Some Partitions of a Skew Matrix (*)

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Summary. - *Explicit formulas are obtained over a finite field for the number of partitions of a skew matrix B into various forms: for example, $B = X'AY' - XA'Y$.*

1. Introduction. - In this journal [4] John H. Hodges found the number of $m \times t$ matrices X over a finite field which satisfy the matric equation $X'A - A'X = B$, where A is $m \times t$ of rank r, B is a $t \times t$ skew matrix and the prime denotes transpose. In this paper we wish to consider a more general partitioning question, and so discuss equations of the form

$$Y_b' \dots Y_1' A X_a' \dots X_1' - X_1 \dots X_a A' Y_1 \dots Y_b = B, \tag{1.1}$$

with A and B as defined above, X_i, $1 \leq i \leq a$, Y_j, $1 \leq j \leq b$, are matrices of arbitrary sizes subject to the condition that the product, difference and equality must be defined. In Th. I we determine the number of partitions of B as described by (1.1) for a, $b \geq 2$. Then, in Th. II and Th. III, we consider the cases $a = 1$, $b \geq 2$ and $a = b = 1$, respectively. Finally, we discuss the number of partitions of a skew matrix B into a sum of h matrices, each in the form of the left side of (1.1)

2. Notation and preliminaries. - Let $F = GF(q)$ be the finite field of $q = p^f$ elements, p odd. Matrices with elements from F will be denoted by Roman capitols A, B, ... $A(s, m)$ will denote a matrix of s rows and m columns and $A(s, m; r)$ a matrix of the same dimensions having rank r. I_r will denote the identity matrix of order r and $I(s, m; r)$ will denote an $s \times m$ matrix having I_r in its upper left hand corner and zeros elsewhere. If $A = A(n, n) = (a_{ij})$ then $\sigma(A) = a_{11} + \dots + a_{nn}$ is the trace of A. Clearly $\sigma(A + B) = \sigma(A) + \sigma(B)$. For a skew matrix C (one so that $C = -C'$), we have $\sigma(A'C) = -\sigma(AC)$, where A' denotes A transpose.

For $\alpha \in F$, we define

$$e(\alpha) = \exp 2\pi i t(\alpha)/p; \qquad t(\alpha) = \alpha + \alpha^p + \dots + \alpha^{p^{f-1}}, \tag{2.1}$$

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from which it follows that

$$(2.2)\qquad \begin{cases} e(\alpha + \beta) = e(\alpha)e(\beta), \\ \sum\limits_{\beta} e(\alpha\beta) = \begin{cases} q, & \alpha = 0, \\ 0, & \alpha \neq 0, \end{cases} \end{cases}$$

where the sum is over all $\beta \in F$. It is noted in [1; (2.3)] that if $A = A(t, t)$ is skew, then

$$(2.3)\qquad \sum_{\beta} e\{\sigma(AB)\} = \begin{cases} q^{t(t-1)/2}, & A = 0, \\ 0. & A \neq 0, \end{cases}$$

where the summation extends over all skew matrices $B = B(t, t)$. The number of skew matrices of order t and rank $2z$ is given [1; (5.6)] to be

$$(2.4)\qquad H(t, 2z) = q^{z(z-1)} \prod_{i=0}^{2z-1} (q^{t-i} - 1) / \prod_{i=1}^{z} (q^{2i} - 1).$$

Following [1; (8.3)], we define

$$(2.5)\qquad W(B, 2z) = \sum_{C} e\{\sigma(BC)\},$$

where $B = B(t, t)$ is skew, and the sum extends over all skew matrices $C = C(t, t; 2z)$. This sum is evaluated [2; Th. I] to be

$$(2.6)\qquad W(B, 2z) = q^{jz} \sum_{k=0}^{z} (-1)^k q^{k(k-2j-1)} \begin{bmatrix} j \\ k \end{bmatrix}' H(t - 2j, 2r - 4k),$$

with

$$\begin{bmatrix} j \\ k \end{bmatrix} = (1 - q^j) \ldots (1 - q^{j-k+1})/(1 - q) \ldots (1 - q^k); \qquad \begin{bmatrix} j \\ 0 \end{bmatrix} = 1,$$

and the prime indicates that q to be replaced by q^2 in the q-binomial coefficients. If D is an arbitrary $m \times n$ matrix, it may be shown that

$$(2.7)\qquad \sum_{E} e\{\sigma(DE)\} = \begin{cases} q^{mn}, & D = 0, \\ 0, & D \neq 0, \end{cases}$$

with the sum over all matrices $E(m, n)$.

We also find need for the number $g(s, w; y)$ of $s \times w$ matrices of rank

y. This is given by LANDSBERG [2] to be

$$g(s, w; y) = q^{y(y-1)/2} \prod_{i=1}^{y} (q^{s-i+1} - 1)(q^{w-i+1} - 1)/(q^i - 1). \tag{2.8}$$

3. The main theorem. - We may now prove

THEOREM 1. - Let a, b be integers ≥ 2; $A = A(m, n; r)$; $B = B(t, t; 2\rho)$ be skew; $X_1 = X_1(t, t_1)$; $X_i = X_i(t_{i-1}, t_i)$ for $1 < i < a$;

$$X_a = X_a(t_{a-1}, n); \quad Y_1 = Y_1(m, t_{a+1}); \quad Y_j = Y_j(t_{a+j-1}, t_{a+j}) \quad \text{for} \quad 1 < j < b,$$

$Y_b = Y_b(t_{a+b-1}, t)$, where m, n, t, t_1, ..., t_{a+b-1} represent arbitrary positive integers, and r, 2ρ integers such that $r \geq 2\rho \geq 0$. Then the number $N = (a, b, m, n, r, \rho, t, t_i)$ of partitions of B as defined by (1.1) is given by

$$N = q^{\gamma} \sum_{2z=0}^{t_0} W(B, 2z) N_{a+b}(z)$$

where $\gamma = t(t_{a+b-1} - [t-1]/2) + t_{a-1}(n - r) + t_{a+1}(m - r)$; $W(B, 2z)$ is given by (2.6); $N_{a+b}(z)$ is given by (3.4); $t_0 = t$ or $t - 1$ according as t is even or odd.

PROOF. - In view of (2.3), the number of solutions of (1.1) is given by

$$N = q^{-t(t-1)/2} \sum_{C} S(X_1, \ldots, X_a, Y_1, \ldots, Y_b) e \{ \sigma[Y'_b \ldots Y'_1 A \cdot$$
$$\cdot X'_a \ldots X'_1 - X_1 \ldots X_a A' Y_1 \ldots Y_b - B] C \},$$

where $S(X_1, \ldots, X_a, Y_1, \ldots, Y_b)$ denotes a summation over all matrices X_i, Y_j, $1 \leq i \leq a$, $1 \leq j \leq b$, and the sum over C is over all skew matrices of order t. If we note (2.2), divide the sum over C into sums over all $C = C(t, t; 2z)$, $0 \leq 2z \leq t_0$, and recall that $\sigma(D'C) = -\sigma(DC)$, we may write the above equation as

$$\left\{ \begin{aligned} & N = q^{-t(t-1)/2} \sum_{2z=0}^{t_0} \sum_{C(t, t; 2z)} e \{ \sigma(BC) \} \\ & S(X_1, \ldots, X_a, Y_1, \ldots, Y_b) e \{ -2\sigma(X_1 \ldots X_a A' Y_1 \ldots Y_b C) \}. \end{aligned} \right. \tag{3.1}$$

There is no loss of generality by taking $A' = I(n, m; r)$, which is the canonical form of A' under equivalence [6; Th. 3·7]. If we let $X_a = [X_{a1}, X_{a2}]$ and $Y_1 = \text{col}[Y_{11}, Y_{12}]$ with $X_{a1} = X_{a1}(t_{a-1}, r)$, $X_{a2} = X_{a2}(t_{a-1}, n - r)$, $Y_{11} = Y_{11}(r, t_{a+1})$, $Y_{12} = Y_{12}(m - r, t_{a+1})$, then after the above substitution for A,

X_{a2} and Y_{12} are multiplied by zero so the value of the inner sum in (3.1) is independent of their choice. Hence, this sum equals

$$\text{(3.2)} \quad \begin{cases} q^T S(X_1, \ldots, X_{a1}, Y_{11}, \ldots, Y_b) e\{-2\sigma(X_1 \ldots X_{a1} Y_{11} \ldots Y_b C)\}, \\ \text{with} \quad T = t_{a-1}(n-r) + t_{a+1}(m-r). \end{cases}$$

It is noted [4; § 2] that for DC square, $\sigma(DC) = \sigma(CD)$, so by making this substitution and then summing over Y_b in accordance with (2.7), the sum in (3.2) may be evaluated as

$$\text{(3.3)} \quad \begin{cases} q^{tt_{a+b-1}}, & \text{if} \quad CX_1 \ldots X_{a1} Y_{11} \ldots Y_{b-1} = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, we must have $CX_1 \ldots X_{a1} Y_{11} \ldots Y_{b-1} = 0$ or else the contribution to the sum in (3.2) and so also to N will be zero. Thus, we seek the number of solutions of this equation. However, this is a special case of a matric equation previously considered by the author, and with rank $C = 2z$, the number of solutions is given [7; Th. III] to be

$$\text{(3.4)} \quad \begin{cases} N_{a+b}(2z) = q^{\beta} \sum\limits_{z_{a+b-1}=0}^{(t_{a+b-1},\, 2z)} g(2z,\, t_{a+b-1};\, z_{a+b-2}) q^{-t_{a+b-2} z_{a+b-2}} \cdot \\ \quad \cdot \prod\limits_{i=2}^{a+b-2} \sum\limits_{z_{a+b-i-1}=0}^{(z_{a+1-i},\, t_{a+b-i})} g(z_{a+b-i},\, t_{a+b-i};\, z_{a+b-i-1}) \cdot \\ \quad \cdot q^{-z_{a+b-i-1} t_{a+b-i-1}}, \\ \text{with} \quad \beta = t_{a+b-1}(t_{a+b-2} - 2z) + tt_1 + \ldots + t_{a+b-3} t_{a+b-2}, \end{cases}$$

where $(u, v) =$ minimum of u and v; $g(s, w; y)$ is defined by (2.8); the sum over any z_k is defined to be 1 when the upper limit is 0; t_a is defined to be r.

If we now combine (2.5) with (3.1) through (3.4), the theorem is established.

4. Two special cases. - It is of some interest to consider (1.1) for $a = 1$, $b \geq 2$ and $a = 1$, $b = 1$. The proofs of the theorems below for these cases are basically the same as for Th. I so will not be included. We note that the case $a \geq 2$, $b = 1$ may be obtained directly from the case $a = 1$, $b \geq 2$ so need not be considered separately.

THEOREM II. - Let $a \geq 2$, $b = 1$ be integers; A, B, X_i, $1 \leq i \leq a$, be as defined in Th. I; $Y_1 = Y_1(m, t)$. Then the number $N_{a1} = (a, 1, m, n, r, e, t, t_i)$ of partitions of B as defined in (1.1) is given by

$$N_{a1} = q^{\delta} \sum_{2z=0}^{t_0} W(B, 2z) N_{a+1}(z),$$

where $\delta = t(m - [t-1]/2) + t_{a-1}(n - r)$; $W(B, z)$ is defined by (2.6); $N_{a+1}(z)$ may be obtained from (3.4) by letting $t_a = r$, $b = 1$, defining $t_{a-3} = 1$ for $a = 3$ and the product over i to be 1 for $a = 2$.

THEOREM III. - Let $a=b=1$; A, B, Y_1 be as defined in Th. II; $X_1 = X_1(t, n)$. Then the number N_{11} of partitions of B as defined by (1.1) is given by

$$N_{11} = q^{t(m+n-[t-1]/2)} \sum_{2z=0}^{t_0} W(B, 2z) q^{-2rz}.$$

5. **The general partition.** - We now let $A_k = A_k(m_k, n_k; r_k)$ and $A_k(X_k, Y_k) = Y'_{kb_k} \dots Y'_{k1} A_k X'_{ka_k} \dots X'_{k1} - X_{k1} \dots X_{ka_k} A'_k Y_{k1} \dots X_{kb_k}$, where $X_{k1} = X_{k1}(t, t_{k1})$; $X_{ki} = X_{ki}(t_{ki-1}, t_{ki})$ for $1 < i < a_k$;

$$X_{ka_k} = X_{ka_k}(t_{ka_k-1}, n_k); \qquad Y_{k1} = Y_{k1}(m_k, t_{ka_k+1});$$

$$Y_{k,j} = Y_{k,j}(t_{k,a_k+j-1}, t_{k,a_k+j}) \quad \text{for} \quad 1 < j < b_k; \qquad Y_{k,b_k} = Y_{k,b_k}(t_{k,a_k+b_k-1}, t),$$

for each $1 \leq k \leq h$. We then seek the number of ways a skew matrix $B = B(t, t; 2\rho)$ may be partitioned as

$$A_1(X_1, Y_1) + \dots + A_h(X_h, Y_h) = B. \tag{5.1}$$

It is possible to prove

THEOREM IV. - The number N_h of partitions of the matrix B as described in (5.1) when $a_k, b_k \geq 2$, $1 \leq k \leq h$, is given by

$$N_h = q^{\beta - t(t-1)/2} \sum_{2z=0}^{t_0} W(B, 2z) \prod_{k=1}^{h} N_{a_k+b_k}(2z),$$

where $\beta = \beta_1 + \dots + \beta_h$ and β_k is defined by (5.5); $W(b, z)$ is defined by (2.5); $N_{a_k+b_k}(2z)$ is defined by (5.4) and (5.5); t_0 is as defined in Th. I.

PROOF. - It is clear that

$$N_h = q^{-t(t-1)/2} \sum_{C} \sum_{X_{ki}, Y_{kj}} e\{\sigma([A_1(X_1, Y_1) + \dots + A_h(X_h, Y_h) - B]C)\},$$

where the sum over C is over all skew matrices of order t, and the sum over X_{ki}, X_{kj} indicates a summation over each X_{ki}, Y_{kj} as these matrices are defined above. If we note (2.2), the properties of trace, divide the sum over C into successive sums over all $C = C(t, t; 2z)$, $0 \leq 2z \leq t_0$, recall (2.5), and note that the sum over X_{ki}, Y_{kj} is distinct for each k, we obtain

$$N_h = q^{-t(t-1)} \sum_{2z=0}^{t_0} W(B,\ 2z) \prod_{k=1}^{h} S_k, \tag{5.2}$$

with

$$S_k = S(X_{k,1}, \ldots, X_{k,a_k}, Y_{k,1}, \ldots, Y_{k,b_k}) e \{ \sigma[A_k(X_k,\ Y_k)] \}, \tag{5.3}$$

and $S(X_{k,1}, \ldots, X_{k,a_k}, Y_{k,1}, \ldots, Y_{k,b_k})$ is as defined in the proof of Th. I.

The value of S_k is given by (3.2) through (3.4), after making appropiate substitutions, to be

$$q^{\beta_k} N_{a_k+b_k}(2z), \tag{5.4}$$

where

$$\beta_k = t_{k,a_k-1}(n_k - r_k) + t_{k,a_k+1}(m_k - r_k) + tt_{k,a_k+b-1}, \tag{5.5}$$

and $N_{a_k+b_k}(2z)$ is obtained from (3.4) by letting $a = a_k$, $b = b_k$, $z = z_k$, $t_j = t_{k,j}$, $z_j = z_{k,j}$, $r_k = t_{k,a_k}$. The theorem follows by substituting (5.4) into (5.2).

It is possible to state results corresponding to Th. IV when some or all of a_k and $b_k = 1$. However, we shall not take the space to do so.

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