Uniqueness and Goursat problems.

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Summary. - *A uniqueness theorem by W. Walter for a second order Goursat problem in to independent variables is generalized. In the resulting theorem there are no restrictions on the order or on the number of indipendent variables.*

1. - Introduction.

Let $f(x, z)$ be a real-valued continuous function in $V \times R$, V being an interval in R starting in zero. Uniqueness theorems for the ordinary differential equation

$$u_x = f(x, u), \qquad u(0) = 0. \tag{1.1}$$

has been brought to a very high degree of perfection through OKAMURA'S uniqueness theorem [4]. See the proof in YOSHIZAWA [10], p. 3-10. Several sufficient conditions have been given by various authors. Among these we like to mention NAGUMO [3], WALTER [9], MOYER [2]. For further references in this field see [2]. GEORGE [1] has pointed out that all are included in Okamura's theorem. OKAMURA proved that (1.1) has at most one continuously differentiable solution if and only if certain Liapunov functions connected with f exist.

The set W in R^2 is defined by

$$W = \{ (x, y); \qquad 0 \leq x \leq a, \qquad 0 \leq y \leq b \}.$$

The function $f(x, y, z_1, z_2, z_3)$ is real-valued and continuous in $W \times R^3$. The question of uniqueness for the Goursat problem

$$u_{xy} = f(x, y, u, u_x, u_y), \qquad u(0, y) = u(x, 0) = 0. \tag{1.2}$$

has been treated by WALTER [9]. The resulting theorems for (1.2) and (1.1) are of the same nature.

The basic theorems for (1.1) and (1.2) say that existance and uniqueness are garanteed if f is Lipschitz continuous in the z-variables. A theorem of this kind for a more general equation than (1.2) is theorem 2 in [6]. The problem treated in [6] has the same main feature as (1.2). A single derivative

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stands to the left and the non-linear right member contains derivatives of lower order. See (3.1) and (3.3) in section 3 of this paper. A local version of theorem 2 in [6] is theorem 4 in [5]. Existence theorems for the problem (3.1) and (3.3) are proved in [6] and (7). A generalization to the case when data are given on hypersurfaces instead of on hyperplanes can be found in [8]. The common point in the theorems in [5]-[8] mentioned above is that the corresponding f is restricted by Lipschitz continuity in some or all z-variables.

There seems to be a closer connection between the uniqueness theorems for (1.1) and (1.2) than between the existence theorems for (1.1) and (1.2). Therefore one might ask if Okamura's theorem has some counterpart for (1.2) or more general equations. We can give no answer to this question. Instead we restrict ourselves to a generalization of the uniqueness theorem in [9] for (1.2).

The necessary notation and some definitions are given in section 2. Section 3 contains the theorem and its proof. The theorem is proved by induction over the number of independent variables that are involved. It is a generalization of the technique used in [9]. In section 4 the theorem is shown to include the Lipschitz continuity case and also a generalization of the Nagumo condition for (1.1). Therefore it is stronger than the uniqueness part of theorem 2 in [6].

2. - Preliminaries.

Let $x = (x_1, \ldots, x_n) \in R^n$ and $z = (z_1, \ldots, z_N) \in R^N$. By $\alpha = (\alpha_1, \ldots, \alpha_n)$ we denote a multi-index with non-negative integers as components. If $D = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$ then we write $D^\alpha = (\partial/\partial x_1)^{\alpha_1}_1 \ldots (\partial/\partial x_n)^{\alpha_n}$. We also write $x^\alpha = x_2^{\alpha_1} \ldots x_n^{\alpha_n}$, $\alpha! = \alpha_1! \ldots \alpha_n!$,

$$\alpha \leq \beta \Leftrightarrow \alpha_j \leq \beta_j, \qquad 1 \leq j \leq n,$$

and $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

DEFINITION. - *Let*

$$K = \{ x \mid 0 \leq x_j \leq a_j,\ 1 \leq j \leq n \}.$$

The function $u(x)$ is real-valued and defined in K aud β is a multi-index. If all derivatiees $D^\alpha u$, $\alpha \leq \beta$, exist and are continuous then we say that u belongs to the functian class $C(\beta, K)$. Let

$$K_0 = \{ x \mid 0 < x_j \leq a_j,\ 1 \leq j \leq n \}.$$

Then $C(\beta, K_0)$ has an obvious sense.

DEFINITION. - *Let* $u \in C(\beta, K)$. *We define* $u = O(x^\beta)$ *by*

$$u = O(x^\beta) \Leftrightarrow D_j^k u(x) = 0, \qquad x_j = 0, \qquad 0 \leq k < \beta_j, \qquad 1 \leq j \leq n.$$

We now make a somewhat lengthy definition of the function classes $H(n, \beta, K_0)$. The definition will explain itself when applied to the proof in section 3.

DEFINITION. - *The function* $h(x, z_1, \ldots, z_N)$ *is defined in* $K_0 \times R^N$. *It is monotonically increasing in each* z_k, $1 \leq k \leq N$, *and* $h(x, 0, \ldots, 0) = 0$, $x \varepsilon K_0$ *Let further* $\beta, \alpha^1, \ldots, \alpha^N$, *be multi-indices such that*

$$\alpha^k \leq \beta, \qquad \alpha^k \neq \beta, \qquad 1 \leq k \leq N.$$

We define $\eta^k = \beta - \alpha^{k}$' $1 \leq k \leq N$.

In the case $n = 1$, *we say that* h *belongs to* $H(1, \beta, K_0)$ *if for every* $\varepsilon > 0$ *there exist a number* $\delta > 0$ *and a function* $v \in C(\beta, K)$ *satisfying the following two conditions,*

$$D^\beta v \leq h(x, D^{\alpha^1} v, \ldots, D^{\alpha^n} v), \qquad x \in K_0. \tag{2.1}$$

$$\delta x^{\eta^k} (\eta^k !)^{-1} < D^{\alpha^k} v < \varepsilon, \qquad x \in K_0, \qquad 1 \leq k \leq N. \tag{2.2}$$

Let $n = n' > 1$. *We assume that we have defined* $H(m, \xi, K_0')$ *for for all* $m < n'$, *and all* $\xi \in R^m$, *and all* $K_0' \subset R^m$. *It is now possible to define* $H(n, \beta, K_0)$ *uniquely in the following way. We define*

$$\beta^j = (\beta_1, \ldots, \beta_{j-1}, \beta_{j+1}, \ldots, \beta_n), \qquad \alpha^{jk} = (\alpha_1^k, \ldots, \alpha_{j-1}^k, \alpha_{j-1}^k, \ldots, \alpha_n^k), \qquad 1 \leq j \leq n.$$

We also define

$$K_0^j = \{ (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \mid 0 < x_t \leq a_t, \ 1 \leq t \leq n, \ t \neq j \},$$

$z_k(j) = 0$, $\alpha_j^k < \beta_j$, *and* $z_k(j) = z_k$, $\alpha_j^k \beta_j$, $1 \leq j \leq n$, $1 \leq k \leq N$.
K^j *is the closure of* K_0^j *in* R^{n-1}.
We say that $h(x, y_1, \ldots, z_N)$ *belongs to* $H(n, \beta, K_0)$ *if to every* $\varepsilon > 0$ *there exist a* $\delta > 0$ *and a function* $v \in C(\beta, K)$ *such that* (2.1), *and* (2.2) *are true, and if*

$$\lim_{x_j \to 0} h(x, z_1(j), \ldots, z_M(j)) = h_j \in H(n-1, \beta^j, K_0^j), \qquad 1 \leq j \leq n.$$

Here h_j *is a function of* $(x_1, \ldots x_{j-1}, x_{j+1}, \ldots, x_n) = x^{(j)} \in R^{n-1}$, *and those* $z_k(j)$ *with* $z_k(j) = z_k$. *The exact meaning of* $h_j \in H(n-1, \beta^j, K_0^j)$ *is that in* (2.1) *we have a* $v^j \in C(\beta^j, K^j)$.

$D^{\beta^j}v^j$ stands to the left and h_j to the right. In h_j $D^{\alpha^{jk}}v^j$ is inserted into the place where $z_k(j) = z_k$. The condition (2.2) *is adjusted in an analogous way. In both* (2.1), *and* (2.2) *we have $x^{(j)} \in K_0^j$.*

3. A uniqueness theorem for Goursat problems.

We start by formulating the theorem

THEOREM. - *The multi-indices β, α^1, ..., α^N, are such that*

$$\beta \geq \alpha^k, \qquad \beta \neq \alpha^k, \qquad 1 \leq k \leq N. \tag{3.1}$$

The real-valued function $f(x, z)$ is continuous in $K \times R^N$. There exists a $h \in H(n, \beta, K_0)$ such that

$$|f(x, z) - f(x, \bar{z})| \leq h(x, |z_1 - \bar{z}_1|, \dots, |z_N - \bar{z}_N|), \qquad (x, z), \quad (x,, \bar{z}) \in K_0 \times R^N. \tag{3.2}$$

It follows then that there exists at most one solution $u \in C(\beta, K)$ of

$$D^\beta u = f(x, D^{\alpha^1}u, \dots, D^{\alpha^N}u), \qquad u = O(x^\beta). \tag{3.3}$$

For the proof of the theorem we need the following lemma.

LEMMA. - *The real-valued function $F(x, z)$ is defined in $K \times R^N$. F is monotonically increasing in the N z-variables. The multi-indices β, α^1, ..., α^N, satisfy* (3.1). *The two functions d and v belong to $C(\beta, K)$. They satisfy the following two conditions.*

$$|D^\beta d| \leq F(x, |D^{\alpha^1}d|, \dots, |D^{\alpha^N}d|), \qquad x \in K. \tag{3.4}$$

$$D^\beta v \geq F(x, D^{\alpha^1}v, \dots, D^{\alpha^N}v), \quad x \in K. \tag{3.5}$$

We define x^k by

$$x_j^k = x_j, \quad \text{if } \alpha_j^k = \beta_j, \quad \text{and } x_j^k = 0 \text{ if } \alpha_j^k < \beta_j, \qquad x \in K, \quad 1 \leq j \leq n. \tag{3.6}$$

If

$$|D^{\alpha^k}d(x^k)| < D^{\alpha^k}v(x^k), \qquad x \in K, \qquad 1 \leq k \leq N, \tag{3.7}$$

then it follows that

$$|D^{\alpha^k}d(x)| < D^{\alpha^k}v(x), \qquad x \in K, \qquad 1 \leq k \leq N. \tag{3.8}$$

PROOF OF THE LEMMA. - It follows from (3.7) and the continuity of the functions that there exists a set

$$K' = \{ x \mid 0 \leq x_j < a_j' \leq a_j , \ 1 \leq j \leq n \} \subset K,$$

such that

$$(3.9) \qquad | D^{\alpha^k} d(x) | < D^{\alpha^k} v(x), \qquad x \in K', \qquad 1 \leq k \leq N.$$

let

$$K'' = \{ x \mid 0 \leq x_j \leq a_j' \leq a_j, \qquad 1 \leq j \leq n \}.$$

We shall prove that

$$(3.10) \qquad | D^{\alpha^k} d(x) | < D^{\alpha^k} v(x), \qquad x \in K'', \qquad 1 \leq k \leq M.$$

If $K'' \neq K$, then it follows from the continuity that we may choose every a_j', $a_j' < a_j$, somewhat bigger. If we choose K' maximal in the sense that no a_j' can be chosen bigger, then we get a contradiction if $K'' \neq K$. It then follows that (3.8) is true. Now to the proof of (3.10).

It follows from (3.4), (3.5), (3.9), the continuity of $D^\beta d$ and $D^\beta v$, and the monotony of F in the z-variables that

$$| D^\beta d(x) | \leq D^\beta v(x), \qquad x \in K''.$$

We let

$$w = v - d, \qquad \text{or } w = v + d.$$

Thus we know that

$$(3.7)' \qquad D^{\alpha^k} w(x^k) > 0, \qquad x \in K'', \qquad 1 \leq k \leq N,$$

and that

$$(3.11) \qquad D^\beta w(x) \geq 0, \qquad x \in K''.$$

We want to prove that

$$(3.10)' \qquad D^{\alpha^k} w(x) > 0, \qquad x \in K'', \qquad 1 \leq k \leq N.$$

We start by a simple. example. Afterwards we prove the general case. Let $\alpha^1 = (\beta_1 - 1, \beta_2, \ldots, \beta_n)$. It follows from (3.11) that

$$0 \leq \int_0^{x_1} D^\beta w(t, x_2, \ldots, x_n) dt = D^{\alpha^1} w(x) - D^{\alpha^1} w(0, x_2, \ldots, x_n) = D^{\alpha^1} w(x) - D^{\alpha^1} w(x^1).$$

Since (3.7)′ says that $D^{\alpha^1}w(x^1) > 0$, we conclude that $D^{\alpha^1}w(x) > 0$, $x \in K''$.

Now to the general case. The indices of α^k are not suited for the proof of the lemma so we define new ones by

$$\alpha^k = (\beta_1 - k_1, \ldots, \beta_n - k_n).$$

Here $x_j^k = x_j$, $k_j = 0$, and $x_j^k = 0$, $k_j > 0$, $1 \leq j \leq n$. For every j, $k_j > 0$, let

$$\alpha^{k(j)} = (\beta_1 - k_1, \ldots, \beta_{j-1} - k_{j-1}, \beta_j - k_j + 1, \beta_{j+1} - k_{j+1}, \ldots, \beta_n - k_n).$$

We assume that for every admissible j, say $j = 1, \ldots, n' \leq n$,

$$D^{\alpha^{k(j)}}w(x) \geq 0, \ x \in K'', \qquad 1 \leq j \leq n'. \tag{3.12}$$

It follows from (3.12) and (3.7)′ that for $x \in K''$

$$0 \leq \int_0^{x_1} D^{\alpha^{k(1)}}w(t, 0 \ldots, 0, x_{n'+1}, \ldots, x_n)dt =$$

$$= D^{\alpha^k}w(x_1, 0, \ldots, 0, x_{n'+1}, \ldots, x_n) - D^{\alpha^k}w(x^k) < D^{\alpha^k}w(x_1, 0, \ldots, 0, x_{n'+1}, \ldots, x_n).$$

From this and from (3.12) it then follows that

$$0 \leq \int_0^{x_2} D^{\alpha^{k(2)}}w(x_1, t, 0, \ldots, 0, x_{n'+1}, \ldots, x_n)dt =$$

$$= D^{\alpha^k}w(x_1, x_2, 0, \ldots, 0, x_{n'+1}, \ldots, x_n) - D^{\alpha^k}w(x_1, 0, \ldots, 0, x_{n'+1}, \ldots, x_n) <$$

$$< D^{\alpha^k}w(x_1, x_2, 0, \ldots, 0, x_{n'+1}, \ldots, x_n), \qquad x \in K''.$$

We repeat the procedure. At last we get

$$D^{\alpha^k}w(x) > 0, \qquad x \in K''.$$

If $|\alpha^k| = |\beta| - 1$, then we have $\alpha^{k(j)} = \beta$ for the unique admissible j. It follows from (3.11) that (3.12) is true in this case. By that we have proved that we can apply the procedure above when $|\alpha^k| = |\beta| - 1$. Since $D^{\alpha^{k(j)}}w(x) > 0$ implies that $D^{\alpha^{k(j)}}w(x) \geq 0$ we can now apply the procedure above to those α^k with $|\alpha^k| = |\beta| - 2$. It is now obvious that (3.10) is true. The lemma is proved.

PROOF OF THE THEOREM. - We shall prove the theorem by induction over the number of variables n. We start by assuming that $n = 1$. We choose a v satisfying (2.1) and (2 2). Let u and $\bar{u}$ be two solutions of (3.3), and let

$d = u - \bar{u}$. Then (3.2) says that

$$(3.13) \qquad |D^\beta d(x)| \leq h(x, |D^{\alpha^1} d|, \ldots, |D^{\alpha^N} d|), \qquad x \in K_0.$$

Since $D^{\alpha^k}(u - \bar{u})(0) = 0$, $1 \leq k \leq N$, it follows from (3.3) that $D^\beta d(0) = D^\beta(u - \bar{u})(0) = 0$.

Let δ be the number used in (2.2). It follows from the continuity of $D^\beta d$ that there exists a number $\varepsilon' > 0$ such that

$$|D^\beta d(x)| < \delta, \qquad 0 \leq x \leq \varepsilon'.$$

Let $\eta^k = \beta - \alpha^k$. It follows from $d(x) = O(x^\beta)$ that

$$(3.14) \qquad |D^{\alpha^k} d(x)| < \delta x^{\eta^k} (\eta^k!)^{-1}, \qquad 0 < x \leq \varepsilon', \qquad 1 \leq k \leq N.$$

The set $K_{\varepsilon'}$ is defined by

$$K_{\varepsilon'} = \{ x \mid \varepsilon' \leq x \leq a_1 \}.$$

We let $F(x, z) = h(x, z)$, $x \in K_{\varepsilon'}$. We also let $x^k = \varepsilon'$ instead of $x^k = 0$. It follows from (3.13) and (2.1) that (3.4) and (3.5) are true with K replaced by $K_{\varepsilon'}$. We also see from (2.2) and (3.14) that

$$|D^{\alpha^k} d(x^k)| = |D^{\alpha^k} d(\varepsilon')| < D^{\alpha^k} v(\varepsilon') = D^{\alpha^k} v(x^k), \; x \in K_{\varepsilon'}, \qquad 1 \leq k \leq N.$$

The lemma applied to $K_{\varepsilon'}$ instead of K now says that

$$|D^{\alpha^k} d(x)| < D^{\alpha^k} v(x) < \varepsilon, \; x \in K_{\varepsilon'}, \qquad 1 \leq k \leq N.$$

We may choose ε' arbitrarily small. Therefore d must be zero in K_0, and thus also in K since d is continuous. The theorem is proved for $n = 1$.

We now assume that the theorem is true for $n = n_0 - 1 \geq 1$. Then we shall prove that the theorem is true for $n = n_0$ too.

Let $x = (x_1, x') \in R^n$ and let $D = (D_1, D')$. We look at

$$(3.15) \qquad D^\beta u(0, x') = f((0, x'), \quad D^{\alpha^1} u(0, x'), \; \ldots, \quad D^{\alpha^N} u(0, x')), \qquad u = O(x^\beta).$$

$D^{\alpha^k} u(0, x') = 0$ when $\alpha_1^k < \beta_1$ since $u = O(x_\beta)$. Let $\beta = (\beta_1, \beta')$ and $\alpha^k = (\alpha_1^k, \alpha'^k)$, $1 \leq k \leq N$. With $\alpha_1^k = \beta_1$ we get

$$D^{\alpha^k} u(0, x') = D'^{\alpha'^k} D_1^{\beta_1} u(0, x').$$

Let the first $\bar{N}$ α^k be such that $\alpha_1^k = \beta_1$ and let $\alpha_1^k < \beta_1$ for the rest. We define

$$w(x') = D_1^{\beta_1} u(0, x').$$

Then (3.15) implies that

$$D'^{\beta'} w(x') = f((0, x'), D'^{\alpha'^1} w, \ldots, D'^{\alpha'^{\bar{N}}} w, 0, \ldots, 0), \qquad w = O(x'^{\beta'}).$$

It follows from the continuity of f and from the definition of $H(n, \beta, K_0)$ that there exist a $h' \in H(n-1, \beta', K_0^1)$ such that

$$| f((0, x'), z_1, \ldots, z_{\bar{N}}, 0, \ldots, 0) - f((0, x'), \bar{z}_1, \ldots, \bar{z}_{\bar{N}}, 0, \ldots, 0) | <$$

$$< h'(x', |z_1 - \bar{z}_1|, \ldots, |z_{\bar{N}} - \bar{z}_{\bar{N}}|), \qquad 0 < x_j \leq a_j, \qquad 2 \leq j \leq n.$$

From the inductive assumption we now conclude that $w(x')$ is uniquely determined. Since

$$D^{\beta} u(0, x') = D'^{\beta'} D_1^{\beta_1} u(0, x) = D'^{\beta'} w(x'),$$

we also see that $D^{\beta} u$ is unique on $x_1 = 0$. In the same way we may prove that $D^{\beta} u$ is unique on $x_j = 0$, $1 \leq j \leq n$.

Let u and $\bar{u}$ be two solutions of (3.3) in K. As in the case $n = 1$ we let $d = u - \bar{u}$. We choose $\varepsilon > 0$ and then we choose δ and v such that (2.1) and (2,2) are true. We have just proved that $D^{\beta} d = 0$, $x_j = 0$, $1 \leq j \leq n$. It follows from the continuity of $D^{\beta} d$ that there exist an $\varepsilon' > 0$ such that

$$| D^{\beta}(x) | < \delta, \qquad x \in K,$$

if at least one $x_j \leq \varepsilon'$. The set $K_{\varepsilon'}$ is defined by

$$K_{\varepsilon'} = \{ x \mid \varepsilon' \leq x_j \leq a_j \}.$$

Since $d = O(x^{\beta})$ we see that with $\eta^k = \beta - \alpha^k$

$$(3.16) \qquad | D^{\alpha^k} d(x) | < \delta x^{\eta^k} (\eta^k !)^{-1}, \; x \in K_0, \quad \text{and at least one } x_j \leq \varepsilon'.$$

Let $F(x, z) = h(x, z)$, $x \in K_{\varepsilon'}$, and let $x_j^k = \varepsilon'$, $\alpha_j^k < \beta_j$ and $x_j^k = x_j$, $\alpha_j^k = \beta_j$. From (3.16) and (2.2) we now get

$$|D^{\alpha^k} d(x^k)| < D^{\alpha^k} v(x^k), \qquad x \in K_{\varepsilon'}, \qquad 1 \leq k \leq N.$$

We note that (3.13) is true in this case too. We now apply the lemma with K replaced by $K_{\varepsilon'}$. Just as in the case $n = 1$ we now conclude that $d = 0$ in K. The theorem is proved.

4. Nagumo conditions and Lipschitz continuity.

In [7] WALTER has shown that the theorem of this paper covers a wide variety of special results when $n = 2$ and $\beta = (1, 1)$. The computation turns out to be increasingly more difficult in the general case. So we give only two examples here. One example shows that the Nagumo condition for ordinary differential equations has its counterpart in the general case too. The other proves that the theorem also covers Lipschitz continuity.

Let $\beta, \alpha^1, \ldots, \alpha^N$, satisfy (3.1). Let $\eta^k = \beta - \alpha^k$, $1 \leq k \leq N$. We shall prove that the function

$$h(x, z_1 \ldots, z_N) = \sum_{k=1}^{N} a_k(x)(x^{\eta^k})^{-1}\eta^k!\, z_k, \qquad (x, z) \in K_0 x R^N,$$

belongs to $H(n, \beta, K_0)$ if $\sum_{k=1}^{N} a_k(x) \leq 1$, $\quad \alpha_k(x) \geq 0, \quad x \in K_0$.

Let $v = Cx^\beta(\beta!)^{-1}$, $C > 0$. It follows that $D^{\alpha^k}v = Cx^{\eta^k}(\eta^k!)^{-1}$, $x \in K$, and $D^\beta v(x) = C$.
It is now obvious that (2.1) is true. We now choose C so small that

$$Cx^{\eta^k}(\eta^k!)^{-1} < \varepsilon, \qquad x \in K_0, \qquad 1 \leq k \leq N.$$

After that we choose $\delta = 2^{-1}C$, and (2.2) is also true. If $n > 1$, then the functions h_j in the definition of $H(n, \beta, K_0)$ in section 2 have the same form as h itself. By that we have proved that $h \in H(n, \beta, K_0)$.

We shall also prove that for $M \geq 1$,

$$h(x, z) = M \sum_{k=1}^{N} z_k, \qquad x \in K_0,$$

belongs to $H(n, \beta, K_0)$. The function v is defined by

$$v = \exp(NMP(x)), \qquad P(\mathrm{x}) = \sum_{j=1}^{N} x_j, \qquad C > 0.$$

Let $\beta, \alpha^1, \ldots, \alpha^N$, satisfy (3.1). Since $NM \geq 1$ and since

$$D^\alpha v(x) = C(NM)^{|\alpha|}\, e^{NMP(x)},$$

we see that

$$M \sum_{k=1}^{N} D^{\alpha^k} v(x) \leq MNC(NM)^{|\beta|-1} \ e^{NMP} = C(NM)^{|\beta|} \ e^{NMP} = D^{\beta} v(x).$$

It is now easy to verify that h belongs to $H(n, \beta, K_0)$.

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