A note on Gevrey classes of functions connected with Goursat problems

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Summary. - *The Gevrey classes $G(\beta, \delta, d)$ have been introduced in connection with non-linear local Goursat problems. A lemma is proved which shows that the definition of the class $G(\beta, \delta, d)$ can be simplified. It then follows that the results in two different papers on Goursat problems are compatible when specialized to formally identical cases.*

The question if certain results for GOURSAT problems in [1] and in [2] are compatible was raised in [2]. The definitions of the classes $G(\beta, \delta, d)$ and $G(0, 0, d)$ in [1] use the existence of a certain real-valued function $b(y)$. It follows from a lemma below that the existence of the function by (y) used in (1.4) in [1] follows from (1.3) in [1] and the continuity of the derivatives involved. The corresponding is true for (1.7) and (1.6). The function $b(y)$ is not used in the definition of the function class $G(\beta, d)$ in [2]. The lemma below shows that to every function of $G(\beta, d)$ there exists a function $b(y)$ satisfying an analogue of (1.4) in [1] and having the other wanted properties too. The lemma is easily extended to the conditions in [1]. It follows that (1.4) can be deleted in the definition of $G(\beta, \delta, d)$ without changing $G(\beta, \delta, d)$.

The same is true for (1.7) and $G(0, 0, d)$ in [1]. Therefore the results in [1] and [2] are compatible when specialized to formally indentical cases.

We shall use $(y, x) \in R^2$ as variables. We define $D_x = \partial/\partial x$ and $D_y = \partial/\partial y$. We repeat definition 1 in [2].

DEFINITION. - *Let $d \geq 1$ be a real number, let β be a positive integer, and let $u(x, y)$ be a realvalued function defined in some neighbourhood of the origin in R^2. The derivatives $D_y^\eta D_x^\alpha u$ are supposed to exist and to be continuous for $\eta \leq \beta$ and for all α. If there exist constants $m > 0$, and r, $0 < r < 1$, such that for every pair (η, α) with $\eta + \alpha d \leq \beta$*

$$(1) \qquad |D_x^\xi D_y^\eta D_x^\alpha u(y, x)| \leq m(2r)^{-\xi}\xi^{\xi d}, \qquad (y, x) \in V, \qquad all\ \xi,$$

then u is said to belong to the function class $G(\beta, d, V)$. We define

$$G(\beta, d) = \bigcup_V G(\beta, d, V).$$

REMARK. - We use $2r$ instead of r in (1) for technical reasons.

(*) Entrata in Redazione il 17 ottobre 1968.

LEMMA. - *If* $u \in G(\beta, d, V)$ *then* (1) *is true and there exists a function* $b(y) \geq 0$ *depending on* $|y|$ *only such that*

$$(2) \qquad |D_x^\xi D_y^\eta D_x^\alpha u(y, x) - D_x^\xi D_y^\eta D_x^\alpha u(0, x)| \leq mb(y) r^{-\xi} \xi^{\xi d}, \quad (y, x) \in V, \quad (0, x) \in V,$$

$$\eta + \alpha d \leq \beta, \qquad \textit{all } \xi.$$

The function $b(y)$ *has the additional property*

$$(3) \qquad y \to 0 \Rightarrow b(y) \to 0 \textit{ monotonically in } |y|.$$

PROOF OF THE LEMMA. - The function $b(y)$ is defined by

$$b(y) = \sup |D_x^\xi D_y^\eta D_x^\alpha u(y', x) - D_x^\xi D_y^\eta D_x^\alpha u(0, x)| r^\xi \xi^{-\xi d}, \qquad |y'| \leq |y|, \qquad (y', x),$$

$$(y, x), \qquad (0, x) \in V, \qquad \eta + \alpha d \leq \beta, \qquad \text{all } \xi.$$

V is supposed to be compact here. It follows from (1) that $b(y) \leq 2m$.
In addition, for a given number $\varepsilon > 0$ there exists a number $K > 0$ such that when $\xi > K$

$$|D_x^\xi D_y^\eta D_x^\alpha u(y', x) - D_x^\xi D_y^\eta D_x^\alpha u(0, x)| r^\xi \xi^{-\xi d} \leq 2m2^{-\xi} \leq m2^{1-K} < \varepsilon.$$

There is only a finite number of derivatives corresponding to (ξ, η, α) with $\xi \leq K$, $\eta + \alpha d \leq \beta$. It follows from the uniform continuity of the derivatives of u in V that there exists a number $\delta > 0$ such that

$$|D_x^\xi D_y^\eta D_x^\alpha u(y', x) - D_x^\xi D_y^\eta D_x^\alpha u(0, x)| r^\xi \xi^{-\xi d} < \varepsilon,$$

$$|y'| \leq |y| < \delta, \qquad \xi \leq K, \qquad \eta + \alpha d \leq \beta.$$

By that the lemma is proved since $b(y) < \varepsilon$ when $|y| \leq \delta$.

REFERENCES

[1] J. PERSSON, *New proofs and generalizations of two theorems by Lednev for Goursat problems*, Math. Ann. 178, 184-208 (1968).

[2] — —, *Exponential majorization applied to a non-linear Cauchy (Goursat) problem for functions of Gevrey nature*, Ann. Mat. pur. appl. LXXVII, 259-268 (1968).