Approximation of set valued functions and fixed point theorems.

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Sommario. - *Il risultato fondamentale della Nota è il seguente: sia* Γ *una multiapplicazione semicontinua superiormente da uno spazio metrico compatto* S *ad uno spazio normato* Y, *tale che* $\Gamma(x)$ *è convesso. Allora per ogni* $\varepsilon > 0$ *esiste una applicazione continua* $f:$ $:S \rightarrow Y$ *tale che* $d^*(F, G) < \varepsilon$, *ove* F *e* G *sono i grafici di* f *e* Γ *e* $d^*(F, G) = \sup \{ d(y, G), y \in F \}$.

Come corollari di questo teorema vengono dimostrati il teorema di punto fisso di Kakutani in uno spazio di Banach ed una sua generalizzazione, che non richiede la compattezza di $\Gamma(x)$. *Viene poi presentato un teorema di punto fisso con condizioni di convessità più deboli di quello di Kakutani.*

Introduction.

One of the problems that has attracted the attention of mathematicians in the last times has been the problem of extracting, or selecting, a single valued mapping with certain properties, like continuity or measurability, from a given multi-valued mapping: see [2], [4], [5], [6], [7].

In [4] was proved, amog other results, that it is always possible to find a continuous selection $f(\cdot)$ from a lower semicontinuous set-valued mapping $\Gamma(\cdot)$, if Γ maps a metric space into the compact convex subsets of a BANACH. space Y.

This result implies the following: if F and G denote respectively the graphs of f and Γ, and if $d^*(F, G)$ denotes the separation of F from G, then under the preceding hypothesis, there exists a continuous single valued function $f(\cdot)$ such that $d^*(F, G) = 0$. If $\Gamma(\cdot)$ is a closed multi-valued mapping, even with the assumption of convexity, is is easy to see that in general there exists no continuous single valued selection $f(\cdot)$ from $\Gamma(\cdot)$, or, what is the same, there exists no continuous function $f(\cdot)$ such that $d^*(F, G) = 0$, where F and G have the same meaning as before. It is therefore natural and of some importance for its many applications to ask whether and under what conditions, if Γ is restricted to a compact set, given arbitrarily a positive ε, there would exist a continuous single value function $f(\cdot)$, depending on ε

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and defined on the domain of $\Gamma(\cdot)$, such that

$$d^*(F, G) < \varepsilon.$$

In this note it is shown that the answer of this question is positive if $\Gamma(x)$ is convex for all x in the domain of Γ, and is still positive if the convexity condition on Γ is replaced by a weaker condition that, roughly speaking, requires that $\Gamma(x)$ has to be the image of a convex set under a continuous mapping.

As immediate corollaries of these theorems first KAKUTANI'S Fixed Point Theorem and a generalization of it, that does not require $\Gamma(x)$ to be compact, are presented.

Then a generalization of KAKUTANI'S Theorem with a weaker convexity condition on $\Gamma(x)$ is proved.

Notations and basic definitions.

If S is a metric space, $x, s \in S$, $d(x, s)$ denotes the distance of x from s. If Z is also a metric space, $S \times Z$ is a metric space with $d((s, z), (x, y)) = \max\{d(s, x), d(z, y)\}$. 2^S is the set of subsets of S and $\emptyset$ is the empty set. For $A \in 2^S$, $d(x, A) = \inf\{d(x, y), y \in A\}$. For A and $B \in 2^S$, $d^*(A, B)$ will denote the *separation* of A from B, i.e., $d^*(A, B) = \sup\{d(x, B), x \in A\}$. $B[x, \varepsilon]$ for $\varepsilon > 0$, is defined to be $\{y \in S : d(y, x) < \varepsilon\}$, and for $A \in 2^S$, $B[A, \varepsilon] = \{y \in S : d(y, A) < \varepsilon\}$ Also the following notations will be used:

$$\mathcal{C}(S) = \{T \in 2^S : T \text{ is closed}\}$$

and, if Y is a normed linear space

$$\mathcal{K}(Y) = \{T \in 2^Y : T \text{ is convex}\}.$$

$\mathcal{CK}(Y)$ denotes $\mathcal{C}(Y) \cap \mathcal{K}(Y)$.

If $A \in 2^Y$, $co\, A$ denotes the *convex hull* of A and $\overline{co}\, A$ the closed convex hull of A. A mapping $\Gamma : S \to 2^X$ will be referred to as a multi-valued mapping. If $A \in 2^S$, $\Gamma(A) = \cup\{\Gamma(s), s \in A\}$. $\Gamma : S \to 2^Y$ is said *lower semi-continuous* if whenever $V \subset Y$ is open in Y, $\{s \in S : \Gamma(s) \cap V \neq \emptyset\}$ is open in S $\Gamma : S \to 2^Y$ is said to be *upper semi-continuous* (u.s.c.) at s if given $\varepsilon > 0$ there exists a $\delta = \delta(s)$ such that $\Gamma(B[s, \delta]) \subset B[\Gamma(s), \varepsilon]$. Γ is said u.s.c. on S if it is u.s.c. at each $s \in S$. The *graph* of a mapping $\Gamma : S \to 2^Y$ is the subset of $S \times Y$ defined by

$$\{(s, y) : s \in S, y \in \Gamma(s)\}.$$

The same definition holds for a single valued mapping $f : S \to Y$. A mapping $\Gamma : S \to 2^Y$ is called *closed*, if its graph is closed.

If $\Gamma : S \to \mathcal{C}(Y)$ is upper semi-continuous, it is closed. If Y is compact, $\Gamma : S \to \mathcal{C}(Y)$ is u.s.c. if and only if it is closed. If $\Gamma : S \to 2^Y$, its *inverse* Γ^{-1} is defined by

$$\Gamma^{-1}(y) = \{x \in S : y \in \Gamma(x)\}.$$

A mapping $\Gamma : S \to 2^Y$ is lower semi-continuous if and only if $\Gamma^{-1}(V)$ is open in S whenever V is open in Y (see [1]). The *range* of Γ, $\mathcal{R}(\Gamma)$ is defined to be $\{y : y \in \Gamma(x)$, some $x \in S\}$.

On a set S we may consider two different metrics d_1, d_2; in this case notations like « S is d_1-compact » indicate the metric with respect to which a certain property is assumed.

We shall say that a metric space S has the *fixed point property* if every continuous mapping of S into itself has a fixed point in S; when we want to emphasize that two metrics are used, we shall say that a mapping from S into S [or 2^S] is (d_1, d_2)-continuous [(d_1, d_2) — semi-continuous] if it is continuous as a mapping from S with metric d_1 to $S[2^S]$ with metric d_2.

Approximation theorem.

The following Proposition will be used for proving the fixed point theorems.

PROPOSITION 1. - Let S be a compact metric space having the fixed point property. Let $\Gamma : S \to 2^S$ be a closed multi-valued mapping. Assume that for arbitrary $\varepsilon > 0$ there exists a single valued continuous mapping $f : S \to S$, depending on ε, such that if F and G denote the graphs of f and Γ, $d^*(F, G) < \varepsilon$. Then Γ has a fixed point in S.

PROOF. - Let $\varepsilon_n \to 0$ and let f_n be the corresponding single valued mappings. Each f_n has a fixed point in S, say y_n. Let $\{y_m\}$ be a subsequence converging to y_0. We want to show that $y_0 \in \Gamma(y_0)$. We have

$$d((y_0, y_0), G) \leq d((y_0, y_0), (y_m, y_m)) + d((y_m, f(y_m)), G) \leq$$
$$\leq d((y_0, y_0), (y_m, y_m)) + d^*(F_m, G)$$

and the right hand side is as small as we please. Since G is a closed set, it follows that $(y_0, y_0) \in G$, or $y_0 \in \Gamma(y_0)$.

THEOREM 1. - Let S be a compact metric space, Y a normed linear space. Let $\Gamma : S \to \mathcal{K}(Y)$ be a upper semi-continuous multi-valued mapping. Then for $\varepsilon > 0$ arbitrary, there exists a continuous single valued mapping $f : S \to B[\mathcal{R}(\Gamma), \varepsilon] \cap co\ \mathcal{R}(\Gamma)$, depending on ε, such that if F and G are the graphs of f and Γ,

$$d^*(F,\ G) \leq \varepsilon.$$

PROOF. - Fix $\varepsilon > 0$. For each $x \in S$, define a real number $\rho(x,\ \varepsilon)$ by

(1) $$\rho(x,\ \varepsilon) = \sup \{ \delta \leq \varepsilon/2 : \exists x' \in B[x,\ \delta] \ni \Gamma(B[x,\ \delta]) \subset B[\Gamma(x'),\ \varepsilon/2] \}.$$

We will show now that $\rho(x,\ \varepsilon)$ is positive and bounded away from zero on S.

Fix x arbitrary. By definition of upper semi-continuity of the multi-valued mapping Γ, given $\varepsilon > 0$, there exists a $\eta = \eta(x) > 0$, such that $\Gamma(B[x,\ \eta] \subset \subset B[\Gamma(x),\ \varepsilon/2]$.

Setting $x' = x$ in definition (1) we see that $\eta_1 = \min \{ \eta(x),\ \varepsilon/2 \}$ is positive and $0 < \eta_1(x) \leq \rho(x,\ \varepsilon)$.

Suppose that $\rho(x,\ \varepsilon)$ is not bounded away from zero on S, i.e., given any $\zeta > 0$, there exists some $x \in S$ such that $\rho(x,\ \varepsilon) < \zeta$.

Let $\zeta_n \downarrow 0$, $\{x_n\}$ such that $\rho(x_n,\ \varepsilon) < \zeta_n$, and let $x_n \to x_0$. Since $x_0 \in S$, there exists a positive $\eta(x_0)$ (that we can assume $\leq \varepsilon/2$) such that

$$\Gamma(B[x_0,\ \eta(x_0)]) \subset B[\Gamma(x_0),\ \varepsilon/2].$$

Then when $d(x_n,\ x_0) < \eta(x_0)/3$

$$\Gamma(B[x_n,\ \eta(x_0)/3]) \subset \Gamma(B[x_0,\ \eta_0]) \subset B[\Gamma(x_0),\ \varepsilon/2]$$

and $x_0 \in B[x_n,\ \eta(x_0)/3]$.

Therefore $\rho(x_n,\ \varepsilon) \geq \eta(x_0)/3$ for n sufficiently large, contradicting the hypothesis.

Let $\zeta_0 > 0$ such that $\rho(x,\ \varepsilon) \geq \zeta_0$ on S, and let $0 < \zeta_1 < \zeta_0$. Define on S a new multi-valued mapping φ by

$$\Phi(x) = \Gamma(B[x,\ \zeta_1]).$$

It is easy to see that the inverse of each open set in $\mathcal{R}(\Gamma)$ is open (in fact the inverse of each point in $\mathcal{R}(\Gamma)$ is open) and therefore the mapping Φ is lower semi-continuous (but not necessarily convex).

The collection $\{ \Phi^{-1}(y);\ y \in \mathcal{R}(\Gamma) \}$ is an open covering of the compact set S. Let $F : \{ \Phi^{-1}(y_i) \}_{i=1}^{s}$ be a finite subcoverig.

Let $\mathcal{S} : \{ p_i(\cdot) \}_{i=1}^{s}$ a partition of unity subordinated to F, and consider the function

$$f(x) = \sum_{i=1}^{s} p_i(x) y_i .$$

$f(x)$ is a continuous function defined on S. Fix x arbitrary; $f(x)$ is the average of a certain number of y_i, corresponding to $p_i(\cdot)$ not vanishing at x, such that $y_i \in \Phi(x)$. By the definition of $\Phi(x)$, there exists a point x' and a positive number δ, $d(x, x') < \delta$, $\zeta_1 < \delta \leq \varepsilon/2$, such that

$$\Phi(x) = \Gamma(B[x, \zeta_1]) \subset \Gamma(B[x, \delta]) \subset B[\Gamma(x'), \varepsilon/2] \cap \mathcal{R}(\Gamma) \subset$$
$$\subset B[\Gamma(x'), \varepsilon/2] \cap co\mathcal{R}(\Gamma).$$

Since $\Gamma(x')$ is convex, so is $B[\Gamma(x'), \varepsilon/2] \cap co\mathcal{R}(\Gamma)$ and therefore $f(x) \in B[\Gamma(x'), \varepsilon/2] \cap co\mathcal{R}(\Gamma)$. The continuous function $f: S \to co\mathcal{R}(\Gamma) \cap B[\mathcal{R}(\Gamma), \varepsilon]$ has the required properties: for x arbitrary and corresponding x' as before, we have

$$d((x, f(x)), G) \leq d((x, f(x)), (x', f(x))) + d(x', f(x)), G) \leq$$
$$\leq d((x, f(x)), (x', f(x))) + d((x', f(x)), (x', \Gamma(x')))$$
$$\leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

COROLLARY 1. (KAKUTANI'S fixed point theorem in a BANACH space [3]) Let S be a compact convex subset of the BANACH space X; let $\Gamma: S \to \mathcal{CK}(S)$ be upper semi-continuous. Then Γ has a fixed point in S.

PROOF. - By Theorem 1 for any $\varepsilon > 0$ there exists a continuous single valued mapping $f: S \to S$ such that $d^*(F, G) < \varepsilon$. By a theorem of SCHAUDER, S has the fixed point property. By Proposition 1 Γ has a fixed point in S.

Now let S be a convex, weakly compact subset of the BANACH space X and Γ a mapping from S into the (strongly) compact, convex subsets of S such that Γ is u.s.c. from the weak to the strong topology. Then Γ has a fixed point in S. In fact when the image of a point is compact, an u.s.c. mapping maps a compact set onto a compact set; therefore $\mathcal{R}(\Gamma)$ is a strongly compact subset of S and $\overline{co\mathcal{R}(\Gamma)}$ is a compact convex subset of S that is mapped into itself by Γ. Moreover Γ is u.s.c. also from the strong to the strong topology. Therefore the preceding assertion follows KAKUTANI'S Theorem.

The next Corollary extends this result to the case where the image of a point is required to be only *closed*, instead of compact: in this case $\mathcal{R}(\Gamma)$ need not be a strongly compact subset of S and the result does not follow from KAKUTANI'S Theorem.

COROLLARY 1'. - Let S be a weakly compact, convex subset of the separable BANACH space X. Let $\Gamma: S \to \mathcal{CK}(S)$ be upper semi-continuous from the weak to the strong topology. Then Γ has a fixed point in S.

PROOF. - Let d_2 be the metric generated by the norm. From the hypothesis also the weak topology on S is metrizable, with metric d_1. Let $\varepsilon_n \to 0$, let $f_n : S \to S$ such that $d^*(F_n, G) < \varepsilon_n$; such a sequence of functions exists by Theorem 1. Let $y_n = f_n(y_n)$ be the fixed points of f_n in S, that exists from TICHONOV'S Theorem, and assume, from the weak compactness of S, that $y_n \to y_0$ in the metric d_1. Given $\varepsilon > 0$ there exists a $\delta > 0$ such that $\Gamma(B_1[y_0, \delta]) \subset B_2\left[\Gamma(y_0), \frac{\varepsilon}{2}\right]$. Set $\eta = \min\left(\frac{\varepsilon}{2}, \delta\right)$. Let N such that $n \geq N$ implies $y_n \in B_1\left[y_0, \frac{\eta}{2}\right]$ and $\varepsilon_n < \frac{\eta}{2}$. Then given any y_n, $n \geq N$, there exists a $y'_n \in B_1\left[y_n, \frac{\eta}{2}\right]$ such that $f_n(y_n) \in B_2\left[\Gamma(y'_n), \frac{\eta}{2}\right]$. Therefore $y_n = f_n(y_n) \in B_2\left[\Gamma(y'_n), \frac{\eta}{2}\right] \subset B_2[\Gamma(y_0), \eta]$ i.e.

$$(y_n, y_n) \in B_1\left[y_0, \frac{\varepsilon}{2}\right] \times B_2\left[\Gamma(y_0), \frac{\varepsilon}{2}\right], \qquad n \geq N.$$

Let c_{nk} such that for each n only a finite number of c_{nk} are non-zero, $\sum_{k=n}^{\infty} c_{nk} = 1$ and $a_n = \sum_{k=n}^{\infty} c_{nk} y_k$ converges to y_0 strongly, i.e. in d_2.

Since the right hand side of (2) is a convex set,

$$(y_n, x_n) \in B_1\left[y_0, \frac{\varepsilon}{2}\right] \times B_2\left[\Gamma(y_0), \frac{\varepsilon}{2}\right]$$

i.e. $d((y_n, x_n), G) < \frac{\varepsilon}{2}$. Then

$$d((y_0, y_0), G) \leq d((y_0, y_0), (y_n, x_n)) + d((y_n, x_n), G) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for $n \geq N$. Since ε was arbitrary and G is closed, $(y_0, y_0) \in G$ or $y_0 \in \Gamma(y_0)$.

With few modifications in the proof the following more general result holds:

COROLLARY 1''. - Let S be a convex subset of the locally convex complete linear topological space Y. Let the topology on S be metrizable, with metric d_2. Let on S be defined another locally convex topology, metrizable with metric d_1, such that S is d_1-compact. Let the d_1-topology on S be coarser than the d_2-topology, but not coarser than the weak topology of d_2. Let Γ be a (d_1, d_2)-u.s.c. mapping from S into $\mathcal{CK}(S)$.

Then Γ has a fixed point in S.

In the next theorems we attempt to substitute to the convexity of $\Gamma(x)$ a more general condition: roughly speaking, it will be assumed that $\Gamma(x)$ is the image of a convex set under a continuous mapping. Precisely we have the following

THEOREM 2. - Let S be a compact convex subset of the BANACH space X. Let Γ and Δ two u.s.c. mappings from S into 2^S such that for each $x \in S$, $\Delta(x)$ is convex and $\Gamma(x)$ is closed. Let D be the graph of Δ and on D let there be defined a continuous single-valued function $\varphi : D \longrightarrow S$ such that

$$\varphi(x, \Delta(x)) = \Gamma(x) \quad \text{for all} \quad x \in S. \tag{2}$$

Then Γ has a fixed point in S.

PROOF. - Let $\varepsilon_n \downarrow 0$. From Theorem 1 for each n there exists a continuous function $f_n : S \longrightarrow S$ such that $d^*(F_n, D) < \varepsilon_n$, where F_n is the graph of f_n. Using DUGUNDJI'S Theorem we can extend the definition of φ to the whole of $S \times S$, keeping its range in S. Set $g_n(x) = \varphi(x, f_n(x))$. Since φ is uniformly continuous on $S \times S$, it is easy to see that $d^*(G_n, G) \longrightarrow 0$, where G_n is the graph of g_n. By Proposition 1 Γ has a fixed point in S.

We remark that, instead of (2) we could assume the following condition

$$\varphi(x, \Delta(x)) \subset \Gamma(x).$$

The following is a trivial example of construction of the mappings Δ, φ for a given Γ.

EXAMPLE. - Let Γ be an u.s.c. multi-valued mapping defined on a region $\Omega \subset E^2$. Let the image of a point $x \in \Omega$ be one of the sets

$$\Gamma_1 = \{(\xi, \eta);\ \xi^2 + \eta^2 = 1\}$$

$$\Gamma_2 = \left\{(\xi, \eta);\ \xi^2 + \eta^2 = \frac{1}{2}\right\}$$

$$\Gamma_3 = \left\{(\xi, \eta);\ \frac{1}{2} \leq \xi^2 + \eta^2 \leq 1\right\}.$$

Define the mapping Δ by

$$\Delta(x) = \begin{cases} \{(\rho, \theta);\ 0 \leq \theta \leq 2\pi \ \text{ and } \ \rho = 1\} & \text{if } \ \Gamma(x) = \Gamma_1 \\ \left\{(\rho, \theta);\ 0 \leq \theta \leq 2\pi \ \text{ and } \ \rho = \frac{1}{2}\right\} & \text{if } \ \Gamma(x) = \Gamma_2 \\ \left\{(\rho, \theta);\ 0 \leq \theta \leq 2\pi \ \text{ and } \ \frac{1}{2} \leq \rho \leq 1\right\} & \text{if } \ \Gamma(x) = \Gamma_3. \end{cases}$$

Then Δ will be clearly u.s.c.. Moreover, $\Delta(x)$ is convex for all $x \in \Omega$, The function

$$\varphi(x,\ y) = \varphi(y) = \begin{cases} \xi = \rho \cos \theta \\ \eta = \rho \sin \theta \end{cases}$$

is such that

$$\varphi(\Delta(x)) = \Gamma(x).$$

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