On a class of partial differential equations of even order (*).

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To Mauro Picone on his 70th birthday.

Summary. - *The general solution for a class of equations of even order is expressed as a sum of solutions of equations of second order.*

1. Introduction.

This paper deals with the general solution for a class of elliptic and hyperbolic partial differential equations of even order which includes some classical equations among which the best known is the equation for polyharmonic functions. As is well known, this equation gave rise to many outstanding papers from ALMANSI to PICONE [1]. For this reason the author hopes that the following modest contribution is not out of place as a tribute to Professor MAURO PICONE on the occasion of his anniversary.

Before going into the general theory let us illustrate our results on the example of the biharmonic equation $\Delta\Delta w = 0$ where w will be, as all functions in the following, considered as a function of the $m+1$ variables $x_1, x_2, \ldots, x_{m+1}$. One of the variables, say x_{m+1}, will play a special role in our considerations and will be denoted by y. We shall also use often the abbreviation x for the set $x_1, x_2, \ldots, x_m$. A classical result of ALMANSI states that every regular solution of the biharmonic equation can be written in the form $w = h + yh_1$, where h and h_1 are harmonic functions satisfying $\Delta u = 0$ in the $m+1$ variables $(x_1, \ldots, x_m, y)$. The interesting feature of this decomposition is that the first term h satisfies a partial differential equation of the second order while the second term yh_1 is itself a biharmonic function of a special type and satisfies an equation of the fourth order. The same remark is true for the other decompositions considered by ALMANSI; for instance $w = h + r^2h_1$ where $r^2 = x_1^2 + \ldots + x_m^2 + y^2$. Let us compare the biharmonic equation to the equation $\Delta(\Delta + \varepsilon)w = 0$ where ε is a constant. The general solution of this equation is $w = h + v$ where $\Delta h = 0$ and $\Delta v + \varepsilon v = 0$ (see [8, p. 15]). This means that the solution w of an equation

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of fourth order appears here as the sum two functions, each satisfying an equation of the second order. This property seemingly disappears if we put ε equal to zero, and it is easily checked that the decomposition of certain solutions of our fourth order equation tends formally to ALMANSI' s decomposition of the biharmonic equation.

It will be one of the purposes of this paper to show that, using some recent results, it is possible to find a decomposition of a biharmonic function into a sum of two functions each satisfying an equation of the second order (see equation (3.3)) which occurs in generalizéd axially symmetric potential theory.

2. Two Differential Operators.

In this paragraph we will consider two differential operators; namely,

$$\frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_m^2} + \frac{\partial^2}{\partial y^2} + \frac{k}{y}\frac{\partial}{\partial y} \tag{2.1}$$

and

$$\frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_m^2} - \frac{\partial^2}{\partial y^2} - \frac{k}{y}\frac{\partial}{\partial y} \tag{2.2}$$

where k is a real parameter, $-\infty < k < \infty$. The first operator is elliptic and occurs in generalized axially symmetric potential theory which we denote fort short by GASPT [2]. The second operator is hyperbolic and appears in the theory of the EULER-POISSON-DARBOUX (abbreviated EPD) equation [3, 4]. While the study of the two operators leads to essentially different problems, we shall use for both of them the same notation L_k because only common properties of both operators will be used in the major part of this paper.

A solution of the partial differential equation of the second order

$$L_k u = 0 \tag{2.3}$$

will be denoted as u^k or $u^{(k)}$. According to what was said previously u^k may be a solution of an equation of GASPT or a solution of an EPD equation. In particular $u^{(0)}$ denotes either a harmonic function or, alternatively, a solution of the wave equation in an $m+1$ dimensional space or space time.

The basic common property of the two operators is that in both cases we have the following fundamental recursion formulas

$$u_y^k(x, y) = y u^{k+2}(x, y) \tag{2.4}$$

$$u^k(x, y) = y^{1-k} u^{2-k}(x, y) \tag{2.5}$$

As the present author stressed many times, these formulas reflect only an elementary property of the expression

$$u_{yy} + \frac{k}{y} u_y \tag{2.6}$$

where u is considered to be a function of a single variable y, the x_i being parameters. For this reason the recursion formulas are valid for the solutions of any differential equation which is obtained by equating (2.6) to any linear differential operator which is free of the variable y. The recursion formula (2.4) defines $u^{(k+2)}$ in terms of $u^{(k)}$. However, if we rewrite (2.4) as

$$u^{(k)}(x, y) = \int_b^y \eta u^{(k+2)}(x, \eta) d\eta + f(x, b) \tag{2.7}$$

where $f(x, b)$ are the values which $u^{(k)}$ takes for $y = b$, we may interpret it as an *integral equation* which yields a function $u^{(k)}$ from a function $u^{(k+2)}$. This integral equation will play an important role in the present paper. A similar equation plays a dominant part in the solution of the CAUCHY problem for the EPD equation as will be briefly indicated in paragraph 3, Remark B.

3. Equations of the Fourth Order Associated with the Operator L_β.

In this paragraph we shall prove the following fundamental theorem.

THEOREM. - *The general solution of the equation*

$$L_\alpha L_\beta w = 0, \qquad \beta \neq \alpha - 2 \tag{3.1}$$

is given by the formula

$$w = u^{(\beta)} + u^{(\alpha-2)}. \tag{3.2}$$

The only arbitrariness in this decomposition is that $u^{\alpha-2}$ and u^{β} can be replaced by $u^{\alpha-2} + h$ and $u^{\beta} - h$, respectively, where h is an arbitrary harmonic function of the variables $x_1, x_2, \ldots, x_m$. Our decomposition (3.2) is valid in any cylindrical domain of $m + 1$ dimensional space with its base in base in the subspace $y = b =$ constant and with generators parallel to the y-axis, this cylinder lying entirely in the domain ef regularity of w. In the special case $\alpha = \beta = 0$ we have the result that every biharmoaic function w admits the decomposition

$$w = u^{(0)} + u^{(-2)}. \tag{3.3}$$

The appearance of axially symmetric functions in this decomposition is remarkable because the biharmonic function w does not in general have any simmetric properties. For $m = 1$, $u^{(0)}$ and $u^{(-2)}$ are STOKES stream functions in a two and a four dimensional space, respectively. By our previous remark the same decomposition holds for the solution of the iterated wave equation.

To prove our theorem let us observe that it follows immediately from (3.1) by our definition of the differential operators that every solution w of (3.1) satisfies the equation

$$L_\beta w = u^{(\alpha)}. \tag{3.4}$$

Conversely, any solution of (3.4) satisfies (3.1). Therefore, the proof of our theorem is reduced to the determination of the general solution w of (3.4) for a *given* function $u^{(\alpha)}$ which satisfies by definition the equation $L_\alpha u^{(\alpha)} = 0$. We shall show as a first step that for $\beta - \alpha + 2 \neq 0$ (3.4) admits a particular solution $w = u^{\alpha-2}$ so that

$$L_\beta u^{\alpha-2} = u^\alpha. \tag{3.5}$$

To this end let us observe that as $\beta = (\alpha - 2) + (\beta - \alpha + 2)$ and as $L_{\alpha-2} u^{\alpha-2} = 0$ we have for any $u^{(\alpha-2)}$ the obvious identity

$$L_\beta u^{(\alpha-2)} \equiv \sum_1^m \frac{\partial^2 u^{\alpha-2}}{\partial x_i^2} + \frac{\partial^2 u^{\alpha-2}}{\partial y^2} + \frac{\alpha-2}{y}\frac{\partial u^{\alpha-2}}{\partial y} + \frac{\beta-\alpha+2}{y}\frac{\partial u^{\alpha-2}}{\partial y} = \frac{\beta-\alpha+2}{y}\frac{\partial u^{\alpha-2}}{\partial y}. \tag{3.6}$$

For the sake of definiteness let us assume that in (3.6) and in the following proof, L_β is the operator in GASPT as the proof for the EPD equation is entirely analogous. In view of (3.6), equation (3.5) can be replaced by the equation

$$\frac{\partial u^{\alpha-2}}{\partial y} = \frac{1}{\beta-\alpha+2}\, y u^\alpha. \tag{3.7}$$

This equation is obviously of the type of the recursion formula (2.4) with u^α being replaced by $\frac{1}{\beta-\alpha+2} u^\alpha$. As in paragraph 2, (3.7) leads to the integral equation

$$u^{\alpha-2}(x, y) = c\int_b^y \eta u^\alpha(x, \eta)\,d\eta + f(x, b) \tag{3.8}$$

where

$$c = \frac{1}{\beta-\alpha+2}.$$

The value $y = b$ is assumed to be in the domain of regularity of w. It can be assumed without loss of generality that $b \neq 0$. The function $f(x, b) = f(x_1, \dots, x_m, b)$ will now be determined by using the differential equation

$$L_{\alpha-2} u^{\alpha-2} \equiv \Delta_x u^{\alpha-2} + u_{yy}^{\alpha-2} + \frac{\alpha-2}{y} u_y^{\alpha-2} = 0 \tag{3.9}$$

where Δ_x denotes the operator $\sum_{i=1}^m \frac{\partial^2}{\partial x_i^2}$. By (3.8) we have

$$\begin{aligned}\Delta_x u^{\alpha-2} &= c\int_b^y \eta \Delta_x u^\alpha(x, \eta)\,d\eta + \Delta_x f(x, b)\\ &= -c\int_b^y \eta\left[u^\alpha_{\eta\eta} + \frac{\alpha}{\eta} u^\alpha_\eta\right] d\eta + \Delta_x f(x, b).\end{aligned}$$

The integration on the right hand side can be performed in an elementary way and leads immediately to the following formula,

$$(3.10)\qquad \begin{aligned}\Delta_x u^{\alpha-2} = &- c\,\{\,y u^{\alpha}_{\eta}(x,\,y) - b u^{\alpha}_{\eta}(x,\,b) \\ &- u^{\alpha}(x,\,y) + u^{\alpha}(x,\,b) + \alpha u^{\alpha}(x,\,y) - \alpha u^{\alpha}(x,\,b)\,\} \\ &+ \Delta_x f(x,\,b).\end{aligned}$$

Further, by (3.8)

$$(3.11)\qquad u^{\alpha-2}_{yy} = c\,\{\,u^{\alpha}(x,\,y) + y u^{\alpha}_{\eta}(x,\,y)\,\}$$

and

$$(3.12)\qquad \frac{\alpha-2}{y}\,u^{\alpha-2}_{y} = c(\alpha-2)u^{\alpha}(x,\,y).$$

The sum of the left hand of (3.10), (3.11), (3.12) is by (3.9) equal to zero. Therefore, we have for the sum of the right hand sides the equation

$$(3.13)\qquad \Delta_x f(x,\,b) + c\,\{\,b u^{\alpha}_{y}(x,\,b) + (\alpha-1)u^{\alpha}(x,\,b)\,\} = 0.$$

This equation is an m-dimensional POISSON equation for the function $f(x_1, \dots, x_m, b)$ on the manifold $y = b$ and determines the unknown function $f(x, b)$ up to a harmonic function h in $x_1, x_2, \dots, x_m$. This last remark is confirmed by the fact that difference, $u_1^{\alpha-2} - u_2^{\alpha-2}$, of the two solutions of (3.7) is a function $u^{\alpha-2}$ which is independent of y and, therefore, by the equation $L_{\alpha-2}u^{\alpha-2} = 0$ is a harmonic function in the x-space.

Once f has been determined it is obvious that the right hand side of (3.8) defines indeed a function $u^{\alpha-2}$ satisfying the equation (3.4). As we have by (3.4) and (3.5)

$$L_\beta w - L_\beta u^{\alpha-2} = L_\beta(w - u^{\alpha-2}) = 0$$

it follows by the definition of a function u^β that $w - u^{\alpha-2} = u^\beta$ which proves our assertion (3.2). As $u^{\alpha-2}$ is determined up to an arbitrary harmonic function h in $x_1, \dots, x_m$, the function u^β for a given w is determined up to the function $-h$.

REMARK (A). - The computation which leads to (3.6) yields for any values of β and γ, the equation $L_\beta u^\gamma = \dfrac{\beta-\gamma}{y}\dfrac{\partial u^\gamma}{\partial y}$. The right hand side is by (2.4), regardless of the value of the constant $\beta - \gamma$, obviously a function $u^{\gamma+2}$ so that we have for any given u^γ the general relation

$$(3.14)\qquad L_\beta u^\gamma = u^{\gamma+2}$$

valid for all values or β and γ.

REMARK (B). CAUCHY PROBLEM. - Let us put in (3.8), $c = 1$, $b = 0$ and write $f(x)$ in place of $f(x, 0)$ so that we have for the EPD equation

$$u^{\alpha-2}(x, y) = \int_0^y \eta u^\alpha(x, \eta) d\eta + f(x). \tag{3.15}$$

Then the computations leading to (3.13) show that the CAUCHY problem for $u^{(\alpha-2)}$, $\alpha \neq 1$, with the data $u^{\alpha-2}(x, 0) = f(x)$, $u_y^{\alpha-2}(x, 0) = 0$ can be solved provided a function $u^\alpha(x, y)$ can be determined from the data $u^\alpha(x, 0) = \dfrac{\Delta f}{1-\alpha}$ and $u_y^\alpha(x, 0) = 0$. This fact was of basic importance for the solution of the CAUCHY problem as given by the present author in 1952 [3a].

4. Generalization to Equations of Higher Order.

Let us consider the differential equation

$$L_{\alpha_1} L_{\alpha_2} \dots L_{\alpha_n} w = 0. \tag{4.1}$$

THEOREM. - *Under the assumption*

$$\alpha_r \neq \alpha_j - 2(r-j), \qquad j < r = 2, 3, \dots, n \tag{4.2}$$

the equation (4.1) *has the general solution*

$$w = u^{\alpha_n} + u^{\alpha_{n-1}-2} + u^{\alpha_{n-2}-4} + \dots + u^{\alpha_1 - 2(n-1)}. \tag{4.3}$$

Each of the functions u on the right hand side of (4.3) is determined up to a harmonic function $h(x_1, x_2, \dots, x_m)$ such that their sum Σh is identically zero. The proof is obviously obtained from the case $n = 2$ by induction. In fact we have from (4.1)

$$L_{\alpha_2} L_{\alpha_3} L_{\alpha_4} \dots L_{\alpha_n} w = u^{\alpha_1}. \tag{4.4}$$

As by assumption $\alpha_2 \neq \alpha_1 - 2$, there is a function u^{α_1-2} such that $u^{\alpha_1} = L_{\alpha_2} u^{\alpha_1 - 2}$. By the same assumption we can put $u^{\alpha_1-2} = L_{\alpha_3} u^{\alpha_1-2\cdot 2}$ so that $u^{\alpha_1} = L_{\alpha_2} L_{\alpha_3} u^{\alpha_1 - 2\cdot 2}$. Continuing in the same way, we obtain the equation $L_{\alpha_2} L_{\alpha_3} \dots L_{\alpha_n} u^{(\alpha_1 - 2(n-1))} = u^{\alpha_1}$. By subtracting this equation from (4.4) we obtain

$$L_{\alpha_2} L_{\alpha_3} \dots L_{\alpha_n} (w - u^{(\alpha_1 - 2(n-1))}) = 0. \tag{4.5}$$

This procedure reduces the solution of (4.1) to the solution of (4.5) which contains only $(n-1)$ operators L. Repeating this process we reduce the solution of (4.1) to the problem treated in Paragraph 3. This proves the formula (4.3) and also the statement about the non-uniqueness of this decomposition.

5. The Polyharmonic Equation.

By our previous result the general solution of the polyharmonic equation of order n, $L_0^n w$ $\left(\text{where according to our notations, } L_0 = \sum_1^m \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y^2}\right)$ is given by (4.3) as

$$w = u^{(0)} + u^{(-2)} + \ldots u^{(-2n+2)}. \tag{5.1}$$

According to ALMANSI. we have the decomposition

$$w = u_0^{(0)} + y u_1^{(0)} + \ldots + y^{n-1} u_{n-1}^{(0)} \tag{5.2}$$

where all functions $u^{(0)}$ are harmonic. However, the corresponding terms of (5.2) are polyharmonic functions of orders 1, 2, ..., n, respectively. Let us compare (5.1) with (5.2), taking for the sake of simplicity $n = 2$. In this case we have with abbreviated notations,

$$w = yP + Q \tag{5.3}$$

where P and Q are harmonic functions in $(x_1, \ldots, x_m, y)$. We shall now compute from P and Q two functions $u_1^{(-2)}$ and $u_1^{(0)}$ such that

$$yP + Q = u_1^{(-2)} + u_1^{(0)}. \tag{5.4}$$

To this end we introduce a harmonic function $u^{(0)}$ defined by the equation

$$u_y^{(0)} = P. \tag{5.5}$$

Such a function exists and can be computed by a procedure similar to that used to solve the equation (3.7) (see also R. J. DUFFIN [5], p. 6). We prove now that we can put in (5.4)

$$u_1^{(-2)} = y u_y^{(0)} - u^{(0)}. \tag{5.6}$$

In fact the right hand side of (5.6) satisfies the equation $L_{-2} u = 0$ as can be checked by the following differentiation of (5.6):

$$y \sum_i u_{y x_i x_i}^{(0)} - \sum_i u_{x_i x_i}^{(0)} + 2u_{yy}^{(0)} + y u_{yyy}^{(0)} - u_{yy}^{(0)} -$$

$$- \frac{2}{y} \{ u_y^{(0)} + y u_{yy}^{(0)} - u_y^{(0)} \} =$$

$$= y \frac{\partial}{\partial y} \left(\sum_i u_{x_i x_i}^{(0)} + u_{yy}^{(0)} \right) - \left(\sum_i u_{x_i x_i}^{(0)} + u_{yy}^{(0)} \right) +$$

$$+ 2u_{yy}^{(0)} - 2u_{yy}^{(0)} = 0.$$

Therefore, by (5.5) and (5.6) we can rewrite (5.3) as follows:

$$w = yP + Q = (y u_y^{(0)} - u^{(0)}) + (u^{(0)} + Q)$$

which shows that in (5.4) $u_1^{(-2)}$ is given by (5.6) and $u_1^{(0)}$ equals $u^{(0)} + Q$.

6. The iterated Wave Equation.

Our decomposition of the solutions of the polyharmonic equations applies also to the solutions of the iterated wave equation. The formula (5.1) remains valid on the understanding that the functions u appearing there are solutions of the EPD equations with the corresponding indices. As in paragraph 5 let us consider here only the case $n = 2$.

The general solution of the iterated wave equation $L_0 L_0 w = 0$, $w = w(x_1, \ldots, x_m, y)$, where y plays the role of the time variable, is given by the formula

$$w = u_1^{(0)} + u_1^{(-2)} \tag{6.1}$$

with arbitrary $u_1^{(0)}$ and $u_1^{(-2)}$.

We shall show now briefly how the formula for the general solution w leads to the solution of the CAUCHY problem for the iterated wave equation with the initial data

$$w = f_0(x), \qquad w_y = f_1(x), \qquad w_{yy} = f_2(x), \qquad w_{yyy} = f_3(x) \tag{6.2}$$

for $y = 0$.

Let us first remark that the formula (6.1) contains not two but four arbitrary functions. In fact (6.1) can be rewritten as follows

$$w = u^{(0)} + y u^{(2)} + u^{(-2)} + y^3 u^{(4)} \tag{6.3}$$

because by (2.5) the additional terms are again solutions of $L_0 u = 0$ and $L_{-2} u = 0$, respectively.

It is possible to satisfy the conditions (6.2) by taking for $u^{(0)}$, $u^{(2)}$, $u^{(-2)}$, $u^{(4)}$ solutions of a singular CAUCHY problem for the corresponding EPD equation, $L_k u^{(k)} = 0$, with prescribed values for $u^{(k)}(x, 0)$ and $u_y^{(k)}(x, 0) = 0$. This last problem was recently solved for all values of k [3, 4] and, therefore the solution is known for $k = 0, 2, -2$, and 4.

For the sake of brevity we shall solve in this way the CAUCHY problem for w only in the special case where f_0, f_1, and f_2 are identically zero while $f_3(x)$ does not vanish identically. Then we see at once from (6.3) that a solution of the CAUCHY ppoblem for w is given by the formula

$$w(x_1, x_2, \ldots, x_m, y) = y^3 u^{(4)}(x_1, x_2, \ldots, x_m, y) \tag{6.4}$$

where u^4 is a solution of a CAUCHY problem, $L_4 u = 0$, with the initial conditions

$$u^{(4)}(x_1, x_2, \ldots, x_m, 1) = \frac{1}{6} f_3(x_1, x_2 \ \ldots, x_m) \tag{6.5}$$

and

$$u_y^{(4)}(x_1, x_2, \ldots, x_m, 0) = 0. \tag{6.6}$$

According to previous results [3] obtained by the present author the solution w for $m = 1, 2, 3$ and 4 is given by the formula

$$w = y^3 \frac{\omega_{5-m}}{6\omega_5} \int \cdots \int_{\sum_1^m \alpha_j^2 \leq 1} f_3(x_1 + \alpha_1 y, \ldots, x_m + \alpha_m y) \cdot \left(1 - \sum_1^m \alpha_j^2\right)^{\frac{3-m}{2}} d\alpha_1 \ldots d\alpha_m \tag{6.7}$$

where $\omega_s = 2(\pi)^{s/2}/\Gamma(s/2)$. For $m = 5$, we have

$$w = \frac{y^3}{6\omega_5} \int \cdots \int_{\sum_1^5 \alpha_j^2 = 1} f(x_1 + \alpha_1 y, \ldots, x_5 + \alpha_5 y) d\omega_5 . \tag{6.8}$$

Finally, if $4 < m - 1$, we select the smallest integer ν such that $4 + 2\nu \geqq m - 1$. Then

$$w = \left(\frac{\partial}{y \partial y}\right)^\nu (y^{2\nu+3} u^{(4+2\nu)}(x, y))$$

where $u^{(4+2\nu)}(x, y)$ satisfies the corresponding EPD equation and the initial conditions

$$u^{(4+2\nu)}(x, 0) = \frac{\frac{1}{6} f_3(x)}{5 \cdot 7 \ldots (3 + 2\nu)}, \tag{6.9}$$

$$u_y^{(4+2\nu)}(x, 0) = 0. \tag{6.10}$$

The function $u^{(4+2\nu)}$ can be determined by a formula of the type (6.7) in the case $4 + 2\nu > m - 1$ or by the formula (6.8) if $4 + 2\nu = m - 1$. From these formulas it is seen that HUYGENS' principle holds only for the values

$$m = 5 + 2\nu, \qquad \nu = 0, 1, 2, \ldots$$

because only in these cases the solution is given by the integral over a hypersurface in the m-dimensional x-space. This last result confirms a remark by GARDING 6, p. 788 and connects HUYGENS' principle for the n-fold iterated wave equation with the fact that HUYGENS' principle holds for the EPD equation of index k each time $m - k$ is an odd positive integer even if k itself is negative, $k \neq -1, -3, -5, \ldots$ [3c, p. 112].

7. The Iterated EPD and GASPT Equation.

Let us consider the equation

$$L_{2n} u = 0, \qquad n = 1, 2, 3, \ldots$$

which is either an EPD equation or a GASPT equation and let L_{2n}^{n+1} denote the $n + 1$ times iterated operator L_{2n}. The we obtain as a special case of (4.3) that the general solution of

$$L_{2n}^{n+1} w = 0 \tag{7.1}$$

is

$$w = u^{(2n)} + u^{(2n-2)} + \ldots + u^{(0)}. \tag{7.2}$$

This result includes ar a special case a theorem of FRIEDRICHS which was extensively discussed in COURANT–HILBERT Vol. II [7, p. 416 ff.]. This teorem states that in the case of one x variable, $m = 1$, the equation (7.1) admits $u^{(0)}$ as a particular solution. It would be, of course, easy to find by our method all equations of the type (4.1) which admit $u^{(0)}$ as a solution. The corresponding results hold obviously for the GASPT equation. The results of this paper were summarized in an abstract, Bull. Amer. Math. Soc., vol. 60, p. 254, 1954.

BIBLIOGRAPHY

[1] M. PICONE, *Nuovi indirizzi di ricerca nella teoria e nel calcolo delle soluzioni di talune equazioni lineari alle derivate parziali della Fisica-matematica*, « Annali della R. Scuola Normale Superiore di Pisa », Serie II, Vol. V (1936-XIV).

[2] A. WEINSTEIN, *Generalized axially symmetric potential theory*, « Bull. Amer. Math. Soc. », Vol. 59 (1953), pp. 20-28.

[3] A. WEINSTEIN, (a) *Sur le problème de Cauchy pour l'équation de Poisson et l'équation des ondes*, « C. R. Acad. Sci. », Paris, Vol. 234 (1952), pp. 2584-2585.

(b) *The singular solutions and the Cauchy problem for generalized Tricomi's equations*, « Communications on Pure and Applied Mathematics », Vol. II, pp. 105-116 (1954).

(c) *On the wave equation and the equation of Euler-Poisson*, Proceedings of Fifth Symposium in Appl. Math., McGraw Hill, 1954, pp. 137-148.

[4] J. B. DIAZ and H. F. WEINBERGER, *A solution of the singular initial value problem for the Euler-Poisson-Darboux equation*, « Proc. Amer. Math. Soc. », Vol. 4 (1923), pp. 703-715.

[5] R. J. DUFFIN, *Continuation of biharmonic functions by reflection II*, Technical Report N°. 13, 1954, Carnegie Institute of Technology.

[6] L. GARDING, *The Solution of Cauchy's problem for two totally hyperbolic linear differential equations by means of Riesz integrals*, « Annals of Math. », Vol. 48, N°. 4 (1942).

[7] R. COURANT and D. HILBERT, *Methoden der mathematischen physik*, Vol. II, Berlin, 1937.

[8] A. WEINSTEIN, *Etude des spectres des équations aux derivées partielles de la theorie des plaques elastiques*, « Memorial Sc. Math. », N°. 88 (1937).

[9] E. K. BLUM, *The Euler-Poisson-Darboux equation in the exceptional Cases*, « Proc. Amer. Math. Soc. », Vol. 4 (1954), pp. 511-520.