}.$$

On the other hand, v_k^1 must satisfy the inequality (12.6.k) for $\arg x=\arg x_1$. Hence, by inspection, we have inequality (12.6.k) for $x\in\Gamma_j'$.

On the rectilinear part Γ_j'', we have $|x|\leqq|x_1|$ and $\arg x$ is constant. Hence by virtue of (12.7.k) inequality (12.6.k) holds for $x\in\Gamma_j''$.
This proves the vectorial inequality (12.3).

To prove the inequality (12.4), it is to be noticed, by a simple calculation, that

$$|V_k(x)|^{-1}\frac{d}{ds_j}|V_k(x)|=Re\left(V_k(x)^{-1}\frac{d}{ds_j}V_k(x)\right)=Re\left(\mu_k x^{-1}\frac{dx}{ds_j}\right).$$

Since $|dx/ds_j|=1$ except for the joint of the curves Γ_j' and Γ_j'', it follows then that

$$\frac{d}{ds_j}|V_k(x)|\geqq-|\mu_k||x|^{-1}|V_k(x)|>-\|\mu\||x|^{-1}\|V(x)\|$$

on $\Gamma_{jx_1}^*$. This proves inequality (12.4).

To prove the inequality (12.5), we observe that $e^{-Re\Omega_j(x)}$ satisfies, by virtue of (8.2.1),

$$\frac{d}{ds_j}e^{-Re\Omega_j(x)}\geqq|x|^{-\sigma-1}e^{-Re\Omega_j(x)}\|\gamma\|'\sin 2\sigma\varepsilon$$

on $\Gamma_{jx_1}^*$. Hence, owing to (12.4), the expression of the left-hand member of (12.5) is not less than the expression

$$|x|^{-\sigma-1}\|V(x)\|^N e^{-Re\Omega_j(x)}(\|\gamma\|'\sin 2\sigma\varepsilon-N\|\mu\||x|^\sigma).$$

Choose ξ_N so small enough to have

$$\|\gamma\|'\sin 2\sigma\varepsilon\geqq 2N\|\mu\|(\xi_N\max\omega^*(\varphi))^\sigma\qquad\text{for }\underline{\Theta}^*\leqq\varphi\leqq\bar{\Theta}^*.$$

Then it is clear that we have inequality (12.5).

§ 13. Proof of Theorem B.

1°. - We make the change of variables

$$\mathcal{Y} = \sum_{|q|<N} V(x)^q f_q(x) + \eta, \quad \mathcal{Z} = \sum_{|q|<N} V(x)^q g_q(x) + \zeta \tag{13.1}$$

in the equations (11.1). Then the transformed equations can be written as

$$x^{\sigma+1}\eta' = 1_\alpha(\gamma)\eta + \tilde{\mathfrak{A}}(x,\ V(x);\ \eta,\ \zeta),\quad x\zeta' = \tilde{\mathfrak{B}}(x,\ V(x);\ \eta,\ \zeta). \tag{13.2}$$

Here $\tilde{\mathfrak{A}}(x,\ v;\ \eta,\ \zeta)$ and $\tilde{\mathfrak{B}}(x,\ v;\ \eta,\ \zeta)$ are holomorphic and bounded vector functions of $(x,\ v,\ \eta,\ \zeta)$ for

$$0 < |x| < \xi',\ \underline{\Theta}^* < \arg\ x < \bar{\Theta}^*,\ \|v\| < \delta',\ \|\eta\| < d',\ \|\zeta\| < d' \tag{13.3}$$

and satisfy there inequalities of the form

$$\begin{cases} \|\tilde{\mathfrak{A}}(x,\ v;\ \eta,\ \zeta)\| \leqq H'(\|\eta\| + \|\zeta\|) + B_N\|v\|^N, \\ \|\tilde{\mathfrak{B}}(x,\ v;\ \eta,\ \zeta)\| \leqq H''(\|\eta\| + \|\zeta\|) + B_N\|v\|^N. \end{cases} \tag{13.4}$$

Moreover, $\tilde{\mathfrak{A}}$ and $\tilde{\mathfrak{B}}$ satisfy LIPSCHITZ's conditions with respect to $(\eta,\ \zeta)$ with LIPSCHITZ's constants H' and H'' respectively. H' and H'' are positive constants independent of N. In particular, owing to Assumption ii) in Section 11, we can assume without loss of generality that H' satisfies

$$8H' < \|\gamma\|' \sin 2\sigma\varepsilon \tag{13.5}$$

for a preassigned positive number ε. B_N may depend on N.

Put

$$\eta = 1_\alpha(e^{\Omega(x)})P, \qquad \zeta = Q, \tag{13.6}$$

so that equations (13.2) are reduced to

$$\begin{cases} P' = x^{-\sigma-1}1_\alpha(e^{-\Omega(x)})\tilde{\mathfrak{A}}(x,\ V(x);\ 1_\alpha(e^{\Omega(x)})P,\ Q), \\ Q' = x^{-1}\tilde{\mathfrak{B}}(x,\ V(x);\ 1_\alpha(e^{\Omega(x)})P,\ Q). \end{cases} \tag{13.7$_N$}$$

We want to solve the following problem:

PROBLEM B. - *If* $8H' < \|\gamma\|' \sin 2\sigma\varepsilon$ *and* $4H'' < N\|Re\,\mu\|'$ *with* $\|Re\,\mu\|' = \min\{Re\,\mu_k\}$, *equations* (13.7) *have a unique solution of the form* $\{\varphi_N(x,\ V(x)),$

$\psi_N(x, V(x))\}$ *satisfying*

$$(13.8)_N \qquad [P] = O(\| V(x)\|^N)[e^{-Re\,\Omega(x)}], \quad \| Q\| = O(\| V(x))\|^N),$$

whenever $(x, V(x))$ *belongs to a domain of the form*

$$(13.9)_N \qquad 0 < |x| < \xi_N \omega^*(\arg x), \ [v] < \delta_N[\chi^*(\arg x)], \ \underline{\Theta}^* < \arg x < \bar{\Theta}^*.$$

Here $\varphi_N(x, v)$ *and* $\psi_N(x, v)$ *are respectively* α- *and* β-*column vectors whose components are holomorphic and bounded functions of* (x, v) *for* $(13.9)_N$.

This problem will be solved in Section 14.

2°. - Assume that Problem B has been solved. Then we can prove Theorem B in the following way. Owing to the transformations (13.1) and (13.6),

$$(13.10) \qquad \begin{cases} \sum_{|q|<N} V(x)^q f_q(x) + 1_\alpha(e^{\Omega(x)})\varphi_N(x, V(x)), \\ \sum_{|q|<N} V(x)^q g_q(x) + \psi_N(x, V(x)) \end{cases}$$

are a solution of equations (11.1) provided that $(x, V(x))$ is in the domain $(13.9)_N$. Let $N' > N$ be any integer. It is easy to see that

$$1_\alpha(e^{-\Omega(x)}) \sum_{N\leq|q|<N'} V(x)^q f_q(x) + \varphi_{N'}(x, V(x)),$$

$$\sum_{N\leq|q|<N'} V(x)^q g_q(x) + \psi_{N'}(x, V(x))$$

are a solution of equations $(13.7)_N$ *satisfying the condition* $(13.8)_N$ if $(x, V(x))$ belongs to the common part of the domains $(13.9)_N$ and $(13.9)_{N'}$. Hence, by the uniqueness of solution, this solution must coincide with the solution $\{\varphi_N(x, V(x)), \psi_N(x, V(x))\}$. From this it follows that the solution expressed by (13.10) is independent of N provided that $4H'' < N\| Re\,\mu\|'$. We write therefore this solution by $\{\mathfrak{Y}(x, V(x)), \mathfrak{W}(x, V(x))\}$. Then by analytic continuation the functions $\mathfrak{Y}(x, v)$ and $\mathfrak{W}(x, v)$ are defined in a domain of the form (11.7).

On the other hand, $v = 0$ is an interior point of the domain (11.7) in which the vector functions $\mathfrak{Y}(x, v)$ and $\mathfrak{W}(x, v)$ are defined.

Therefore, by CAUCHY's theorem, $\mathfrak{Y}(x, V(x))$ and $\mathfrak{W}(x, V(x))$ can be developed in uniformly convergent power series of $V(x)$ whenever $(x, V(x))$ belongs to the domain (11.7). Clearly, $\mathfrak{Y}(x, V(x))$ and $\mathfrak{W}(x, V(x))$ admit the asymptotic expansions (11.5). By the uniqueness of expansions, these asymptotic expansions must coincide with the uniformly convergent expansions. This proves the uniform convergence of the formal solutions (11 5).

Thus the proof of Theorem B has been completed.

§ 14 Solution of Problem B.

1°. FAMILY $\mathcal{F}$. - Let $\mathcal{F}=\{\varphi(x, v), \psi(x, v)\}$ be the family of α-column vectors $\varphi(x, v)$ and β-column vectors $\psi(x, v)$ whose components are holomorphic and bounded functions of (x, v) for $(13.9)_N$ and satisfy there inequalities of the form

$$[\varphi(x, v)] \leqq K_N\|v\|^N[e^{-Re\,\Omega(x)}],\ \|\psi(x, v)\| \leqq K_N\|v\|^N. \tag{14.1}$$

Here K_N is a certain positive constant.

Let (x_1, v^1) be an arbitrary point in the domain $(13.9)_N$ and choose the integration constant C'' so that $V(x_1)=v^1$. We define then the vectors $\Phi(x_1, v^1)$ and $\Psi(x_1, v^1)$ by

$$\Phi(x_1, v^1)=\int_0^{x_1}\mathcal{H}(x, V(x))dx,\ \Psi(x_1, v^1)=\int_0^{x_1}\mathcal{K}(x, V(x))dx, \tag{14.2}$$

where

$$\mathcal{H}(x, v)=x^{-\sigma-1}\mathbf{1}_\alpha(e^{-\Omega(x)})\tilde{\mathcal{A}}(x, v;\ \mathbf{1}_\alpha(e^{\Omega(x)})\ \varphi(x, v),\ \psi(x, v)),$$

$$\mathcal{K}(x, v)=x^{-1}\tilde{\mathcal{B}}(x, v;\ \mathbf{1}_\alpha(e^{\Omega(x)})\ \varphi(x, v),\ \psi(x, v)).$$

The integration of the j^{th} component of the first equation of (14.2) must be carried out along the curve $\Gamma^*_{jx_1}$ which was already defined in Section 8. The integration of the second equation of (14.2) must be carried out along the segment $\overline{0x_1}$ joining x_1 with the origin.

2°. MAPPING $\mathcal{T}$. - By virtue of (12.3) in Lemma B, the values of $(x, V(x))$ remain in the domain $(13.9)_N$ as x moves on the curve $\Gamma^*_{jx_1}$. Hence the j^{th} component of the integrand of the first equation of (14.2), say $\mathcal{H}_j(x, V(x))$, is a holomorphic function of x for $x\in\Gamma^*_{jx_1}$. Since $\|V(x)\|$ is a monotone increasing function of $|x|$ as x moves on the segment $\overline{0x_1}$, the values of x, $V(x)$ remain in domain $(13.9)_N$ for $x\in\overline{0x_1}$. Hence the integrand $\mathcal{K}(x, V(x))$ is a holomorphic function of x for $x\in\overline{0x_1}$.

The inequalities (13.4) imply that

$$\begin{cases} [\mathcal{H}(x, V(x))] \leqq (2H'K_N+B_N)\ |x|^{-\sigma-1}\ \|V(x)\|^N\ [e^{-Re\,\Omega(x)}], \\ \|\mathcal{K}(x, V(x))\| \leqq (2H''K_N+B_N)\ |x|^{-1}\ \|V(x)\|^N. \end{cases} \tag{14.3}$$

As we have already seen, the functions $e^{-Re\Omega_j(x)}$ tend to 0 exponentialls as x approaches 0 along the curves $\Gamma^*_{jx_1}$ respectively and $\|V(x)\|$ tends to 0 ay

x approaches 0 along the curves $\Gamma^*_{jx_1}$ or the segment $\overline{0x_1}$. Hence if $N\|Re\,\mu\|'>0$ the integrals (14.2) are convergent. This proves that the mapping $\mathfrak{T}$:

$$\{\varphi(x,\ v),\ \psi(x,\ v)\}\rightarrow\{\Phi(x,\ v),\ \Psi(x,\ v)\}$$

is well defined.

3°. EXISTENCE OF A FIXED-POINT. - Our solution of Problem B is based on a fixed-point theorem [2]. Since $\{0,\ 0\}\in\mathfrak{F}$, the family $\mathfrak{F}$ is not empty. Moreover, it is clear that $\mathfrak{F}$ is closed, normal and convex.

Therefore, in order to conclude the existence of a fixed-point of $\mathfrak{T}$, it is necessary to prove the following assertions:

a) $\mathfrak{T}$ *maps* $\mathfrak{F}$ *into itself, i.e.* $\mathfrak{T}\{\mathfrak{F}\}\subset\mathfrak{F}$.

b) $\mathfrak{T}$ *is a continuous mapping with respect to the topology of uniform convergence on compact subsets.*

We shall prove first Assertion a). This assertion is equivalent to the facts that:

a$_1$) *The vectors* $\Phi(x_1,\ v^1)$ *and* $\Psi(x_1,\ v^1)$ *satisfy the inequalities*

$$[\Phi(x_1,\ v^1)]\leqq K_N\|v^1\|^N[e^{-Re\Omega(x_1)}],\quad \|\Psi(x_1,\ v^1)\|\leqq K_N\|v^1\|^N. \tag{14.4}$$

a$_2$) $\Phi(x,\ v)$ *and* $\Psi(x,\ v)$ *are holomorphic and bounded functions of* $(x,\ v)$ *for* $(13.9)_N$.

In order to prove Assertion a$_1$), let s^1_j be the arc length of $\Gamma^*_{jx_1}$. By virtue of (14.3), the j^{th} component of the vector $\Phi(x_1,\ v^1)$ does not exceed

$$(2H'K_N+B_N)\int_0^{s^1_j}|x|^{-\sigma-1}\|V(x)\|^N e^{-Re\Omega_j(x)}ds_j$$

$$\leqq\frac{2(2H'K_N+B_N)}{\|\gamma\|'\ \sin\ 2\sigma\varepsilon}\|v^1\|^N e^{-Re\Omega_j(x_1)} \qquad \text{(by (12.5))}.$$

The last expression will be bounded by $K_N\|v^1\|^N e^{-Re\Omega_j(x_1)}$ if we can choose K_N large enough to have

$$K'\equiv 2B_N(\|\gamma\|'\ \sin\ 2\sigma\varepsilon-4H')^{-1}\leqq K_N.$$

Since $4H'<\|\gamma\|'\ \sin\ 2\sigma\varepsilon$, this choice of K_N is obviously possible. This proves the first inequality of (14.4).

Since

$$|V_k(x)|=\left|\frac{x}{x_1}\right|^{Re\mu_k}e^{(Im\mu_k)(\arg x_1-\arg x)},$$

we have

$$\frac{d}{d|x|}\|V(x)\| = \frac{d}{d|x|}|V_k(x)| = (Re\,\mu_k)\,|x|^{-1}\,|V_k(x)| \qquad \text{for some } k$$

$$= (Re\,\mu_k)\,|x|^{-1}\,\|V(x)\| \geqq \|Re\,\mu\|'\,|x|^{-1}\,\|V(x)\|$$

with $\|Re\,\mu\|' = \min\,\{Re\,\mu_k\}$ and, consequently,

$$\frac{d}{d|x|}\|V(x)\|^N \geqq N\|Re\,\mu\|'\,|x|^{-1}\,\|V(x)\|^N. \tag{14.5}$$

Hence we see by virtue of (14.3) that $\|\Psi(x_1,\ v^1)\|$ is not larger than

$$(2H''K_N + B_N)\int_0^{|x_1|}|x|^{-1}\,\|V(x)\|^N\,d|x|$$

$$\leqq \frac{2H''K_N + B_N}{N\|Re\,\mu\|'}\,\|v^1\|^N.$$

Since $N\|Re\,\mu\|' > 4H''$, if we take K_N large enough to have

$$K'' \equiv B_N(N\|Re\,\mu\|' - 2H'')^{-1} \leqq K_N,$$

$\|\Psi(x_1,\ v^1)\|$ will be bounded by $K_N\|v^1\|^N$, which proves the second inequality of (14.4).

Let $K_N = \max\,\{K',\ K''\}$. Then we get Assertion a$_1$). Concerning the quantity δ_N appearing in $(13.9)_N$, we have to take δ_N so small that

$$K_N(\delta_N\|\chi^*(\varphi)\|)^N < d' \qquad \text{for } \underline{\Theta}^* \leqq \varphi \leqq \bar{\Theta}^*.$$

To prove Assertion a$_2$), assume that, for x_0 sufficiently near to x_1, the relations

$$\begin{cases} \Phi(x_1,\ v^1) = \displaystyle\int_0^{x_0}\mathcal{H}(x,\ V(x))dx + \int_{x_0}^{x_1}\mathcal{H}(x,\ V(x))dx, \\ \Psi(x_1,\ v^1) = \displaystyle\int_0^{x_0}\mathcal{K}(x,\ V(x))dx + \int_{x_0}^{x_1}\mathcal{K}(x,\ V(x))dx \end{cases} \tag{14.6}$$

hold. In the first equation of (14.6), the j^{th} component of the first integral must be carried out along the path $\Gamma^*_{jx_0}$ and that of the second integral along the segment $\overline{x_0x_1}$. In the second equation of (14.6), the first and the second integrals must be carried out along the segments $\overline{0x_0}$ and $\overline{x_0x_1}$ respectively.

Then the relations (14.6) show that the vector functions $\Phi(x, v)$ and $\Psi(x, v)$ are holomorphic at $x = x_1$ with respect to x for each v^1. On the other hand, the inequalities (14.4) imply that the integrals (14.2) are uniformly convergent with respect to v^1 for each x_1. Hence, by HARTOGOS' theorem, the vector functions $\Phi(x, v)$ and $\Psi(x, v)$ are holomorphic at (x_1, v^1) with respect to (x, v). This proves Assertion a_2) since (x_1, v^1) is an arbitrary point in the domain $(13.9)_N$. Therefore, to get Assertion a_2), it is sufficient to prove the relations (14.6).

For the proof of the first relation of (14.6), it is sufficient to prove that we have

$$\Phi_j(x_1, v^1) = \int_0^{x_0} \mathcal{H}_j(x, V(x))dx + \int_{x_0}^{x_1} \mathcal{H}_j(x, V(x))dx \tag{14.7.j}$$

for each index j, where Φ_j is the j^{th} component of Φ.

Let t_0 and t_1 be respectively the intersection points of the paths $\Gamma^*_{jx_0}$ and $\Gamma^*_{jx_1}$ with a circle $|x| = \rho$ of small radius ρ. Then the relation (14.7.j) will be an immediate consequence of

$$\left| \int_{t_0}^{t_1} \mathcal{H}_j(x, V(x))dx \right| \to 0 \qquad \text{as } \rho \to 0. \tag{14.8.j}$$

Here the path of integration must be taken on the circle $|x| = \rho$. Since $Re\,\Omega_j(x) > 0$ for $x \in \widehat{t_0t_1}$, we see by virtue of (14.3) that the expression appearing in the left-hand member of (14.8.j) tends to 0 exponentially as $\rho \to 0$. This proves the relation (14.7.j).

Similarly the second relation of (14.6) can be proved and we omit therefore the proof.

In order to prove Assertion b), it will be sufficient to prove that, if $\{\varphi^k(x, v), \psi^k(x, v)\}$ be any sequence which converges to $\{\varphi(x, v), \psi(x, v)\}$, then the corresponding sequence $\{\Phi^k(x, v), \Psi^k(x, v)\}$ converges to the corresponding $\{\Phi(x, v), \Psi(x, v)\}$. However, this assertion is almost evident because the vector functions $\tilde{\mathcal{A}}(x, v; \eta, \zeta)$ and $\tilde{\mathfrak{B}}(x, v; \eta, \zeta)$ satisfy LIPSCHITZ'S conditions with respect to (η, ζ). Hence it is concluded by a fixed-point theorem (See for example [2]) that there exists a member $\{\varphi_N(x, v), \psi_N(x, v)\}$ of $\mathcal{F}$ that corresponds to a fixed-point of the mapping $\mathcal{T}$.

4°. EXISTENCE OF SOLUTION. - We assert that:

c) *The pair* $\{\varphi_N(x, V(x)), \psi_N(x, V(x))\}$ *is a solution of equations* $(13.7)_N$ *whenever* $(x, V(x))$ *belongs to* $(13.9)_N$.

To prove this assertion, we write $V(x)$ as $W(x, x_1, v^1)$. Then it is sufficient to prove that

$$\frac{d}{dx_0}\Phi(x_0, v^0) = \mathcal{H}(x_0, v^0), \quad \frac{d}{dx_0}\Psi(x_0, v^0) = \mathcal{K}(x_0, v^0), \tag{14.9}$$

where v^0 is a vector function of x_0 given by $W(x_0, x_1, v^1)$.

We shall prove the first equation of (14.9). Since $W(x, x_0, v^0) = W(x, x_1, v^1)$, the first equation of (14.2) implies that

$$\frac{d}{dx_0}\Phi(x_0, v^0) = \mathcal{H}(x_0, v^0) +$$

$$+ \int_0^{x_0} \frac{\partial \mathcal{H}(x, W)}{\partial W}\left\{\frac{\partial W(x, x_0, v^0)}{\partial x_0} + \frac{\partial W(x, x_0, v^0)}{\partial v^0}\frac{\partial W(x_0, x_1, v^1)}{\partial x_0}\right\}dx.$$

As is well known, for any constant ξ, $W(\xi, x, V)$ is an integral of the equation $xv' = \mathbf{1}_n(\mu)\, v$. Hence, the expression appearing in the braces of the above integrand is zero identically and we have the first equation of (14.9).

Similarly we can prove the second equation of (14.9).

5°. UNIQUENESS. - For the complete solution of Problem B, it remains only to prove that:

d) *A solution of equations* $(13.7)_N$ *satisfying the condition* $(13.8)_N$ *is unique.*

Suppose that there exist two solutions satisfying the same conditions. Let $\{P(x, V(x)), Q(x, V(x))\}$ be the difference of these two solutions. By assumption, $\tilde{\mathfrak{A}}(x, v; \eta, \zeta)$ and $\tilde{\mathfrak{B}}(x, v; \eta, \zeta)$ satisfy LIPSCHITZ'S conditions with respect to (η, ζ) with LIPSCHITZ'S constants H' and H'' respectively. Hence the j^{th} component of $P(x, v)$, say $P_j(x, v)$, and $Q(x, v)$ satisfy

$$|P_j(x_1, v^1)| \leqq H' \int_0^{s_j^1} M(x, V(x))\,|x|^{-\sigma-1}\, e^{-Re\,\Omega_j(x)} ds_j$$

and

$$\|Q(x_1, v^1)\| \leqq H'' \int_0^{|x_1|} M(x, V(x))\,|x|^{-1} d|x|$$

with

$$M(x, v) = \|\mathbf{1}_\alpha(e^{\Omega(x)})\, P(x, v)\| + \|Q(x, v)\|.$$

If we put

$$K = \sup_{(x, v)} \{\|v\|^{-N}\|\mathbf{1}_\alpha(e^{\Omega(x)})\, P(x, v)\|,\ \|v\|^{-N}\|Q(x, v)\|\}$$

when x and v move in $(13.9)_N$, then we have

$$M(x, v) \leqq 2K\|v\|^N, \qquad 0 \leqq K < +\infty$$

since $P(x, v)$ and $Q(x, v)$ satisfy inequalities similar to (14.1).

If we could prove that $K=0$, the proof of uniqueness would be completed. Suppose that $K \neq 0$. By virtue of (12.5), we have

$$[P(x_1, v^1)] \leqq \frac{4H'K}{\|\gamma\|' \sin 2\sigma\varepsilon}\|v^1\|^N[e^{-Re\Omega(x_1)}],$$

$$\|Q(x_1, v^1)\| \leqq \frac{2H''K}{N\|Re\,\mu\|'}\|v^1\|^N.$$

The definition of K yields

$$K \leqq K \max\left\{\frac{4H'}{\|\gamma\|' \sin 2\sigma\varepsilon}, \frac{2H''}{N\|Re\,\mu\|'}\right\}.$$

By the assumption imposed on Problem B, the last expression is not larger that $K/2$, which is a contradiction. Hence K must be zero.

Thus Problem B has been completely solved.

CHAPTER III.

Proof of Theorem 1.

I. Formal Solution (S_1).

§ 15. Formal Solutions. - Assumptions 1) ~ 6) in Section 1 and Assumptions I, II in Section 3 imply that the right-hand members of equations

$$\text{(A)} \qquad x^{\sigma+1}y' = f(x, y, z), \qquad xz' = g(x, y, z)$$

satisfy the following conditions:

$$\text{(15.1)} \quad \begin{cases} f_y(0, 0, 0) = \mathfrak{A},\ f_z(0, 0, 0) = 0,\ f_x(0, 0, 0) = 0,\ \det\mathfrak{A} \neq 0, \\ g_y(0, 0, 0) = 0,\ g_z(0, 0, 0) = \mathbf{1}_n(\mu),\ g_x(0, 0, 0) = 0. \end{cases}$$

Let $\Lambda_j(x) = -\nu_j/\sigma x^\sigma$ and let $\Theta_1 < \arg x < \bar{\Theta}_1$ be a sector which has Property-$\mathfrak{S}$ with respect to $\{\Lambda_1(x), \dots, \Lambda_m(x)\}$ and contains the positive real axis.

We shall prove the following proposition:

PROPOSITION 1.1. - *Let* $V(x) = 1_n(x^\mu)C'$. *Under Assumptions* I *and* II *equations* (A) *have a formal solution of the form*

$$(S_1)\qquad \begin{cases} y \sim \varphi[x;\ V(x)] + x^{\sigma+1}(a(x) + \sum_{|q|=1}^{\infty} V(x)^q a_q(x)), \\ z \sim \psi[x;\ V(x)] + x^{\sigma+1}(b(x) + \sum_{|q|=1}^{\infty} V(x)^q b_q(x)) \end{cases}$$

with the properties that

i) $\varphi[x;\ v]$ *and* $\psi[x;\ v]$ *are respectively m- and n-column vector functions with Property-*σ *with respect to* x *for* $\|v\| < \delta'$.

In particular, we have

$$(15.2)\qquad \psi_v[0;\ 0] = 1_n .$$

ii) $a(x)$ *and* $a_q(x)$ *are m-column vectors,* $b(x)$ *and* $b_q(x)$ *are n-column vectors, and the components of these vectors are functions which belong to class* $\mathcal{C}(\Theta_1,\ \bar{\Theta}_1;\ \xi^1)$.

If we replace φ, ψ, a, a_q, b and b_q by their corresponding convergent or asymptotic expansions, (S_1) is reduced to double power series of x and $V(x)$. The existence of such a formal solution can be verified by the fact that, if we apply a formal transformation of the form

$$y \sim Y + \sum_{l+|q|=2}^{\infty} x^l Z^q P_{lq},\quad z \sim Z + \sum_{l+|q|=2}^{\infty} x^l Z^q Q_{lq}$$

to equations (A) and determine the coefficient vectors in such a way that the formally transformed equations take as simple a form as possible, then the formally simplified equations have $Y = 0$, $Z = V(x)$ as a particular solution. From the formal point of view the formal solution (S_1) results from a suitable rearrangement of these double power series of x and $V(x)$.

However, in order to simplify the arguments, we want to prove directly the existence of a formal solution of the form (S_1). It is to be noticed that, if our purpose is only to construct an analytic expression for a solution with Property-$\mathcal{U}$ with respect to $V(x)$, the arguments for the construction of such a formal solution become much simpler than those which are going to be developed here.

§ 16. - Proof of Proposition 1.1 (Part I).

We shall determine first the vectors $\varphi[x;\, v]$ and $\psi[x;\, v]$. We expect these vectors to have the form

$$(16.1)\qquad \varphi[x;\, v] = \sum_{\varkappa=0}^{\sigma} x^{\varkappa}\varphi_{\varkappa}(v),\quad \psi[x;\, v] = \sum_{\varkappa=0}^{\sigma} x^{\varkappa}\psi_{\varkappa}(v),$$

where $\varphi_{\varkappa}(v)$ and $\psi_{\varkappa}(v)$ are holomorphic functions of v at $v = 0$.

The vectors $f(x, y, z)$ and $g(x, y, z)$ have unique representation of the form

$$(16.2)\qquad \begin{cases} f(x, y, z) = f[x;\, y, z] + x^{\sigma+1}f^{0}(x, y, z), \\ g(x, y, z) = g[x;\, y, z] + x^{\sigma+1}g^{0}(x, y, z), \end{cases}$$

where $f[x;\, y, z]$ and $g[x;\, y, z]$ have Property-σ with respect to x for $\|y\| < d$, $\|z\| < d$, and $f^{0}(x, y, z)$ and $g^{0}(x, y, z)$ have Property-$\mathfrak{A}$ with respect to (y, z) in $|x| < \xi$, $\|y\| < d$, $\|z\| < d$. By virtue of (15.1) we have at once

$$(16.3)\qquad \begin{cases} f_{y}[0;\, 0, 0] = \mathcal{A},\ f_{z}[0;\, 0, 0] = 0,\ \det \mathcal{A} \neq 0, \\ g_{y}[0;\, 0, 0] = 0,\ g_{z}[0;\, 0, 0] = 1_{n}(\mu). \end{cases}$$

If we substitute (S_1) for (y, z) into the right-hand members of equations (A) and omit all the terms containing $x^{\sigma+1}$ as factors, we have relations of the form, abbreviating the independent variable of $V(x)$,

$$(16.4)\qquad x^{\sigma+1}y' \sim \{f[0;\, \varphi_0(V), \psi_0(V)]\} + x\{K(V)\varphi_1(V) + H(V)\psi_1(V) + \mathfrak{R}_1(V)\} + \\ + \dots + x^{\sigma}\{K(V)\varphi_{\sigma}(V) + H(V)\psi_{\sigma}(V) + \mathfrak{R}_{\sigma}(V)\} + x^{\sigma+1}(\ \dots\ ),$$

$$(16.5)\qquad xz' \sim g[0;\, \varphi_0(V), \psi_0(V)]\} + x\{M(V)\varphi_1(V) + E(V)\psi_1(V) + \mathfrak{S}_1(V)\} + \\ + \dots + x^{\sigma}\{M(V)\varphi_{\sigma}(V) + E(V)\psi_{\sigma}(V) + \mathfrak{S}_{\sigma}(V)\} + x^{\sigma+1}(\ \dots\ ).$$

Here

$$(16.6)\qquad \begin{cases} K(v) \equiv f_{y}[0;\, \varphi_0(v), \psi_0(v)],\ H(v) \equiv f_{z}[0;\, \varphi_0(v), \psi_0(v)], \\ M(v) \equiv g_{y}[0;\, \varphi_0(v), \psi_0(v)],\ E(v) \equiv g_{z}[0;\, \varphi_0(v), \psi_0(v)]. \end{cases}$$

$\mathfrak{R}_h(v)$ is a linear form of m-column vectors $f_x[0;\, \varphi_0(v), \psi_0(v)], \dots, \dfrac{\partial^h}{\partial x^h} f[0;\, \varphi_0(v), \psi_0(v)]$ with polynomial coefficients of $\varphi_k(v)$ and $\psi_k(v)$ $(1 \leqq k < h)$ and $\mathfrak{S}_h(v)$ has the property similar to $\mathfrak{R}_h(v)$.

In particular, we have

$$\mathfrak{R}_1(v) = f_x[0;\ \varphi_0(v),\ \psi_0(v)],\ \ \mathfrak{S}_1(v) = g_x[0;\ \varphi_0(v),\ \psi_0(v)].$$

On the other hand, if we differentiate (S_1) term by term and pick up only the terms not containing $x^{\sigma+1}$ as factors, we have equations of the form

$$x^{\sigma+1}y' \sim x^{\sigma}\{x\varphi_0'(V)\} + x^{\sigma+1}(\ \dots\ ), \qquad \left(' = \frac{d}{dx}\right), \tag{16.7}$$

$$\begin{aligned} xz' \sim\ & \{x\psi_0'(V)\} + x\{x\psi_1'(V) + \psi_1(V)\} + \\ & + \dots + x^{\sigma}\{x\psi_\sigma'(V) + \sigma\psi_\sigma(V)\} + x^{\sigma+1}(\ \dots\ ). \end{aligned} \tag{16.8}$$

Since V is a solution of $xv' = \mathbf{1}_n(\mu)v$, we have

$$x\varphi'(V) = \frac{\partial}{\partial V}\varphi(V)\cdot \mathbf{1}_n(\mu)V,$$

which shows that the functions appearing in the braces $\{\ \dots\ \}$ depend on V alone.

From ((16.4), (16.7)) and ((16.5), (16.8)) we can easily derive the following differential equations which determine the vector functions $\{\varphi_h(V(x)),\ \psi_h(V(x))\}$ $(h = 0,\ 1,\ \dots,\ \sigma)$:

$$f[0;\ \varphi_0,\ \psi_0] = 0,\ \ x\psi_0' = g[0;\ \varphi_0,\ \psi_0] \tag{16.9}$$

and

$$\left\{\begin{aligned} & K(V)\varphi_h + H(V)\psi_h + \mathfrak{R}_h(V) = \frac{\partial}{\partial V}\varphi_{h-\sigma}(V)\cdot \mathbf{1}_n(\mu)V, \\ & x\psi_h' + h\psi_h = M(V)\varphi_h + E(V)\psi_h + \mathfrak{S}_h(V) \\ & \qquad\qquad\qquad\qquad (h = 1,\ 2,\ \dots,\ \sigma), \end{aligned}\right. \tag{16.10.h}$$

where we put $\varphi_k(v) = 0$ if $k < 0$.

§ 17. - Proof of Proposition 1.1 (Part II).

We shall solve differential equations (16.9) and (16.10.h).

1°. By solving the first equation of (16.9) with respect to φ_0, we have a unique equation of the form

$$\varphi_0 = F(\psi_0), \tag{17.1}$$

where $F(\psi_0)$ is a holomorphic vector function of ψ_0 at 0 and, by virtue of (16.3), satisfies

$$(17.2) \qquad F(0)=0, \qquad F_{\psi_0}(0)=0.$$

Substituting $F(\psi_0)$ for φ_0 into the second equation of (16.9), we have a differential equation of the form

$$(17.3) \qquad x\psi_0' = g(\psi_0).$$

$g(\psi_0)$ is an n-column vector whose components are holomorphic functions of ψ_0 for $\|\psi_0\| \leqq d'$, d' being a positive constant. By virtue of (16.3) and (17.2) it is easy to verify that $g(\psi_0)$ has a uniformly convergent expansion of the form

$$g(\psi_0) = \mathbf{1}_n(\mu)\psi_0 + \sum_{|q|=2}^{\infty} \psi_0^q g_q,$$

g_q being n-column constant vectors.

Since, by Assumption II, we have

$$\det(\mathbf{1}_n(\mu) - q\cdot\mu\mathbf{1}_n) \neq 0 \quad \text{for} \quad 2 \leqq |q|,$$

it can be verified that equation (17.3) possesses a formal solution of the form

$$(17.4) \qquad \psi_0 \sim V(x) + \sum_{|q|=2}^{\infty} V(x)^q Q_{0q}, \quad V(x) \equiv \mathbf{1}_n(x^{\mu})C'',$$

where Q_{0q} are n-column constant vectors. Therefore we see that the equation (17.3) has a form similar to equations (11.1) with $\alpha=0$, $\beta=n$.

By applying Theorem B in Section 11 to equation (17.3), it is concluded that:

The formal solution (17.4) *is uniformly convergent for* $\|V(x)\| \leqq \delta'$ *and the sum* $\psi_0(V(x))$ *is a solution of equation* (17.3) *for* $\|V(x)\| \leqq \delta'$, δ' *denoting a suitably chosen positive constant.*

Clearly $\psi_0(v)$ is an n-column vector function holomorphic in v for $\|v\| \leqq \delta'$ and satisfies

$$(17.5) \qquad \psi_0(0)=0, \ \frac{\partial}{\partial v}\psi_0(v)\Big|_{v=0} = \mathbf{1}_n.$$

Let $\varphi_0(v) = F(\psi_0(v))$. Then $\varphi_0(v)$ is an m-column vector function holomorphic in v for $\|v\| \leqq \delta'$ and $\{\varphi_0(V(x)), \psi_0(V(x))\}$ is a holomorphic solution of equations (16.9) whenever $\|V(x)\| \leqq \delta'$.

It is to be noticed that we have, by (17.2),

$$\varphi_0(0) = 0, \ \frac{\partial}{\partial v}\varphi_0(v)\Big|_{v=0} = 0. \tag{17.6}$$

2°. - By definition, $K(v)$, $H(v)$, $M(v)$ and $E(v)$ are holomorphic matrix functions of v for $\|v\| \leqq \delta'$. Furthermore, owing to (16.3), we have

$$\begin{cases} K(0) = \mathfrak{A}, \ H(0) = 0, \ \mathfrak{R}_1(0) = 0, \\ M(0) = 0, \ E(0) = \mathbf{1}_n(\mu), \ \mathfrak{S}_1(0) = 0 \end{cases} \tag{17.7}$$

In order to solve equations (16.10.h) by induction, we assume that the vectors $\varphi_\varkappa(v)$ and $\psi_\varkappa(v)$ have been already determined for $\varkappa \leqq h - 1$ in such a way that they are holomorphic vector functions of v for $\|v\| \leqq \delta'$ and the pairs $\{\varphi_\varkappa(V(x)),\ \psi_\varkappa(V(x))\}$ are holomorphic solutions of equations (16.10.$\varkappa$). Then $\mathfrak{R}_h(v)$ and $\mathfrak{S}_h(v)$ are both holomorphic vector functions of v for $\|v\| \leqq \delta'$.

From (16.10.h) we get an equation of the form

$$\varphi_h = F(V(x))\psi_h + f(V(x)), \ (F(v) \equiv -K(v)^{-1}H(v)), \tag{17.8}$$

where $F(v)$ and $f(v)$ are respectively $m \times m$ and $m \times 1$ matrices whose components are holomorphic functions of v for $\|v\| \leqq \delta'$, and they satisfy

$$F(0) = 0 \qquad \text{for any } h, \ f(0) = 0 \qquad \text{for } h = 1.$$

Substituting (17.8) for φ_h into the second equation of (16.10.h), we have an equation of the form

$$x\psi'_h = (G(V(x)) - h\mathbf{1}_n)\psi_h + g(V(x)), \tag{17.9.h}$$

where $G(v)$ and $g(v)$ are respectively $n \times n$ and $n \times 1$ matrices whose components are holomorphic functions of v for $\|v\| \leqq \delta'$, and

$$G(0) = \mathbf{1}_n(\mu) \qquad \text{for any } h, \ g(0) = 0 \qquad \text{for } h = 1.$$

Assumption II implies that

$$\det\left((h + q \cdot \mu)\mathbf{1}_n - \mathbf{1}_n(\mu)\right) \neq 0 \qquad \text{for } 2 \leqq h + |q|.$$

By using these inequalities we can verify that equation (17.9.h) admits a formal solution of the form

$$\psi_h \sim \sum V(x)^q Q_{hq}, \ (Q_{10} = 0), \tag{17.10.h}$$

where Q_{hq} are n-column constant vectors. Hence we see that equation (17.9.h) has a form similar to equations (11.1) with $\alpha = 0$, $\beta = n$. By applying Theorem B in Section 11 to equation (17.9.h), it is concluded that:

Equation (17.9.h) *has a solution* $\psi_h(V(x))$ *for* $\|V(x)\| \leqq \delta'$ *which admits there the uniformly convergent expansion* (17.10.h).

Hence equations (16.10.h) have a solution $\{\varphi_h(V(x)), \psi_h(V(x))\}$ for $\|V(x)\| \leqq \delta'$. Here $\varphi_h(v)$ and $\psi_h(v)$ are respectively m- and n-column vectors whose components are holomorphic functions of v for $\|v\| \leqq \delta'$ and vanish at $v = 0$ if, in particular, $h = 1$. The precise form of $\varphi_h(v)$ will be clear from (17.8).

Thus Assertion i) of Proposition 1.1 has been proved.

§ 18. - Proof of Proposition 1.1 (Part III).

In order to simplify the arguments, we apply first a transformation of the form

$$(\tilde{\mathrm{T}}) \qquad y = \varphi[x;\ V(x)] + x^{\sigma+1}\tilde{Y},\quad z = \psi[x;\ V(x)] + x^{\sigma+1}\tilde{Z}$$

to equations (A). By virtue of (16.2) we have

$$x^{\sigma+1}\tilde{Y}' = -(\sigma+1)x^{\sigma}\tilde{Y} + f^0(x;\ \varphi[x;\ V] + x^{\sigma+1}\tilde{Y},\ \psi[x;\ V] + x^{\sigma+1}\tilde{Z})$$
$$-\, x^{-\sigma-1}\{x^{\sigma+1}\varphi'[x;\ V] - f[x;\ \varphi[x;\ V] + x^{\sigma+1}\tilde{Y},\ \psi[x;\ V] + x^{\sigma+1}\tilde{Z}]\}.$$

and we have a similar equation for $x\tilde{Z}'$.

On the other hand, by the determination of the vectors $\varphi[x;\ v]$ and $\psi[x;\ v]$, we see that both of the vectors

$$x^{\sigma+1}\varphi'[x;\ V(x)] - f[x;\ \varphi[x;\ V(x)],\ \psi[x;\ V(x)]],$$
$$x\psi'[x;\ V(x)] - g[x;\ \varphi[x;\ V(x)],\ \psi[x;\ V(x)]]$$

contain $x^{\sigma+1}$ as factors and have Property-$\mathfrak{A}$ with respect to $V(x)$ in $|x| < \xi'$, $\|V(x)\| < \delta'$. From this fact it is concluded that:

The equations satisfied by $\{\tilde{Y},\ \tilde{Z}\}$ are written as

$$(\tilde{\mathrm{A}}) \qquad x^{\sigma+1}\tilde{Y}' = A(x,\ V(x),\ \tilde{Y},\ \tilde{Z}),\quad x\tilde{Z}' = B(x,\ V(x),\ \tilde{Y},\ \tilde{Z}),$$

where $A(x,\ v,\ \tilde{Y},\ \tilde{Z})$ and $B(x,\ v,\ \tilde{Y},\ \tilde{Z})$ are respectively m- and n-column vectors whose components are holomorphic and bounded functions of $(x,\ v,\ \tilde{Y},\ \tilde{Z})$ for

$$(18.1) \qquad |x| < \xi',\ \|v\| < \delta',\ \|\tilde{Y}\| < d',\ \|\tilde{Z}\| < d'.$$

ξ', δ', d' are positive constants and we can give a value to the quantity d' as large as we want provided that $(\xi')^{\sigma+1}d' < d$. Moreover we have

$$(18.2)\qquad \begin{cases} A_{\tilde{Y}}(0,\,0,\,0,\,0) = \mathfrak{A},\; A_{\tilde{Z}}(0,\,0,\,0,\,0) = 0, \\ B_{\tilde{Y}}(0,\,0,\,0,\,0) = 0,\; B_{\tilde{Z}}(0,\,0,\,0,\,0) = \mathbf{1}_n(\mu) - (\sigma+1)\mathbf{1}_n. \end{cases}$$

In order to complete the proof of Proposition 1.1, it will be sufficient to prove that equations $(\tilde{\mathrm{A}})$ possess a formal solution of the form

$$(\tilde{\mathrm{S}}_1)\qquad Y \sim a(x) + \sum_{|q|=1}^{\infty} V(x)^q a_q(x),\; Z \sim b(x) + \sum_{|q|=1}^{\infty} V(x)^q b_q(x).$$

We have to look for differential equations which determine these coefficients.

1°. - We see at once that the equations satisfied by $\{a(x), b(x)\}$ are given by

$$(18.3)\qquad x^{\sigma+1}a' = A(x,\,0,\,a,\,b),\; xb' = B(x,\,0,\,a,\,b).$$

Since $\det \mathfrak{A} \neq 0$ and $\det(\mathbf{1}_n(\mu) - (l+\sigma+1)\mathbf{1}_n) \neq 0$ (by Assumption II), we can easily verify that equations (18.3) possess a formal solution of the form

$$(18.4)\qquad a \sim \sum x^l P_{l0},\; b \sim \sum x^l Q_{l0},$$

where P_{l0} and Q_{l0} are m- and n-column constant vectors respectively. We can assume that $\|P_{00}\| < d'$ and $\|Q_{00}\| < d'$.

Hence equations (18.3) have a form similar to equations (6.1) with

$$\alpha = m,\; \beta = n,\; \mathbf{1}_\alpha(\gamma) = \mathbf{1}_m(\nu).$$

It follows then that a sector with Property-$\mathcal{S}$ with respect to $\{\Omega_1(x), \ldots, \Omega_\alpha(x)\}$ has Property-$\mathcal{S}$ with respect to $\{\Lambda_1(x), \ldots, \Lambda_m(x)\}$. By applying Theorem A in Section 6 to equations (18.3), we have the following conclusion:

The vectors $a(x)$ and $b(x)$ are uniquely determined as a solution of equations (18.3) *in such a way that they belong to class $\mathfrak{C}(\underline{\Theta}_1, \bar{\Theta}_1;\, \xi^1)$ and admit asymptotic expansions of the form* (18.4) *as x tends to 0 through the sector $\underline{\Theta}_1 < \arg x < \bar{\Theta}_1$, ξ^1 being a positive number.*

2°. - To simplify our calculation, we make the change of variables

$$Y = \eta + a(x),\; Z = \zeta + b(x)$$

to equations $(\tilde{\mathrm{A}})$. Then the transformed equations can be written as

$$
(18.5)\quad \begin{cases} x^{\sigma+1}\eta' = E(x,\ V(x))\eta + F(x,\ V(x))\zeta + \displaystyle\sum_{|p|+|q|=2}^{\infty} \eta^p\zeta^q f_{pq}(x,\ V(x)) + f(x,\ V(x)), \\ x\zeta' = G(x,\ V(x))\eta + H(x,\ V(x))\zeta + \displaystyle\sum_{|p|+|q|=2}^{\infty} \eta^p\zeta^q g_{pq}(x,\ V(x)) + g(x,\ V(x)), \end{cases}
$$

where the power series in the right-hand members are uniformly convergent for

$$0<|x|<\xi^1,\ \underline{\Theta}_1<\arg x<\bar{\Theta}_1,\ \|V(x)\|<\delta^1,\ \|\eta\|<d^1,\ \|\zeta\|<d^1$$

and the coefficients have Property-$\mathfrak{A}$ with respect to $V(x)$ in

$$0<|x|<\xi^1,\ \underline{\Theta}_1<\arg x<\ddot{\Theta}_1,\ \|V(x)\|<\delta^1,$$

ξ^1, δ^1, d^1 being suitably chosen positive constants. Moreover, we have

$$
(18.6)\quad \begin{cases} E(0,\ 0)=\mathcal{A},\ F(0,\ 0)=0,\ f(x,\ 0)\equiv 0, \\ G(0,\ 0)=0,\ H(0,\ 0)=\mathbf{1}_n(\mu)-(\sigma+1)\mathbf{1}_n,\ g(x,\ 0)\equiv 0. \end{cases}
$$

We assert that equations (18.5) possess a formal solution of the form

$$
(18.7)\quad \eta \sim \sum_{|q|=1}^{\infty} V(x)^q a_q(x),\ \zeta \sim \sum_{|q|=1}^{\infty} V(x)^q b_q(x).
$$

Indeed, differentiation of (18.7) term by term yields

$$
(18.8)\quad \begin{cases} x^{\sigma+1}\eta' \sim \sum V(x)^q x^{\sigma+1}a_q'(x) + \sum x\ V(x)^q (q\cdot\mu)a_q(x), \\ x\zeta' \sim \sum V(x)^q x b_q'(x) + \sum V(x)^q (q\cdot\mu) b_q(x). \end{cases}
$$

Substituting the power series (18.7) for $(\eta,\ \zeta)$ into the right-hand members of (18.5) and rearranging formally the resulting equations in the form of a single power series of $V(x)$, we have equations of the form

$$
(18.9)\quad \begin{cases} x^{\sigma+1}\eta' \sim \displaystyle\sum_{|q|=1}^{\infty} V(x)^q \{E(x,\ 0)a_q(x) + F(x,\ 0)b_q(x) + \mathfrak{R}_q(x)\}, \\ x\zeta' \sim \displaystyle\sum_{|q|=1}^{\infty} V(x)^q \{G(x,\ 0)a_q(x) + H(x,\ 0)b_q(x) + \mathfrak{S}_q(x)\}. \end{cases}
$$

Here $\mathfrak{R}_q(x)$ is a linear form of *known* m-column vector functions belonging to class $\mathcal{C}(\Theta_1, \bar{\Theta}_1; \xi^1)$ with polynomial coefficients of $a_{q'}(x)$ and $b_{q'}(x)$ for $|q'| < |q|$ and $\mathfrak{S}_q(x)$ has the property similar to $\mathfrak{R}_q(x)$.

From (18.8) and (18.9) we get the following linear differential equations which determine the vector functions $a_q(x)$ and $b_q(x)$:

$$(18.10)\qquad \begin{cases} x^{\sigma+1}a'_q = \{E(x, 0) - x^{\sigma}(q\cdot\mu)\mathbf{1}_m\}a_q + F(x, 0)b_q + \mathfrak{R}_q(x), \\ xb'_q = G(x, 0)a_q + \{H(x, 0) - (q\cdot\mu)\mathbf{1}_n\}b_q + \mathfrak{S}_q(x). \end{cases}$$

Since $\det E(0, 0) = \det \mathfrak{A} \neq 0$ and $\det(\mathbf{1}_n(\mu) - (l+1+q\cdot\mu)\mathbf{1}_n) \neq 0$ (by Assumption II), we can prove, by using (18.6), that equations (18.10) have formal solutions of the form

$$(18.11)\qquad a_q \sim \sum_l x^l P_{lq},\ b_q \sim \sum_l x^l Q_{lq},$$

where P_{lq} and Q_{lq} are m- and n-column constant vectors respectively. Hence equations (18.10) have a form similar to equations (6.1) with

$$\alpha = m,\ \beta = n,\ \mathbf{1}_{\alpha}(\gamma) = \mathbf{1}_m(\nu).$$

Since (18.10) are linear differential equations, Theorem A in Section 6 says that:

The vectors $a_q(x)$ and $b_q(x)$ are determined successively as a unique solution of equations (18.10) *in such a way that they belong to class $\mathcal{C}(\Theta_1, \bar{\Theta}_1; \xi^1)$ and admit asymptotic expansions of the form* (18.11).

Thus Proposition 1.1 has been completely proved.

II. Uniform Convergence of Formal Solution (S_1).

§ 19. **Proof of Theorem 1.** – By virtue of Proposition 1.1 in Section 15 we have the formal solution (S_1) for equations (A). In order to prove Theorem 1, namely to prove uniform convergence of the formal solution (S_1), we make the change of variables

$$(\tilde{\mathrm{T}})\qquad y = \varphi[x;\ V(x)] + x^{\sigma+1}\tilde{Y},\ z = \psi[x;\ V(x)] + x^{\sigma+1}\tilde{Z}$$

to equations (A). As we have already proved in Section 18, the equations satisfied by $\{\tilde{Y}, \tilde{Z}\}$ take the form

$$(\tilde{\mathrm{A}})\qquad x^{\sigma+1}\tilde{Y}' = A(x,\ V(x),\ \tilde{Y},\ \tilde{Z}),\ x\tilde{Z}' = B(x,\ V(x),\ \tilde{Y},\ \tilde{Z}),$$

where $A(x, v, \tilde{Y}, \tilde{Z})$ and $B(x, v, \tilde{Y}, \tilde{Z})$ are holomorphic and bounded vector functions of $(x, v, \tilde{Y}, \tilde{Z})$ for

$$|x| < \xi^0, \ \|v\| < \delta^0, \ \|\tilde{Y}\| < d^0, \ \|\tilde{Z}\| < d^0$$

and we have

$$A_{\tilde{Y}}(0, 0, 0, 0) = \mathfrak{A}, \ A_{\tilde{Z}}(0, 0, 0, 0) = 0. \tag{19.1}$$

ξ^0, δ^0 and d^0 are suitably chosen positive constants and satisfy

$$(\xi^0)^{\sigma+1} d^0 < d.$$

Moreover equations $(\tilde{\mathrm{A}})$ possess a formal solution of the form

$$\tilde{Y} \sim a(x) + \sum_{|q|=1}^{\infty} V(x)^q a_q(x), \ \tilde{Z} \sim b(x) + \sum_{|q|=1}^{\infty} V(x)^q b_q(x) \tag{$\tilde{\mathrm{S}}_1$}$$

with coefficient vectors belonging to class $\mathcal{C}(\underline{\Theta}_1, \bar{\Theta}_1; \xi')$. We can assume that $\|a(x)\| < d^0$, $\|b(x)\| < d^0$ since $a(x)$ and $b(x)$ are bounded.

From these facts we see that equations $(\tilde{\mathrm{A}})$ have quite a similar form to equations (11.1). By applying Theorem B in Section 11 to equations $(\tilde{\mathrm{A}})$, we have at once the following conclusion:

THEOREM 1′. - *Equations* $(\tilde{\mathrm{A}})$ *have a solution of the form* $\{\Phi^0(x, V(x)), \Psi^0(x, V(x))\}$ *with* $V(x) \equiv \mathbf{1}_n(x^{\mu})C''$, *whenever* $(x, V(x))$ *belongs to a domain of the form*

$$0 < |x| < \xi_1'', \ \underline{\Theta}_1 < \arg x < \bar{\Theta}_1, \ \|v\| < \delta_1'' \tag{19.2}$$

Here $\Phi^0(x, v)$ *and* $\Psi^0(x, v)$ *are respectively* m- *and* n-*column vector functions with Property*-$\mathfrak{A}$ *with respect to* v *in* (19.2) *and moreover admit there uniformly convergent expansions of the form* $(\tilde{\mathrm{S}}_1)$ *with* $V(x) = v$.

We define $\Phi(x, v)$ and $\Psi(x, v)$ by the formulas

$$\Phi(x, v) = \varphi[x; v] + x^{\sigma+1}\Phi^0(x, v), \ \Psi(x, v) = \psi[x; v] + x^{\sigma+1}\Psi^0(x, v).$$

Then, owing to the transformation $(\tilde{\mathrm{T}})$, the pair $\{\Phi(x, \mathbf{1}_n(x^{\mu})C''), \Psi(x, \mathbf{1}_n(x^{\mu})C'')\}$ is a solution of equations (A) provided that the values of x and $\mathbf{1}_n(x^{\mu})C''$ stay in the domain (19.2). Obviously the vectors $\Phi(x, v)$ and $\Psi(x, v)$ thus defined satisfy conditions stated in Theorem 1 in Section 3. Thus the proof of Theorem 1 has been completed.

CHAPTER IV

Proof of Theorem 2.

§ 20. - Equations (B_1) and Reduction of Linear Parts.

As we have already seen in Section 4, equations (B_1) have the form, by virtue of Assumption IV,

$$(B_1)\qquad \begin{cases} x^{\sigma+1}\tilde{y}' = 1_m(\nu)\tilde{y} + C(x,\ \tilde{z})\tilde{y} + \sum\limits_{|p|=2}^{\infty} \tilde{y}^p F_p(x,\ \tilde{z}), \\ x\tilde{z}' = 1_n(\mu)\tilde{z} + D(x,\ \tilde{z})\tilde{y} + \sum\limits_{|p|=2}^{\infty} \tilde{y}^p G_p(x,\ \tilde{z}), \end{cases}$$

where the power series in the right-hand members are uniformly convergent for

$$(20.1)\qquad 0 < |x| < \xi_1, \qquad \underline{\Theta}_1 < \arg x < \bar{\Theta}_1, \qquad \|\tilde{y}\| < d_1, \qquad \|\tilde{z}\| < d_1.$$

Here $C(x,\ \tilde{z})$, $D(x,\ \tilde{z})$, $F_p(x,\ \tilde{z})$ and $G_p(x,\ \tilde{z})$ are respectively $m \times m$, $n \times m$, $m \times 1$ and $n \times 1$ matrices whose components are functions with Property-$\mathfrak{A}$ with respect to $\tilde{z}$ in

$$(20.2)\qquad 0 < |x| < \xi_1, \qquad \underline{\Theta}_1 < \arg x < \bar{\Theta}_1, \qquad \|\tilde{z}\| < d_1$$

and, in particular, we have

$$(20.3)\qquad C(0,\ 0) = 0, \qquad D(0,\ 0) = 0.$$

Moreover, the matrix $C(x,\ \tilde{z})$ has a unique representation of the form

$$(20.4)\qquad C(x,\ \tilde{z}) = c[x;\ \tilde{z}] + x^{\sigma+1}C^0(x,\ \tilde{z}),$$

where $c[x;\ \tilde{z}]$ is an $m \times m$ matrix function with Property-σ with respect to x for $\|\tilde{z}\| < d_1$ and $C^0(x,\ \tilde{z})$ is an $m \times m$ matrix whose components have Property-$\mathfrak{A}$ with respect to $\tilde{z}$ in (20.2).

To construct a transformation of the form (T_2) appearing in Theorem 2 in Section 4, we consider first a transformation of the form

$$(T_2')\qquad \tilde{y} = \eta + \tilde{A}(x,\ \zeta)\eta, \qquad \tilde{z} = \zeta + x^{\sigma}\tilde{B}(x,\ \zeta)\eta,$$

where $\tilde{A}(x,\ \zeta)$ is an $m \times m$ matrix *with off-diagonal form* and $\tilde{B}(x,\ \zeta)$ is an $n \times m$ matrix.

We impose upon $\tilde{A}(x,\ \zeta)$ and $\tilde{B}(x,\ \zeta)$ the conditions that they have Property-$\mathfrak{A}$ with respect to ζ in a domain of the form

$$(20.5)\qquad 0<|x|<\xi_2'',\quad \underline{\Theta}_2<\arg x<\bar{\Theta}_2,\quad \|\zeta\|<d_2'',$$

where $\underline{\Theta}_2$ and $\bar{\Theta}_2$ are the same as those that appeared in 3° in Section 4.

Then we want to prove first the following proposition:

PROPOSITION 2.1. - *We can determine the matrices $\tilde{A}(x, \zeta)$ and $\tilde{B}(x, \zeta)$ of the transformation* (T_2') *in such a way that equations* (B_1) *are transformed into equations of the form*

$$(20.6)\qquad x^{\sigma+1}\eta' = \mathbf{1}_m(F(x, \zeta))\eta + [\eta]_2,\qquad x\zeta' = \mathbf{1}_n(\mu)\zeta + [\eta]_2.$$

Here $F(x, \zeta)$ is an m-column vector function with a unique representation of the form

$$(20.7)\qquad F(x, \zeta) = f[x;\ \zeta] + x^{\sigma+1}F^0(x, \zeta),$$

where $f[x;\ \zeta]$ has Property-σ with respect to x for $\|\zeta\|<d_2''$ while $F^0(x, \zeta)$ has Property-$\mathfrak{A}$ with respect to ζ in (20.5).

The symbol $[\eta]_2$ represents a uniformly convergent power series of η for

$$(20.8)\qquad 0<|x|<\xi_2'',\quad \underline{\Theta}_2<\arg x<\bar{\Theta}_2,\quad \|\eta\|<d_2'',\quad \|\zeta\|<d_2''$$

which satisfies the condition $[\eta]_2 = O(\|\eta\|^2)$ and whose coefficients are vector functions having Property-$\mathfrak{A}$ with respect to ζ in (20.5).

The proof of this proposition will be given in Section 21.

Next we want to prove the following proposition.

PROPOSITION 2.2. - *Put*

$$(20.9)\qquad \lambda[x;\ Z] = f[x;\ Z] - \frac{x^\sigma}{\sigma!}\cdot\frac{\partial^\sigma}{\partial x^\sigma}(f[x;\ Z]-f[x;\ 0])\Big|_{x=0}.$$

Then there exists a transformation of the form

$$(T_2'')\qquad \eta = Y + \mathbf{1}_m(R(x, Z))Y,\qquad \zeta = Z,$$

by which equations (20.6) *are changed to equations* (B_2) *appearing in Theorem* 2 *in Section* 4. *Equations* (B_2), *picking up the linear terms only, are written as*

$$(20.10)\qquad x^{\sigma+1}Y' = \mathbf{1}_m(\lambda[x;\ Z])Y + [Y]_2,\qquad xZ' = \mathbf{1}_n(\mu)Z + [Y]_2.$$

Here $R(x, Z)$ is an m-column vector with elements $\{R_j(x, Z)\}$, where $R_j(x, Z)$ have Property-$\mathfrak{A}$ with respect to Z in the domain (20.8) *and* $R_j(0, 0) = 0$.

This proposition will be proved in Section 22.

Assume that these two propositions have been established. By combining (T_2'') with (T_2') we have a transformation from $(\tilde{y}, \tilde{z})$ to (Y, Z) of the form

$$(T_2) \qquad \tilde{y} = Y + A(x, Z)Y, \qquad \tilde{z} = Z + x^\sigma B(x, Z)Y,$$

where

$$(20.11) \qquad \begin{cases} A(x, Z) = (\mathbf{1}_m + \tilde{A}(x, Z))\mathbf{1}_m(R(x, Z)) + \tilde{A}(x, Z), \\ B(x, Z) = \tilde{B}(x, Z)(\mathbf{1}_m + \mathbf{1}_m(R(x, Z))). \end{cases}$$

This completes the proof of Theorem 2 in Section 4.

§ 21. - Proof of Proposition 2.1.

In order to prove the proposition, first of all we have to look for differential equations which determine the matrices $\tilde{A}(x, \zeta)$ and $\tilde{B}(x, \zeta)$.

Differentiating (T_2') and replacing $\{x^{\sigma+1}\eta', x\zeta'\}$ by (20.6), we have equations of the form

$$(21.1) \qquad \begin{cases} x^{\sigma+1}\tilde{y}' = \Big\{\mathbf{1}_m(F(x, \zeta)) + x^{\sigma+1}\dfrac{\partial \tilde{A}}{\partial x} + x^\sigma \sum\limits_{k=1}^{n} \dfrac{\partial \tilde{A}}{\partial \zeta_k}\mu_k\zeta_k + \\ \qquad + \tilde{A}\mathbf{1}_m(F(x, \zeta))\Big\}\eta + [\eta]_2, \\ x\tilde{z}' = \mathbf{1}_n(\mu)\zeta + \Big\{x^{\sigma+1}\dfrac{\partial \tilde{B}}{\partial x} + x^\sigma \sum \dfrac{\partial \tilde{B}}{\partial \zeta_k}\mu_k\zeta_k + \\ \qquad + \tilde{B}\mathbf{1}_m(F(x, \zeta)) + \sigma x^\sigma \tilde{B}\Big\}\eta + [\eta]_2. \end{cases}$$

In the other direction, a substitution of (T_2') for $(\tilde{y}, \tilde{z})$ into the right-hand members of equations (B_1) yields equations of the form

$$(21.2) \qquad \begin{cases} x^{\sigma+1}\tilde{y}' = (\mathbf{1}_m(\nu) + C(x, \zeta))(\mathbf{1}_m + \tilde{A}(x, \zeta))\eta + [\eta]_2, \\ x\tilde{z}' = \mathbf{1}_n(\mu)\zeta + \{D(x, \zeta)(\mathbf{1}_m + \tilde{A}(x, \zeta)) + x^\sigma \mathbf{1}_n(\mu)\tilde{B}(x, \zeta)\}\eta + [\eta]_2. \end{cases}$$

By equating the coefficients of the linear terms of η appearing in the right-hand members of the first equations of (21.1) and (21.2), we have the equation:

$$x^{\sigma+1}\frac{\partial \tilde{A}}{\partial x} + x^\sigma \sum_{k=1}^{n} \frac{\partial \tilde{A}}{\partial \zeta_k}\mu_k\zeta_k = (\mathbf{1}_m(\nu) + C(x, \zeta))(\mathbf{1}_m + \tilde{A}) - (\mathbf{1}_m + \tilde{A})\mathbf{1}_m(F(x, \zeta)).$$

Then we see that $\tilde{A}(x, V(x))$, with $V(x) = \mathbf{1}_n(\lambda^{\mu})C''$, satisfies the equation

$$(21.3) \qquad x^{\sigma+1}\tilde{A}' = (\mathbf{1}_m(\nu) + C(x, V(x))(\tilde{A} + \mathbf{1}_m) - (\tilde{A} + \mathbf{1}_m)\mathbf{1}_m(F(x, V(x))).$$

Similarly we can verify that $\tilde{B}(x, V(x))$ satisfies the equation:

$$(21.4) \quad x^{\sigma+1}\tilde{B}' = -\tilde{B}\mathbf{1}_m(F(x, V(x))) + x^{\sigma}(\mathbf{1}_n(\mu) - \mathbf{1}_n)\tilde{B} + D(x, V(x))(\mathbf{1}_m + \tilde{A}(x, V(x))).$$

1°. DETERMINATION OF $\tilde{A}(x, v)$. - Since, by hypothesis, $\tilde{A}$ *is of off-diagonal form*, we see that equations (21.3) are equivalent to m^2 equations of the form

$$(21.3') \qquad F_k(x, V(x)) = \sum_{h \neq k} C_{kh}(x, V(x))\tilde{A}_{hk} + \nu_k + C_{kk}(x, V(x))$$

and

$$(21.3'') \quad x^{\sigma+1}\tilde{A}'_{jk} = \nu_j\tilde{A}_{jk} + \sum_{h \neq k} C_{jh}(x, V(x))\tilde{A}_{hk} - \tilde{A}_{jk}F_k(x, V(x)) + C_{jk}(x, V(x)) \qquad (j \neq k),$$

$$(21.3''') \quad \tilde{A}_{jj} \equiv 0.$$

$\tilde{A}_{jk}$ and C_{jk} are the (j, k)-elements of $\tilde{A}$ and C, and F_j is the j^{th} component of F. Inserting (21.3') for F_k into equations (21.3′), we have equations of the form

$$(21.5) \qquad x^{\sigma+1}\tilde{A}'_{jk} = (\nu_j - \nu_k)\tilde{A}_{jk} + \sum_{h \neq k} C_{jh}(x, V(x))\tilde{A}_{hk} -$$

$$- \tilde{A}_{jk}C_{kk}(x, V(x)) - \tilde{A}_{jk}\sum_{h \neq k} C_{kh}(x, V(x))\tilde{A}_{hk} + C_{jk}(x, V(x)) \qquad (j \neq k),$$

which determine the components $\tilde{A}_{jk}(x, v)$ for $j \neq k$.

Notice that:

i) $\nu_j - \nu_k \neq 0$ for $j \neq k$ and $C_{jk}(0, 0) = 0$ for each (j, k) (See Assumption IV and (20.3)).

ii) For each (j, k), $C_{jk}(x, v)$ has a unique representation of the form

$$(21.6) \qquad C_{jk}(x, v) = c_{jk}[x; v] + x^{\sigma+1}C^0_{jk}(x, v) \qquad \text{(See (4.4))},$$

where $c_{jk}[x; v]$ has Property-σ with respect to x for $\|v\| < d_1$ while $C^0_{jk}(x, v)$ has Property-$\mathfrak{A}$ with respect to v in

$$0 < |x| < \xi_1, \qquad \underline{\Theta}_1 < \arg x < \bar{\Theta}_1, \qquad \|v\| < d_1.$$

By applying the arguments in Sections 16, 17, 18, which were used to construct the formal solution (S_1) of equations (A), to equations (21.5), we can prove that equations (21.5) have a formal solution of the form

$$(21.7) \qquad \tilde{A}_{jk} \sim a_{jk}[x;\ V(x)] + x^{\sigma+1}\left(a_{jk}(x) + \sum_{|q|=1}^{\infty} V(x)^q a_{q,\,jk}(x)\right).$$

Here $a_{jk}[x;\ v]$ have Property-σ with respect to x for $\|v\| < d_1$, $a_{jk}(x)$ and $a_{q,\,jk}(x)$ belong to class $\mathcal{C}\,\underline{\Theta}_2,\ \bar{\Theta}_2;\ \xi_1')$.

In order to prove uniform convergence of the formal solution (21.7), we have to change the dependent variables from $\{\tilde{A}_{jk}\}$ to $\{Y_{jk}\}$, where

$$Y_{jk} = x^{-\sigma-1}(\tilde{A}_{jk} - a_{jk}[x;\ V(x)]).$$

By using the reasonings in Section 18, which were used to derive equations $(\tilde{\mathrm{A}})$ from (A) by transformation $(\tilde{\mathrm{T}})$, we see that the equations on $\{Y_{jk}\}$ have a form similar to equations (11.1) with

$$\alpha = m^2 - m, \qquad \beta = 0, \qquad \mathfrak{l}_\alpha(\gamma) = \sum_{j \neq k} \oplus (\nu_j - \nu_k)\mathbf{1}_1,$$

where the symbol $\oplus$ denotes the direct sum.

By applying Theorem B in Section 11 to the equations on $\{Y_{jk}\}$, it is concluded that:

The equations on $\{Y_{jk}\}$ *have a solution* $\{A^0_{jk}(x,\ V(x))\}$ *whenever* $(x,\ V(x))$ *belongs to a domain of the form*

$$(21.8) \qquad 0 < |x| < \xi_2'', \qquad \underline{\Theta}_2 < \arg x < \bar{\Theta}_2, \qquad \|v\| < d_2''.$$

$A^0_{jk}(x,\ V(x))$ *admit uniformly convergent expansions of the form*

$$A^0_{jk}(x,\ V(x)) = a_{jk}(x) + \sum_{|q|=1}^{\infty} V(x)^q a_{q,\,jk}(x)$$

provided that $(x,\ V(x))$ *is in* (21.8), *so that* $A^0_{jk}(x,\ v)$ *are functions with Property-*$\mathfrak{A}$ *with respect to* v *in* (21.8).

Hence if we put

$$(21.9) \qquad \tilde{A}_{jk}(x,\ v) = a_{jk}[x;\ v] + x^{\sigma+1} A^0_{jk}(x,\ v),$$

$\{\tilde{A}_{jk}(x,\ V(x))\}$ *are a solution of equations* (21.5) *and admit uniformly convergent expansions* (21.7) *whenever the values of* x, $V(x)$ *stay in* (21.8).

It is clear that $\tilde{A}_{jk}(x,\ v)$ have Property-$\mathfrak{A}$ with respect to v in (21.8).

2°. DETERMINATION OF $F(x,\ v)$. - Substituting (21.9) for $\tilde{A}_{jk}$ into equations (21.3′) and replacing $V(x)$ by v, we have at once the following conclusion:

The components $F_k(x, v)$ *of the m-column vector* $F(x, v)$ *are given by*

$$F_k(x, v) = \sum_{h \neq k} C_{kh}(x, v)\tilde{A}_{hk}(x, v) + \nu_k + C_{kk}(x, v) \tag{21.10}$$

and, by virtue of (21.6) *and* (21.9), *have unique representations of the form*

$$F_k(x, v) = f_k[x; v] + x^{\sigma+1}F_k^0(x, v), \qquad f_k[0; 0] = \nu_k,$$

where $f_k[x; v]$ *and* $F_k^0(x, v)$ *are the same functions as those that appeared in Proposition* 2.1.

3°. DETERMINATION OF $\tilde{B}(x, v)$. – We consider linear differential equations (21.4) in which $F(x, V(x))$ and $\tilde{A}(x, V(x))$ are *known functions.*

Observe that $\mathbf{1}_m(F(0, 0)) = \mathbf{1}_m(\nu)$ and $\det \mathbf{1}_m(\nu) \neq 0$. By applying the reasonings in Section 18, we can easily prove that equations (21.4) possess a formal solution of the form

$$\tilde{B} \sim B(x) + \sum_{|q|=1}^{\infty} V(x)^q B_q(x), \tag{21.11}$$

where $B(x)$ and $B_q(x)$ are both $n \times m$ matrix functions belonging to class $\mathcal{C}(\Theta_2, \bar{\Theta}_2; \xi_2'')$.

We now introduce an *mn-column* vector $\mathcal{Y}$ with elements $(\tilde{B}_{11}, \ldots, \tilde{B}_{n1}; \ldots; \tilde{B}_{1m}, \ldots, \tilde{B}_{nm})$. Then we see that equations (21.4) have quite a similar form to equations (11.1) with

$$\alpha = mn, \qquad \beta = 0, \qquad \mathbf{1}_\alpha(\gamma) = -\sum_{j=1}^{m} \oplus \nu_j \mathbf{1}_n.$$

By applying Theorem B in Section 11 to equations (21.4), we have the following conclusion:

Equations (21.4) *have a solution* $\tilde{B}(x, V(x))$ *whenever* x *and* $V(x)$ *are in* (21.8). *Here* $\tilde{B}(x, V(x))$ *is an* $n \times m$ *matrix admitting uniformly convergent expansion* (21.11) *for* $(x, V(x))$ *in* (21.8), *so that* $\tilde{B}(x, v)$ *has Property-*$\mathfrak{A}$ *with respect to* v *in* (21.8).

Thus Proposition 2.1 has been proved.

§ 22. – Proof of Proposition 2.2.

By virtue of (20.7), $F(x, z)$ has a unique representation of the form

$$F(x, z) = f[x; z] + x^{\sigma+1}F^0(x, z).$$

We define $\lambda[x; Z]$ by the formula (20.9). Then it is clear that $\lambda[x; z]$ has Property-σ with respect to x for $\|z\| < d_2''$. Moreover we have $\lambda[0; 0] = \nu$ and

$\frac{\partial^\sigma}{\partial x^\sigma}\lambda[0;\ z]$ is an m-column constant vector.

Substitute (T_2''), appearing in Proposition 2.2, for $(\eta,\ \zeta)$ into both sides of the first equation of (20.6) and equate the coefficients of the linear terms with respect to Y in the resulting equation. Then we see that, for each index j, $R_j(x,\ Z)$ satisfies a partial differential equation of the form

$$x^{\sigma+1}\frac{\partial R_j}{\partial x}+x^{\sigma}\sum_{k=1}^{n}\frac{\partial R_j}{\partial Z_k}\mu_k Z_k=(F_j(x,\ Z)-\lambda_j[x;\ Z])(1+R_j). \tag{22.1}$$

It follows from (20.9) and (20.7) that $\tilde{F}_j(x,\ v)=x^{-\sigma}(F_j(x,\ v)-\lambda_j[x;\ v])$ have Property-$\mathfrak{A}$ with respect to v in the domain (21.8).

Hence, for each index j, $R_j(x,\ V(x))$ with $V(x)=\mathbf{1}_n(x^{\mu})C''$ satisfies the linear ordinary differential equation

$$xR_j'=\tilde{F}_j(x,\ V(x))R_j+\tilde{F}_j(x,\ V(x)),\qquad \tilde{F}_j(0,\ 0)=0. \tag{22.2}$$

It is easy to prove that equation (22.2) possesses a formal solution of the form

$$R_j\sim R_j(x)+\sum_{|q|=1}^{\infty}V(x)^q R_{q,j}(x) \tag{22.3}$$

with coefficients belonging to class $\mathfrak{C}(\Theta_2,\ \bar{\Theta}_2;\ \xi_2'')$.

Therefore, for each index j, equation (22.2) has a form similar to equations (11.1) with $\alpha=0$, $\beta=1$. By applying Theorem B in Section 11 to equations (22.2) we have the following conclusion:

Equations (22.2) *have a solution* $\{R_j(x,\ V(x))\}$ *whenever* x *and* $V(x)$ *are in the domain* (21.8). *This solution admits uniformly convergent expansions* (22.3) *for* $(x,\ V(x))$ *in* (21.8), *so that* $R(x,\ v)$ *is an* m*-column vector function with Property-*$\mathfrak{A}$ *with respect to* v *in* (21.8).

This proves Proposition 2.2 and, consequently, Theorem 2 has been completely proved.

CHAPTER V

Proof of Theorem 3.

I. Formal Solution $(\mathfrak{S}_2)$.

§ 23. - Equations (B_2).

As we have already seen in Theorem 2 in Section 4, equations (B_2) have the form

$$(\mathrm{B}_2)\qquad \begin{cases} x^{\sigma+1}Y' = \mathbf{1}_m(\lambda[x;\, Z])Y + \sum_{|p|=2}^{\infty} Y^p F_p(x,\, Z), \\ xZ' = \mathbf{1}_n(\mu)Z + \sum_{|p|=2}^{\infty} Y^p G_p(x,\, Z), \end{cases}$$

where the power series in the right-hand members are uniformly convergent for

$$(23.1)\qquad 0 < |x| < \xi_2'', \qquad \underline{\Theta}_2 < \arg x < \bar{\Theta}_2, \qquad \| Y \| < d_2'', \qquad \| Z \| < d_2''$$

and $F_p(x,\, Z)$ and $G_p(x,\, Z)$ are respectively m– and n–column vector functions with Property-$\mathfrak{A}$ with respect to Z in

$$(23.2)\qquad 0 < |x| < \xi_2'', \qquad \underline{\Theta}_2 < \arg x < \bar{\Theta}_2, \qquad \| Z \| < d_2''.$$

Let $\underline{\Theta}_3 < \arg x < \bar{\Theta}_3$ be the sector that appeared in Theorem 3 in Section 4. Then we shall prove the following proposition:

PROPOSITION 3.1. - *Assume that Assumptions* III *and* V *in Section* 4 *are satisfied. The equations* (B_2) *admit a formal transformation of the form*

$$(\mathrm{T}_3)\qquad Y \sim u + \sum_{|p|=2}^{\infty} u^p A_p(x,\, v), \qquad Z \sim v + x^{\sigma} \sum_{|p|=2}^{\infty} u^p B_p(x,\, v)$$

which transforms formally equations (B_2) *into equations of the form*

$$(\mathrm{B})\qquad x^{\sigma+1}u' = \mathbf{1}_m(\lambda[x;\, v])u, \quad xv' = \mathbf{1}_n(\mu)v.$$

Here $A_p(x,\, v)$ *and* $B_p\,(x,\, v)$ *are respectively* m*– and* n*–column vectors whose components are functions having Property-*$\mathfrak{A}$ *with respect to* v *in a domain of the form*

$$(23.3)\qquad 0 < |x| < \tilde{\xi}_2'', \ \underline{\Theta}_3 < \arg\, x < \bar{\Theta}_3, \ \|v\| < \tilde{\delta}_2''.$$

The proof of this proposition will be given in Section 24.

By virtue of Proposition 3.1, we have a formal general solution of equations (B_2). As we have proved in Section 5, equations (B) can be integrated by quadratures and their general solution $\{U(x),\, V(x)\}$ is given by the formula (5.1). If we substitute $\{U(x),\, V(x)\}$ for $\{u,\, v\}$ into the formal transformation (T_3), we have a formal solution for (B_2) of the form

$$(\mathrm{S}_2)\qquad Y \sim U(x) + \sum_{|p|=2}^{\infty} U(x)^p A_p(x,\, V(x)), \qquad Z \sim V(x) + x^{\sigma} \sum_{|p|=2}^{\infty} U(x)^p B_p(x,\, V(x)).$$

§ 24. **Proof of Proposition 3.1.** - Differentiation of the formal solution (S_2) term by term yields

$$x^{\sigma+1}Y' \sim \mathbf{1}_m(\lambda[x;\ V(x)])U(x) + \sum_{|p|=2}^{\infty} U(x)^p \{ x^{\sigma+1}A'_p(x,\ V(x)) + \\ + p \cdot \lambda[x;\ V(x)]A_p(x,\ V(x))\},$$

$$xZ' \sim \mathbf{1}_n(\mu)V + \sum_{|p|=2}^{\infty} U^p \{ x^{\sigma+1}B'_p(x,\ V) + (p \cdot \lambda[x;\ V] + \sigma x^{\sigma})B_p(x,\ V)\}.$$

On the other direction, inserting (S_2) for $(Y,\ Z)$ into the right-hand members of equations (B_2) and rearranging formally the resulting expressions in the form of a single power series of $U(x)$, we have after a simple calculation equations of the form

$$x^{\sigma+1}Y' \sim \mathbf{1}_m(\lambda[x;\ V(x)])U(x) + \\ + \sum_{|p|=2}^{\infty} U(x)^p \{ \mathbf{1}_m(\lambda[x;\ V(x)])A_p(x,\ V(x)) + \mathfrak{R}_p(x,\ V(x))\},$$

$$xZ' \sim \mathbf{1}_n(\mu)V + \sum_{|p|=2}^{\infty} U^p \{ x^{\sigma}\mathbf{1}_n(\mu)B_p(x,\ V) + \mathfrak{S}_p(x,\ V)\}.$$

Here $\mathfrak{R}_p(x,\ v)$ is a linear form of ***known*** m-column vector functions which have Property-$\mathfrak{A}$ with respect to v in a domain of the form

$$0 < |x| < \xi''_2,\ \underline{\Theta}_2 < \arg x < \bar{\Theta}_2,\ \|v\| < d''_2,$$

with polynomial coefficients of $A_{p'}(x,\ v)$ and $B_{p'}(x,\ v)$ for $|p'| < |p|$. $\mathfrak{S}_p(x,\ v)$ has the property similar to $\mathfrak{R}_p(x,\ v)$.

From the above power series representations for $\{x^{\sigma+1}Y',\ xZ'\}$ of two kinds, we can easily derive ordinary differential equations which determine $\{A_p(x,\ V(x)),\ B_p(x,\ V(x))\}$:

(24.1.p) $$x^{\sigma+1}A'_p = (\mathbf{1}_m(\lambda[x;\ V(x)]) - p \cdot \lambda[x;\ V(x)]\mathbf{1}_m)A_p + \mathfrak{R}_p(x,\ V(x)),$$

(24.2.p) $$x^{\sigma+1}B'_p = -p \cdot \lambda[x;\ V(x)]B_p + x^{\sigma}(\mathbf{1}_n(\mu) - \sigma\mathbf{1}_n)B_p + \mathfrak{S}_p(x,\ V(x)).$$

We determine inductively $A_p(x,\ v)$ and $B_p(x,\ v)$ in the following way. Assume that $A_p(x,\ v)$ and $B_p(x,\ v)$ have been determined for $2 \leqq |p| < N$ in such a way that they have Property-$\mathfrak{A}$ with respect to v in a domain of the form (23.3) and $A_p(x,\ V(x))$ and $B_p(x,\ V(x))$ are solutions of the equations (24.1.p) and (24.2.p) respectively for $(x,\ V(x))$ in (23.3). Then the vectors $\mathfrak{R}_p(x,\ v)$ and $\mathfrak{S}_p(x,\ v)$ for $|p| = N$ are functions with Property-$\mathfrak{A}$ with respect to v in (23.3).

Since $\det\{1_m(\lambda[0;0]) - p\cdot\lambda[0;0]1_m\} \neq 0$ and $p\cdot\lambda[0;0]\neq 0$ respectively by Assumptions V and III, we can prove by applying Theorem A in Section 6 that:

Equations (24.1.p) *and* (24.2.p) *for* $|p| = N$ *have formal solutions of the form*

$$A_p \sim A_p(x) + \sum_{|q|=1}^{\infty} V(x)^q A_{pq}(x) \tag{24.3.p}$$

and

$$B_p \sim B_p(x) + \sum_{|q|=1}^{\infty} V(x)^q B_{pq}(x). \tag{24.4.p}$$

Here $A_p(x)$, $A_{pq}(x)$ *are m-column vectors and* $B_p(x)$, $B_{pq}(x)$ *are n-column vectors which are uniquely determined as solutions of linear ordinary differential equations belonging to class* $\mathfrak{C}(\Theta_3, \bar{\Theta}_3;\ \xi_2'')$.

Hence equations (24.1.p) and (24.2.p) for each p such that $|p| = N$ have a form similar to equations (11.1) with

$$\alpha = m,\ \beta = 0,\ 1_\alpha(\gamma) = \sum_{j=1}^{m} \oplus\, (\nu_j - p\cdot\nu)1_1$$

and

$$\alpha = n,\ \beta = 0,\ 1_\alpha(\gamma) = -(p\cdot\nu)1_n$$

respectively. By applying Theorem B in Section 11, we have the following conclusion:

The m-column vectors $A_p(x, v)$ *and the n-column vectors* $B_p(x, v)$ *are uniquely determined in such a way that* $A_p(x, V(x))$ *and* $B_p(x, V(x))$ *respectively are solutions of equations* (24.1.p) *and* (24.2.p) *and, moreover, admit uniformly convergent expansions of the forms* (24.3.p) *and* (24.4.p) *whenever the values of* $(x, V(x))$ *belong to* (23.3).

Obviously $A_p(x, v)$ and $B_p(x, v)$ have Property-$\mathfrak{A}$ with respect to v in the domain (23.3).

II. Investigation of the Growth of a General Solution of Equations (B).

§ **25. Fundamental Lemma.** - We noticed that Lemmas A and B (in Sections 8 and 12) played a fundamental role in the proof of Theorems A and B (in Sections 6 and 11). For the proof of Theorem 3 also we need a corresponding lemma which we are going to establish.

By virtue of Assumption III in Section 4 all the real parts of the monomials $\Lambda_j(x) = -\nu_j/\sigma x^\sigma$ are negative valued for x on the positive real axis. If we denote by θ'_{j+} and θ'_{j-} singular directions of $\Lambda_j(x)$ which are immediately above and below the positive real axis respectively, then θ'_{j+} is an ascending singular direction while θ'_{j-} is a descending singular direction. As was explained in Section 7, we can choose arg ν_j so that

$$\theta'_{j-}=\frac{1}{\sigma}\left(\arg\ \nu_j-\frac{\pi}{2}\right),\ \theta'_{j+}=\frac{1}{\sigma}\left(\arg\ \nu_j+\frac{\pi}{2}\right). \tag{25.1}$$

Put

$$\theta'_{-}=\max_{j=1}^{m}\ \{\theta'_{j-}\},\qquad \theta'_{+}=\min_{j=1}^{m}\ \{\theta'_{j+}\}. \tag{25.2}$$

Then we can assume without loss of generality that the angles θ'_{j-}, θ'_{j+} satisfy

$$0\leqq\theta'_{-}-\theta'_{j-}\leqq\frac{\pi}{\sigma}-5\varepsilon,\ 0\leqq\theta'_{j+}-\theta'_{+}\leqq\frac{\pi}{\sigma}-5\varepsilon \tag{25.3}$$

and the angles $\underline{\Theta}_3$, $\bar{\Theta}_3$ appearing in Theorem 3 satisfy

$$\theta'_{-}-\frac{\pi}{\sigma}+5\varepsilon\leqq\underline{\Theta}_3<\bar{\Theta}_3\leqq\theta'_{+}+\frac{\pi}{\sigma}-5\varepsilon \tag{25.4}$$

for a preassigned sufficiently small positive constant ε.

We define $L(\varphi)$ by

$$L(\varphi)=\begin{cases}\sigma(\varphi-\theta'_{+}+3\varepsilon), & \theta'_{+}-2\varepsilon\leqq\varphi\leqq\bar{\Theta}_3,\\ \dfrac{\pi}{2}, & \theta'_{-}+2\varepsilon\leqq\varphi\leqq\theta'_{+}-2\varepsilon,\\ \sigma(\varphi-\theta'_{-}-3\varepsilon)+\pi, & \underline{\Theta}_3\leqq\varphi\leqq\theta'_{-}+2\varepsilon.\end{cases} \tag{25.5}$$

By virtue of (25.4) we see that $L(\varphi)$ satisfies the inequality

$$\sigma\varepsilon\leqq L(\varphi)\leqq\pi-\sigma\varepsilon\qquad \text{for } \underline{\Theta}_3\leqq\varphi\leqq\bar{\Theta}_3. \tag{25.6}$$

Let

$$\omega(\varphi)=\exp\int_{\theta_0}^{\varphi}\cot\ L(\tau)d\tau, \tag{25.7}$$

$$\chi_k(\varphi)=\exp\left\{(Re\,\mu_k)\int_{\theta_0}^{\varphi}\cot\ L(\tau)d\tau+(Im\,\mu_k)(\theta_0-\varphi)\right\}, \tag{25.8}$$

where θ_0 is a fixed angle satisfying $\underline{\Theta}_3 \leqq \theta_0 \leqq \bar{\Theta}_3$. Since we have (25.6), the functions $\omega(\varphi)$ and $\chi_k(\varphi)$ $(k = 1, 2, \ldots, n)$ are *strictly positive valued, bounded and continuous* for $\underline{\Theta}_3 \leqq \varphi \leqq \bar{\Theta}_3$.

Then our lemma which we are going to establish can be stated as follows:

LEMMA 3.1. – *Let* $\{U(x), V(x)\}$ *be a general solution of equations* (B) *given by the formula* (5.1). *Let* x_1, u^1, v^1 *be arbitrary values belonging to a domain of the form*

$$\left\{ \begin{array}{ll} 0 < |x| < \xi_N \omega(\arg x), & [v] < \delta_N[\chi(\arg x)], \\ \underline{\Theta}_3 < \arg x < \bar{\Theta}_3, & \|u\| < \delta_N, \end{array} \right. \tag{25.9}$$

where $\chi(\varphi)$ *is an n-column vector with elements* $\{\chi_k(\varphi)\}$. *Choose the integration constants* C' *and* C'' *being involved in* $\{U(x), V(x)\}$ *so that* $U(x_1) = u^1$, $V(x_1) = v^1$.

Then there exists a curve Γ_{x_1}, *which joins the point* x_1 *with the origin, such that*

i) *The curve* Γ_{x_1} *is entirely contained in the domain*

$$0 < |x| < \xi_N \omega(\arg x), \qquad \underline{\Theta}_3 < \arg x < \bar{\Theta}_3 \tag{25.10}$$

except for the origin.

ii) *As* x *moves on the curve* Γ_{x_1}, *we have the following three inequalities:*

$$[V(x)] < \delta_N[\chi(\arg x)], \qquad \underline{\Theta}_3 < \arg x < \bar{\Theta}_3, \tag{25.11}$$

$$\frac{d}{ds}\|U(x)\| \geqq \frac{\|\nu\|' \sin \sigma\varepsilon}{2} |x|^{-\sigma-1} \|U(x)\|, \tag{25.12}$$

$$\frac{d}{ds}\left(\|U(x)\|^N e^{-Re\Lambda_j(x)}\right) \geqq \frac{N\|\nu\|' \sin \sigma\varepsilon}{4} |x|^{-\sigma-1} \|U(x)\|^N e^{-Re\Lambda_j(x)} \tag{25.13}$$

$$(N\|\nu\|' \sin \sigma\varepsilon \geqq 4\|\nu\|)$$

with $\|\nu\|' = \min |\nu_j|$. s *is the arc length of the curve* Γ_{x_1} *measured from the origin to the variable point* x.

§ 26. **Proof of Lemma 3.1.** – To prove our lemma, we denote by (ρ, φ) the polar coordinate of the variable point x on Γ_{x_1}. Then the curve Γ_{x_1} is defined as follows:

If $\underline{\Theta}_3 < \arg x_1 < \theta'_- + 2\varepsilon$ or $\theta'_+ - 2\varepsilon < \arg x_1 < \bar{\Theta}_3$, Γ_{x_1} consists of a curvilinear part Γ':

$$\rho = |x_1| \exp \int_{\arg x_1}^{\varphi} \cot L(\tau) d\tau \tag{26.1}$$

for $\arg x_1 \leqq \varphi \leqq \theta'_- + 2\varepsilon$ or $\theta'_+ - 2\varepsilon \leqq \varphi \leqq \arg x_1$
and of a rectilinear part Γ'':

$$0 \leqq \rho \leqq |x_1| \exp \int_{\arg x_1}^{\varphi} \cot L(\tau) d\tau, \qquad \varphi = \theta'_- + 2\varepsilon \text{ or } \theta'_+ - 2\varepsilon.$$

If $\theta'_- + 2\varepsilon \leqq \arg x_1 \leqq \theta'_+ - 2\varepsilon$, Γ_{x_1} consists of a rectilinear part Γ'' only:

$$0 \leqq \rho \leqq |x_1|, \ \varphi = \arg x_1.$$

1°. - By the definition of the curve Γ_{x_1}, Assertion i) of our lemma is almost evident. An application of the reasonings which were used to prove the inequality (12.3) in Lemma B in Section 12 proves easily the inequality (25.11).

2°. - We want to prove the inequality (25.12). Since $U(x)$ is a solution of the equation

$$x^{\sigma+1} u' = 1_m(\lambda[x;\ V(x)])u, \ \lambda[0;\ 0] = \nu,$$

a simple calculation shows that

$$\text{(26.2)} \qquad \|U(x)\|^{-1} \frac{d\|U(x)\|}{ds} = |U_j(x)|^{-1} \frac{d|U_j(x)|}{ds} \qquad \text{(for some index } j)$$

$$= Re\left(x^{-\sigma-1} \lambda_j[x;\ V(x)] \frac{dx}{ds}\right)$$

$$= Re\left(x^{-\sigma-1} \nu_j \frac{dx}{ds} \frac{\lambda_j[x;\ V(x)]}{\nu_j}\right).$$

The index j depends naturally on the choice of the point x, i.e. s.

Since $\|V(x)\|$ is uniformly bounded for $x \in \Gamma_{x_1}$ no matter how we choose the point x_1, we can assume without loss of generality that:

For each index j, as x moves on Γ_{x_1}, we have

$$\text{(26.3)} \qquad \frac{1}{2}|\nu_j| \leqq |\lambda_j[x;\ V(x)]|, \quad \left|\arg \frac{\lambda_j[x;\ V(x)]}{\nu_j}\right| \leqq \sigma\varepsilon$$

On the curvilinear part Γ', ρ is a functions of φ given by (26.1). An easy computation shows that we have for $x \in \Gamma'$

$$\text{(26.4)} \qquad \frac{dx}{ds} = -e^{i(L(\varphi)+\varphi)} \quad \text{or} \quad + e^{i(L(\varphi)+\varphi)}$$

according as φ satisfies $\arg x_1 \leqq \varphi \leqq \theta'_- + 2\varepsilon$ or $\theta'_+ - 2\varepsilon \leqq \varphi \leqq \arg x_1$. Hence an inequality of the form

$$(26.5)\qquad \|U(x)\|^{-1}\frac{d\|U(x)\|}{ds} \geqq \frac{1}{2}|x|^{-\sigma-1}|\nu_j| \sin \sigma\varepsilon \geqq \frac{1}{2}|x|^{-\sigma-1}\|\nu\|' \sin \sigma\varepsilon$$

would follow from (26.2) and (26.3) if we could prove that $L(\varphi)$ satisfies

$$(26.6)\qquad \cos\left(L(\varphi) - \sigma\varphi + \arg \nu_j + \arg \frac{\lambda_j[x;\ V(x)]}{\nu_j}\right) < -\sin \sigma\varepsilon \quad \text{or} \quad > \sin \sigma\varepsilon$$

according as φ satisfies

$$\arg x_1 \leqq \varphi \leqq \theta'_- + 2\varepsilon \text{ or } \theta'_+ - 2\varepsilon \leqq \varphi \leqq \arg x_1.$$

Hence, in order to have inequality (25.12) on Γ', it sufficies that (26.6) holds in the desired interval. We shall prove (26.6). The definitions of $L(\varphi)$ and θ'_{j+} imply that

$$\arg \nu_j + L(\varphi) - \sigma\varphi = \sigma(\theta'_{j+} - \theta'_+ + 3\varepsilon) - \frac{\pi}{2}$$

for $\theta'_+ - 2\varepsilon \leqq \varphi \leqq \bar{\Theta}_3$. This relation yields by (25.3)

$$(26.7)\qquad |\arg \nu_j + L(\varphi) - \sigma\rho| \leqq \frac{\pi}{2} - 2\sigma\varepsilon \quad \text{for } \theta'_+ - 2\varepsilon \leqq \varphi \leqq \bar{\Theta}_3.$$

Since $|\arg \lambda_j[x;\ V(x)] - \arg \nu_j| \leqq \sigma\varepsilon$ for $x \in \Gamma'$, we have (26.6) from (26.7) for $\theta'_+ - 2\varepsilon \leqq \varphi \leqq \arg x_1$.

Similarly we can prove that

$$(26.8)\qquad |\arg \nu_j + L(\varphi) - \sigma\varphi - \pi| \leqq \frac{\pi}{2} - 2\sigma\varepsilon \quad \text{for } \underline{\Theta}_3 \leqq \varphi \leqq \theta_- + 2\varepsilon,$$

from which inequality (26.6) follows for $\arg x_1 \leqq \varphi \leqq \theta'_- + 2\varepsilon$ since we have (26.3). Thus we have inequality (25.12) for $x \in \Gamma'$.

On the rectilinear part Γ'', we have $|x| = s$ and $\arg x = \arg x_1$. Hence $dx/ds = \exp(i \arg x_1)$ and $\theta'_- + 2\varepsilon \leqq \arg x_1 \leqq \theta'_+ - 2\varepsilon$, which, by virtue of (25.1), yields

$$-\frac{\pi}{2} + \sigma(\theta'_{j+} - \theta'_+ + 2\varepsilon) \leqq \arg \nu_j - \sigma \arg x_1 \leqq \frac{\pi}{2} + \sigma(\theta'_{j-} - \theta'_- - 2\varepsilon).$$

Then it follows from the definitions of θ'_- and θ'_+ that

$$|\arg \nu_j - \sigma \arg x_1| \leqq \frac{\pi}{2} - 2\sigma\varepsilon.$$

Hence, by using (26.3), we can derive from (26.2) inequality (25.12) for $x \in \Gamma''$.

3°. - To prove the inequality (25.13), we take N so large that

$$4\|\nu\| \leqq N\|\nu\|' \sin \sigma\varepsilon .$$

Then, by using the fact that the inequalities (25.12) and $|dx/ds| = 1$ are satisfied for $x \in \Gamma_{x_1}$, we have by an elementary calculation

$$\frac{d}{ds}\left(\| U(x)\|^N e^{-Re\Lambda_j(x)}\right) = \left(N\| U(x)\|^{-1}\frac{d\| U(x)\|}{ds} - Re\,\frac{\nu_j}{x^{\sigma+1}}\frac{dx}{ds}\right)\| U(x)\|^N e^{-Re\Lambda_j(x)}$$

$$\geqq \left(\frac{N\|\nu\|' \sin \sigma\varepsilon}{2} - |\nu_j|\right)\| U(x)\|^N e^{-Re\Lambda_j(x)}\, |x|^{-\sigma-1}$$

$$\geqq \frac{N\|\nu\|' \sin \sigma\varepsilon}{4}\| U(x)\|^N e^{-Re\Lambda_j(x)}\, |x|^{-\sigma-1}.$$

This completes the proof of Lemma 3.1.

III. Outline of Proof of Convergence of Formal Solution (S₂).

§ 27. **Problem to Prove Theorem 3.** - By virtue of Proposition 3.1 in Section 23 equations (B_2) have a formal solution of the form

$$(S_2)\qquad Y \sim U(x) + \sum_{|p|=2}^{\infty} U(x)^p A_p(x,\ V(x)),\ Z \sim V(x) + x^\sigma \sum_{|p|=2}^{\infty} U(x)^p B_p(x,\ V(x)),$$

where $A_p(x, v)$ and $B_p(x, v)$ have Property-$\mathfrak{A}$ with respect to v in

$$(27.1)\qquad 0 < |x| < \tilde{\xi}_2'',\ \underline{\Theta}_3 < \arg x < \bar{\Theta}_3,\ \|v\| < \tilde{\delta}_2''.$$

To prove uniform convergence of (S_2), put

$$(27.2)\qquad \begin{cases} P_N(x,\ u,\ v) = u + \sum_{2 \leqq |p| < N} u^p A_p(x,\ v), \\ Q_N(x,\ u,\ v) = v + x^\sigma \sum_{2 \leqq |p| < N} u^p B_p(x,\ v). \end{cases}$$

We make the change of variables

$$(27.3)\qquad Y = P_N(x,\ U(x),\ V(x)) + \eta\,,\ Z = Q_N(x,\ U(x),\ V(x)) + \zeta$$

to equations (B_2). Since we have an identity of the form

$$x^{\sigma+1}\frac{d}{dx}P_N(x,\ U(x),\ V(x)) = x^{\sigma+1}\frac{\partial}{\partial x}P_N(x,\ U(x),\ V(x))$$

$$+\frac{\partial}{\partial U(x)}P_N(x,\ U(x),\ V(x))\cdot \mathbf{1}_m(\lambda[x;\ V(x)])U(x)$$

$$+x^{\sigma}\frac{\partial}{\partial V(x)}P_N(x,\ U(x),\ V(x))\cdot \mathbf{1}_n(\mu)V(x)$$

and since $\{U(x),\ V(x)\}$ is a general solution, we see that $x^{\sigma+1}\dfrac{d}{dx}P_N(x,\ U(x),\ V(x))$ is determined as a function of $(x,\ U(x),\ V(x))$ in a unique way. Similarly we can prove that $x\dfrac{d}{dx}Q_N(x,\ U(x),\ V(x))$ is uniquely determined as a function of $(x,\ U(x),\ V(x))$. Hence, if we write the equations satisfied by $\{\eta,\ \zeta\}$ as

$$\text{(27.4)}\qquad \begin{cases} x^{\sigma+1}\eta' = \mathbf{1}_m(\nu)\eta + F(x,\ U(x),\ V(x);\ \eta,\ \zeta), \\ x\zeta' = G(x,\ U(x),\ V(x);\ \eta,\ \zeta), \end{cases}$$

$F(x,\ u,\ v;\ \eta,\ \zeta)$ and $G(x,\ u,\ v;\ \eta,\ \zeta)$ are respectively m- and n-column vector functions holomorphic and bounded in $(x,\ u,\ v;\ \eta,\ \zeta)$ for a domain of the form

$$\text{(27.5)}\qquad \begin{cases} 0<|x|<\xi_N',\ \underline{\Theta}_3<\arg x<\overline{\Theta}_3,\ \|u\|<\delta_N',\ \|v\|<\delta_N', \\ \|\eta\|<d_N',\ \|\zeta\|<d_N' \end{cases}$$

for suitably chosen positive constants ξ_N', δ_N' and d_N'.

Since equations (27.4) possess a formal solution of the form

$$\text{(27.6)}\qquad \eta\sim\sum_{|p|=N}^{\infty}U(x)^pA_p(x,\ V(x)),\qquad \zeta\sim x^{\sigma}\sum_{|p|=N}^{\infty}U(x)^pB_p(x,\ V(x)),$$

an easy computation shows that F and G satisfy both an inequality of the form

$$\text{(27.7)}\qquad \|F(x,\ u,\ v;\ \eta,\ \zeta)\|,\ \|G(x,\ u,\ v;\ \eta,\ \zeta)\| \leqq A(\|\eta\|+\|\zeta\|)+B_N\|u\|^N$$

for $(x,\ u,\ v;\ \eta,\ \zeta)$ in (27.5). Moreover, F and G satisfy LIPSCHITZ'S condition with respect to $(\eta,\ \zeta)$ with the same LIPSCHITZ's constant A. Here A is a positive constant independent of N while B_N may depend on N.

We make a further transformation of the form

$$\eta = \mathbf{1}_m(e^{\Lambda(x)})P, \quad \zeta = Q, \tag{27.8}$$

so that equations (27.4) are reduced to

$$\begin{cases} P' = x^{-\sigma-1}\mathbf{1}_m(e^{-\Lambda(x)})\,F(x,\ U(x),\ V(x);\ \mathbf{1}_m(e^{\Lambda(x)})P,\ Q), \\ Q' = x^{-1}G(x,\ U(x),\ V(x);\ \mathbf{1}_m(e^{\Lambda(x)})P,\ Q). \end{cases} \tag{27.9}$$

We shall solve the following problem:

PROBLEM. - *Let* $N \geqq 16A/\|\nu\|' \sin\sigma\varepsilon$ *and* $N \geqq 4\|\nu\|/\|\nu\|' \sin\sigma\varepsilon$. *Then equations* (27.9) *have a unique solution* $\{\varphi_N(x,\ U(x),\ V(x)),\ \psi_N(x,\ U(x),\ V(x))\}$ *such that*

$$[P] = O(\|U(x)\|^N)[e^{-\Lambda(x)}], \qquad \|Q\| = O(\|U(x)\|^N), \tag{27.10$_N$}$$

whenever the values of x, $U(x)$, $V(x)$ *belong to a domain of the form*

$$\begin{cases} 0 < |x| < \xi''_N\omega(\arg x),\ \underline{\Theta}_3 < \arg x < \bar{\Theta}_3, \\ \|u\| < \delta''_N,\ [v] < \delta''_N[\chi(\arg x)]. \end{cases} \tag{27.11$_N$}$$

Here $\varphi_N(x,\ u,\ v)$ *and* $\psi_N(x,\ u,\ v)$ *are respectively* m*- and* n*-column vectors whose components are holomorphic and bounded functions of* (x, u, v) *for* $(27.11)_N$.

If we assume that Problem has been solved, an application of the arguments in 2° in Section 13 proves uniform convergence of the formal solution (S_2) when x, $U(x)$, $V(x)$ belong to a domain of the form

$$0 < |x| < \xi''_3,\ \underline{\Theta}_3 < \arg x < \bar{\Theta}_3,\ \|u\| < \delta''_3,\ \|v\| < \delta''_3$$

and, consequently, we have Theorem 3 in Section 4.

Therefore, in order to prove Theorem 3, it is sufficient to solve Problem.

§ 28. **Solution of Problem.** - To solve Problem, let $\mathcal{F} = \{\varphi(x,\ u,\ v),\ \psi(x,\ u,\ v)\}$ be the family of m-column vectors $\varphi(x,\ u,\ v)$ and n-column vectors $\psi(x,\ u,\ v)$ which are holomorphic and bounded functions of $(x,\ u,\ v)$ for $(27.11)_N$ and satisfy there inequalities of the form

$$\begin{cases} [\varphi(x,\ u,\ v)] \leqq K_N\|u\|^N[e^{-Re\Lambda(x)}], \\ \|\psi(x,\ u,\ v)\| \leqq K_N\|u\|^N, \end{cases} \tag{28.1}$$

K_N being a certain positive constant.

Let (x_1, u^1, v^1) be values arbitrarily chosen from the domain $(27.11)_N$ and determine the integration constants C' and C'' being involved in $\{U(x), V(x)\}$ so that $U(x_1)=u^1$ and $V(x_1)=v^1$.

We define the vector functions $\Phi(x, u, v)$ and $\Psi(x, u, v)$ by

$$(28.2)\qquad \begin{cases} \Phi(x_1, u^1, v^1) = \displaystyle\int_0^{x_1} \mathcal{H}(x, U(x), V(x))dx, \\ \Psi(x_1, u^1, v^1) = \displaystyle\int_0^{x_1} \mathfrak{K}(x, U(x), V(x))dx, \end{cases}$$

where

$$\mathcal{H}(x, u, v) \equiv x^{-\sigma-1}\mathbf{1}_m(e^{-\Lambda(x)})F(x, u, v; \mathbf{1}_m(e^{\Lambda(x)})\varphi(x, u, v), \psi(x, u, v)),$$

$$\mathfrak{K}(x, u, v) \equiv x^{-1}G(x, u, v; \mathbf{1}_m(e^{\Lambda(x)})\varphi(x, u, v), \psi(x, u, v)).$$

The integration must be carried out along the curve Γ_{x_1} which was already defined in Section 26.

By virtue of Lemma 3.1 in Section 25, $\|U(x)\|$ is a monotone increasing function of s as x moves on the curve Γ_{x_1}. By combining this fact with the inequality (25.11), we see that the values of x, $U(x)$, $V(x)$ belong to the domain $(27.11)_N$ as x is on Γ_{x_1} and, consequently, the integrands of integrals (28.2) are holomorphic functions of x for $x\in\Gamma_{x_1}$ except for $x=0$. On the other hand, we see by (27.7) and (28.1) that the integrands of integrals (28.2) satisfy

$$(28.3)\qquad \begin{cases} [\mathcal{H}(x, U(x), V(x))] \leqq (2AK_N+B_N)\,|x|^{-\sigma-1}\|U(x)\|^N[e^{-Re\Lambda(x)}], \\ \|\mathfrak{K}(x, U(x), V(x))\| \leqq (2AK_N+B_N)|x|^{-1}\|U(x)\|^N. \end{cases}$$

Since $\|U(x)\|$ tends to 0 exponentially as x approaches the origin along Γ_{x_1}, the integrals (28.2) are convergent. Hence the mapping $\mathfrak{T}$:

$$\{\varphi(x, u, v), \psi(x, u, v)\} \to \{\Phi(x, u, v), \Psi(x, u, v)\}$$

is well defined.

Since $\{0, 0\}\in\mathfrak{F}$, $\mathfrak{F}$ is non-empty. Moreover it is clear that $\mathfrak{F}$ is a closed, convex and normal family. Hence, in order to solve Problem by using a fixed-point theorem (see [2]), we must prove first the following assertions:

1° - *$\mathfrak{T}$ maps $\mathfrak{F}$ into itself, namely we have $\mathfrak{T}\{\mathfrak{F}\}\subset\mathfrak{F}$.*

2°. – *$\mathcal{T}$ is a continuous mapping with respect to the topology of uniform convergence on compact subsets.*

If we assume that these two assertions have been proved, then there exists a member of $\mathcal{F}$ that corresponds to a fixed-point of $\mathcal{T}$. We denote this member by $\{\varphi_N(x, u, v), \psi_N(x, u, v)\}$. Then we must prove the following assertions:

3°. – *$\{\varphi_N(x, U(x), V(x)), \psi_N(x, U(x), V(x))\}$ is a solution of equations* (27.9).

4°. – *A solution of equations* (27.9) *satisfying condition* $(27.10)_N$ *is unique.*

These four assertions can be proved by applying almost exactly the same arguments as in Section 14 which were used to solve Problem B in Section 11.

For example the proof of the inequalities

$$(28.4)\qquad \begin{cases} [\Phi(x_1, u^1, v^1)] \leqq K_N \|u^1\|^N [e^{-Re\Lambda(x_1)}], \\ \|\Psi(x_1, u^1, v^1)\| \leqq K_N \|u^1\|^N. \end{cases}$$

is carried out as follows.

By virtue of (28.3), it will be sufficient, to have (28.4), to prove that:

$$(28.5)\qquad (2AK_N + B_N) \int_0^{s_1} |x|^{-\sigma-1} \|U(x)\|^N e^{-Re\Lambda_j(x)}\, ds \leqq K_N \|u^1\|^N e^{-Re\Lambda_j(x_1)},$$

$$(28.6)\qquad (2AK_N + B_N) \int_0^{s_1} |x|^{-1} \|U(x)\|^N\, ds \leqq K_N \|u^1\|^N,$$

where s_1 is the arc length of the curve Γ_{x_1}.

By using inequality (25.13), we see at once that the expression of the left-hand member of (28.5) does not exceed

$$(28.7)\qquad \frac{4(2AK_N + B_N)}{N\|\nu\|' \sin \sigma\varepsilon} \|u^1\|^N e^{-Re\Lambda_j(x_1)}$$

if $4\|\nu\| \leqq N\|\nu\|' \sin \sigma\varepsilon$. Since $16A \leqq N\|\nu\|' \sin \sigma\varepsilon$, we can take K_N large enough to have

$$(28.8)\qquad 4B_N(N\|\nu\|' \sin \sigma\varepsilon - 8A)^{-1} \leqq K_N.$$

Then the expression (28.7) is bounded by the expression of the right-hand member of (28.5), which proves inequality (28.5) and, consequently, we have the first inequality of (28.4).

Similarly, by using (25.12), we see that the expression of the left-hand member of (28.6) does not exceed

$$\frac{2(2AK_N+B_N)}{N\|\nu\|' \sin \sigma\varepsilon}\|u^1\|^N. \tag{28.9}$$

By virtue of (28.8) the expression (28.9) is obviously bounded by $K_N\|u^1\|^N$, which proves (28.6) and we have the second inequality of (28.4).

REFERENCES

[1] M. HUKUHARA, *Sur les points singuliers des équations différentielles linéaires*, III, Mem. Fac. Sci. Kyusyu Univ , 2 (1942), pp. 125-137.

[2] ——, Renzokuna Kansu no Zoku to Syazo, Ibid. **5** (1950). pp. 61-63.

[3] M. IWANO, *Intégration analytique d'un système d'équations différentielles non linéaires dans le voisinage d'un point singulier*, I, Annali Mat. Pura Appl. Serie IV, **44** (1957), pp. 261-292.

[4] ——, *Intégration analytique d'un système d'équations différentielles non linéaires dans le voisinage d'un point singulier*, II, Ibid. **47** (1959) pp. 91-150.

[5] ——, *On a singular point of Briot-Bouquet type of a system of ordinary nonlinear differential equations*, Comment. Math. Univ. Sancti Pauli, **11** (1963), pp. 37-78.

[6] ——, *A method to construct analytic expressions for bounded solutions of non-linear ordinary differential equations with an irregular singular point*, Funkcialaj Ekvacioj **10** (1967), pp. 75-105.