On a property of the stable or conditionally stable solutions of systems of non-linear differential equations.

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A Giovanni Sansone nel suo 70° compleanno.

Summary. - Qualitatively expressed we assert the following. If the equation (1.1)-(1.2) fulfills the restriction (1.6)-(1.7)-(1.8) which are less restrictive than usual, then all solutions with norms tending to 0 with $\frac{1}{t}$ cannot be « too small » in an arbitrary interval in $t \geq 0$ with fixed length.

1. As well-known, the stability-problem of a non-linear system can be reduced to the investigation of the stability of $X(t) \equiv 0$ concerning the system

$$(1.1) \quad \frac{dX}{dt} = AX + W(X, t)$$

with

$$(1.2) \quad W(0, t) \equiv 0;$$

here stand $X = X(t)$ and $W(X, t)$ for the $n \times 1$ column-vectors

$$X(t) \equiv \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad W(Y, t) \equiv \begin{pmatrix} w_1(y_1, \dots, y_n, t) \\ w_2(y_1, \dots, y_n, t) \\ \vdots \\ w_n(y_1, \dots, y_n, t) \end{pmatrix}$$

and A stands for an $n \times n$ matrix with complex elements

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} = (a_{ih}).$$

It was discovered for certain systems by LIAPOUNOFF and POINCARÉ and for more general systems extended by COTTON and PERRON ⁽¹⁾ that e.g. if all solutions $X_i(t)$ with sufficiently small $\|X_i(0)\|$ of the system (1.1) - (1.2) tend to 0 for $t \rightarrow +\infty$ then they cannot be « too near » to 0 for $t > t_0$,

⁽¹⁾ O. PERRON, *Über Stabilität und asymptotisches Verhalten der Integrale von Differentialgleichungssysteme*, « Math. Zeitschrift », Bd. 29, p. 129-160.

whatever large we choose t_0 . More exactly they found that in this case for all solutions $X(t)$ of the system (1.1)-(1.2) the relation

$$\lim_{t \rightarrow +\infty} \frac{\log \|X(t)\|}{t} = \operatorname{Re} \lambda_\nu$$

holds with a ν where λ_ν stand for the characteristic roots of A and

$$\|X(t)\| = \left(\sum_{\nu=1}^n |x_\nu(t)|^2 \right)^{1/2}.$$

Hence in this case for each such solutions and an arbitrary small $\varepsilon > 0$ there is a sequence

$$t_1 < t_2 < \dots \rightarrow +\infty$$

such that

$$(1.3) \quad \|X(t_\nu)\| > e^{(\operatorname{Re} \lambda_\nu - \varepsilon)t_\nu} \quad \nu = 1, 2, \dots$$

where

$$(1.4) \quad (0 >) \operatorname{Re} \lambda_1 \geq \operatorname{Re} \lambda_2 \geq \dots \geq \operatorname{Re} \lambda_n.$$

These results however give no information about the distribution of these t_ν -values so that the picture it furnishes about the «Endverlauf» of solutions «near» to a stable solution remains rather incomplete. Even the possibility cannot be excluded that for a solution $X_1(t) \neq 0$ of the system (1.1)-(1.2) for arbitrary large segments $X_1(t)$ vanishes. A more refined theory would be beside the possible technical applications the more desirable since a systematic *negative* use of it would mean a standard device in the treatment of such problems that a certain system

$$x_1(t), x_2(t), \dots, x_n(t)$$

of functions cannot satisfy certain types of systems of differential equations. For the treatment of these questions I shall return elsewhere.

In this note I shall show that essentially the inequality (1.3) is satisfied in every interval of a fixed length of the positive t -axis and this holds for a more general class of systems (1.1)-(1.2). More exactly we consider systems (1.1)-(1.2) for which — denoting the characteristic roots of A by $\lambda_1, \lambda_2, \dots, \lambda_n$ — we have

$$(1.5) \quad \operatorname{Re} \lambda_n = \min_j \operatorname{Re} \lambda_j = L < 0.$$

Let Δ be so large that $\Delta > 1$ and

$$(1.6) \quad 5n(2e(\Delta + 1))^{2n} e^{2L\Delta} \leq 1$$

and the $W(Z, t)$ in the equations (1.1) such that in a halfcylinder of the (Z, t) -space

$$(1.7) \quad \|Z\| \leq c_1, \quad t \geq 0 \quad (c_1 \text{ positive constant})$$

the inequality

$$(1.8) \quad \|W(Z, t)\| \leq \frac{e^{-2n\|A\|(\Delta^2 + \Delta)}}{3\sqrt[n]{n\sqrt[n]{\Delta} + \Delta}} \|Z\|$$

holds; here $\|A\|$ stands for $\max_{i,k} |a_{ik}|$. Then we assert the following

THEOREM. - If $Y(t)$ with $\lim_{t \rightarrow +\infty} \|Y(t)\| = 0$ is a solution of the system (1.1)-(1.2) satisfying the restrictions (1.6)-(1.8), then we have for all sufficiently large α 's the inequality

$$\max_{\alpha \leq t \leq \alpha + \Delta^2 + \Delta} \|Y(t)\| e^{(1 + \frac{2}{\Delta})|L|t} \geq c_2$$

with a positive constant c_2 .

The proof will be based on the following theorem ⁽²⁾.

If for the complex $w_1, w_2, \dots, w_n$ -numbers we have

$$(1.9) \quad \min_j \operatorname{Re} w_j = 0$$

then for $a > 0, d > 0$ we have for arbitrary complex w_j 's the inequality

$$(1.10) \quad \begin{aligned} \max_{\alpha \leq t \leq \alpha + d} \left| \sum_{j=1}^n b_j e^{w_j t} \right| &\geq \\ &\geq \left(\frac{d}{2e(a+d)} \right)^n \left| \sum_{j=1}^n b_j \right| \end{aligned}$$

holds.

As the simple example $\frac{dy}{dt} + Ly = 0$ (i.e. $n = 1$) shows, the theorem is in a certain sense best-possible.

2. Next we turn to the proof of our theorem.

With our $Y(t)$ we consider first the system

$$(2.1) \quad \frac{dU}{dt} = AU, \quad U(w) = Y(w)$$

⁽²⁾ See my book, *Eine neue Methode in der Analysis und deren Anwendungen*, Budapest, Akad. Kiadó, 1953, p. 42. A completely rewritten and enlarged English edition is in preparation.

with a fixed $\omega \geq 0$ (which will be chosen later suitably). We suppose first that the characteristic roots of A are simple. If

$$U = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix},$$

then as well-known, we have for $v = 1, 2, \dots, n$

$$u_v(t) = \sum_{j=1}^n b_{jv} e^{\lambda_j t}$$

with constant b_{jv} 's. Application of the theorem (1.9)-(1.10) with the ω in (2.1) to

$$u_v(t + \omega) e^{-Lt} = \sum_{j=1}^n (b_{jv} e^{\lambda_j \omega}) e^{(\lambda_j - L)t}$$

and

$$\alpha = \Delta^2, \quad d = \Delta$$

gives for $v = 1, 2, \dots, n$

$$\begin{aligned} \max_{\omega + \Delta^2 \leq t \leq \omega + \Delta^2 + \Delta} |u_v(t)| e^{-Lt} &= \max_{\Delta^2 \leq t \leq \Delta^2 + \Delta} |u_v(t + \omega)| e^{-L(t + \omega)} \geq \\ &\geq \frac{e^{-L\omega}}{(2e(\Delta + 1))^n} \left| \sum_{j=1}^n b_{jv} e^{\lambda_j \omega} \right| = \frac{1}{(2e(\Delta + 1))^n} |u_v(\omega)| e^{-L\omega} = \\ &= \frac{1}{(2e(\Delta + 1))^n} |Y_v(\omega)| e^{-L\omega} \end{aligned}$$

i.e. for $v = 1, 2, \dots, n$ (L is negative !)

$$(2.2) \quad \max_{\omega + \Delta^2 \leq t \leq \omega + \Delta^2 + \Delta} |u_v(t)| \geq \frac{e^{L(\Delta^2 + \Delta)}}{(2e(\Delta + 1))^n} |Y_v(\omega)|.$$

This was deduced under the supposition all the λ_j 's were simple but a trivial limiting process in (2.2) gives its validity without any restrictions upon the λ_j 's. Since

$$\|Y(\omega)\|^2 = \sum_{v=1}^n |Y_v(\omega)|^2,$$

there is an index v_0 with

$$|Y_{v_0}(\omega)| \geq \frac{1}{\sqrt{n}} \|Y(\omega)\| =$$

and thus from (2.2) we get

$$(2.3) \quad \begin{aligned} \max_{\omega + \Delta^2 \leq t \leq \omega + \Delta^2 + \Delta} \|U(t)\| &\geq \max_{\omega + \Delta^2 \leq t \leq \omega + \Delta^2 + \Delta} |u_{v_0}(t)| \geq \\ &\geq \frac{e^{L(\Delta^2 + \Delta)}}{(2e(\Delta + 1))^n} |Y_{v_0}(\omega)| \geq \frac{e^{L(\Delta^2 + \Delta)}}{\sqrt{n}(2e(\Delta + 1))^n} \|Y(\omega)\|. \end{aligned}$$

3 We shall need two well-known lemmas. If E stands for the $n \times n$ unit-matrix and $E^{(k)}$ the $n \times 1$ column-vector whose k^{th} component is 1, the others 0, further

$$(3.1) \quad V(t) = \begin{pmatrix} v_{11}(t) \dots v_{1n}(t) \\ \vdots \\ v_{n1}(t) \dots v_{nn}(t) \end{pmatrix}$$

where for $k = 1, 2, \dots, n$ the column-vector $V^{(k)}(t)$ (with the elements of the k^{th} column of (3.1) as components) satisfies the equation

$$(3.2) \quad \frac{dV^{(k)}(t)}{dt} = A V^{(k)}(t)$$

with

$$(3.3) \quad V^{(k)}(\omega) = E^{(k)},$$

then we need the Lemma due to LIOUVILLE.

LEMMA I. - With the $U(t)$ of (2.1) we have for $t \geq \omega$ the (vector) equation ⁽³⁾

$$(3.4) \quad Y(t) = U(t) + \int_{\omega}^t V^*(t + \omega - x) W(Y(x), x) dx.$$

Further we need PERRON'S well-known

LEMMA II. - If $x_1 \geq 0$, $x_2 \geq 0$ and for the vector $R(t)$ the equation

$$\frac{dR}{dt} = G(t)R$$

holds, then we have

$$\|R(x_2)\| \leq \|R(x_1)\| \exp \{ n |x_2 - x_1| \max_{\min(x_1, x_2) \leq t \leq \max(x_1, x_2)} \|G(t)\| \}.$$

⁽³⁾ The star denotes the transpose of the matrix in (3.1).

A simple consequence of Lemma II. which we need is the

LEMMA III. - For the $U(t)$ of (2.1) the inequality

$$\begin{aligned} & \max_{\omega \leq t \leq \omega + \Delta^2 + \Delta} \|U(t)\| \leq \\ & \leq \exp \{ n(\Delta^2 + \Delta) \|A\| \} \max_{\omega + \Delta^2 \leq t \leq \omega + \Delta^2 + \Delta} \|U(t)\|. \end{aligned}$$

holds.

Let namely $\|U(t)\|$ assume its absolute maximum with respect to $[\omega, \omega + \Delta^2 + \Delta]$ at $t = t_0$ and with respect to $[\omega + \Delta^2, \omega + \Delta^2 + \Delta]$ at $t = t_1$. We have to apply Lemma II. to $R(t) = U(t)$ with

$$G(t) = A, \quad x_1 = t_1, \quad x_2 = t_0.$$

This proves the lemma since

$$|t_1 - t_0| \leq \Delta^2 + \Delta.$$

4. Now we can turn to the proof of our theorem. Introducing for an $n \times n$ matrix $H = (h_{ik}(t))$ a second norm $\|H\|$ by

$$(4.1) \quad \|H\| = (\sum_{i,k} |h_{ik}(t)|^2)^{1/2}$$

(3.4) gives for $\omega \leq t \leq \omega + \Delta^2 + \Delta$ the estimation

$$\begin{aligned} (4.2) \quad \|Y(t) - U(t)\|^2 & \leq \int_{\omega}^t \|V^*(t + \omega - x)\|^2 \cdot \|W(Y(x), x)\|^2 dx = \\ & = \int_{\omega}^t \|V(t + \omega - x)\|^2 \cdot \|W(Y(x), x)\|^2 dx. \end{aligned}$$

Since applying Lemma II. to the vector $V^{(k)}(t)$ in (3.2)-(3.3) with $x_1 = \omega$, $x_2 = y \geq \omega$ we get

$$\|V^{(k)}(y)\|^2 \leq e^{2n\|A\|(y-\omega)}$$

i.e. summing over $k = 1, 2, \dots, n$

$$\|V(y)\|^2 \leq ne^{2n\|A\|(y-\omega)}.$$

Owing to $y = t + \omega - x$ we have now

$$y - \omega = t - x \leq t - \omega \leq \Delta^2 + \Delta,$$

and thus we obtained from (4.2) for our t 's

$$(4.3) \quad \|Y(t) - U(t)\|^2 \leq ne^{2n\|A\|(\Delta^2 + \Delta)} \int_{\omega}^{\omega + \Delta^2 + \Delta} \|W(Y(x), x)\|^2 dx.$$

Owing to $\lim_{t \rightarrow +\infty} \|Y(t)\| = 0$ there is a $c_3 > 0$ such that for $t > c_3$

$$\|Y(t)\| < c_1$$

with c_1 from (1.7); choosing

$$(4.4) \quad \omega > c_3$$

we get using (1.8) since $(a+b)^2 \leq 2(a^2 + b^2)$

$$(4.5) \quad \begin{aligned} \|Y(t) - U(t)\|^2 &\leq \frac{1}{9} \frac{e^{-2n(\Delta^2 + \Delta)} \|A\|}{\Delta^2 + \Delta} \int_{\omega}^{\omega + \Delta^2 + \Delta} \|Y(x)\|^2 dx \leq \\ &\leq \frac{2}{9} \frac{e^{-2n(\Delta^2 + \Delta)} \|A\|}{\Delta^2 + \Delta} \int_{\omega}^{\omega + \Delta^2 + \Delta} \|U(x)\|^2 dx + \frac{2}{9(\Delta^2 + \Delta)} \int_{\omega}^{\omega + \Delta^2 + \Delta} \|Y(x) - U(x)\|^2 dx. \end{aligned}$$

This holds for $\omega \leq t \leq \omega + \Delta^2 + \Delta$; if

$$\max_{\omega \leq t \leq \omega + \Delta^2 + \Delta} \|Y(x) - U(x)\| = \|Y(t_2) - U(t_2)\|,$$

then applying (4.5) with $t = t_2$ results

$$\begin{aligned} \|Y(t_2) - U(t_2)\|^2 &\leq \frac{2}{9} e^{-2n(\Delta^2 + \Delta)} \|A\| \max_{\omega \leq t \leq \omega + \Delta^2 + \Delta} \|U(t)\|^2 + \\ &+ \frac{2}{9} \|Y(t_2) - U(t_2)\|^2. \end{aligned}$$

Hence

$$\|Y(t_2) - U(t_2)\|^2 \leq \frac{2}{7} e^{-2n(\Delta^2 + \Delta)} \|A\| \max_{\omega \leq t \leq \omega + \Delta^2 + \Delta} \|U(t)\|^2$$

and using Lemma III. we obtained

$$(4.6) \quad \|Y(t_2) - U(t_2)\|^2 \leq \frac{2}{7} \max_{\omega + \Delta^2 \leq t \leq \omega + \Delta^2 + \Delta} \|U(t)\|^2.$$

If this last maximum is attained for $t = t_3$, then from (4.6) we get

$$(4.7) \quad \begin{aligned} \max_{\omega + \Delta^2 \leq t \leq \omega + \Delta^2 + \Delta} \|Y(t)\|^2 &\geq \|Y(t_3)\|^2 \geq \\ &\geq \frac{1}{2} \|U(t_3)\|^2 - \|Y(t_3) - U(t_3)\|^2 \geq \frac{1}{2} \|U(t_3)\|^2 - \|Y(t_2) - U(t_2)\|^2 \geq \\ &\geq \frac{1}{2} \|U(t_3)\|^2 - \frac{2}{7} \|U(t_3)\|^2 = \frac{3}{14} \max_{\omega + \Delta^2 \leq t \leq \omega + \Delta^2 + \Delta} \|U(t)\|^2 > \\ &> \frac{1}{5n} \frac{e^{2L(\Delta^2 + \Delta)}}{(2e(\Delta + 1))^{2n}} \|Y(\omega)\|^2 \end{aligned}$$

owing to (2.3). Using (1.6) this gives finally from (4.7) and (1.6)

$$(4.8) \quad \max_{\omega + \Delta^2 \leq t \leq \omega + \Delta^2 + \Delta} \|Y(t)\| > e^{L(\Delta^2 + 2\Delta)} \|Y(\omega)\|.$$

5. From (4.8) the theorem follows at once. Namely if $Y(t)$ does not vanish identically for $t > c_s$, then there is an $\omega_0 > c_s$ such that

$$\|Y(\omega_0)\| > 0.$$

We apply first (4.8) with $\omega = \omega_0$. This gives the existence of a $t = \omega_1$ with

$$\omega_0 + \Delta^2 \leq \omega_1 \leq \omega_0 + \Delta^2 + \Delta$$

and

$$\|Y(\omega_1)\| \geq e^{L(\Delta^2 + 2\Delta)} \|Y(\omega_0)\|.$$

Next we apply (4.8) with $\omega = \omega_1$. This gives the existence of a $t = \omega_2$ with

$$\omega_1 + \Delta^2 \leq \omega_2 \leq \omega_1 + \Delta^2 + \Delta$$

and

$$\begin{aligned} \|Y(\omega_2)\| &> e^{L(\Delta^2 + 2\Delta)} \|Y(\omega_1)\| > \\ &> e^{2L(\Delta^2 + 2\Delta)} \|Y(\omega_0)\|. \end{aligned}$$

Continuing this process we get a sequence $(0 \leq) \omega_0 < \omega_1 < \omega_2 < \dots$ with

$$(5.1) \quad \omega_\nu + \Delta^2 \leq \omega_{\nu+1} \leq \omega_\nu + \Delta^2 + \Delta$$

and

$$(5.2) \quad \|Y(\omega_\nu)\| > e^{\nu L(\Delta^2 + 2\Delta)} \|Y(\omega_0)\|$$

for $\nu = 0, 1, 2, \dots$. Since from (5.1) we have

$$\omega_\nu \geq \omega_0 + \nu \Delta^2$$

$$(\Delta^2 + 2\Delta)\nu \leq \left(1 + \frac{2}{\Delta}\right)(\omega_\nu - \omega_0)$$

and, since L is negative,

$$\nu L(\Delta^2 + 2\Delta) \geq \left(1 + \frac{2}{\Delta}\right)L(\omega_\nu - \omega_0).$$

Hence for $\alpha > \omega_0$ every interval

$$\alpha \leq t \leq \alpha + \Delta^2 + \Delta$$

contains at least one of the ω_ν 's and

$$\|Y(\omega_\nu)\| \geq \left(\|Y(\omega_0)\| e^{-\left(1 + \frac{2}{\Delta}\right)L\omega_0}\right) e^{\left(1 + \frac{2}{\Delta}\right)L\omega_\nu}$$

i.e.

$$c_2 = \|Y(\omega_0)\| e^{\left(1 + \frac{2}{\Delta}\right)|L|\omega_0}$$

can be chosen in the theorem.