Non continuous Liapunov functions.

GIORGIO P. SZEGÖ and GIULIO TRECCANI (Milano) (*) (**)

Summary. - *Theorems which give sufficient conditions for various kinds of qualitative behaviours of flows defined by ordinary differential equations are proved. These theorems are based upon suitable properties of non continuous real-valued functions and their lower-right-hand-side Dini derivatives along the trajectories of the system.*

§ 0. - Notations.

In this work we shall use the following notations. Small latin letters will denote vectors (exceptions v, w, t which are scalars), greek letters will denote scalars and maps, while capital letters sets. With R^n we shall denote the n-dimensional euclidean space. If $A \subset R^n$, we shall denote with $\bar{A}$, ∂A, $\mathfrak{C}A$, $\mathfrak{I}A$ its closure, boundary, complement and interior respectively. If $x \in R^n$ and $\delta > 0$ we shall denote with $S(x, \delta)$, $S[x, \delta]$ the open and closed ball with center x.

For a map φ: $R^n \to R$ we shall write

$$\liminf_{x \to x^0} \varphi(x) = \lim_{\varepsilon \to 0} \inf \{[\varphi(x) : x \in S(x^0, \varepsilon)]\}$$

and similarly define $\limsup_{x \to x^0} \varphi(x)$.

In this paper we shall often use lower and upper semicontinuous real-valued functions on a set $U \subset R^n$. If $U \subset R^n$, a function φ: $U \to R$ is lower-semicontinuous if and only if:

$$\liminf_{y \to x} \varphi(y) \geq \varphi(x) \qquad \text{for all } x \in U, \tag{0.1}$$

and upper semicontinuous if and only if:

$$\limsup_{y \to x} \varphi(y) \leq \varphi(x) \qquad \text{for all } x \in U. \tag{0.2}$$

If φ: $R^n \to R$ is a lower-semicontinuous real-valued function and $\dot{x} = f(x)$, $x \in R^n$, f: $R^n \to R^n$, is an ordinary differential equation and f is a continuous

(*) Entrata in Redazione il 10 febbraio 1969.

(**) Lavoro eseguito nell'ambito del Gruppo N. 11 del C. N. R.

function, we shall consider the following extended real valued function:

$$\psi(x) = \varphi^*(x) = \liminf_{\substack{\tau \to 0^+ \\ \|y\| \to 0}} \tau^{-1}[\varphi(x + \tau f(x) + \tau y) - \varphi(x)], \tag{0.3}$$

which is the lower-right-hand-side DINI derivative of the function $\varphi(x)$ along the solutions of the differential equation $\dot{x} = f(x)$.

§ 1. - Introduction.

In this work we shall give some preliminary results on the characterization of the flow defined by the solutions of the ordinary differential equation

$$\dot{x} = f(x) \qquad f(0) = 0, \qquad x \in R^n \tag{1.1}$$

in the neighborhood of the rest point $x = 0$, which will be assumed to be isolated.

It will be assumed that the differential equation (1.1) defines a dynamical system, i.e. it has properties of existence, uniqueness and global extendability of solutions, through each point $x \in R^n$.

Here we shall use the customary notations of the theory of dynamical systems [1] and denote with xt the solution of the equation (1.1) such that $x0 = x$ and with $\gamma(x)$ the trajectory through x, i.e. the set $\bigcup_{t \in R} xt$.

The problem of the characterization of the qualitative behaviour of the solutions of the differential equation (1.1) in a neighborhood of the isolated rest point $x = 0$ is composed by the sequence of the following different problems:

a) Classification of the flow in a neighborhood of the rest point $x = 0$.

b) Classification of the particolar behaviour of the solutions of a given differential equation in the neighborhood of the rest point $x = 0$, by means of suitable auxiliary functions (LIAPUNOV functions).

c) Estimation of the region Γ in which the behaviour shown near the rest point $x = 0$ is invariant.

d) In the case in which $\Gamma \neq R^n$, estimation of $\partial\Gamma$.

A possible classification of the flow in the neighborhood of a compact invariant set $M \subset R^n$, and in particular of the rest point $x = 0$, is the following.

(1.2) Classification of the flow.

Consider a dynamical system in the space R^n. Then in a sufficiently small neighborhood of the rest point $x = 0$ at least one of the following holds:

(1.3) The point $x = 0$ is positively asymptotically stable.

(1.4) The point $x = 0$ is negatively asymptotically stable.

(1.5) For each $\delta > 0$ there exists a point $x \neq 0$ such that $\gamma(x) \subset S(0, \delta)$ and $\gamma(x)$ is recurrent.

(1.6) There exist two points $x, y \neq 0$ such that $\Lambda^+(x) = \Lambda^-(y) = \{0\}$.

(1.7) For each $\delta > 0$ there exist $x \in S(0, \delta)/\{0\}$ such that $\{0\} \in \Lambda^+(x) \cap \Lambda^-(x)$, $\gamma(x) \subset S(0, \delta)$ and $x \in J^+(x)$.

The classification given above is a refinement of the one proposed by BHATIA [2].

We shall proceed next with the discussion of the available methods for the characterization of the flow induced by the solutions of a given differential equation in the neighborhood of the rest point $x = 0$.

The most powerful method for the solution of this problem is the so-called second method of LIAPUNOV. This «method» is usually based upon the construction of a sufficiently smooth real-valued function $v = \varphi(x)$ (the LIAPUNOV function), such that the real valued function:

$$\psi_1(x) = < \text{grad } \varphi(x), f(x) > = \left[\frac{d\varphi(xt)}{dt}\right]_{t=0}$$

is continuous and sign-definite.

It has been proved [3] that, under our hypothesis on the differential equation (1.1), in the cases (1.3) and (1.4), there always exists in a neighborhood of $x = 0$ a real valued function $v = \varphi(x)$ with continuous partial derivatives of arbitrarily high order and with total time-derivative along the solutions of (1.1) which is sign-definite.

It can also be proved that there exists a continuously differentiable LIAPUNOV function with total time-derivative which is sign-semidefinite, also in the case, (1.5), provided that (1.6) and (1.7) do not hold.

This type of LIAPUNOV function may also be constructed in a neighborhood of $x = 0$ in the case (1.6), but it will not exists in the case (1.7) as well as in the case (1.6) when $x = y$.

On the other hand the existence of continuous first partial derivatives is not strictly needed for characterizing the stability properties of the rest point $x = 0$.

The first aim of this paper will be to develop conditions based upon the use of lower-semicontinuous and upper-semicontinuous real-valued functions which may exists in a neighborhood of the rest point $x = 0$ and allow a classification of the flow in situations in which there cannot exist a continuously differentiable Liapunov function.

The types of conditions that we impose on the functions we use, are different from the one used by Yorke [4] and therefore the next section will be devoted to the proof of some conditions for a lower-semicontinuous real-valued function to be decreasing along the trajectories of the dynamical system (1.1).

Notice, however, that at least, for the local results (Section 3) we could have used the same type of conditions as the one used by Yorke.

We do however believe that the one that we propose here are more convenient to use in the applications.

In the remaining sections of the paper (Sec. 4 and 5) we respectively prove global and local extension theorems [1,5] for the case of semicontinuous real-valued functions.

§ 2. - Definitions and basic inequalities.

In this section we shall give conditions for a lower-semicontinuous real-valued function to be decreasing along the trajectories of the dynamical system (1.1). The first result (Theorem 2.1) is due to Yorke.

2.1. Theorem. - *Let $v = \varphi(x)$, $\varphi\colon U \to R$, be a lower semicontinuous real-valued function defined on a set $U \subset R^n$. Assume that there exists a real-valued function $w = \xi(x)$, $\xi\colon U \to R^n$, which is defined and continuous in U and such that the extended real-valued function* (0.3) *satisfies to the condition*

$$\psi(x \leq \xi(x) \qquad \text{for all } x \in U. \tag{2.2}$$

Then

$$\varphi(xt) - \varphi(x) \leq \int_0^t \xi(x\tau)d\tau \tag{2.3}$$

for all $x \in U$ and $t \in R$ such that $x[0, t] \subset U$.

Assume next that the point $x = 0$ is an isolated rest point. From definition (0.1) and (0.3) it follows that $\psi(x) = 0$. Indeed we have:

$$\psi(0) = \liminf_{\substack{\tau \to 0^+ \\ \|y\| \to 0}} \tau^{-1}\varphi(\tau y) \geq 0 \tag{2.4}$$

but

$$\psi(0) = \liminf_{\substack{\tau \to 0^+ \\ \|y\| \to 0}} \tau^{-1}\varphi(\tau y) \leq \liminf_{\tau \to 0^+} \tau^{-1}\varphi(0\tau) = 0.$$

Which proves the assertion.

Theorem 2.1. suggests the following definition.

2.5. DEFINITION. - Given a lower-semicontinuous real-valued function $v = \varphi(x)$, $\varphi\colon U \to R$, $\{0\} \in U \subset R^n$, we shall say that $\varphi(x)$ has a lower-right-hand side Dini derivative $\psi(x)$ (definition (0.3)), which is negative definite on the set U, if and only if $\psi(0) = 0$ and for each $\varepsilon > 0$ there exists a real valued function $\alpha_\varepsilon(\mu)$, $\alpha_\varepsilon(0) = 0$, $\alpha_\varepsilon(\mu) > 0$ for $\mu > 0$, which is non-decreasing and continuous on $[\varepsilon, +\infty)$, and for each $x \in U$:

$$\psi(x) \leq -\alpha_\varepsilon(\|x\|). \tag{2.6}$$

Following this definition, then, if $\psi(x)$ is negative definite, from (2.3) it follows that:

$$\varphi(xt) - \varphi(x) \leq -\int_0^t \alpha_\varepsilon(\|x\tau\|)d\tau \tag{2.7}$$

for $x[0, t] \subset U / S\,(0, \varepsilon)$.

We shall give next a theorem which allows us to prove an integral inequality of the type (2.6) with different assumptions on the real valued function $\varphi(x)$.

These new assumptions are directly suggested by the special applications we want to make. We shall consider functions $v = \varphi(x)$ which are piecewise differentiable and satisfy suitable properties on the set $D \subset R^n$ on which they are not continuously differentiable.

2.8. PROPERTY. - Consider a lower-semicontinuous real-valued function $v = \varphi(x)$, $\varphi\colon U \subset R^n \to R$, which is not continuously differentiable on the set $D \subset U$.

We shall say that $\varphi(x)$ has the property (2.8) if and only if for each $x \in D^* \equiv \bar{D} / \{0\}$ there exists a real number $\delta > 0$ and a real-valued function $\omega_{x,\delta}(y)$, which is lower semicontinuous in the sphere $S(x, \delta)$, and such that:

$$\omega^*_{x\delta}(y) = \liminf_{\substack{\tau \to 0^+ \\ \|y\| \to 0}} \tau^{-1}[\omega_{x\delta}(y + \tau f(y) + \tau z) - \omega_{x,\delta}(y)] \leq \tag{2.9}$$

$$\leq -\varepsilon, \ \varepsilon > 0 \qquad \text{for all } y \in S(x, \delta)$$

and moreover:

$$z \in D^* \cap S(x, \delta),\ z \neq x \text{ implies } \varphi(\tau) = \varphi(x).$$

From the definitions given it easily follows that:

2.10 PROPERTY. - If $v = \varphi(x)$ is a lower-semicontinuous real-valued function which has the property (2.8) in the set $U \subset R^n$, then for each $x \in U$ the set $\gamma(x) \cap D^*$ is a set of isolated points.

We shall prove next that if a real valued function has the property (2.8), then some properties of the function which hold in the set U/D^*, hold for the whole set U.

2.11. THEOREM. - *Assume that the lower-semicontinuous real-valued function* $v = \varphi(x)$, $\varphi: U \to R$ *has the property* (2.8) *in the set* $U \subset R^n$ *and that for each* $\varepsilon > 0$ *there exists a real-valued function* $\alpha_\varepsilon(\mu)$, $\alpha_\varepsilon(0) = 0$, $\alpha_\varepsilon(\mu) > 0$ *for* $\mu > 0$, *non decreasing and continuous on* $[\varepsilon, +\infty)$, *such that for all* $x \in U/D^*$:

$$\psi(x) \leq -\alpha_\varepsilon(\|x\|)$$

while for each $x \in D^*$, $\psi(x)$ *is finite.*
Then

$$\varphi(xt) - \varphi(x) \leq -\int_0^t \alpha_\varepsilon(\|x\tau\|)d\tau \tag{2.12}$$

for all $x \in U$ *and* $t \in R^+$ *such that* $x[0, t] \subset U/S(0, \varepsilon)$.

PROOF. - Clearly in the interval $[0, t]$ there exists (property 2.10) at the most a finite number of points τ_k such that $x\tau_k \in D^*$. Let this number be n.

For each $x\tau_k$ there exists then a real number $\sigma_k(\tau_k)$ such that:

$$\{x(\tau_k - \sigma_k,\ \tau_k) \cup x(\tau_k,\ \tau_k + \sigma_k)\} \cap D^* = \emptyset .$$

Let $\sigma' = \min\limits_{[0,\, t]} \sigma_k$ and $K = \max\limits_{[0,\, t]} \psi(x\tau_k)$.
By applying the inequality (2.6) we have then:

$$\varphi(xt) - \varphi(x) \leq \sum_k \left\{ \int_{\tau_k - \sigma'}^{\tau_k + \sigma'} K d\tau \right\} + \sum_k \left\{ \int_{\tau_k + \sigma'}^{\tau_{k+1} - \sigma'} -\alpha_\varepsilon(\|x\|)d\tau \right\} + \int_0^{\tau_1 - \sigma'} -\alpha_\varepsilon(\|x\|)d\tau + \int_{\tau_n + \sigma'}^{t} -\alpha_\varepsilon(\|x\|)d\tau. \tag{2.13}$$

Now by letting $\sigma' \to 0^+$ from (2.13) we obtain

$$\varphi(xt) - \varphi(x) \leq -\int_0^t \alpha_\varepsilon(\|x\tau\|)d\tau$$

for all $x[0, t] \subset U/S(0, \varepsilon)$, which proves the theorem.

In spite of the necessity of constructing the local function $\omega_{x,\delta}(y)$ for each $x \in D^*$ and some $\delta > 0$, the condition (2.8) is not difficult to be applied. Its use will be shown in the following example.

2.14. EXAMPLE. - Consider the following planar system of ordinary differential equations, described in polar coordinates:

$$\dot{r} = -r \sin \theta \qquad (2.15)$$
$$\dot{\theta} = r$$

where r and θ are the radial and angular coordinates respectively.

We want to investigate the stability properties of the rest point $r = 0$ by means of a lower semicontinuous real-valued function $v = \varphi(x)$ which is continuously differentiable on the whole plane (r, θ) except the axis $\theta = 0$. Consider for each point $(r_0, 0)$ $r_0 \neq 0$, the neighborhood:

$$s = \{|r - r_0| < \sigma;\ 2\pi - \varepsilon < \theta < 2\pi;\ 0 \leq \theta < \varepsilon\}. \qquad (2.16)$$

On each set s we shall then define the following function:

$$\omega(r, \theta) = \begin{cases} -\theta & \text{for } 0 \leq \theta < \varepsilon \\ e^{-\theta} & \text{for } 2\pi - \varepsilon < \theta < 2\pi \end{cases} \qquad (2.17)$$

The real valued function (2.17) is lower semicontinuous and has the value 0 for $\theta = 0$, $|r - r_0| < \sigma$.

For each point of the set s (2.16), in which $\theta > 0$, the function $\omega(r, \theta)$ is continuously differentiable with $\dot{\omega} = <\text{grad } \omega, f> = -r$, while for each point of the set s in which $\theta < 2\pi$, we have $\dot{\omega} = -r\, e^{-\theta}$. Consider then a point in the set $s(|r - r_0| < \sigma)$ in which $\theta = 0$, i.e. the point $(r, 0) \in S$. Then:

$$\omega^*(r, 0) = \liminf_{\substack{\tau \to 0^+ \\ \tilde{r} \to 0}} \tau^{-1}[\omega(r + \tau\tilde{r},\ \tau r + \tau\tilde{\theta})] =$$
$$= \lim_{\alpha \to 0^+} [\inf_{0 \leq \tau < \alpha} \tau^{-1}\omega(\tau r + \tau\tilde{\theta})].$$

Now if

$$\alpha < \frac{\varepsilon}{r + 2\pi}, \text{ we have } 0 < \tau(r + \tilde{\theta}) < \varepsilon$$

and hence $\tau^{-1}\omega(\tau r + \tau\tilde{\theta}) = -(r + \theta)$.

Thus

$$\omega^*(r, 0) = -(r + 2\pi). \qquad (2.18)$$

In synthesis for each point $(r, \theta) \in S$ we have:

$$\omega^*(r, \theta) \leq -e^{-2\pi}r = -k_1, \; k_1 > 0. \tag{2.19}$$

while for each point of the axis $\varepsilon = 0$, we have

$$\omega(r, 0) = 0. \tag{2.20}$$

Thus each lower-semicontinuous real-valued function which is continuously differentiable on the whole plane (r, θ) but on the axis $\theta = 0$, has the property (2.8) for the system (2.17).

Notice that in this example the condition on ω^* holds indeed only locally (for each $\alpha < \varepsilon/r + 2\pi$) and not globally on the axis $\theta = 0$.

All the results above suggest the following definition:

2.21. DEFINITION. - Let $U \subset R^n$ be a set such that $\{0\} \in U$. Consider the dynamical system defined by the differential equation (1.1).

Let $v = \varphi(x)$: $\varphi = U \to R$ be a lower-semicontinuous real-valued function. Let $D \subset U$ be the set on which $\varphi(x)$ is not continuously differentiable. Let $D^* = \bar{D}/\{0\}$.

We shall say that $\varphi(x)$ is a LIAPUNOV function for the rest point $x = 0$ of the differential equation (1.1) on the set U, with a negative definite lower-right-hand side Dini derivative $w = \psi(x)$, if and only if $\varphi(x)$ has the property (2.8) and in addition for each $\varepsilon > 0$ there exists a real function $\alpha_\varepsilon(\mu)$ $\alpha_\varepsilon : R \to R$, with $\alpha_\varepsilon(0) = 0$ and $\alpha_\varepsilon(\mu) > 0$ for $\mu > 0$, continuous and non-decreasing on $[\varepsilon, +\infty)$, such that the inequality:

$$\psi(x) = \liminf_{\substack{\tau \to 0^+ \\ \|y\| \to 0}} \tau^{-1}[\varphi(x + \tau f(x) + \tau y) - \varphi(x)] \leq -\alpha_\varepsilon(\|x\|) \tag{2.22}$$

holds for each $x \in U/D^*$, while for each $x \in D^*$, $\psi(x)$ is finite.

If and only if the real valued extended function $w = \psi(x)$ has the properties stated above, it will be said to have the property (2.21).

A similar definition can be given for upper semicontinuous real-valued functions.

2.23. REMARK. - It is well known that for the case of a continuous real-valued function $v = \varphi(x)$; φ: $U \to R$, $U \subset R^n$, $\{0\} \in U$, the condition:

(2.24) there exists a continuous, non decreasing real-valued function $\alpha(\mu)$, α: $R \to R$, $\alpha(0) = 0$ and $\alpha(\mu) > 0$ for $\mu > 0$, and such that $\varphi(x) \leq -\alpha(\|x\|)$ for each $x \in U$,

is equivalent to the condition:

(2.25) $\varphi(x) < 0$ for $x \in U/\{0\}$ and for $\{x^n\} \subset U$, $\varphi(x^n) \to 0$ implies $x^n \to \{0\}$.

This equivalence is not generally true for semicontinuous functions. In this case however, condition (2.25) on $\psi(x)$ is equivalent to the condition (2.21) on the set U / D^*.

§ 3. - Local results.

We shall next apply the results of the previous section to the study of some local properties of the flow in the neighborhood of the rest point $x = 0$ of the dynamical system (1.1). Here we shall give some results on properties which cannot be characterized by continuously differentiable LIAPUNOV functions.

3.1. THEOREM. - *If $U \in R^n$ is an open neighborhood of the rest point $x = 0$ $\bar{U}$ is compact and $v = \varphi(x)$ is a lower-semicontinuous Liapunov function with lower-right-hand-side Dini derivative which is negative definite on U with respect to the dynamical system* (1.1) (*def.* 2.21), *then the set $U / \{0\}$ does not contain any recurrent trajectory.*

PROOF. - Let $\gamma(x) \subset U / \{0\}$ be a recurrent trajectory. Then $\overline{\gamma(x)}$ is a compact minimal set. Thus $\rho[\overline{\gamma(x)}, \{0\}] = \sigma > 0$. We can choose $\alpha_\sigma(\mu)$ such that (2.22) holds. By theorem (2.11) we have then for each $t > 0$:

$$(3.2) \qquad \varphi(xt) - \varphi(x) \leq -\int_0^t \alpha_\sigma(\|x\|)d\tau \leq -\int_0^t \alpha_\sigma(\sigma)d\tau = -\alpha_\sigma(\sigma)t$$

On the other hand if $\gamma(x)$ is recurrent then $x \in \Lambda^+(x)$ hence there exists a sequence $\{t^n\} \subset R^+$, $t^n \to +\infty$, such that $x\, t^n \to x$. Thus from (3.2):

$$(3.3) \qquad \varphi(xt^n) \leq \varphi(x) - \alpha_\sigma(\sigma)t^n$$

and $\varphi(xt^n) \to -\infty$ for $n \to +\infty$ which implies that $v = \varphi(x)$ is not lower-semicontinuous in the point $x \in U$ and violates the hypothesis.

3.4. THEOREM. - *Let $v = \varphi(x)$; $\varphi\colon U \to R$ be a lower-semicontinuous real-valued function defined on an open set $U \subset R^n$, with $\{0\} \in U$, and such that:*

i) The extended real-valued function $\psi(x)$ (*def.* 0.3) *has the property* (2.21) *on the set U.*

ii) $\varphi(0) = 0$, $\varphi(x) > 0$ for $x \in U / \{0\}$

iii) The component containing $\{0\}$ of the set $\{x \in U\colon \varphi(x) \leq \beta;\ \beta > 0\}$ is either compact or is U.

Then $\{0\}$ is a positive attractor and $A(0) \subset U$.

Proof. - Let $y \in U/\{0\}$ and assume $\Lambda^+(y) = \emptyset$. It must be $\varphi(y) <$ Sup $\{\varphi(x): x \in U\}$: as U is open, there is $\tau < 0$ such that $y\tau \in U$. By applying the inequality (2.12) to the point $y\tau$, we have that $\varphi(y\tau) > \varphi(y)$. It follows that $\{x \in U: \varphi(x) \leq \varphi(y)\} \neq U$ and so it is compact. Then $\Lambda^+(y) \neq 0$ and compact.
We shall prove that $\Lambda^+(y) = \{0\}$.

Assume $z \in \Lambda^+(y)$, $z \neq \{0\}$. There is then a sequence $\{t^n\} \subset R^+$, $t^n \to +\infty$ such that $yt^n \to z$. Clearly we can construct two sequences, $\{\tau^n\}$ and $\{s^n\}$ such that $y\tau^n \in H(z, \|z\|/3)$ and $ys^n \in \left(z, \frac{2}{3}\|z\|\right)$ and for each n, $s^n < \tau^n < t^n$.

We may also assume that for $s^n < t < \tau^n$, $yt \in S\left(z, \frac{2}{2}\|z\|\right)$ and that $\tau^n - s^n \geq \sigma > 0$ for each n.

From the inequality (2.12) it follows that:

$$\varphi(yt^n) - \varphi(y) < \sum_1^n (\varphi(y\tau^n) - \varphi(ys^n)) \leq$$

$$\leq -\sum_1^n \int_{s^n}^{\tau^n} 2\alpha_{\frac{2}{3}\|z\|}\left(2\frac{\|z\|}{3}\right) d\tau = -\alpha_{\frac{2}{3}\|z\|}\left(\frac{2}{3}\|z\|\right) \sum_1^n (\tau^n - s^n)$$

$$\leq -n\alpha_{\frac{2}{3}\|z\|}\left(\frac{2}{3}\|z\|\right)\sigma.$$

Then for sufficiently large n, $\varphi(yt^n) < \frac{1}{2}\varphi(z)$ and so $\varphi(x)$ cannot be lower semicontinuous in $z \in U$.

We have proved that for each $y \in U$, $\Lambda^+(y) = \{0\}$ and so $\{0\}$ is an attractor and its region of attraction contains U.

§ 4. - Global extension theorem.

In this section we shall prove the global extension theorem i.e. a theorem which allows to extend to the whole space the local stability properties of the rest point $x = 0$, which are supposed to be known.

4.1. Theorem. - *Let $v = \varphi(x)$, $\varphi: R^n \to R$ be a real-valued function which satisfies the following conditions:*

i) $v = \varphi(x)$ *is lower-semicontinuous i.e. if* $x^n \to x$, *then* $\liminf_{x^n \to x} \varphi(x^n) \geq \varphi(x)$.

ii) $\varphi(0) = 0$.

iii) The extended real-valued function $\psi(x)$ (*def.* 0.3) *is negative definite no* R^n, *as in definition* (2.21).

Then if $\{0\}$ *is a positive attractor, it is a positive global attractor.*

Proof. - Assume that the rest point $x = 0$ is a positive attractor and A is its region of attraction. Let $x^0 \in \partial A$, since ∂A is invariant, $\gamma(x^0) \subset \partial A$. Now:

$$\inf \{\|y\|: y \in \gamma(x^0)\} = k > 0.$$

As $\psi(x)$ has the property (2.21), we can choose $\alpha_k(\mu)$ such that:

$$\varphi(x^0 t) - \varphi(x^0) \leq -\int_0^t \alpha_k(\|x^0\tau\|)d\tau \leq -\alpha_k(k)t.$$

Thus there exists $\tau > 0$, such that $x^0\tau \in \partial A$ and $\varphi(x^0\tau) < 0$.

Since $\varphi(x)$ has the property (2.8), for each $\tau' > \tau$, $|\tau' - \tau| < \varepsilon, \varepsilon > 0$ sufficiently small, $\varphi(x)$ is continuous in $x^0\tau'$ and $\varphi(x^0\tau') \leq \varphi(x^0\tau) < 0$.

Since $x^0\tau' \in \partial A$, there exists $z \in A$ such that $\varphi(z) < 0$. Hence:

$$\varphi(zt) < \varphi(z) \qquad \text{for all } t > 0.$$

Consider now a sequence $\{t^n\}$, $t^n \to +\infty$, such that $zt^n \to \{0\}$.

We have $\varphi(zt^n) < \varphi(\tau) < 0$ and therefore

$$\liminf_{n\to+\infty} \varphi(zt^n) < \varphi(0)$$

which contraddicts the assumption *i*) and proves that $\partial A = \emptyset$.

Thus $A = R^n$. Q.E.D.

4.2. Remark. - The most interesting ca e in which theorem (2.1) can be applied is the case of un unstable attractor. This is a case in which it is not possible to construct a continuous Liapunòv function with sign-definite total time derivative.

On the other hand it is possible to construct a lower-semicontinuous real-valued function $v = \varphi(x)$ which satisfies the hypothesis of Theorem 2.1. It is to be noticed that such a real-valued function must be discontinuous at the point $x = 0$. This follows from the fact that the assumption of lower-semicontinuity together with the conditions on $\psi(x)$, do not allow the function to «jump away from $x = 0$» along any trajectory, not even in the points of discontinuity of $v = \varphi(x)$ for $x \in R^n / \{0\}$.

A theorem analogous to theorem (2.1) can be proved also for upper-semicontinuous real-valued function $v = \varphi(x)$ such that, if $x^n \to x$, then

(4.3) $$\limsup_{x^n \to x} \varphi(x^n) \leq \varphi(x).$$

Clearly if $v = \varphi(x)$ is upper-semicontinuous, then the real-valued function $\bar{\varphi}(x) = -\varphi(x)$ is lower-semicontinuous.

In addition for an upper-semicontinuous function we define

$$\psi^*(x) = \limsup_{\substack{\tau \to 0^+ \\ |y| \to 0}} \tau^{-1}[\varphi(x + \tau f(x) + \tau y) - \varphi(x)], \tag{4.4}$$

then

$$\psi^*(x) = -\liminf_{\substack{\tau \to 0^+ \\ |y| \to 0}} \tau^{-1}[\bar{\varphi}(x + \tau f(x) + \tau y) - \bar{\varphi}(x)] = -\bar{\psi}(x) \tag{4.5}$$

where $\bar{\varphi}(x) = -\varphi(x)$ and $\bar{\psi}(x)$ is as defined as in (0.3), for $\bar{\varphi}(x)$.

Thus, if $\varphi(x)$ is such that in an open set $U \subset R^n$ it is

$$\psi^*(x) \geq \xi(x) \tag{4.6}$$

where the real-valued function $\xi(x)$ is continuous in U, then $\bar{\varphi}(x)$ satisfies the inequality

$$\bar{\psi}(x) = -\psi^*(x) \leq -\xi(x). \tag{4.7}$$

Since $\bar{\varphi}(x)$ is lower-semicontinuous, then

$$\bar{\varphi}(xt) - \bar{\varphi}(x) \leq \int_0^t -\xi(x\tau)d\tau \quad \text{for all } t \text{ with } x[0, t] \subset U. \tag{4.8}$$

Hence

$$\varphi(xt) - \varphi(x) \geq \int_0^t \xi(x\tau)d\tau. \tag{4.9}$$

Thus:

4.10. THEOREM. - *Let* $v = \varphi(x)$; φ: $R^n \to R$ *be a real-valued function, which satisfies the following conditions:*

i) $v = \varphi(x)$ *is upper-semicontinuous.*

ii) $\varphi(0) = 0$.

iii) The extended real-valued function $\psi^*(x)$ *defined in* (4.5) *is positive definite on* R^n *(def.* 2.21).

Then if $\{0\}$ *is a positive attractor, it is a positive global attractor.*

§ 5. - Local extension theorem for discontinuous Liapunov functions.

Let $v = \varphi(x)$, φ: $R^n \to R$ be a real-valued, lower-semicontinuous function and let $\varphi(0) = 0$.

Define the following sets:

(5.1) $N'(\beta) = \{x \in R^n\colon \varphi(x) < \beta\}$, β real.

(5.2) $N(\beta)$ is the component of $N'(\beta)$ which contains $\{0\}$.

(5.3) $$N_c(\beta) = \begin{cases} \emptyset \text{ if } \overline{N}(\beta) \text{ is not compact} \\ N(\beta) \text{ if } \overline{N(\beta)} \text{ is compact.} \end{cases}$$

If $\beta \leq 0$, $N(\beta) = N_c(\beta) = \emptyset$, while if $\beta > 0$, $N(\beta) \neq \emptyset$. Then the following properties hold.

(5.4) If $v = \varphi(x)$ is continuous at the point $\{0\}$ and $\beta > 0$, then $N(\beta)$ is a neighborhood of $\{0\}$.

(5.5) If $v = \varphi(x)$ is bounded in a neighborhood of $\{0\}$, then there exists $\beta > 0$, such that $N(\beta)$ contains an open neighborhood of $\{0\}$.

(5.6) If $v = \varphi(x)$ is discontinuous at the point $\{0\}$, then there exists a real number $\beta > 0$ such that $\{0\} \subset N(\beta)$; we may even have $\{0\} = N(\beta)$.

(5.7) $N(\beta)$ may not be open.

Consider next the differential equation (1.1) where $f\colon R^n \to R^n$ satisfies usual conditions for uniqueness and global existence of solutions for each point $x \in R^n$. Consider next the real-valued function $w = \psi(x)$, defined in (0.3).

From inequality (2.3) follows that if $\psi(x) \leq 0$ for all $x \in N(\beta)$, the set $N(\beta)$ is positively invariant.

Consider next the sets

(5.8) $0'(\beta) = \{x \in R^n\colon \varphi(x) > \beta\}$

(5.9) $0(\beta)$ is the component of $0'(\beta)$ which contains $\{0\}$; since $v = \varphi(x)$ is lower-semicontinuous then the set $0(\beta)$ is an open set.

In this case, however, if $\psi(x) \leq 0$ for all $x \in 0(\beta)$, it does not follows that $0(\beta)$ is positively invariant.

Finally it also follows that:

(5.10) If $v = \varphi(x)$ is continuous on R^n, then $\varphi(x) = \beta$ for $x \in \partial N(\beta)$.

(5.11) If $v = \varphi(x)$ lower-semicontinuous on R^n, then $v = \varphi(x) < \beta$ for $x \in \partial N(\beta)$.

We are now in the position of proving the main result.

5.12. THEOREM. - Consider the differential equation (1.1) with the usual hypothesis. Let $x = 0$ be an isolated rest point which is a positive attractor. Let A^+ be its region of attraction. Let $v = \varphi(x)$ be a lower-semicontinuous real-valued function, such that:

i) $\varphi(0) = 0$.

ii) there exists a real-number $\beta^c > 0$ such that the extended real-valued function, $w = \psi(x)$, defined in (0.3) has the property (2.21) in $N(\beta^c)$.

Then $A^+ \supset N(\beta^c)$.

PROOF. - Assume by absurd that $x^0 \in (\partial A^+ \cap N(\beta^c))$. From the hypothesis made, the set $N(\beta^c) \cap \partial A^+$ is positively invariant, hence $\gamma^+(x^0) \subset [N(\beta^c) \cap \partial A^+]$.

Now by proceding as in the proof of theorem (2.1), we reach a contradiction which proves $\partial A^+ \cap N(\beta^c) = \emptyset$ and $N(\beta^c) \subset A^+$. Q.E.D.

5.13. THEOREM. - *If in addition to the hypothesis of theorem* (5.12) *it is also true that*

iii) for each $x \in \partial N(\beta^c)$ *we can find a* $\sigma > 0$ *such that* $\psi(z) < -\sigma$ *for each* $z \in S(x, \delta) \cap N(\beta^c)$, *with* $\delta > 0$, *sufficiently small.*

Then $A^+ \supset \overline{N(\beta^c)}$.

PROOF. - If $N(\beta^c)$ is positively invariant, then also $\overline{N(\beta^c)}$ has this property. Let now $x^0 \in \partial N(\beta^c) \cap \partial A^+$. From the hypothesis *iii)* it then follows that there exists $\tau > 0$, such that $x^0\tau \in N(\beta^c)$, which contraddicts the fact that $x^0\tau \in \partial A^+$ and proves the theorem.

5.14. REMARK. - In the same fashion as in theorem (4.10), theorems analogous to theorem (5.12) and (5.13) can be proved also with upper-semicontinuous real-valued functions. Clearly now the hypothesis of the theorem will be on the sets $O(\beta)$, instead of on the sets $N(\beta)$.

Theorems which are the dual of Theorem (5.12) and (5.13) can be proved for the case of negative attractor, with the natural changes of hypothesis. In addition to the characterization of instable attractors, it is conceivable that the theorems that we have presented in this paper may be applied to the numerical construction of LIAPUNOV functions which are made by different families of elementary polynomial forms with suitably matched boundaries.

§ 6. - **Conclusions.**

The preliminary results that we have given prove the usefulness of semicontinuous real-valued functions for characterizing the behaviour of flow for which a continuously differential LIAPUNOV functions cannot exist. Many questions however still remain open in particular regarding the converse theorems.

Another important application of this theory is in the numerical construction of LIAPUNOV functions, in the case in which the rest point is asymptotically stable, but not globally asymptotically stable and one wishes to identify ∂A. This identification can be made withe semicontinuous functions which are made by different families of continuously differentiable function with very simple analytic expressions.

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