Bounded perturbations of forced harmonic oscillators at resonance.

A. C. LAZER (*) and D. E. LEACH (Cleveland) (**)

Summary. - *Let e be continuous and 2π-periodic, h continuous and bounded, aud $n > 0$ an integer. Sufficient conditions for the existence of 2π-periodic solutions of $x'' + n^2 x + h(x) = e(t)$ are given. The proofs are based on a modification of Cesari's method and the Schauder fixed point theorem.*

Introduction.

Let $e(t)$ be continuous and 2π-periodic. It is well known that if ω is not an integer, then the differential equation

$$x'' + \omega^2 x = e(t)$$

always has a 2π-periodic solution. In extending a recent result due to LOUD [7], the second author, in his dissertation, has established the following:

If g is continuously differentiable, if for some integer n

$$(n-1)^2 < k_1 \leq g'(x) \leq k_2 < n^2$$

holds for all x, and if h is continuous and bounded, then the differential equation

$$x'' + g(x) + h(x) = e(t)$$

has a 2π-periodic solution.

This result has led us to consider the differential equation

$$x'' + n^2 x + h(x) = e(t). \tag{S}$$

where h is as above and n is a positive integer. The case $n = 0$ has already been considered by the first author. It follows from the result in [4] that if there exists a number b such that $x(h(x) - m) \geq 0$ for $|x| \geq b$, where m is the mean value of e, then for $n = 0$ (S) has a 2π-periodic solution. The technique used in the proof of this result will also be used here. It is closely related to a techique used by the first author in [5] which in turn was motivated

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by a method developed by Cesari and his co-workers (see [1], [3]).

In the following we give conditions which are sufficient and conditions which are necessary for (S) to have a 2π-periodic solution. If it is assumed that $\lim_{x\to\infty} h(x)$ and $\lim_{x\to-\infty} h(x)$ exist and that $h(-\infty)\leq h(x)\leq h(\infty)$, these conditions will coincide to yield a necessary and sufficient condition. We will also give sufficient conditions for (S) to possess odd 2π-periodic solutions and even 2π-periodic solutions. We also consider uniqueness.

The hypothesis of each of our theorems will involve the quantities

$$A=\int_0^{2\pi} e(s)\cos ns\, ds, \qquad B=\int_0^{2\pi} e(s)\sin ns\, ds.$$

This is not too surprising since for the case $h(x)\equiv 0$, (S) will possess 2π-periodic solutions if and only if $A=B=0$. In fact, if $h(x)\equiv 0$ and this condition is not satisfied, no solution of (S) is bounded (the phenomena of resonance); while if this condition holds, every solution is 2π-periodic.

In the paper [8], mainly due to P. O. FREDERICKSON, perturbations of the harmonic oscillator involving derivative terms are considered. In the proof of Theorem 1.2 we borrow a technique from this paper.

1. - The General Case.

THEOREM 1.1. - *Let $e(t)$ be a continuous 2π-periodic fuuction. Assume that $h(x)$ is a continuous, bounded and nonconstant functiou and that there exist numbers c, d, C and D $(c<d)$ such that*

$$h(x)\leq C \quad \text{for} \quad x\leq c \tag{1}$$

and

$$h(x)\geq D \quad \textit{for} \quad x\geq d. \tag{2}$$

For any positive integer n, there exists a 2π-periodic solution of the differential equation

$$x''+n^2x+h(x)=e(t) \tag{S}$$

if the condition

$$\sqrt{A^2+B^2}<2(D-C) \tag{3}$$

holds where

$$A=\int_0^{2\pi} e(s)\cos ns\, ds,\quad B=\int_0^{2\pi} e(s)\sin ns\, ds.$$

PROOF. - Let us write equation (S) as the system

$$(4)\qquad \begin{aligned} x_1' &= x_2 \\ x_2' &= -n^2 x_1 - h(x_1) + e(t) \end{aligned}$$

and introdnce new variables z_1 and z_2 by means of the transformation

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos nt & \sin nt \\ -n \sin nt & n \cos nt \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

The transformed system is

$$(5)\qquad \begin{aligned} z_1' &= [h(z_1 \cos nt + z_2 \sin nt) - e(t)] \frac{\sin nt}{n} \\ z_2' &= [e(t) - h(z_1 \cos nt + z_2 \sin nt)] \frac{\cos nt}{n} . \end{aligned}$$

Let R denote the reals and define

$$P = \{\bar\theta / \bar\theta \in C(R, R^2) \quad \text{and} \quad \bar\theta(t) \equiv \bar\theta(t + 2\pi)\}.$$

For $\bar\theta \in P$, $\bar\theta = (\varphi, w)$, set

$$\|\bar\theta\| = \max_t \sqrt{\varphi(t)^2 + w(t)^2} .$$

Define $V = P \times R^2$ and for $(\bar\theta, \bar a) \in V$, set

$$|(\bar\theta, \bar a)| = \|\bar\theta\| + |\bar a|,$$

where $|\bar a| = \sqrt{a^2 + b^2}$ if $\bar a = (a, b)$. Now, if for any $(\bar\theta_1, \bar a_1)$, $(\bar\theta_2, \bar a_2) \in V$ and $\lambda_1, \lambda_2 \in R$ we define

$$\lambda_1(\bar\theta_1, \bar a_1) + \lambda_2(\bar\theta_2, \bar a_2) = (\lambda_1\bar\theta_1 + \lambda_2\bar\theta_2, \lambda_1\bar a_1 + \lambda_2\bar a_2),$$

then $(V, |\,|)$ is a real normed linear space.

Let us define a mapping F of V into V as follows:

For $(\varphi, w) \in P$, set

$$M(\varphi, w) = \frac{1}{2\pi}\int_0^{2\pi} [h(\varphi(s) \cos ns + w(s) \sin ns) - e(s)] \frac{\sin ns}{n} ds$$

and

$$N(\varphi, w) = \frac{1}{2\pi}\int_0^{2\pi}[e(s) - h(\varphi(s)\cos ns + w(s)\sin ns)]\frac{\cos ns}{n}\,ds.$$

Now for $(\bar{\theta}, \bar{a}) \in V$, $\bar{\theta} = (\varphi, w)$, $\bar{a} = (a, b)$ define

$$F(\bar{\theta}, \bar{a}) = (\bar{\theta}^*, \bar{a}^*), \ \bar{\theta}^* = (\varphi^*, w^*), \ \bar{a}^* = (a^*, b^*)$$

where

$$\varphi^*(t) \equiv a + \int_0^t \left([h(\varphi(s)\cos ns + w(s)\sin ns) - e(s)]\frac{\sin ns}{n} - M(\varphi, w)\right)ds,$$

$$w^*(t) \equiv b + \int_0^t \left([e(s) - h(\varphi(s)\cos ns + w(s)\sin ns)]\frac{\cos ns}{n} - N(\varphi, w)\right)ds,$$

$$a^* = a + N(\varphi^*, w^*),$$

and

$$b^* = b - M(\varphi^*, w^*).$$

Since φ^* and w^* are primitives of continuous 2π-periodic functions with mean zero, φ^* and w^* are continuous 2π-periodic functions and hence, F maps V into V. Furthermore, it is easily shown that F is continuous with respect to the norm $\|\ \|$.

Suppose $(\bar{\bar{\theta}}, \bar{\bar{a}}) = ((\widehat{\varphi}, \widehat{w}), (\widehat{a}, \widehat{b}))$ is a fixed point of F. Since $\widehat{a} = \widehat{a}^*$ and $\widehat{b} = \widehat{b}^*$, $M(\widehat{\varphi}^*, \widehat{w}^*) = N(\widehat{\varphi}^*, \widehat{w}^*) = 0$ and so

$$\widehat{\varphi}(t) \equiv \varphi^*(t) \equiv \widehat{a} + \int_0^t [h(\widehat{\varphi}(s)\cos ns + \widehat{w}(s)\sin ns) - e(s)]\frac{\sin ns}{n}\,ds$$

and

$$\widehat{w}(t) \equiv \widehat{w}^*(t) \equiv \widehat{b} + \int_0^t [e(s) - h(\widehat{\varphi}(s)\cos ns + \widehat{w}(s)\sin ns)]\frac{\cos ns}{n}\,ds.$$

Consequently, $\mathrm{col}(\widehat{\varphi}, \widehat{w})$ is a 2π-periodic solution of (5) and $\widehat{x}(t) = \widehat{\varphi}(t) \cos nt + \widehat{w}(t) \sin nt$ is a 2π-periodic solution of equation (S). Hence, to prove the theorem, it is sufficient to show that F has a fixed point. To this end we shall establish the existence of numbers r_1 and r_2 such that if

$$K = \{(\bar{\theta}, \bar{a}) \in V / \|\bar{\theta}\| \leq r_1 \quad \text{and} \quad |\bar{a}| \leq r_2\},$$

then $F(K) \subseteq K$ and $F(K)$ is a relatively compact set. Since K is obviously closed, bounded and convex, it will follow from Schauder's Fixed Point Theorem as given in [2] that F has a fixed point.

Now,

$$|\bar{a}|^2 =$$
$$|\bar{a}|^2 + 2(aN(\varphi^*, w^*) - bM(\varphi^*, w^*)) + N(\varphi^*, w^*)^2 + M(\varphi^*, w^*)^2.$$

But $(N(\varphi^*, w^*)^2 + M(\varphi^*, w^*)^2) \leq \dfrac{2H^2}{n^2}$, where

$$(|h(x)| + |e(t)|) \leq H \quad \text{for} \quad (x, t) \in (-\infty, \infty) \times [0, 2\pi],$$

and so

$$|\bar{a}^*|^2 \leq |\bar{a}|^2 + 2(aN(\varphi^*, w^*) - bM(\varphi^*, w^*)) + \frac{2H^2}{n^2}.$$

By definition

$$aN(\varphi^*, w^*) - bM(\varphi^*, w^*) =$$
$$\frac{1}{2\pi n} \int_0^{2\pi} [e(s) - h(\varphi^*(s) \operatorname{con} ns + w^*(s) \sin ns)] (a \cos ns + b \sin ns)\, ds.$$

However, $\varphi^*(t) = a + \alpha(t)$ and $w^*(t) = b + \beta(t)$, where $\alpha(t)$ and $\beta(t)$ are continuous, 2π-periodic and bounded functions and since;

$$(a \cos ns + b \sin ns) = |\bar{a}| \sin (ns + \xi_0),$$

$\xi_2 = \tan^{-1}\left(\dfrac{a}{b}\right)$, we have

$$aN(\varphi^*, w^*) - bM(\varphi^*, w^*) =$$
$$\frac{1}{2\pi n} \int_0^{2\pi} [e(s) - h(|\bar{a}| \sin (ns + \xi_0) + \gamma(s)] |\bar{a}| \sin (ns + \xi_0)\, ds$$

where $\gamma(s) = \alpha(s) \operatorname{con} ns + \beta(s) \sin ns$. By the change of variable $n\mu = ns + \xi_0$,

(6) $aN(\varphi^*, w^*) - bM(\varphi^*, w^*) =$

$$= \frac{1}{2\pi n} \int_{\frac{\xi_0}{n}}^{2\pi + \frac{\xi_0}{n}} \left[e\left(\mu - \frac{\xi_0}{n}\right) - h\left(|\bar{a}| \sin n\mu + \gamma\left(\mu - \frac{\xi_0}{n}\right)\right) |\bar{a}| \sin n\mu \, d\mu = \right.$$

$$= \frac{|\bar{a}|}{2\pi n} \left[\int_0^{2\pi} e\left(\mu - \frac{\xi_0}{n}\right) \sin n\mu \, d\mu - \int_0^{2\pi} h\left(|\bar{a}| \sin n\mu + \gamma\left(\mu - \frac{\xi_0}{n}\right)\right) \sin n\mu \, d\mu \right]$$

because of periodicity.

To prove the existence of a suitable r_2, we shall first prove the existence of numbers $\sigma > 0$ and $m_1 > 0$ such that

$$aN(\varphi^*, w^*) - bM(\varphi^*, w^*) < -\sigma |\bar{a}|$$

whenever $|\bar{a}| \geq m_1$. For this purpose we note that by the change of variable $nr = n\mu - \xi_0$,

$$\left| \int_0^{2\pi} e\left(\mu - \frac{\xi_0}{n}\right) \sin n\mu \, d\mu \right| = \left| \int_{-\frac{\xi_0}{n}}^{2\pi - \frac{\xi_0}{n}} e(r) \sin (nr + \xi_0) dr \right| =$$

$$= \left| \int_0^{2\pi} e(r) \sin nr \cos \xi_0 dr + \int_0^{2\pi} e(r) \cos nr \sin \xi_0 \, dr \right| =$$

$$= \left| B \cos \xi_0 + A \sin \xi_0 \right| .$$

Hence,

$$\left| \int_0^{2\pi} e\left(\mu - \frac{\xi_0}{n}\right) \sin n\mu \, d\mu \right| \leq \sqrt{A^2 + B^2} \tag{7}$$

by the Schwarz inequality.

By (2), for $0 < \delta < \dfrac{\pi}{2n}$

$$\int_{2k\frac{\pi}{n}+\delta}^{(2k+1)\frac{\pi}{n}-\delta} h\left(|\bar{a}|\sin n\mu+\gamma\left(\mu-\frac{\xi_0}{n}\right)\right)\sin n\mu\, d\mu \geq$$

$$\int_{2k\frac{\pi}{n}+\delta}^{(2k+1)\frac{\pi}{n}+\delta} D\sin n\mu\, d\mu$$

for $k=0, 1, \ldots, (n-1)$ whenever $|\bar{a}|\geq\dfrac{d+L}{\sin n\delta}$, where $L=\max\limits_{\mu}|\gamma(\mu)|$. (*) Similarly, by (1), for all such δ

$$\int_{(2k+1)\frac{\pi}{n}+\delta}^{(2k+)2\frac{\pi}{n}-\delta} h\left(|\bar{a}|\sin n\mu+\gamma\left(\mu-\frac{\xi_0}{n}\right)\right)\sin n\mu\, d\mu \geq$$

$$\int_{(2k+1)\frac{\pi}{n}+\delta}^{(2k+2)\frac{\pi}{n}-\delta} C\sin n\mu\, d\mu$$

or $k=0, 1, \ldots, (n-1)$ whenever $|\bar{a}|\geq\dfrac{L-c}{\sin n\delta}$.

Thus, for all δ small and positive,

$$\int_0^{2\pi} h\left(|\bar{a}|\sin n\mu+\gamma\left(\mu-\frac{\xi_0}{n}\right)\right)\sin n\mu\, d\mu=$$

$$=\sum_{k=0}^{n-1}\int_{2\pi\frac{\pi}{n}}^{(2k+1)\frac{\pi}{n}} h\left(|\bar{a}|\sin n\mu+\gamma\left(\mu-\frac{\xi_0}{n}\right)\right)\sin n\mu\, d\mu+$$

$$+\sum_{k=0}^{n-1}\int_{(2k+1)\frac{\pi}{n}}^{(2k+1)\frac{\pi}{n}} h\left(|\bar{a}|\sin n\mu+\gamma\left(\mu-\frac{\xi_0}{n}\right)\right)\sin n\mu\, d\mu\geq$$

$$\geq n\int_0^{\frac{\pi}{n}} D\sin u\mu\, d\mu+n\int_{\frac{\pi}{n}}^{\frac{2\pi}{n}} C\sin n\mu\, d\mu+Q(\delta)$$

(*) It is easily shown that L can be chosen indipendent of φ aud w.

whenever $|a| \geq \max\left\{\frac{(d+L)}{\sin n\delta}, \frac{(L-c)}{\sin n\delta}\right\}$, where $Q(\delta)$ is a continuous function which vanishes at $\delta = 0$. By (3) there exists $m_2 > 0$ such that

$$\sqrt{A^2 + B^2} = 2(D - C) - m_2. \tag{8}$$

Thus, if we choose δ_0 so small that $|Q(\delta_0)| < \frac{m_2}{2}$,

$$\int_0^{2\pi} h\left(|\bar{a}| \sin n\mu + \gamma\left(\mu - \frac{\xi_0}{n}\right)\right) \sin n\mu \, d\mu < \tag{9}$$

$$> n\left(\frac{2}{n}\right) D + n\left(-\frac{2}{n}\right) C - \frac{m_2}{2} = 2(D - C) - \frac{m_2}{2}$$

whenever $|\bar{a}| \geq \max\left\{\frac{(d+L)}{\sin n\xi_0}, \frac{(L-c)}{\sin n\delta_0}\right\} = m_1$, so by (6), (7), (8) and (9),

$$aN(\varphi^*, w^*) - bM(\varphi^*, w^*)$$

$$< |\bar{a}|\left(\frac{1}{2\pi n}\right)\left[\sqrt{A^2 + B^2} - 2(D - C) + \frac{m_2}{2}\right] = -|\bar{a}|\sigma$$

whenever $|\bar{a}| \geq m_1$, where $\sigma = \frac{m_2}{4\pi n}$.

Now

$$|\bar{a}^*|^2 \leq |\bar{a}|^2 + 2(aN(\varphi^*, w^*)) - bM(\varphi^*, a^*)) + \frac{2H^2}{n^2},$$

but

$$2(aN(\varphi^*, w^*) - bM(\varphi^*, w^*)) + \frac{2H^2}{n^2} < 0$$

whenever $|\bar{a}| \geq \max\left\{m_1, \frac{H_2}{n^2\sigma}\right\} = m_3$. Thus, $|\bar{a}^*| \leq |\bar{a}|$ whenever $|\bar{a}| \geq m_3$.

$$|a^*| \leq \left[n_3^2 + 4\left(\frac{1}{n}\right) H m_3 + \frac{2H^2}{n^2}\right]^{\frac{1}{2}} = m_4$$

for all $|\bar{a}|$ such that $|\bar{a}| \leq m_3$, and therefore we have

$$|\bar{a}^*| \leq r_2$$

whenever $|\bar{a}| \leq r_2$ if we set $r_2 = (m_3 + m_4)$.

Considering $\bar{\theta}^*$, we see that for $|\bar{a}| \leq r_2$,

$$\|\bar{\theta}^*\| \leq \left\{ r_2^2 + \frac{4r_2(2\pi)2H}{n} + 2\left(\frac{(2\pi)2H}{n}\right)^2 \right\}^{\frac{1}{2}} = r_1$$

for all $\bar{\theta} \in P$. Hence if we define

$$K = \{(\bar{\theta}, \bar{a}) \in V / \|\theta\| \leq r_1 \text{ and } |\bar{a}| \leq r_2\}$$

where r_1 and r_2 are as above, K will be a closed, bounded and convex subset of V such that $F(K) \subseteq K$. In order to show that $F(K)$ is a relatively compact set, we need only show that if

$$\{F(\bar{\theta}_n, \bar{a}_n) = \{(\bar{\theta}_n^*, \bar{a}_n^*)\}$$

is any sequence of $F(K)$, there exists $(\bar{l}, \bar{\nu}) \in V$ and a subsequence $\{(\bar{\theta}_{n_k}^*, \bar{a}_{n_k}^*)\}$ of $\{(\bar{\theta}_n^*, \bar{a}_n^*)\}$ such that

$$\lim_{k \to \infty} \|(\bar{\theta}_{n_k}^*, \bar{a}_{n_k}^*) - (\bar{l}, \bar{\nu})\| = 0.$$

The sequence $\{\bar{\theta}_n^*\}$ has the property that

$$\|\bar{\theta}_n^*\| \leq r_1$$

and

$$\left\| \frac{d\bar{\theta}_n^*}{dt} \right\| \leq \frac{\sqrt{8}H}{n}$$

for $n = 1, 2, \ldots$. Hence, the sequence $\{\bar{\theta}_n^*\}$ is equicontinuous and uniformly bounded, and therefore there exists a subsequence $\{\bar{\theta}_{n_k}^*\}$ of $\{\bar{\theta}_n^*\}$ and an $\bar{l} \in P$ such that

$$\lim_{k \to \infty} \|\bar{\theta}_{n_k} - \bar{l}\| = 0$$

by Ascoli's lemma. Since $|\bar{a}_{n_k}^*| \leq r_2$ for all k, there exists a subsequence $\{\bar{a}_{n_{k_r}}\}$ of $\{\bar{a}_{n_k}^*\}$ and $\bar{\nu} \in R^2$ such that

$$\lim_{r \to \infty} |\bar{a}_{n_{k_r}}^* - \bar{\nu}| = 0.$$

Thus,

$$\lim_{r \to \infty} \|(\bar{\theta}_{n_{k_r}}^*, \bar{a}_{n_{k_r}}^*) - (\bar{l}, \bar{v})\| = 0$$

and the set $F(K)$ is relatively compact. By a previous remark, the proof is

THEOREM 1.2. - *Let $e(t)$ be a continuous 2π-periodic function. Assume that $h(x)$ is a continuous, bounded, and nonconstant function. For any positive integer n, there does not exists a 2π-periodic solution of the differential equation*

$$x'' + n^2 x + h(x) = e(t) \tag{S}$$

if the condition

$$\sqrt{A^2 + B^2} \geq 2(\sup_x h(x) - \inf_x h(x)) \tag{10}$$

holds where

$$A = \int_0^{2\pi} e(s) \cos ns\, ds, \quad B = \int_0^{2\pi} e(s) \sin ns\, ds\,.$$

PROOF. - If we choose α_0 such that

$$\cos\alpha_0 = \frac{B}{\sqrt{A^2 + B^2}} \quad \text{and} \quad \sin\alpha_0 = \frac{A}{\sqrt{A^2 + B^2}},$$

then

$$\int_0^{2\pi} e(s) \sin(ns + \alpha_0) ds = \sqrt{A^2 + B^2}.$$

Suppose $x(t)$ is a 2π-periodic solution of (S). Then $y(t) \equiv x\left(t - \frac{\alpha_0}{n}\right)$ is a 2π-periodic solution of

$$y'' + n^2 y + h(y) = e\left(t - \frac{\alpha_0}{n}\right),$$

and hence

$$\int_0^{2\pi} \left[e\left(t - \frac{\alpha_0}{n}\right) - h(y(t))\right] \sin nt\, dt = 0,$$

or equivalently,

$$\int_0^{2\pi} e\left(t - \frac{\alpha_0}{n}\right) \sin nt\, dt = \int_0^{2\pi} h(y(t)) \sin nt\, dt\,.$$

But, by the change of variable $ns = nt - \alpha_0$,

$$\int_0^{2\pi} e\left(t-\frac{\alpha_0}{n}\right)\sin nt\,dt=\int_{-\frac{\alpha_0}{n}}^{2\pi-\frac{\alpha^0}{n}} e(s)\sin(ns+\alpha_0)\,ds=\int_0^{2\pi} e(s)\sin(ns+\alpha_0)\,ds$$

Thus, by the choice of α_0,

$$\int_0^{2\pi} e\left(t-\frac{\alpha_0}{n}\right)\sin nt\,dt=\sqrt{A^2+B^2}.$$

Now,

$$\tag{11}\int_{2k\frac{\pi}{n}}^{(2k+1)\frac{\pi}{n}} h(y(t))\sin nt\,dt\leq$$

$$\leq\sup_x h(x)\int_{2k\frac{\pi}{n}}^{(2k+1)\frac{\pi}{n}}\sin nt\,dt=\left(\frac{n}{2}\right)\sup_x h(x)$$

and

$$\tag{12}\int_{(2k\infty1)\frac{\pi}{n}}^{(2k+2)\frac{\pi}{n}} h(y(t))\sin nt\,dt\leq$$

$$\leq\inf_x h(x)\int_{(2k+1)\frac{\pi}{n}}^{(2k+2)\frac{\pi}{2}}\sin nt\,dt=\;-\left(\frac{2}{n}\right)\inf_x h(x)$$

for $k=0,\,1,\,\ldots,\,(n\quad1)$. But if equality in (11) and (12) hold simultaneously for some integer k_0, $0\leq k_0\leq(n-1)$, then $h(y(t))$ must be discontinuous at $t=\dfrac{(2k_0+1)\pi}{n}$ since $h(x)$ is non-constant. Consequently, we have that

$$\sqrt{A^2+B^2}=\int_0^{2\pi} h(y(t))\sin nt\,dt=$$

$$=\sum_{k=0}^{(n-1)}\int_{k2\frac{\pi}{n}}^{(2k+1)\frac{\pi}{n}} h(y(t))\sin nt\,dt+$$

$$+\sum_{k=0}^{(n-1)}\int\limits_{(2k+1)\frac{\pi}{n}}^{(2k+2)\frac{\pi}{n}} h(y(t))\sin nt\,dt <$$

$$< n\left(\frac{2}{n}\right)\sup_x h(x) + n\left(-\frac{2}{n}\right)\inf_x h(x) =$$

$$= 2(\sup_x h(x) - \inf_x h(x)).$$

But this contradicts (10), and hence the theorem is proven.

From Theorem 1.1 and 1.2 we have the following:

COROLLARY. - *Let $e(t)$ be a continuous 2π-periodic function. Assume that $h(x)$ is a continuous function such that the limits*

$$\lim_{x\to\infty} h(x) = h(\infty)$$

and

$$\lim_{x\to-\infty} h(x) = h(-\infty)$$

exists and are finite. Assume further that

$$h(-\infty) \leq h(x) \leq h(\infty)$$

holds for all x and that

$$h(\infty) - h(-\infty) > 0.$$

There exists a 2π-periodic solution of the differential equation

$$(S) \qquad s'' + n^2 x + h(x) = e(t)$$

if and only if

$$\sqrt{A^2 + B^2} < 2\,(h(\infty) - h(-\infty))$$

where

$$A = \int_0^{2\pi} e(s)\cos ns\,ds, \qquad B = \int_0^{2\pi} e(s)\sin ds.$$

2. - Equations with Symmetries.

THEOREM 2.1. - *Let $e(t)$ be an odd, continuous and 2π-periodic function. Assume that $h(x)$ is an odd, continuous, bouuded and nonconstant function and that there exist numbers c, d, C and D $(c < d)$ such that the inequalities* (1) *and* (2) *hold. For any positive integer n, there exists an odd 2π-periodic solution of the differential equation*

$$x'' + n^2 x + h(x) = e(t) \tag{S}$$

if the condition

$$|B| < 2(D - C) \tag{13}$$

holds where

$$B = \int_0^{2\pi} e(s) \sin ns\, ds.$$

PROOF. - Again, let us write equation (S) as the system (4) an introduce new variables z_1 and z_2 by means of the transformation

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos nt & \sin nt \\ -n \sin nt & n \cos nt \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

As before, the transformed system is

$$\begin{aligned} z_1' &= [z_1 \cos nt + z_2 \sin nt) - e(t)] \frac{\sin nt}{n} \\ z_2' &= [e(t) - h(z_1 \cos nt + z_2 \sin nt)] \frac{\cos nt}{n}. \end{aligned} \tag{5}$$

Define the sets

$$0 = \{\varphi / \varphi \in C(R, R), \quad \varphi \quad \text{is odd and} \quad \varphi(t) \equiv \varphi(t + 2\pi)\}$$

and

$$P = \{w / w \in C(R, R), \quad w \quad \text{is even and} \quad w(t) \equiv w(t + 2\pi)\},$$

where R is the set of real numbers. For any $f(t) \in 0 \cup P$, set

$$\|f\| = \max_t |f(t)|.$$

Define $V = 0 \times P \times R$ and for $(\varphi, w, a) \in V$, set

$$|(\varphi, w, a)| = \|\varphi\| + \|w\| + |a|.$$

If for any $(\varphi_1, w_1, a_1), (\varphi_2, w_2, a_2) \in V$ and $\lambda_1, \lambda_2 \in R$ we define

$$\lambda_1(\varphi_1, w_1, a_1) + \lambda_2(\varphi_2, w_2, a_2) = (\lambda_1\varphi_1 + \lambda_2\varphi_2, \lambda_1 w_1 + \lambda_2 w_2, \lambda_1 a_1 + \lambda_2 a_2),$$

$(V, |\,|)$ is a real normed linear space.

Let us define a mapping F of V into V as follows:

For $(\varphi, w) \in 0 \times P$, set

$$N(\varphi, w) = \frac{1}{2\pi}\int_0^{2\pi} [h(\varphi(s)\cos ns + w(s)\sin ns) - e(s)]\frac{\sin ns}{n}\, ds.$$

Now for $(\varphi, w, a) \in V$, define $F(\varphi, w, a) = (\varphi^*, w^*, a^*)$ where

$$\varphi^*(t) = \int^t \left([h(\varphi(s)\cos ns + w(s)\sin ns) - e(s)]\frac{\sin ns}{n} - N(\varphi, w)\right) ds,$$

$$w^*(t) = a + \int_0^t [e(s) - h(\varphi(s)\cos ns + w(s)\sin ns)]\frac{\cos ns}{n}\, ds,$$

and

$$a^* = a - N(\varphi^*, w^*). \tag{14}$$

Since $\varphi^*(0) = 0$ and φ^* is the primitive of an even continuous 2π-periodic function with mean zero, $\varphi^* \in 0$. Similarly, since w^* is the primitive of an odd continuous 2π-periodic function, $w^* \in P$ and hence, F maps V into V. As before F is continuous with respect to the norm $|\,|$.

Now, proceeding in a manner analogous to that used in the proof of

Theorem 1.1, we shall find numbers r_1, r_2 and r_3 such that if

$$K = \{(\varphi,\ w,\ a) \in V / \|\varphi\| \leq r_1,\ \|w\| \leq r_2 \quad \text{and} \quad |a| \leq r_3\}.$$

then $F(K) \subseteq K$ and $F(K)$ is relatively compact.

To show the existence of a suitable r_3, we shall first show that $N(\varphi^*,\ w^*) > 0$ if $a > 0$ and large, and that $N(\varphi^*,\ w^*) < 0$ if $a < 0$ and negatively large. Now

$$N(\varphi^*,\ w^*) =$$

$$\frac{1}{2\pi n} \int_0^{2\pi} [h(\varphi^*(s) \cos ns + w^*(s) \sin ns) - e(s)] \sin ns \, ds.$$

But $w^*(s) = a + \alpha(s)$, where $\alpha(s)$ is a continuous 2π-periodic function with $|\alpha(s)| \leq \dfrac{2\pi H}{n}$, where $H = \max\limits_{(x,\ t) \in R^2} (|h(x)| + |e(t)|)$. Thus, since

$$|\varphi^*(t)| \leq \frac{4\pi H}{n} \quad \text{for all} \quad t,$$

$$N(\varphi^*,\ w^*) = \frac{1}{2\pi n} \int_0^{2\pi} [h(a \sin ns + \tilde{\alpha}(s)) - e(s)] \sin ns \, ds = \tag{15}$$

$$= \frac{1}{2\pi n} \left[\int_0^{2\pi} h(a \sin ns + \tilde{\alpha}(s)) \sin ns \, ds - B \right]$$

where $\tilde{\alpha}(s) = \varphi^*(s) \cos ns + \alpha(s) \sin ns$ is continuous, 2π-periodic and bounded.

Now, by (2), for $0 < \delta < \dfrac{\pi}{2n}$ and $a > 0$

$$\int_{2{,}\pi\frac{\pi}{n}-\delta}^{(2k+1)\frac{\pi}{n}-\delta} h(a \sin ns + \alpha(s)) \sin ns \, ds \geq \int_{2k\frac{\pi}{n}+\delta}^{(2k+1)\frac{\pi}{n}-\delta} D \sin ns \, ds$$

for $k = 0,\ 1,\ \ldots,\ (n-1)$ whenever $a \geq \dfrac{d+L}{\sin n\delta}$, where $L = \max |\tilde{\alpha}(s)|$. (*) By (1), for such $\delta > 0$ and $a > 0$

$$\int_{(2k-1)\frac{\pi}{n}+\delta}^{(2k+2)\frac{\pi}{n}-\delta} h(a \sin ns + \tilde{\alpha}(s)) \sin ns \, ds \geq \int_{(2k+2)\frac{\pi}{n}+\delta}^{(2k+2)\frac{\pi}{n}-\delta} C \sin ns \, ds$$

(*) L can be chosen indipendent of φ and w

for $k = 0, 1, \ldots, (n-1)$ whenever $a \geq \dfrac{L-c}{\sin n\delta}$. Hence for $\delta > 0$ and small and $a > 0$,

$$\int_0^{2\pi} h(a \sin ns + \tilde{\alpha}(d)) \sin ns \, ds =$$

$$= \sum_{k=0}^{n-1} \int_{(2k+1)\frac{\pi}{n}}^{(2k+2)\frac{\pi}{n}} h(a \sin ns + \tilde{\alpha}(s)) \sin ns \, ds +$$

$$+ \sum_{k=0}^{n-1} \int_{(2k+1)\frac{\pi}{n}}^{(2k+2)\frac{\pi}{n}} h(a \sin ns + \tilde{\alpha}(s)) \sin ns \, ds \geq$$

$$\geq n \int_0^{\frac{\pi}{n}} D \sin ns \, ds + n \int_{\frac{\pi}{n}}^{\frac{2\pi}{n}} C \sin ns \, ds + Q_1(\delta)$$

whenever $a \geq \max\left\{\dfrac{(d+L)}{\sin n\delta}, \dfrac{(L-c)}{\sin n\delta}\right\}$, where $Q_1(\delta)$ is a continuous function vanishing at $\delta = 0$. By (13), there exists $m > 0$ such that

(16) $$m - 2(D - C) < B < 2(D - C) - m.$$

Choosing δ_1 such that $|Q_1(\delta_1)| < \dfrac{m}{2}$, we have

(17) $$\int_0^{2\pi} h(a \sin ns + \tilde{\alpha}(s)) \sin ns \, ds$$

$$> n\left(\frac{2}{n}\right) D + n\left(-\frac{2}{n}\right) C - \frac{m}{2}$$

$$= 2(D - C) - \frac{m}{2}$$

whenever $a \geq \max\left\{\delta_1, \dfrac{(d+L)}{\sin n\delta_1}, \dfrac{(L-c)}{\sin n\delta_1}\right\} = b_1$. Thus, by (15), (16), and (17)

(18) $$N(\varphi^*, w^*) > \frac{1}{2\pi 1}\left[2(D - C) - \frac{m}{2} - 2(D - C) + m\right] =$$

$$= \frac{1}{2\pi n}\left(\frac{1}{m}\right) > 0$$

whenever $a \geq b_1$.

On the other hand, by (1), for $0 < \delta < \frac{\pi}{2n}$ and $a < 0$

$$\int_{2k\frac{\pi}{n}+\delta}^{(2k+1)\frac{\pi}{n}-\delta} h(a \sin ns + \tilde{\alpha}(s)) \sin ns \, ds \leq \int_{2k\frac{\pi}{n}+\delta}^{(2k+1)\frac{\pi}{k}-\delta} C \sin ns \, ds$$

for $k = 0, 1, \ldots, (n-1)$ whenever $a \leq \frac{c-L}{\sin n\delta}$. By (2), for such $\delta < 0$ and $a < 0$

$$\int_{(2k-1)\frac{\pi}{n}+\delta}^{(2k+2)\frac{\pi}{n}-\delta} h(a \sin ns + \tilde{\alpha}(s)) \sin ns \, ds \leq \int_{(2k+1)\frac{\pi}{n}+\delta}^{(2k+2)\frac{\pi}{n}+\delta} D \sin ns \, ds$$

for $k = 0, 1, \ldots, (n-1)$ whenever $a \leq \frac{d+L}{\sin(-n\delta)}$. Hence, for $\delta > 0$ and small and $a < 0$,

$$\int_0^{2\pi} h(a \sin ns + \tilde{\alpha}(s)) \sin ns \, ds \leq n \int_0^{\frac{\pi}{n}} C \sin ns \, ds + n \int_{\frac{\pi}{n}}^{\frac{2\pi}{n}} D \sin ns \, ds + Q_2(\delta)$$

whenever $a \leq \min\left\{\frac{(c-L)}{\sin n\delta}, \frac{(d+L)}{\sin(-n\delta)}\right\}$, where $Q_2(\delta)$ is a continuous function vanishing at $\delta = 0$. Choosing δ_2 such that $|Q_2(\delta_2)| < \frac{m}{2}$, we have

$$(19) \quad \int_0^{2\pi} h(a \sin sn + \tilde{\alpha}(s)) \sin ns \, ds < n\left(\frac{2}{n}\right) C + n\left(-\frac{n}{2}\right) D + \frac{m}{2} = -2(D-C) + \frac{m}{2}$$

whenever $a \leq \min\left\{-\delta_2, \frac{(c-L)}{\sin n\delta_2}, \frac{(d+L)}{\sin(-n\delta_2)}\right\} = b_2$. Thus, by (15), (16) and (19)

$$(20) \qquad N(\varphi^*, w^*) < \frac{1}{2\pi n}\left[-2(D-C) + \frac{m}{2} - m + 2(D-C)\right] = \frac{1}{2\pi n}\left(-\frac{m}{2}\right) < 0$$

whenever $a \leq b_2$. Defining

$$b^* = \max\left(|b_1|, |b_2|, \frac{H}{n} + 1\right),$$

we see by (14), (18), and (20)

$$|a^*| \leq |a|$$

whenever $|a| \geq b^*$. But $|a^*| \leq |a| + |N(\varphi^*, w^*)| \leq | + \frac{H}{n}$ for all 'a', and thus if we define $r_3 = b^* + \frac{H}{n}$, then

$$|a^*| \leq r_3$$

for $|a| \leq r_3$. Hence, if we set $r_1 = \frac{4\pi H}{n}$ and $r_2 = r_3 + \frac{2\pi H}{n}$ and define

$$K = \{(\varphi, w, a) \in V / \|\varphi\| \leq \cdot r_1, \ \|w\| \leq r_2 \quad \text{and} \quad |a| \leq r_3\},$$

then $F(K) \subseteq K$. That $F(K)$ is a relatively compact set follows as before by Ascoli's lemma. Hence, since K is a closed, bounded and convex set, we can apply Schauder's Fixed Point Theorem and obtain a fixed point of F. This gives ss an odd 2π-periodic solution of equation (S) which completes the proof.

THEOREM 2.2. - *Let $e(t)$ be an even, continuous and 2π-periodic function. Assume that $h(x)$ is a continuous, bounded, and nonconstant function and that there exist numbers c, d, C and D $(c < d)$ such that the inequalities* (1) *and* (2) *hold. For any positive integer n, there exists an even 2π-periodic solution of the differential equation*

$$(S) \qquad x'' + n^2 x + h(x) = e(t)$$

if the condition

$$|A| < 2(D - C)$$

holds where

$$A = \int_0^{2\pi} e(s) \cos ns \, ds .$$

PROOF. - Let *us* write equation (S) as the system (4) and then transform system (4) into

$$(5) \qquad \begin{aligned} z_1' &= [h(z_1 \cos nt + z_2 \sin nt) - e(t)] \frac{\sin nt}{n} \\ z_2' &= [e(t) - h(z_1 \cos nt + z_2 \sin nt)] \frac{\cos nt}{n} \end{aligned}$$

as it was done in the proof of Theorem 2.1. Further, let us define the real normed linear space $(V, \|\ \|)$ as we did in the previous proof.

Now, we shall define a mapping F of V into V as follows:

For $(\varphi, w) \in 0 \times P$, set

$$M(\varphi, w) = \frac{1}{2\pi}\int_0^{2\pi} [e(s) - h(w(s)\cos ns + \varphi(s)\sin ns)] \frac{\cos ns}{n}\, ds.$$

For $(\varphi, w, a) \in V$, define $F(\varphi, w, a) = (\varphi^*, w^*, a^*)$ where

$$w^*(t) = a + \int_0^t [h(w(s)\cos ns + \varphi(s)\sin ns) - e(s)] \frac{\sin ns}{n}\, ds,$$

$$\varphi^*(t) = \int_0^t \left([e(s) - h(w(s)\cos ns + \varphi(s)\sin ns)] \frac{\cos ns}{n} - \varphi, w)\right) ds,$$

and

$$a^* = a + M(\varphi^*, w^*).$$

Since w^* is the primitive of an odd continuous 2π-periodic function, $w^* \in P$. Similarly, since $\varphi^*(0) = 0$ and φ^* is the primitive of an even continuous 2π-periodic function with mean zero, $\varphi^* \in 0$ and hence, F maps V into V. Moreover, F is continuous.

Now, making only slight modifications in the prof of Theorem 1.1, we can find a set

$$K = \{(\varphi, w, a) \in V / \|\varphi\| \leq r_1, \quad \|w\| \leq r_2 \text{ and } |a| \leq r_3\}$$

such that $F(K) \subseteq K$ and that $F(K)$ is relatively compact. Thus, since K is a closed, bounded and convex set, we can apply Schauder's Fixed Point Theorem, which will yield the existence of the desired solution and hence completes the proof.

3. - A Uniqueness Condition.

Applying the same argument as given in [6], we obtain

THEOREM 3.1. *Assume that the hypothesis of Theorem* 1.11 *and the condition* (3) *hold. If h is continuously differentiable and*

$$0 < h'(x) < 2n + 1 \tag{21}$$

holds for all x, then there exists a nnique 2π-periodic solution of (S).

PROOF. - The existence of at least one 2π- periodic solution of (S) follows from Theorem 1.1. If $x_1(t)$ and $x_2(t)$ were two distinct 2π-periodic solutions of (S), the difference $y(t) \equiv x_2(t) - x_1(t)$ would be a nontrivial 2π-periodic solution of the linear differential equation

$$y'' + p(t)y = 0,$$

where

$$p(t) = n^2 + \int_0^1 h'(x_1(t) + s(x_2(t) - x_1(t)))ds.$$

From (21)

$$n^2 < p(t) < (n+1)^2, \tag{22}$$

and since $p(t) = p(t + 2\pi)$, $p(t)$ has a positive lower bound. Therefore there exists a number c such that $y(c) = y(c + 2\pi) = 0$, $y'(c) \neq 0$. By (22) and the Sturm comparison theorem, $y(t)$ has exactly $2n$ zeros on the open interval $(c, c + 2\pi)$ contradicting $y'(c) = y'c + 2\pi)$. This contradiction proves the theorem.

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