A Theory of Extended Lie Transformation Groups.

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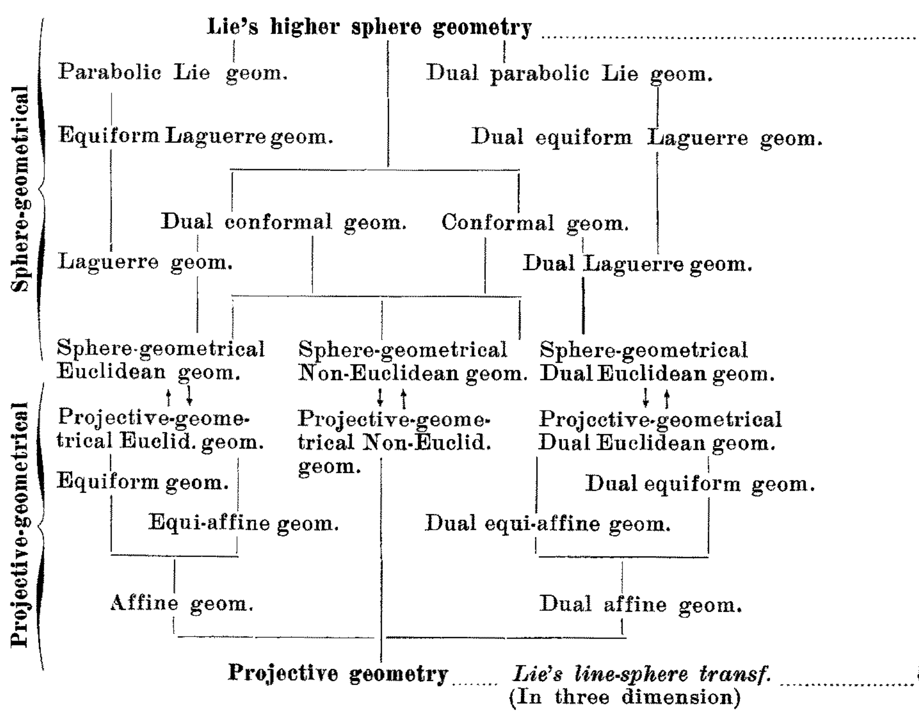
Summary. - *The theory of Lie transformation groups is extended to a theory of extended Lie transformation groups by extending the group parameters to functions of coordinates in the base manifolds. The result is global both in the group manifolds (the O. Schreier's fundamental theorems being not taken into account) as well as in the base differentiable manifolds owing to the introduction of the author's* Π*-geodesic parallel coordinates. The Lie's fundamental theorems are extremely simplified.*

The transformation parameters hitherto considered have been exclusively of the nature of variable constants. But the present author has succeeded in extending all the branches of the following table by extending respective group parameters to functions of coordinates [1, 2, ..., 11], the invariants being retained:



Thereby I considered the combined manifold:

$$\{x^p\} + \{a^l_m(x^p)\}, \quad (\,|\,a^l_m(x^p)\,| \neq 0;\ l,\ m,\ p = 1,\ 2,\ \dots,\ n)$$

of the base manifold $\{x^p\}$ and the extended group manifold $\{a^l_m(x^p)\}$, the x^p being the local coordinates in the

differentiable manifolds | classical spaces

and the Π-geodesic curves

$$\frac{d}{dt}\frac{\omega^l}{dt} = 0, \quad (\omega^l \overset{\text{def}}{=} \omega^l_m(x^p)dx^m = a^l_m(x^p)dx^m),$$

which exist in the

differentiable manifolds | classical spaces

owing to the fact that ω^l are written in invariant forms and behave as for meet and join like straight lines, play the important rôles and the global Π-*geodesic parallel coordinates* ξ^l such that

$$d\xi^l = \omega^l = a^l dt$$

were introduced by introducing at least one system of $\omega^l_m(x^p) \in C^v$, $(v =$ $=$ positive integer $k)$ | $= \infty)$, | $= \omega)$, such that

$$|\,\omega^l_m(x^p)\,| \neq 0.$$

Now the present author is in the situation *to extend his extension of group parameters to functions of coordinates of the base manifolds to the general case,* and this will be done in the following lines, being led *to extend the theory of Lie transformation groups by extending the group parameters to functions of coordinates. The abstract theory itself of the Lie groups remains however thereby unaltered, although the domain of validity is enlarged therewith.* Thereby the following *combined manifolds* $M + G$ are considered:

[the base manifold M: $\{x^p\}$] + [the extended Lie transformation group manifold G: $\{a^t(x)\}$], $(p = 1,\ 2,\ \dots,\ n;\ t = 1,\ 2,\ \dots,\ r)$.

The famous Fundamental Theorems of OTTO SCHREIER [13, 14] have hitherto enabled us to reduce the *global theory of Lie groups to the case of that of the vicinity of unit element.*

The present author has introduced the global Π*-geodesic parallel coordinates* ξ^l *not only in the base differentiable manifolds* M (¹) *but also in the transformation group space* ($\{\alpha^\lambda\}$ *in notation*). Thus they enabled us to establish *the theory of the extended Lie*

groups | *transformation groups*

in the large without taking the Otto Schreier's Fundamental Theorems into account.

The resulting theory of extended LIE transformation groups includes the various extended geometries hitherto considered by the present author as special cases ($r = n^2$), the above parameter t (cf. Art. 12) being a special canonical parameter.

Just as we have obtained $d\xi^l = \omega^l_m(x^p)dx^m$, the present author has rendered the usual notation

$$X_i = \xi_i^j(x)\frac{\partial}{\partial x^j} \qquad | \qquad Z_l = \alpha_l^k(a)\frac{\partial}{\partial a^k}$$

in the

differentiable manifold $\{x^l\}$ | group manifold $\{a^l\}$

into the form

$$\frac{\partial}{\partial \xi^i}, \qquad | \qquad \frac{\partial}{\partial \alpha^l},$$

where

$$(\xi^i) \qquad | \qquad (\alpha^l)$$

are the Π-geodesic parallel coordinates corresponding to

$$\xi_i^j(x). \qquad | \qquad \alpha_l^k(a).$$

Thus the fundamental theorems of the extended LIE transformation groups are made extremely simple as the following underlying formulas suggest:

$$X_i = \frac{\partial}{\partial \xi^i}, \ (X_i, X_j) = 0, \qquad | \qquad Z_i = \frac{\partial}{\partial \alpha^i}, \ (Z_i, Z_j) = 0,$$

(¹) Usually the Euclidean space E^n only is treated as the base manifold.

the structure constants

$$C^i_{jk} = 0, \qquad \bar{C}^i_{jk} = 0,$$

$$\mathbf{d}(\omega^l_m(x)dx^m) = 0, \qquad \mathbf{d}(b^i_j(a)da^j) = 0,$$

$$\alpha^k_i \frac{\partial f}{\partial a^k} = \xi^i_j \frac{\partial f}{\partial x^j} \quad \longrightarrow \quad \frac{\partial f}{\partial \alpha^l} = \frac{\partial f}{\partial \bar{\xi}^l}.$$

In Art. 19, *E. Cartan's theories in his "géométrie des groupes"* [15] *concerning "equipollence des vecteurs", "parallélisme des vecteurs" and "géodésique" will be extended to the case where the groups are the extended ones in the present author's sense,* the fact that *his geodesics are* Π-*geodesic in the present author's sense* being shown.

§ 1. - Otto Schreier's Two Fundamental Theorems.

1. **Recapitulation of the Otto Schreier's Two Fundamental Theorems.** The study of the global LIE groups has hitherto been based on the following principles.

First Fundamental Theorem of Otto Schreier [13, 14]. If U be an arbitrary vicinity of the unit element of a connected topological space G; then every element of G is expressible * as the product of a finite number of elements $a_1, a_2, \ldots, a_n$ belonging to U.

COR. - Connected r-dimensional continuous group G may be covered by at most enumerable open sets of the forms $a_r U$, $(r = 1, 2, \ldots, n)$, where U is an arbitrary vicinity of the unit element of G.

Second Fundamental Theorem of Otto Schreier [13, 14]. If we divide a connected r-dimensional continuous group into subsets by the equivalence relations of locally continuous isomorphism, then each subset contains only one simply connected group, provided that we do not distinguish the subsets, which are continuously isomorphic to one another. Every continuous group belonging to one of the subsets is continuously isomorphic to the coset group of the simply connected group (belonging to the subset) formed with its isolated invariant subgroup as modulus.

And conversely, such a coset group is a continuous group belonging to one and the same subset as its simply connected group.

In the First Fundamental Theorem of OTTO SCHREIER, the expressibility * holds only except local continuous isomorphism and by the continuous group, locally continuously isomorphic subset only come into our consideration. Hence we see that *the study of connected continuous groups is reducible to that of the*

vicinity of the unit element	*group germ (local group)*

only.

§ 2. - The Theory of Lie Groups in the Large by Extending the Group Parameters to Functions of Coordinates.

2. **Differentiable Manifolds.** - In order to fix our notion, we will recapitulate some definitions of terms etc. under consideration.

Let R^n be au n-dimensional Cartesian space with the real coordinates (x_λ). We call the topological representation of an open subset U_α of an n-dimensional manifold $M = V^n$ on an open subset $x(U_\alpha)$ of R^n a *system of local coordinates* (or a *local chart*) of M. U_α is called the *domain of the chart* (or the *domain of the coordinate system*). To each point P of $U_\alpha \subset M$, there corresponds a point of R^n, which is represented by (x^λ) called the *coordinates of P in the chart* under consideration.

DEFINITION. - A differentiable manifold M of the class $C^v (v =$ positive integer or $v = \infty$ or $v = \omega)$ is an n-dimensional manifold [2], to which a system A(*atlas*) of charts satisfying the following conditions are associated:

A_1. $M = \bigcup\limits_{\alpha} U_\alpha$.

A_2. $P \in U_1 \cap U_2$, (U_1, U_2: two domains of charts of A), and (x^λ) and (y^λ) are the local coordinates having U_1 and U_2 as the domains respectively, then

$y^\lambda = y^\lambda(x^\nu)$	$x^\lambda = x^\lambda(y^\nu)$

are functions of class C^v such that

$\dfrac{\partial(y^1, \ldots, y^n)}{\partial(x^1, \ldots, x^n)} \neq 0.$	$\dfrac{\partial(x^1, \ldots, x^n)}{\partial(y^1, \ldots, y^n)} \neq 0.$

DEFINITION. - Two atlas A and B are said to be *equivalent*, when their reunion is also an atlas of class C^v.

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(2) A topological space is said to be locally Euclidean at a point P, if there exists a chart A on a vicinity of P. A HAUSDORFF space which is locally Euclidean at each point is called a *manifold*.

THEOREM. – In order that two atlas A and B of one and the same differentiable manifold M may be equivalent, it is necessary and sufficient that A, B satisfy the axiom A_2.

DEFINITION. – Two equivalent atlas are said to define one and the same *structure of differentiable manifold of class C^v on M.*

DEFINITION – A system of local coordinates of M is said to be *compatible* with the structure of differentiable manifold (or to be *admissible*) when the reunion with an atlas defining M as differentiable manifold is also an atlas of the same class.

THEOREM. – *Every compact differentiable manifold can be covered by a finite number of domains of the charts.*

3. **The Lie Groups are r-Dimensional Differentiable Manifolds of Class C^3.** At the end of Art. 1, we have seen that *the study of connected continuous group is reducible to that of the*

vicinity of the unit element | *group germ (local group)*

only.

Now we have succeeded *in introducing global* Π*-geodesic parallel coordinates* $\{\xi^l\}$ *into differentiable manifolds and any point of a differentiable manifold may be considered as the origin by virtue of the extended affine transformation group.*

THEOREM. – *The Lie group is a differentiable manifold of class C^3.*

In order to prove this fact, we begin with the definition of the r-dimensional LIE group germ.

DEFINITION. – A set G of elements

$$S_a = S(a^1, a^2, \dots, a^r)$$

having points $a = (a^1, a^2, \dots, a^r)$ belonging to a vicinity U_0 of the origin (O) of the r-dimensional Euclidean space as parameters, is called an *r-dimensional Lie group germ,* when it is characterized by the following conditions:

(i) If we take a vicinity $U_1 \subset U_0$ of the origin appropriately, then for

$$a = (a^1, a^2, \dots, a^r) \in U,$$

and

$$b = (b^1, b^2, \dots, b^r) \in U_1,$$

the product

$$S_a \cdot S_b = S_c, \quad (c = (c^1, c^2, \ldots, c^r) \in U_0)$$

is defined, where the composition function

$$c^i = \varphi^i(a^1, a^2, \ldots, a^r; b^1, b^2, \ldots, b^r), \qquad (i = 1, 2, \ldots, r)$$

are of class C^3.

(*ii*) For arbitrary $a \in U_0$, the relation

$$S_a \cdot S_0 = S_0 \cdot S_a = S_a$$

i. e.

$$\varphi^i(a^1, a^2, \ldots, a^r; 0, \ldots, 0) = \varphi^i(0, \ldots, 0; a^1, \ldots, a^r) = a^i, \quad (i = 1, 2, \ldots, r) \tag{3.1}$$

holds.

(*iii*) If $a, b, c \in U_2$ for sufficiently small vicinity U_2 of the origin, then the associative law

$$S_a \cdot (S_b \cdot S_c) = (S_a \cdot S_b) \cdot S_c$$

i. e.

$$\varphi^i(a; \varphi(b; c)) = \varphi^i(\varphi(a; b); c), \qquad (i = 1, 2, \ldots, r) \tag{3.2}$$

holds.

Lemma. – If a and b be sufficiently near the origin, then

$$\frac{\partial(\varphi^1(a; b), \ldots, \varphi^r(a; b))}{\partial(a^1, a^2, \ldots, a^r)} \neq 0, \quad \frac{\partial(\varphi^1(a; b), \ldots, \varphi^r(a; b))}{\partial(b^1, b^2, \ldots, b^r)} \neq 0,$$

so that we can solve

$$c^i = \varphi^i(a; b), \qquad (i = 1, 2, \ldots, r)$$

with respect to a or b. In particular, $S_x = S_a^{-1}$ such that

$$S_x \cdot S_a = S_a \cdot S_x = S_0$$

is determined for arbitrary S_a.

Proof. – $\dfrac{\partial \varphi^i(a; b)}{\partial a^j}$ and $\dfrac{\partial \varphi^i(a; b)}{\partial b^j}$ and thus the fundamental determinants $\dfrac{\partial(\varphi)}{\partial(a)}$ and $\dfrac{\partial(\varphi)}{\partial(b)}$ are continuous functions in the vicinity of the origin.

If we set $b=0$ resp. $a=0$, then, by (3.1), we have

$$\left(\frac{\partial(\varphi)}{\partial(a)}\right)_{b=0}=\left(\frac{\partial(\varphi)}{\partial(b)}\right)_{a=0}=|\,\delta_{ij}\,|=1$$

and thus

$$\frac{\partial(\varphi)}{\partial(a)}\neq 0,\quad \frac{\partial(\varphi)}{\partial(b)}\neq 0$$

in the vicinity of the origin.

If, in particular, we solve $S_x\cdot S_a=S_0$, we have

$$S_x=(S_x\cdot S_a)\cdot S_x=S_x\cdot(S_a\cdot S_x)$$

by the associative law. Comparing this with $S_x=S_x\cdot S_0$, we obtain $S_a\cdot S_x=S_0$. Thus $S_x=S_a^{-1}$ exists.

PROOF of the THEOREM. - I. When a vicinity of the unit element of a topological group G is an r–dimensional LIE group germ, the topological group G is called an *r–dimensional Lie group.*

II. A topological group G is an *r–dimensional continuous group*, when G is provided with a vicinity of the unit element of G, which is homeomorphic to an open hypersphere of the r–dimensional Euclidean space.

From I and II, we see that *the r–dimensional Lie group G is an r–dimensional continuous group*, since for the LIE group germ, the existence of the vicinity of the unit element of G, which is homeomorphic to an open hypersphere of the r–dimensional Euclidean space, is preassumed.

Now

III. *an r–dimensional continuous group* is a topological group whose group space is an *r–dimensional manifold.*

Hence *the r–dimensional Lie group G is an r–dimensional manifold.*

By the Cor. above, this r–dimensional manifold is a differentiable manifold of class C^3, since, by the Cor. of the First Fundamental Theorem of OTTO SCHREIER, Axiom A_1 of Art. 2 is satisfied and by the Theorem above, the Axim A_2 of Art. 2 is satisfied.

Hence the r–dimensional LIE group is an r–dimensional differentiable manifold of class C^3.

4. Realization of the Present Author's Extended Affine Geometry in the

n–Dimensional Base Differentiable Manifolds.	r–Dimensional Lie Group Spaces.

Since the r-dimensional LIE Group is an r-dimensional differentiable manifold of class C^3, the author's extended affine geometry [4, 11] is realizable in it. In the following lines, a realization of the present author's extended affine geometry will be exposed in the

n-dimensional differentiable manifold M.	r-dimensional LIE group space G.

5. **II-Geodesic Curves.** Take

(5.1) $\omega^l \overset{\text{def}}{=} \omega^l_\mu(x^\nu)dx^\mu,$	$\alpha^\lambda \overset{\text{def}}{=} \alpha^\lambda_l(a^p)da^l,$
$(\lambda, \mu, \nu, \ldots = 1, 2, \ldots, n),$	$(l, m, n, \ldots = 1, 2, \ldots, r), \ (r \geqq n),$

where the Pfaffians

ω^l	α^λ

are assumed to be anholonomic in general and to be *of rank r*, so that the condition

(5.2) $\lVert \omega^l_\mu(x^\nu) \rVert^2 \neq 0$ in M	$\lVert \alpha^\lambda_l(a^p) \rVert^2 \neq 0$ in G

is satisfied.

Since (5.1) is written in an invariant form,

ω^l	α^λ

are global in $\bigcup_\alpha U_\alpha$.

For the given

$\omega^l_\mu(x^\nu),$	$\alpha^\lambda_l(a^p),$

we introduce

$\Omega^\lambda_l(x^\nu)$	$\beta^l_\lambda(a^p)$

by the condition:

(5.3) $\Omega^\lambda_l \omega^l_\mu = \delta^\lambda_\mu \iff \Omega^\lambda_m \omega^l_\lambda = \delta^l_m,$	$\beta^l_\lambda \alpha^\lambda_m = \delta^l_m \iff \beta^l_\mu \alpha^\lambda_l = \delta^\lambda_\mu,$

where δ's are Kronecker deltas.

We define the connection parameters

$\Lambda^\lambda_{\mu\nu}$	Λ^l_{mn}

by

(5.4) $\Lambda^{\lambda}_{\mu\nu} \overset{\text{def}}{=} \Omega^{\lambda}_{l} \dfrac{\partial \omega^{l}_{\mu}}{\partial x^{\nu}} \equiv -\omega^{l}_{\mu} \dfrac{\partial \Omega^{\lambda}_{l}}{\partial x^{\nu}}$, | $\Lambda^{l}_{mn} \overset{\text{def}}{=} \alpha^{l}_{\lambda} \dfrac{\partial \beta^{\lambda}_{m}}{\partial a^{n}} \equiv -\beta^{\lambda}_{m} \dfrac{\partial \alpha^{l}_{\lambda}}{\partial a^{n}}$,

the last identily arising from (5.3).

Consider a parameterized curve

$x^{\lambda} = x^{\lambda}(t),$ | $a^{l} = a^{l}(t),$

where it is assumed that the t is *invariant*.

We can easily prove the identity:

(5.5) $\dfrac{d}{dt}\dfrac{\omega^{l}}{dt} = \omega^{l}_{\lambda}\left(\dfrac{d^{2}x^{\lambda}}{dt^{2}} + \Lambda^{\lambda}_{\mu\nu}\dfrac{dx^{\mu}}{dt}\dfrac{dx^{\nu}}{dt}\right).$ | $\dfrac{d}{dt}\dfrac{\alpha^{\lambda}}{dt} = \alpha^{\lambda}_{l}\left(\dfrac{d^{2}a^{l}}{dt^{2}} + \Lambda^{l}_{rt}\dfrac{da^{r}}{dt}\dfrac{da^{t}}{dt}\right).$

We consider the combined manifold:

$\{x^{\lambda}\} + \{\omega^{l}_{\mu}(x^{\nu})\}$ | $\{a^{l}\} + \{\alpha^{\lambda}_{m}(a^{p})\}$

forming a principal fibre bundle, the

$\{\omega^{l}_{\mu}(x^{\nu})\} = \{\Omega^{\lambda}_{l}(x^{\nu})\}$ | $\{\alpha^{\lambda}_{m}(a^{p})\} = \{\beta^{l}_{m}(a^{p})\}$

making the *structure group*. (This group

$\{\omega^{l}_{\mu}(x^{\nu})\}$ | $\{\alpha^{\lambda}_{m}(a^{p})\}$

will afterwards be enlarged to

$\{\omega^{l}_{\mu}(x^{\nu}),\ \omega^{l}_{0}\}).$ | $\{\alpha^{\lambda}_{m}(a^{p}),\ \alpha^{\lambda}_{0}\}).$

Although the group elements

$\omega^{l}_{\mu}(x^{\nu})$ | $\alpha^{\lambda}_{m}(a^{p})$

contain the local coordinates

$(x^{\nu}),$ | $(a^{p}),$

the function forms make the group elements (in a certain sense) independent of the local coordinates

$(x^{\nu}).$ | $(a^{p}).$

From (5.5), we have

(5.6) $$\frac{d}{dt}\frac{\omega^l}{dt}=0 \rightleftarrows \frac{d^2x^\lambda}{dt^2}+\Lambda^\lambda_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}=0.$$ | $$\frac{d}{dt}\frac{\alpha^\lambda}{dt}=0 \rightleftarrows \frac{d^2a^l}{dt^2}=\Lambda^l_{rt}\frac{da^r}{dt}\frac{da^t}{dt}=0.$$

Indeed, we can convert (5.5) into

(5.6') $$\Omega^\sigma_l\frac{d}{dt}\frac{\omega^l}{dt}=\frac{d^2x^\sigma}{dt^2}+\Lambda^\sigma_{\mu\nu}\frac{da^\mu}{dt}\frac{dx^\nu}{dt}.$$ | $$\beta^s_\lambda\frac{d}{dt}\frac{\alpha^\lambda}{dt}=\frac{d^2a^s}{dt^2}+\Lambda^s_{rt}\frac{da^r}{dt}\frac{da^t}{dt}.$$

The differential equations on the right-hand side of (5.6) define the autoparallel curves of the teleparallelism (E. CARTAN (1926), WEITZENBÖCK (1928)). *The left-hand side* (3) *is convenient for the study of the global properties and is integrated readily:*

(5.7) $$\omega^l=e^l dt,\ e^l=\text{const.}),$$ | $$\alpha^\lambda=e^\lambda dt,\ (e^\lambda=\text{const.}),$$

(5.8) $$\int\frac{\omega^l}{dt}dt=e^lt+c^l,(c^l=\text{const.}),$$ | $$\int\frac{\alpha^\lambda}{dt}dt=e^\lambda t+c^\lambda,\ (c^\lambda=\text{const.}),$$

the (5.8) being guided by the simple character of the right-hand side of (5.7). Noticing again the simple character of the right–hand side of (5.8), we set

$$\xi^l\stackrel{\text{def}}{=}e^lt+c^l,$$ | $$\eta^\lambda\stackrel{\text{def}}{=}e^\lambda t+c^\lambda,$$

so that

(5.9) $$\xi^l=\int\frac{\omega^l}{dt}dt=e^lt+c^l.$$ | $$\eta^\lambda=\int\frac{\alpha^\lambda}{at}dt=e^\lambda t+c^\lambda.$$

This means that *we adopt such curves as*

ξ^l — *axes.* | η^λ — *axes.*

From (5.9), we see that *the curves represented by* (5.6) *or* (5.9) *behave as for meet and join like straight lines in the large.* We will call such curves (4) II-*geodesic curves* (read: geodesic curves of the second kind!).

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(3) A glimpse is found (for the group manifold (a_h)) in: E. CARTAN, [15], p. 62.

(4) In the group manifolds, such curves have been called *geodesic curves* (E. CARTAN, [15], p. 14 and p. 62). The author has just found that *the* II-*geodesics are geodesics for* ω^1.

Although the

$$\omega^l \qquad\Big|\qquad \alpha^\lambda$$

are anholonomic in general, we may write it in the form of differentials:

$$(5.10)\qquad d\xi^l = \omega^l = a^l_\mu(x^\nu(t))dx^\mu(t) \qquad\Big|\qquad d\mu^\lambda = \alpha^\lambda = \alpha^\lambda_m(a^p(t))da^m(t)$$

for II-*geodesic line-elements,* where

$$(5.11)\qquad \begin{array}{l} a^l_\mu(x^\nu) \overset{\text{def}}{=} \omega^l_\mu(x^\nu), \\ \| a^l_\mu(x^\nu) \| \neq 0 \text{ in } M. \end{array} \qquad\Big|\qquad | \alpha^\lambda_m(a^p) | \neq 0 \text{ in } G$$

The expressions (5.7) and (5.10) tell us that, for the given

$$a^l_\mu(x^\nu)dx^\mu, \qquad\Big|\qquad \alpha^\lambda_m(a^p)da^m,$$

there exists a curve

$$x^\lambda(t), \qquad\Big|\qquad a^l(t),$$

whose line-element

$$\{dx^\mu\} \qquad\Big|\qquad \{da^m\}$$

with direction

$$\{e^l\} \qquad\Big|\qquad \{e^\lambda\}$$

is given by the differential

$$d\xi^l. \qquad\Big|\qquad d\eta^\lambda.$$

This is the case for *all the directions*

$$\{e^l\}. \qquad\Big|\qquad \{e^\lambda\}.$$

Thus in (5.10), *we may omit* t *and write down as follows:*

$$(5.12)\qquad d\xi^l = a^l_\mu(x^\nu)dx^\mu, \qquad\Big|\qquad d\eta^\lambda = \alpha^\lambda_m(a^p)da^m,$$

notwithstanding the right-hand side is anholonomic in general. (Hence (5.12) will lead us afterwards to

(5.13) $\xi^l = a^l_\mu(x^\nu)x^\mu + a^l_0 = \xi^l(x^\nu)$, $(a^l_0 = \text{const.})$, | $\eta^\lambda = \alpha^\lambda_m(a^p)a^m + \alpha^\lambda_0 = \eta^\lambda(a^p)$, $(\alpha^\lambda_0 = \text{const.})$,

cf. (6.6)). That the anholonomic Pfaffian

$$a^l_\mu(x^\nu)dx^\mu \quad \Big| \quad \alpha^\lambda_m(a^p)da^m$$

is expressible in the form of the differential

$$d\xi^l \quad \Big| \quad d\eta^\lambda$$

is an *unexpected* consequence of the superior quality of the Π-geodesic line-elements. This point is the *primary difficulty* encountered by the readers, who are apt to overlook the differential equation ([5]) (5.6):

(5.14) $$da^l_\mu(x^\nu)dx^\mu = 0 \quad \Big| \quad d\alpha^\lambda_m(a^p)da^m = 0$$

for the Π-*geodesic line-elements.*

The first differential equation of (5.6) may be rewritten as follows:

(5.15) $$\frac{d^2\xi^l}{dt^2} = 0. \quad \Big| \quad \frac{d^2\eta^\lambda}{dt^2} = 0.$$

From (5 13) and (5.12), we obtain

(5.16) $$da^l_\mu(x^\nu)x^\mu = 0 \quad \Big| \quad d\alpha^\lambda_m(a^p)a^m = 0$$

along Π-*geodesic line-elements.*

Multiplying (5.7) with

$$\Omega^\lambda_m \quad \Big| \quad \beta^l_\mu$$

and taking (5.3) into account, we see that the relations ([5])

(5.17) $$\frac{dx^\lambda}{dt} = e^l\Omega^\lambda_l \quad \Big| \quad \frac{da^l}{dt} = e^\lambda\beta^l_\lambda$$

hold along the Π-*geodesic line-elements.*

([5]) This condition is lacking for the general Pfaffians.

We will call the

$\{\xi^l\}$ | $\{\eta^\lambda\}$

the Π-*geodesic parallel coordinates corresponding to*

$a^l_\mu(x^\nu)$ | $\alpha^\lambda_m(a^p)$

referred to Π-*geodesic coordinate axes.* The $\{\xi^l\}$ are global in the atlas $\bigcup_\alpha U_\alpha$.

6. **Extension of the Affine Transformation Groups by Extending the Group Parameters to Functions of Coordinates.** When the differentiable manifold

M | G

is the classical affine space and the

$\{x^\nu\}$ | $\{a^p\}$

are the ordinary parallel coordinates, the atlas $\bigcup_\alpha U_\alpha$ reduces to a single chart U_α, whose map is the classical affine space.

In general case, the Π-geodesic parallel coordinates

(ξ^l) | (η^λ)

can stand for

$\{x^\nu\}$, | $\{a^p\}$,

so that the atlas $\bigcup_\alpha U_\alpha$ may be considered to consist of a single chart U_α and in place of (5.12), we come to consider

(6.1) $d\bar{\xi}^l = a^l_m(\xi^p)d\xi^m,$ $\quad(\,|\,a^l_m(\xi^p)\,| \neq 0$ in $M)$ | $d\bar{\eta}^\lambda = \alpha^\lambda_\mu(\eta^\nu)d\eta^\mu,$ $\quad(\,|\,\alpha^\lambda_\mu(\eta^\nu)\,| \neq 0$ in $G)$

for Π-*geodesic line-elements corresponding to*

$a^l_m(\xi^p)$. | $\alpha^\lambda_\mu(\eta^\nu)$.

We take Π-*geodesic curves corresponding to*

$a^l_m(\xi^p)$ | $\alpha^\lambda_\mu(\eta^\nu)$

as tangents to the curves.

We consider a transformation

(6.2) $\bar{\xi}^l = a^l_m(\xi^p)\xi^m + a^l_0,$ $\quad (| a^l_m(\xi^p) | \neq 0 \text{ in } M)$ | $\bar{\eta}^\lambda = \alpha^\lambda_\mu(\eta^\nu)\eta^\mu + \alpha^\lambda_0,$ $\quad (| \alpha^\lambda_\mu(\eta^\nu) | \neq 0 \text{ in } G)$

accompanying (6.1). We will call the transformations (6.2), which transform Π-geodesic curves

$\xi^m(t)$ | $\eta^\lambda(t)$

into Π-geodesic curves corresponding to

$a^l_m(\xi^p),$ | $\alpha^\lambda_\mu(\eta^\nu),$

extended affine transformations. By such a transformation, Π-geodesic curves

(6.3) $\dfrac{d^2\xi^l}{dt^2} = 0$ | $\dfrac{d^2\eta^\lambda}{dt^2} = 0$

are transformed into Π-geodesic curves

(6.4) $\dfrac{d^2\bar{\xi}^l}{dt^2} = 0.$ | $\dfrac{d^2\bar{\eta}^\lambda}{dt^2} = 0.$

Now by (6.1), we have

$$\frac{d^2\bar{\xi}^l}{dt^2} = \frac{d}{dt} a^l_m(\xi^p)\frac{d\xi^m}{dt} + a^l_m(\xi^p)\frac{d^2\xi^m}{dt^2}. \quad \Big| \quad \frac{d^2\bar{\eta}^\lambda}{dt^2} = \frac{d}{dt}\alpha^\lambda_\mu(\eta^\nu)\frac{d\eta^\mu}{dt} + \alpha^\lambda_\mu(\eta^\nu)\frac{d^2\eta^\mu}{dt^2}.$$

Hence by the demands (6.3) and (6.4), we must have

(6.5) $$da^l_m(\xi^p)\,d\xi^m = a^l_s(\xi^p)\left\{\frac{d^2\xi^s}{dt^2} + \Lambda^s_{rt}(\xi^p)\frac{d\xi^r}{dt}\frac{d\xi^t}{dt}\right\}dt^2 = 0$$

$$d\alpha^\lambda_\mu(\eta^\nu)\,d\eta^\mu = \alpha(\eta^\nu)\left\{\frac{d^2\eta^\sigma}{dt^2} + \Lambda^\sigma_{\tau\omega}\frac{d\eta^\tau}{dt}\frac{d\eta^\omega}{dt}\right\}dt^2 = 0$$

for the Π-*geodesic line-elements.*

Integrating (6.1) along the Π-geodesic

$\bar{\xi}^l$-axis, | $\bar{\eta}^\lambda$-axis, we have

$$\bar{\xi}^l = a^l_m(\xi^p)\xi^m + \int \xi^m \frac{da^l_m(\xi^p)}{dt} dt. \qquad \bar{\eta}^\lambda = \alpha^\lambda_\mu(\eta^\nu)\eta^\mu + \int \eta^\mu \frac{d\alpha^\lambda_\mu(\eta^\nu)}{dt} dt.$$

Now

$$\int \xi^m \frac{da^l_m(\xi^p)}{dt} dt = \int \frac{da^l_m(\xi^p)}{dt} dt \int d\xi^m \qquad \int \eta^\mu \frac{d\alpha^\lambda_\mu(\eta^\nu)}{dt} dt = \int \frac{d\alpha^\lambda_\mu(\eta^\nu)}{dt} dt \int d\eta^\mu$$

$$= \iint \left\{ \frac{da^l_m(\xi^p)}{dt} dt d\xi^m \right\} \qquad = \iint \left\{ \frac{d\alpha^\lambda_\mu(\eta^\nu)}{dt} dt d\eta^\mu \right\}$$

$$= \text{const.}$$

by (6.5), (the indication of the domain of integration is here omitted), and the condition for that the repeated integral may be converted into the double integral being evidently satisfied. Hence for the

$$a^l_0 \qquad \alpha^\lambda_0$$

in (6.2), we have

$$a^l_0 = \text{const.}, \qquad \alpha^\lambda_0 = \text{const.},$$

being led to

$$(6.6) \qquad \bar{\xi}^l = a^l_m(\xi^p)\xi^m + a^l_0, \qquad \bar{\eta}^\lambda = \alpha^\lambda_\mu(\eta^\nu)\eta^\mu + \alpha^\lambda_0,$$

$$(\mid a^l_m(\xi^p) \mid \neq 0 \text{ in } M,\ a^l_0 = \text{const.}). \qquad (\mid \alpha^\lambda_\mu(\eta^\nu) \mid \neq 0 \text{ in } G,\ \alpha^\lambda_0 = \text{const}).$$

From (6.2) and (6.5), we see that

$$(6.7) \qquad da^l_m(\xi^p)\xi^m = 0 \qquad d\alpha^\lambda_\mu(\eta^\nu)\eta^\mu = 0$$

for the Π-*geodesic line-elements.*

The totality of the extended affine transformations forms a group,

$\mathfrak{G}$, say. | $\mathfrak{F}$, say.

In order to show this analytically, it suffices to show that the product of (6.6) with

$$(6.8) \qquad \bar{\bar{\xi}}^l = \bar{a}^l_m(\bar{\xi}^p)\bar{\xi}^m + \bar{a}^l_0, \qquad \bar{\bar{\eta}}^\lambda = \bar{\alpha}^\lambda_\mu(\bar{\eta}^\nu)\bar{\eta}^\mu + \bar{\alpha}^\lambda_0,$$

$$(\mid \bar{a}^l_m(\bar{\xi}^p) \mid \neq 0 \text{ in } M,\ \bar{a}^l_0 = \text{const.}) \qquad (\mid \bar{\alpha}^\lambda_\mu(\bar{\eta}^\nu) \mid \neq 0 \text{ in } G,\ \bar{\alpha}^\lambda_0 = \text{const.})$$

is of the form (6.6):

$$(6.9)\qquad \tilde{\xi}^l = b^l_m(\xi^p)\xi^m + b^l_0, \qquad\Big|\qquad \tilde{\eta}^\lambda = \beta^\lambda_\mu(\eta^\nu)\eta^\mu + \beta^\lambda_0,$$

$$(|b^l_m(\xi^p)| \neq 0 \text{ in } M,\ b^l_0 = \text{const.}), \qquad\Big|\qquad (|\beta^\lambda_\mu(\eta^\nu)| \neq 0 \text{ in } G,\ \beta^\lambda_0 = \text{const.}),$$

where

$$(6.10)\qquad b^l_k(\xi^p) = \bar{a}^l_m(a^p_h(\xi^r)\xi^h + a^p_0)a^m_k(\xi^r), \qquad\Big|\qquad \beta^\lambda_\varkappa(\eta^\nu) = \bar{\alpha}^\lambda_\mu(\alpha^\nu_\tau(\eta^\sigma)\eta^\tau + \alpha^\nu_0)\alpha^\mu_\varkappa(\eta^\sigma),$$

$$(6.11)\qquad b^l_0 = \bar{b}^l_m(\xi^p)a^m_0 + \bar{a}^l_0, \qquad\Big|\qquad \beta^\lambda_0 = \bar{\beta}^\lambda_\mu(\eta^\nu)\alpha^\mu_0 + \bar{\alpha}^\lambda_0,$$

$$(6.12)\qquad \bar{b}^l_m(\xi^p) = (\bar{a}^l_m(a^q_k(\xi^p)\xi^k + a^q_0). \qquad\Big|\qquad \bar{\beta}^\lambda_\mu(\eta^\nu) = \bar{\alpha}^\lambda_\mu(\alpha^\sigma_\varkappa(\eta^\nu)\eta^\varkappa + \bar{\alpha}^\sigma_0).$$

We shall see that

$$(6.13)\qquad b^l_0 = \bar{b}^l_m(\xi^p)a^m_0 + \bar{a}^l_0 = \text{const.}, \qquad\Big|\qquad \beta^\lambda_0 = \bar{\beta}^\lambda_\mu(\eta^\nu)\alpha^\mu_0 + \bar{\alpha}^\lambda_0 = \text{const.},$$

for which it suffices to prove that

$$(6.14)\qquad a^m_0 d\bar{b}^l_m(\xi^p) = 0 \qquad\Big|\qquad \alpha^\mu_0 d\bar{\beta}^\lambda_\mu(\eta^\nu) = 0$$

on summation with respect to m. For (6.9), the condition (6.7) for that the

$$\bar{\xi}^l\text{-axes} \qquad\Big|\qquad \bar{\eta}^l\text{-axes}$$

may be Π-geodesic curves corresponding to

$$\bar{a}^l_m(\bar{\xi}^p) \qquad\Big|\qquad \bar{\alpha}^\lambda_\mu(\bar{\eta}^\nu)$$

becomes

$$(6.15)\qquad \xi^m d\bar{a}^l_m(\bar{\xi}^p) = 0. \qquad\Big|\qquad \bar{\eta}^\mu d\bar{\alpha}^\lambda_\mu(\bar{\eta}^\nu) = 0.$$

We shall show that (6.14) follows from (6.15). The (6.15) becomes

$$\{a^m_k(\xi^p)\xi^k + a^m_0\}\, d\bar{a}^l_m(\bar{\xi}^p) = \{a^m_k(\xi^p)\xi^k + a^m_0\}\, d\bar{b}^l_m(\xi^p) = 0, \qquad\Big|\qquad \{\alpha^\mu_\varkappa(\eta^\nu)\eta^\varkappa + \alpha^\mu_0\}\, d\bar{\alpha}^\lambda_\mu(\bar{\eta}^\nu) = \{\alpha^\mu_\varkappa(\eta^\nu)\eta^\varkappa + \alpha^\mu_0\}\, d\bar{\beta}^\lambda_\mu(\eta^\nu) = 0,$$

so that

(6.16) $a_0^m d\bar{a}^l_m(\xi^p) = a_0^m d\bar{b}^l_m(\xi^p)$	$\alpha_0^\mu d\bar{\alpha}^\lambda_\mu(\eta^\nu) = \alpha_0^\mu d\bar{\beta}^\lambda_\mu(\eta^\nu)$
$= - a_k^m(\xi^p)\, d\bar{b}^l_m(\bar{\xi}^p)\,\xi^k$	$= - \alpha_\varkappa^\mu(\eta^\nu) d\bar{\beta}^\lambda_\mu(\eta^\nu)\eta^\varkappa$
$= - a_k^m(\xi^p) d\bar{b}^l_m(\xi^p)\xi^k$	$= - \alpha_\varkappa^\mu(\eta^\nu)\, d\bar{\beta}^\lambda_\mu(\eta^\nu)\eta^\varkappa$
$- \{\xi^k d a_k^m(\xi^p)\}\, \bar{b}^l_m(\xi^p)$	$- \{\eta^\varkappa d\alpha^\mu_\varkappa(\eta^\nu)\}\, \bar{\beta}^\lambda_\mu(\eta^\nu)$

by the differential equation

(6.17) $\xi^k d a^m(\xi^p) = 0$	$\eta^\varkappa d\alpha^\mu_\varkappa(\eta^\nu) = 0$

of the Π-geodesic curves corresponding to

$a_k^m(\xi^p)$.	$\alpha^\mu_\varkappa(\eta^\nu)$.

Thus we have

$a_0^m d\bar{a}^l_m(\bar{\xi}^p) = - \xi^k d\{ a_k^m(\xi^p)\bar{b}^l_m(\xi^p)\}$	$\alpha_0^\mu d\bar{\alpha}^\lambda_\mu(\bar{\eta}^\nu) = - \eta^\varkappa d\{ \alpha^\mu_\tau(\eta^\nu)\bar{\beta}^\lambda_\mu(\eta^\nu)\}$
$= - \xi^k d\{ a_k^m(\xi^p)\bar{a}^l_m(\xi^p)\}$	$= - \eta^\varkappa d\{ \alpha^\mu_\varkappa(\eta^\nu)\bar{\alpha}^\lambda_\mu(\eta^\nu)\}$
$= - \xi^k d b^l_k(\xi^p) = 0$	$= - \eta^\varkappa d\beta^\lambda_\varkappa(\eta^\nu) = 0$

by the differential equations

(6.18) $\xi^k d b^l_k(\xi^p) = 0$	$\eta^\varkappa d\beta^\lambda_\varkappa(\eta^\nu) = 0$

of the Π-geodesic line-elements corresponding to

$b^l_m(\xi^p)$.	$\beta^\lambda_\varkappa(\eta)$.

The (6.17) shows (6.14). We have called the

$\mathfrak{G}$	$\mathfrak{F}$

the *extended affine group. The most general extended affine group*

$\mathfrak{G}$	$\mathfrak{F}$

contains the ordinary affine group

$$\mathfrak{C} \qquad\qquad \Big| \qquad\qquad \mathfrak{R}$$

(in the abstract sense) as a subgroup. The totality of the elements of

$$\mathfrak{G}, \qquad\qquad \Big| \qquad\qquad \mathfrak{F},$$

which are free from

$$\mathfrak{C} \qquad\qquad \Big| \qquad\qquad \mathfrak{R}$$

together with the unit transformation, forms a subgroup,

$$\mathfrak{H}, \qquad\qquad \Big| \qquad\qquad \mathfrak{I},$$

say, of

$$\mathfrak{G}, \qquad\qquad \Big| \qquad\qquad \mathfrak{F},$$

so that

$$(6.19) \qquad \mathfrak{G} = \mathfrak{C}\mathfrak{H} + \mathfrak{H}\mathfrak{C}. \qquad\qquad \Big| \qquad\qquad \mathfrak{F} = \mathfrak{R}\mathfrak{I} + \mathfrak{I}\mathfrak{R}.$$

The geometry under the extended affine group has been called by me the *extended affine geometry.*

7. **Realization of the Extended Affine Geometry in the Differentiable Manifolds.** - Our results of Art. 3 - 6 show us that *the author's extended affine geometry is realized in the differentiable manifolds.*

8. **The Fundamental Pfaffians fer the Lie Group (Germs).** - *The ordinary theory of the fundamental Pfaffians for the Lie group germs applies still when the elements*

$$a^l, \quad (l = 1, 2, \ldots, r; \;\; i = 1, 2, \ldots, n)$$

of the Lie group germs are extended to appropriate functions of the coordinates of the base manifold. Such a theory will be exposed in the following lines, writing a^l in place of $a^l(x) = a^l(x^i)$. *We assume moreover the coordinates* (x^i) *to be* Π*-geodesic parallel coordinates* (ξ^i), *which are global. Then we may omit the term "germ" without relying upon the Otto Schreier's Fundamental Theorems.*

We have assumed in Art. 3 that the composition functions

$$c^i = \varphi^i(a^1, \ldots, a^r; b^1, \ldots, b^r), \qquad (i = 1, 2, \ldots, r) \tag{8.1}$$

are such that

$$\varphi^i \in C^3. \tag{8.2}$$

We form the matrix

$$\alpha^i_j(a) = \left(\frac{\partial \varphi^i(a; b)}{\partial b^j}\right)_{b=O}, \qquad (i, j = 1, 2, \ldots, r). \tag{8.3}$$

Since

$$|\alpha^i_j(O)| = |\delta^i_j| = 1, \tag{8.4}$$

we introduce the inverse $\beta^i_j(a)$ by the conditions

$$\alpha^i_k(a)\beta^k_j(a) = \delta^i_j, \quad \rightleftarrows \quad \alpha^k_j(a)\beta^i_k(a) = \delta^i_j, \tag{8.5}$$

where α^i_j are KRONECKER deltas.

DEFINITION. - We call

$$\omega^i(a, da) = \beta^i_j(a)da^j, \quad a^l = a^l(x^i), \quad \omega^i \in A^{(1)}(C^2) \tag{8.6}$$

the *fundamental Pfaffians* of the extended LIE group (germ), where $A^{(1)}(C^2)$ is a LIE algebra having $\omega^i(a, da)$ as base.

Multiplying (8.6) with $\alpha^j_i(a)$, we obtain

$$da^j = \alpha^j_i(a)\,\omega^i. \tag{8.7}$$

THEOREM. - *The necessary and sufficient condition for that the differential form*

$$\Phi = \sum_{i_1 < \ldots < i_p} g_{i_1 \ldots i_p}(a) da^{i_1} \wedge \ldots \wedge da^{i_p} \in A(C^0) \tag{8.8}$$

may be invariant:

$$\bar{\Phi} = \sum_{i_1 < \ldots < i_p} g_{i_1 \ldots i_p}(\bar{a})\, d\bar{a}^{i_1} \wedge \ldots \wedge d\bar{a}^{i_p} = \Phi \tag{8.9}$$

for all the transformations

$$\bar{a}^i = \varphi^i(k^1, \ldots, k^r; a^1, \ldots, a^r), \qquad (i = 1, 2, \ldots, r) \tag{8.10}$$

with parameters $(k^1(x^i), \ldots, k^r(x^i))$ *belonging to a vicinity of the origin* (O) *is that for*

$$\Phi = \sum_{i_1 < \ldots < i_p} h_{i_1 \ldots i_p} \omega^{i_1} \wedge \ldots \wedge \omega^{i_p}, \tag{8.11}$$

the coefficients $h_{i_1 \ldots i_p}$ *are all constants.*

PROOF. - We will begin with the proof for that (8.8) are invariant for (8.10). Apply the transformation (8.10) to (8.7); then we have

$$d\bar{a}^i = \alpha^i_j(\bar{a})\bar{\omega}^j,$$

i. e.

$$\frac{\partial \varphi^i(k; a)}{\partial a^l} da^l = \alpha^i_j(\varphi(k; a))\bar{\omega}^j \tag{8.12}$$

on one hand and

$$\alpha^i_j(\varphi(k, a)) = \left(\frac{\partial \varphi^i(k; a); c)}{\partial c^j}\right)_{c=O} = \left(\frac{\partial \varphi^i(k; \varphi(a; c))}{\partial c^j}\right)_{c=O} \tag{8.13}$$

$$= \left(\frac{\partial \varphi^i(k; b)}{\partial b^l} \frac{\partial \varphi^l(a; c)}{\partial c^j}\right)_{c=O} = \frac{\partial \varphi^i(k; a)}{\partial a^l} \alpha^l_j(a)$$

on the other hand, where $b^i = \varphi^i(a; c)$. Apply the inverse of

$$\left(\frac{\partial \varphi^i(k; a)}{\partial a^l}\right)$$

to (8.12). Then it results that

$$da^l = \alpha^l_j(a)\bar{\omega}^j.$$

Thus we have

$$\omega^j = \beta^j_l(a) da^l = \beta^j_l(a)\alpha^l_k(a)\bar{\omega}^k = \delta^j_k \bar{\omega}^k = \bar{\omega}^j.$$

Secondly, in order that Φ may be invariant, the relation

$$h_{i_1 \ldots i_p}(a) = h_{i_1 \ldots i_p}(\varphi(k; a))$$

must hold for all values of k. If we take $a \longrightarrow O$, since $\varphi^i(k; O) = k^i$, we must have

$$h_{i_1 \dots i_p}(O) = h_{i_1 \dots i_p}(k).$$

Hence $h_{i_1 \dots i_p}$ must all be constants. Q.E.D.

THEOREM. - *For the fundamental Pfaffians of r-dimensional extended Lie group (germ), it holds that*

$$\mathbf{d}\omega^i = \frac{1}{2} C^i_{jk}\omega^j \wedge \omega^k, \tag{8.14}$$

where the r^3 constant coefficients C^i_{jk} obey the rules

$$\begin{cases} C^i_{jk} = -C^i_{kj}, \\ C^i_{jj} = O, \end{cases} \tag{8.15}$$

$$C^h_{ij}C^l_{hk} + C^h_{jk}C^l_{hi} + C^h_{ki}C^l_{hj} = O. \tag{8.16}$$

PROOF. - Since ω^i are invariant, $\mathbf{d}\omega^i$ must also be invariant, since the operator $\mathbf{d}$ and coordinate transformation are commutative. Hence, by the last Theorem, we must have constants C^i_{jk} such that

$$\mathbf{d}\omega^i = C^i_{jk}\omega^j \wedge \omega^k.$$

If we set (8.15):

$$C^i_{jk} = -C^i_{kj}, \quad (j > k), \qquad C^i_{jj} = O,$$

we have

$$\mathbf{d}\omega^i = \frac{1}{2} C^i_{jk}\omega^j \wedge \omega^k, \ \omega^i \in A(C^2), \ \mathbf{d}\omega^i \in A(C^1), \tag{8.17}$$

$$\mathbf{d}(\mathbf{d}\omega^i) = 0. \tag{8.18}$$

Therefore

$$\begin{aligned} \mathbf{d}(\mathbf{d}\omega^i) &= \frac{1}{2} C^i_{kl}\mathbf{d}\omega^k \wedge \omega^l - \frac{1}{2} C^i_{kl}\omega^k \wedge \mathbf{d}\omega^l \\ &= C^i_{kl}\mathbf{d}\omega^k \wedge \omega^l - \frac{1}{2} C^i_{kl}C^k_{pq}\omega^q \wedge \omega^q \wedge \omega^l = O. \end{aligned}$$

Hence

$$C^h_{ij}C^l_{hk} + C^h_{jk}C^l_{hi} + C^h_{ki}C^l_{hj} = O, \qquad (i, j, k = 1, 2, \dots, r).$$

DEFINITION. - The r^3 constants C^i_{jk} are called the *structure constants* of the r-dimensional extended LIE group (germ).

If we develop $\varphi^i(a(x^i);\, b(x^i))$, by virtue of (3.1), then we obtain

$$\varphi^i(a;b) = a^i + b^i + d^i_{jk}a^j b^k + \varepsilon^i, \tag{8.19}$$

where ε^i is an infinitesimal higher than the second order in the vicinity of the origin. From (8.19), it results that

$$\alpha^i_j(a) = \delta^i_j + d^i_{kj}a^k + \varepsilon^2,$$

$$\beta^i_j(a) = \delta^i_j - d^i_{kj}a^k + \varepsilon^3,$$

where ε^2 and ε^3 are infinitesimals. Hence

$$\omega^i(a, da) = da^i - d^i_{kj}a^k da^j + \varepsilon_{4j}\, da^j,$$

where ε_{4j} is an infinitesimal. Hence it results that

$$\mathbf{d}\omega^i = - d^i_{kj}da^k \wedge da^j + d\varepsilon_{4j} \wedge da^j = C^i_{jk}\omega^j \wedge \omega^k.$$

Comparing the coefficients of $da^k \wedge da^j$, we obtain

$$C^i_{jk} = d^i_{jk} - d^i_{kj}. \tag{8.20}$$

N. B. - (i) In order to deduce (8.16) in terms of d^i_{jk} directly, we utilize (3.2) having written out the terms of the third degree in (8.19) [16].

(ii) As for the class C^v in the ordinary case, L. PONTRIAGIN [16] has taken $v = 3$. L. van der WAERDEN [17] has assumed that (1) $\varphi^i(a; b)$ is once differentiable, (2) $\varphi'_a(a;b)$ satisfies the Lipschitz's condition for b and (3) its converse. G. BIRKHOFF [18] has assumed the existence of the total differential of $\varphi^i(a; b)$ and its continuity in the origin. P. A. SMITH [19] has proved that when for $\varphi^i(a; b) = a^i + b^i + \psi^i(a; b)$, the condition $\dfrac{\psi_i(a;\, b)}{|\, a \,|} \to 0$, $(a \to 0,\ b \to 0)$, where $(\,|\, a \,| = a^{12} + \ldots + a^{r2})$, is satisfied, the LIE group (germ) may be rendered into an analytic LIE group (germ).

In our case, we have assumed "$\varphi^i \in C^3$". This condition is fully utilized in (8.18). But, it will be seen that the result of Art. 8 hold good also for $\varphi^i \in C^2$, if we notice the following fact. Indeed, if $\varphi^i \in C^2$, then we have $\omega^i \in A(C^1)$, $\mathbf{d}\omega^i \in A(C^0)$. Thus the first Theorem of Art. 8 is still applicable, so that (8.17) holds. Consequently we see that $\mathbf{d}\omega^i \in A(C^1)$, so that $\mathbf{d}(\mathbf{d}\omega^i)$

exists and the fact $\mathbf{d}(\mathbf{d}\omega^i) = 0$ is a consequence of $\omega^i \in A(C^2)$. Hence it suffices to deduce $\mathbf{d}(\mathbf{d}\omega^i) = 0$ from $d\omega^i \in A(C^1)$ in another way. For this purpose we utilize the generalized STOKE's theorem. When $\omega^r \in A(C^v)$, $(v \geqq 1)$ is an homogeneous expression of r-*th* degree and C^{r+1} be an algebraic complex composed of curved simplex of μ-*th* class $(\mu \geqq 2)$, then the relation

$$\int_{\Delta C^{r+1}} \omega^r = \int_{C^{r+1}} \mathbf{d}\omega^r$$

holds. Thus for an arbitrary 3-dimensional curved simplex C^3, we have

$$(C^3, \mathbf{d}(\mathbf{d}\omega^i)) = (\Delta C^3, \mathbf{d}\omega^i) = (\Delta(\Delta C^3), \omega^i) = 0,$$

where

$$\int_{C^r} \omega^r = (C^r, \omega^r).$$

Hence we have

$$\mathbf{d}(\mathbf{d}\omega^i) = O.$$

(iii) The name "fundamental Pfaffians" arises from the following theorem.

THEOREM. - *When r fundamental Pfaffians are invariant for*

$$a^i \longrightarrow \bar{a}^i = \psi^i(a), \qquad (i = 1, 2, \ldots, r),$$

which maps the points of a vicinity U of the origin into a vicinity U_0 of the origin:

$$\omega^i(a, da) = \omega^i(\bar{a}, d\bar{a}), \qquad (i = 1, 2. \ldots, r), \tag{8.21}$$

the $\psi^i(a)$ coincides with the composition function $\varphi^i(k; a)$:

$$\psi^i(a) = \varphi^i(k; a), \qquad (i = 1. 2, \ldots, r) \tag{8.22}$$

for

$$\psi^i(O) = k^i, \qquad (i = 1, 2, \ldots, r), \tag{8.23}$$

that is to say, the extended Lie group (germ) is determined uniquely by r given fundamental Pfaffians.

PROOF. - Consider the simultaneous total differential equations

$$\bar{\omega}^i - \omega^i = 0, \qquad (i = 1, 2, \dots, r), \tag{8.24}$$

by putting

$$\bar{\omega}^i = \beta^i_j(\bar{a}) d\bar{a}^j.$$

These are completely integrable. For.

$$d(\bar{\omega}^i - \omega^i) = \sum_{j<k} C^i_{jk}(\bar{\omega}^j \wedge \bar{\omega}^k - \omega^j \wedge \omega^k)$$

$$= \sum_{j<k} C^i_{jk} \{\bar{\omega}^j \wedge (\bar{\omega}^k - \omega^k) + (\bar{\omega}^j - \omega^j) \wedge \omega^k\}.$$

$$\equiv 0, \quad (\text{mod. } \bar{\omega}^1 - \omega^1, \dots, \bar{\omega}^r - \omega^r),$$

and since

$$| \beta^i_j(\bar{a}) | \neq 0,$$

the solutions such that

$$\begin{cases} \bar{a}^i = f^i(k^1, \dots, k^r; a^1, \dots, a^r), \\ k^i = f^i(k^1, \dots, k^r; 0, \dots, 0), \qquad (i = 1, 2, \dots, r) \end{cases} \tag{8.25}$$

exist on one hand. $\bar{a}^i = \psi^i(a)$ are solutions of (8.24) for the initial conditions (8.23) so that, by the uniqueness of the solutions we have

$$\psi^i(a) = f^i(k; a), \qquad (i = 1, 2, \dots, r).$$

On the other hand

$$\bar{a}^i = \varphi^i(k^1, \dots, k^r; a^1, \dots, a^r)$$

are also the solutions of (8.24) for the same initial conditions by the First Theorem above. Therefore we must have

$$\varphi^i(k; a) = f^i(k; a) = \psi^i(a), \qquad (i = 1, 2, \dots, r). \tag{8.26}$$

9. Abstract Lie Ring. - In order to make the structure of the extended LIE groups clear, we give the definition of the abstract LIE ring.

DEFINITION. - A vector space R of rank r with

real | complex

coefficients is called an *abstract Lie ring,* when the following conditions (i) and (ii) are satisfied:

(i) For $A, B \in R$, a *commutator product* $(A, B) \in R$ is defined uniquely;

(ii) $(\lambda_1 A_1 + \lambda_2 A_2, B) = \lambda_1(A_1, B) + \lambda_2(A_2, B)$,

$$(A, B) = -(B, A), \tag{9.1}$$

$$((AB), C) + ((B, C), A) + ((C, A), B) = 0. \tag{9.2}$$

THEOREM. - *For given basès $E_1, E_2, \dots, E_r$ of a vector space, there exists r-dimensional abstract Lie ring R having the structure constants of an r-dimensional (extended) Lie group (germ) G as coefficients of*

$$(E_i, E_j) = C_{ij}^k E_k. \tag{9.3}$$

PROOF. - Since $E_1, E_2, \dots, E_r$ form a basis of a vector space, we may set (9.3). Then from (9.1) and (9.2), we obtain

$$\begin{cases} C_{ij}^k = -C_{ji}^k, \\ C_{ij}^h C_{hk}^l + C_{jk}^h C_{hi}^l + C_{ki}^h C_{hj}^l = 0. \end{cases} \tag{9.4}$$

Conversely, if (9.4) holds for certain r constants C_{jk}^i, we can determine, the basis $E_1, E_2, \dots, E_r$ so that the commutator product of them is (9.3) and introduce the definition

$$(\alpha^i E_i, \beta^j E_j) = \alpha^i \alpha^j (E_i, E^j),$$

then (9.1) and (9.2) hold. Hence the theorem.

N. B. - When a property of an extended LIE group (germ) is given, we shall express it in terms of the corresponding abstract LIE ring.

10. Coordinate Transformation.

DEFINITION. - When the relations

$$\begin{cases} \bar{g}^i(\varphi(a;\ b)) = \bar{\varphi}^i(\bar{g}(a);\ \bar{g}(b)), \\ g^i(\bar{\varphi}(\bar{a};\ \bar{b})) = \varphi^i(g(\bar{a});\ g(\bar{b})), \end{cases} \qquad (i = 1, 2, \dots, r) \tag{10.1}$$

hold by a certain one-to-one transformation

$$(10.2)\quad \begin{cases} a^i = g^i(\bar{a}^1, \ldots, \bar{a}^r), & O = g^i(O, \ldots, O), \\ \bar{a}^i = \bar{g}^i(a^1, \ldots, a^r), & O = \bar{g}^i(O, \ldots, O), \qquad (i = 1, 2, \ldots, r), \end{cases}$$

$$g^i, \ \bar{g}^i \in C^1$$

between certain vicinities U, $\bar{U}$ of respective origin of two r-dimensional extended LIE group (germs) G and $\bar{G}$ hold, G and $\bar{G}$ are said to be *isomorphic* to each other. Thereby $\varphi(a; b)$ and $\bar{\varphi}(\bar{a}; \bar{b})$ are respective composition functions in G and $\bar{G}$.

The (10.2) may also be expressed as follows:

$$(10.3)\quad \begin{aligned} &\text{If } S_a \cdot S_b = S_c, \text{ then } \bar{S}_{\bar{g}(a)} \cdot \bar{S}_{\bar{g}(b)} = \bar{S}_{\bar{g}(c)}, \\ &\text{if } \bar{S}_{\bar{a}} \cdot \bar{S}_{\bar{b}} = \bar{S}_{\bar{c}}, \text{ then } S_{g(\bar{a})} \cdot S_{g(\bar{b})} = S_{g(\bar{c})}, \end{aligned}$$

$$(S_a, \ S_b, \ldots \in G, \ \ \bar{S}_{\bar{a}}, \ \bar{S}_{\bar{b}}, \ldots \in \bar{G}).$$

When g^i and $\bar{g}^i$ are, in particular, analytic functions, G and $\bar{G}$ are said to be *analytically isomorphic.*

If we transform the extended parameters $(a^1, \ldots, a^r)$ of an r-dimensional extended LIE group (germ) G into $(\bar{a}^1, \ldots, \bar{a}^r)$ by $\bar{g}^1, \ldots, g^r \in C^1$ such that

$$(10.4)\quad \bar{a}^i = g^i(a^1, \ldots, a^r), \ O = g^i(O, \ldots, O), \qquad (i = 1, 2, \ldots, r),$$

$$\frac{\partial(\bar{g}^1, \ldots, \bar{g}^r)}{\partial(a^1, \ldots, a^r)} \neq O,$$

then it results that

$$S_a = \bar{S}_{\bar{a}},$$

which is a special case $G = \bar{G}$ of the above definition for isomorphism. Thus a treatment of the isomorphism consequences a transformation of the extended parameters.

If G and $\bar{G}$ be isomorphic to each other, then introducing

$$d\bar{a}^i = d\bar{g}^i(a) = \frac{\partial \bar{g}^i}{\partial a^k} da^k$$

and

$$\left(\frac{\partial\bar{\varphi}^i(\bar{a};\bar{c})}{\partial\bar{c}^j}\right)_{\bar{c}=0}=\frac{\partial\bar{g}^i}{\partial a^k}\left(\frac{\partial\varphi^k(a;c)}{\partial c^l}\right)_{c=0}\left(\frac{\partial g^l(\bar{c})}{\partial\bar{c}^j}\right)_{\bar{c}=0}$$

obtained by differentiation of

$$\bar{\varphi}^i(\bar{a};\ \bar{c})=\bar{\varphi}^i(\bar{g}(a);\ \bar{g}(c))=g^i(\varphi(a;c)),$$

into

$$d\bar{a}^i=\left(\frac{\partial\bar{\varphi}^i(\bar{a};\ \bar{c})}{\partial\bar{c}^j}\right)_{\bar{c}=0}\bar{\omega}^j(\bar{a},d\bar{a})$$

and solving the resulting equations with respect to da^k, we obtain

$$da^k=\left(\frac{\partial\varphi^k(a;c)}{\partial c^l}\right)_{c=0}\left(\frac{\partial g^l(\bar{c})}{\partial\bar{c}^j}\right)_{\bar{c}=0}\bar{\omega}^j(\bar{a},d\bar{a}).$$

Comparing this with the fundamental Pfaffians $\omega^j(a,da)$, we obtain

(10.5) $$\omega^i(a,da)=h^i_j\bar{\omega}^j(\bar{a},d\bar{a}),$$

where

(10.6) $$h^i_j=\left(\frac{\partial g^i(\bar{c})}{\partial\bar{c}^j}\right)_{\bar{c}=0}.$$

Thus the fundamental Pfaffians undergo a linear transformation with constant coefficients.

We introduce this into

$$\mathbf{d}\omega^i=\frac{1}{2}\ C^i_{kl}\omega^k\wedge\omega^l.$$

Then it results that

$$\mathbf{d}(h^i_j\bar{\omega}^j)=\frac{1}{2}\ C^i_{kl}h^k_p h^l_q\bar{\omega}^p\wedge\bar{\omega}^q.$$

Set

$$|\ \bar{h}^i_j\ |=|\ h^i_j|^{-1},\quad\left(\bar{h}^i_j=\left(\frac{\partial\bar{g}^i(c)}{\partial c^j}\right)_{c=0}\right).$$

Then we have

$$\mathbf{d}\bar{\omega}^j = \frac{1}{2}\, C^i_{kl}\bar{h}^j_i h^k_p h^l_q \bar{\omega}^p \wedge \bar{\omega}^q.$$

Comparing this with

$$\mathbf{d}\bar{\omega}^j = \frac{1}{2}\, \bar{C}^j_{pq}\bar{\omega}^p \wedge \bar{\omega}^q,$$

we see that

$$\bar{C}^j_{pq} = (\bar{h}^j_i h^k_p h^l_q) C^i_{kl}. \tag{10.7}$$

Taking this result with the converse, we shall prove the following theorem.

THEOREM. - *The necessary and sufficient condition for that two r-dimensional (extended) Lie group (germs) G and $\bar{G}$ may be isomorphic to each other, is that the structure constants of G and $\bar{G}$ are transformed by matrix* (10.7), *where* (h^i_j) *is a matrix of constants such that* $|h^i_j| \neq 0$ *and* $(\bar{h}^j_i)$ *its reciprocal matrix.*

PROOF - Setting

$$\bar{\theta}^i(\bar{a}, d\bar{a}) = h^i_j \bar{\omega}^j(\bar{a}, d\bar{a}), \tag{10.8}$$

$$\mathbf{d}\theta^i = \frac{1}{2}\, C^i_{jk}\bar{\theta}^j \wedge \bar{\theta}^k$$

as in the case of $\mathbf{d}\bar{\omega}^j$ above. Hence

$$\bar{\theta}^i(\bar{a}, d\bar{a}) - \omega^i(a, da) = 0, \qquad (i = 1, 2, \dots, r)$$

is completely integrable as in the case of (8.24) and the solution may be given by

$$\bar{a}^i = \bar{g}^i(a^1, \dots, a^r), \quad O = \bar{g}^i(O, O, \dots, O), \qquad (i = 1, 2, \dots, r).$$

Since these are one and the same integral, we must have

$$g^i(\bar{g}^i(a)) = a^i, \quad \bar{g}^i(g(\bar{a})) = \bar{a}^i, \qquad (i = 1, 2, \dots, r), \tag{10.9}$$

$$\begin{cases} \omega^i(g(\bar{a}), dg(\bar{a})) = \bar{\theta}^i(\bar{a}, d\bar{a}), \\ \theta^i(\bar{g}(a), d\bar{g}(a)) = \omega^i(a, da), \qquad (i = 1, 2, \dots, r). \end{cases}$$

Now the composition functions $\bar{\varphi}(\bar{a};\bar{b})$ of $\bar{G}$ makes $\bar{\omega}^1, \dots, \bar{\omega}^r$ invariant for $\bar{a} \to \bar{\varphi}(\bar{k};\bar{a})$ and consequently it makes also their linear combinations $\bar{\theta}^1, \dots, \bar{\theta}^r$ invariant. Hence, for the transformation

$$a^i \to \bar{g}^i(a) \to \bar{\varphi}^i(\bar{g}(k);\bar{g}(k)) \to g^i(\bar{\varphi}(\bar{g}(k);g(a)), \qquad (i=1,2,\dots,r).$$

we obtain

$$\omega(a,da) \to \bar{\theta}(\bar{a},d\bar{a}) \to \bar{\theta}(\bar{a},d\bar{a}) \to \omega(a,da)$$

together with

$$O \to \bar{\varphi}^i(O) = O \to \bar{\varphi}^i(\bar{g}(k);O) = \bar{g}^i(k) - g^i(\bar{g}(k)) = k^i, \qquad (i=1,2,\dots,r)$$

in particular.

Now by the Theorem concerning (8.22), we must have

$$g^i(\bar{\varphi}(\bar{g}(k);\ \bar{g}(a)) = \varphi^i(k;\ a), \qquad (i=1,2,\dots,r)$$

i. e.

$$\bar{\varphi}^i(\bar{g}(k);\ \bar{g}(a)) = \bar{g}^i(\varphi(k;\ a)), \qquad (i=1,2,\dots,r)$$

by (10.9).

A similar result will be obtained, when we interchange the situations of G and $\bar{G}$.

Taking these two results together, we arrive at (10.1).

If hereby $\varphi^i,\ \bar{\varphi}^i \in C^3$, then $\omega^i,\ \bar{\omega}^i, \bar{\theta}^i \in A(C^2)$ and so we see that

$$g^i,\ \bar{g}^i \in C^2. \qquad \text{Q. E. D.}$$

Restating the last Theorem in terms of the abstract LIE ring, we obtain the following theorem.

THEOREM. - *In order that two r-dimensional (extended) Lie group (germs) G and $\bar{G}$ may be isomorphic to each other, it is necessary and sufficient that the corresponding abstract Lie rings R and $\bar{R}$ become ring-isomorphic by an appropriate linear transformation between their bases, that is to say, that to $A \in R$ there corresponds $f(A) = \bar{A} \in \bar{R}$ uniquely and that the relations*

$$\begin{cases} f(\lambda A_1 + \mu A_2) = \bar{\lambda} f(A_1) + \mu f(A_2), \\ f((A,\ B)) = (f(A),\ f(B)) \end{cases}$$

hold, the linear transformation being

$$f(E_i) = h_i^j \bar{E}_j, \qquad (i = 1, 2, \dots, r).$$

11. Inner Automorphic Transformations.

DEFINITION. - The isomorphism $G \longrightarrow G$ of the type

$$S_{\bar{a}} \longrightarrow S_{\bar{a}} = S_x S_a S_x^{-1}, \ (S_x \in G) \tag{11.1}$$

is called an *inner automorphism* of G.

The transformation

$$\bar{a}^i = \bar{g}^i(a), \qquad (i = 1, 2, \dots, r)$$

transforms a vicinity of the origin into a vicinity of the origin in one-to-one manner and since $\bar{g}^i \in C^3$, the first theorem of Art. 10 applies, so that we have

$$\bar{\omega}^i(\bar{a}, d\bar{a}) = h_k^i(x)\omega^k(a, da), \qquad (i = 1, 2, \dots, r), \tag{11.2}$$

where the matrix $(h_k^i(x))$ is obtained as follows. Since from (11.1) follows:

$$S_{\bar{a}} S_x = S_x S_a,$$

the relation

$$\varphi^i(\bar{a}; x) = \varphi^i(x; a), \qquad (i = 2, \dots, r)$$

holds and consequently

$$\left(\frac{\partial \varphi^k(\bar{a}; x)}{\partial \bar{a}^l}\right)_{\bar{a}=0} \left(\frac{\partial \bar{g}^l(a)}{\partial a^j}\right)_{a=0} = \left(\frac{\partial \varphi^k(x; a)}{\partial a^j}\right)_{a=0}. \tag{11.3}$$

We set

$$\alpha_j^{*k}(a) = \left(\frac{\partial \varphi^k(c; a)}{\partial c^j}\right)_{c=0}, \ \alpha_j^{*k}(a)\beta_k^{*i}(a) = \delta_j^i \tag{11.4}$$

according to (8.3) and (8.5) and multiply (11.3) with β_k^{*i}, then it results that

$$h_j^i(x) = \left(\frac{\partial \bar{g}^i(a)}{\partial a^j}\right)_{a=0} = \alpha_j^k(x)\beta_k^{*i}(x).$$

Next, for

$$S_{\bar{\bar{a}}} = S_y S_{\bar{a}} S_y^{-1} = (S_y S_x) S_a (S_y S_x)^{-1},$$

we have

$$\bar{\bar{\omega}}^i(\bar{\bar{a}}, d\bar{\bar{a}}) = h^i_j(y)\bar{\omega}^j(\bar{a}, d\bar{a}) = h^i_j(y)h^j_k(x)\omega^k(a, da),$$

whence follows:

$$h^i_k(\varphi(x;\ y)) = h^j_k(x)h^i_j(y). \tag{11.5}$$

Thus, if we set

$$H(S_x) \overset{\text{def}}{=} |\ h^j_k(x)\ |,$$

from (11.5), we obtain

$$H(S_x \cdot S_y) = H(S_x) \cdot H(S_y). \tag{11.6}$$

This tells us that the set

$$\{(h^j_k(x));\ \ x \in U_0\} \tag{11.7}$$

forms a group (germ), which is *homomorphic* to the r-dimensional extended LIE group (germ) G.

DEFINITION. - We call (11.7) the *adjoint extended group* of G.

N. B. - *The adjoint extended group is an extended Lie group (germ).*

12. Existence Conditions and Canonical Parameter.

DEFINITION. - An r-dimensional group (germ) is said *to have a canonical parameter*, when the following two conditions are satisfied:

(i) it is an extended analytic LIE group (germ) i.e. $\varphi^i(a;\ b)$ are analytic functions of a and b; (ii) for sufficiently small real values of s and t, the relation

$$a^i(s+t) = \varphi^i(a^1s,\ \dots,\ a^rs;\ a^1t,\ \dots,\ a^rt), \qquad (i = 1,\ 2,\ \dots,\ r) \tag{12.1}$$

in $a \in U_1$, i.e.

$$S_a\colon\ a^i = a^i_0 t,\ \ |\ t\ | < \varepsilon, \qquad (i = 1,\ 2,\ \dots,\ r) \tag{12.2}$$

forms a one-dimensional extended subgroup (germ). The (12.2) is called a *one-parametric extended subgroup (germ).*

THEOREM 1°. - *It is possible to make any (extended) Lie group (germ) G have a normal parameter by an appropriate change of parameter, retaining the structure constants.*

This theorem implies also that there exist an analytic (extended) LIE group (germ) $\overline{G}$ having the structure constants with an arbitrary given extended LIE group (germ) G in common and the G and the $\overline{G}$ being isomorphic to each other.

This theorem is an immediate consequence of the following existence theorem having a stronger content.

THEOREM 2°. - *If r^3 constants*

$$C^i_{jk}, \qquad (i, j, k = 1, 2, \ldots, r) \tag{12.3}$$

have the properties (8.15) *and* (8.16), *there exists an r-dimensional* (*extended*) *Lie group* (*gevm*) $\overline{G}$ *having the canonical parameter and the* (12.3) *as structure constants.*

For, if we form an r-dimensional extended LIE group (germ) of canonical parameter having the structure constants C^i_{jk} of the given r-dimensional LIE group (germ) as structure constants, the G and the $\overline{G}$ are isomorphic to each other by the first theorem of Art. 10.

N. B. - The Theorem 2° shows us the complete correspondence between an r-dimensional LIE group (germ) and an abstract LIE ring of rank r. Thus taking the first theorem of Art. 10 together, we have the

THEOREM 3°. - *There exists an r-dimensional extended Lie group* (*germ*) *corresponding to an arbitrary given abstract Lie ring of rank r. Consequently a class of mutually isomorphic r-dimensional extended Lie group* (*s*) (*germs*) *and a class of mutually ring-isomorphic extended abstract Lie ring of rank r have one-to-one correspondence.*

Let us now prove Theorem 3° in three steps I, II, III.

I. If analytic functions $b^i_j(a)$ such that for constants C^i_{jk} the relations

$$d\omega^i = \frac{1}{2} C^i_{jk} \omega^j \wedge \omega^k, \tag{12.4}$$

$$\left\{ \begin{array}{ll} \omega^i = b^i_j(a) da^j, & (i = 1, 2, \ldots, r). \\ \delta^i = b^i_j\,(O, \ldots, O), & \end{array} \right. \tag{12.5}$$

hold, then there exists an r-dimensional analytic (extended) LIE group (germ) G, for whose composition functions φ the relation

$$(b^i_j(a))^{-1} = \left(\left[\frac{\partial \varphi^i(a; c)}{\partial c^j} \right]_{c=O} \right) \tag{12.6}$$

holds, so that the C^i_{jk} become the structure constants for this G.

PROOF. (i) The simultaneous total differential equations

(12.7) $$\bar{\omega}^i - \omega^i = 0, \qquad (i = 1, 2, \dots, r)$$

for $2r$ independent variables $a^1, \dots, a^r; \bar{a}^1, \dots, \bar{a}^r$ formed after (12.5) as in the case of (8.24) are completely integrable.

Taking their solutions such that

(12.8) $$\begin{cases} \bar{a}^i = \varphi(k^1, \dots, k^r;\ a^1, \dots, a^r), & (i = 1, 2, \dots, r). \\ k^i = \varphi^i(k^1, \dots, k^r;\ O, \dots, O), & (i = 1, 2, \dots r). \end{cases}$$

we define the product

$$S_a \cdot S_b = S_c, \qquad (c_0^i = \varphi^i(a;\ b)), \qquad (i = 1, 2, \dots, r)$$

for sufficiently small vicinity of the origin. Let us examine if an (extended) LIE group (germ) $\bar{G}$ is formed.

(ii) By (12.8), we have

$$\varphi^i(k; O) = k^i, \qquad (i = 1, 2, \dots, r).$$

It is further seen that

$$\varphi^i(O; a) = a^i, \qquad (i = 1, 2, \dots, r)$$

from the fact that both sides are solutions of (12.7) for the initial condition $\varphi^i(O;\ O) = 0$.

(iii) Since under the two transformations

$$a^i \longrightarrow \bar{a}^i = \varphi^i(l; a) \longrightarrow \bar{\bar{a}}^i = \varphi^i(k; \varphi(l; a)), \qquad (i = 1, 2, \dots, r),$$

the Pfaffians $\omega^1, \dots, \omega^r$ are invariant,

$$\bar{a}^i = \varphi^i(k; \varphi(l; a)), \qquad (i = 1, 2, \dots, r)$$

are solutions of (12.7) and satisfy

$$\varphi^i(k; \varphi(l; O)) = \varphi^i(k; l), \qquad (i = 1, 2, \dots, r).$$

Hence by the uniqueness of the solution, they coincide with $\varphi^i(\varphi(k; l); a)$ taking the same values in $a = O$:

$$\varphi^i(k; \varphi(l;\ a)) = \varphi^i(\varphi(k; l);\ a), \qquad (i = 1, 2, \dots, r).$$

Finally, comparing

$$\begin{pmatrix} d\bar{a}^1 \\ \vdots \\ d\bar{a}^r \end{pmatrix} = (b_k^i(\bar{a}))^{-1}(b_j^k(a)) \begin{pmatrix} da^1 \\ \vdots \\ da^r \end{pmatrix}$$

deduced from (12.7) with

$$d\bar{a}^i = \frac{\partial \varphi^i(k;\, a)}{\partial a^j}\, da^j$$

deduced from (12.8), we see that $\bar{a}^i = k^i$ on putting $a = O$, so that we obtain (12.6). Q. E. D.

II. Since the solutions $b^i_j(a)$ such that (12.4), (12.5) hold are determinable not uniquely, we shall solve the problem under an additional demand (12.11) below.

If we introduce (12.5) into (12.4), then it results that

$$\left(\frac{\partial b^i_k}{\partial a^l}\, da^l\right) \wedge da^k = \frac{1}{2}\, C^i_{pq} b^p_l b^q_k da^l \wedge da^k.$$

Comparing the coefficients of $da^k \wedge da^l$, $(k < l)$, we are led to solve

$$\text{(12.9)} \qquad \frac{\partial b^i_k}{\partial a^l} - \frac{\partial b^i_l}{\partial a^k} = C^i_{pq} b^p_l b^q_k, \qquad (i,\ l,\ k = 1,\ 2,\ \dots,\ r).$$

(These equations are called *Maurer-Cartan differential equations*).

Let us prove:

There exist analytic functions $b^i_j(a^1, \dots, a^r)$ satisfying the MAURER-CARTAN differential equations such that

$$\text{(12.10)} \qquad b^i_j(O, \dots, O) = \delta^i_j, \qquad (i,\ j = 1,\ 2,\ \dots,\ r),$$

$$\text{(12.11)} \qquad b^i_j(a) a^j = a^i.$$

PROOF [6]. - Before all we shall solve the simultaneous ordinary differential equations of the first order

$$\text{(12.12)} \qquad \frac{df^i_l}{dt} = \delta^i_j + C^i_{pq} a^p f^q_l, \qquad (i,\ l = 1,\ 2,\ \dots,\ r)$$

having $a^1, \dots, a^r$ as parameters, under the initial condition

$$\text{(12.13)} \qquad f^i_l = 0, \quad \text{in} \quad t = 0.$$

Their solutions

$$\text{(12.14)} \qquad f^i_l(a^1, \dots, a^r;\, t),$$

(6) Substantially due to F. Schur. Another substantial solution will be found in: J. H. C. Whitehead, Note on Maurer's equations. Jour. of London Math. Soc., 7(1932).

are analytic functions of $a^1, \ldots, a^r$ and t. Setting

$$b^i_j(a^1, \ldots, a^r) = f^i_j(a^1, \ldots, a^r;\ 1),$$

we shall see that (12.9) holds. For it, we set

$$F^i_{kl} = \frac{\partial f^i_k}{\partial a^l} - \frac{\partial f^i_l}{\partial a^k} - C^i_{pq} f^p_l f^q_k, \qquad (i, k, l = 1, 2, \ldots, r). \tag{12.15}$$

Since

$$f^p_l = f^q_k = 0, \frac{\partial f^i_l}{\partial a^k} = \frac{\partial f^i_k}{\partial a^l} = 0$$

for $t = 0$, we have $F^i_{kl} = 0$ for $t = 0$.

If we could show

$$\frac{dF^i_{lk}}{dt} = C^i_{pz} a^p F^z_{lk}, \qquad (i, k, l = 1, 2, \ldots, r), \tag{12.16}$$

by virtue of $F^i_{lk}(0), = 0$, it would follow that

$$F^i_{lk} \equiv 0,$$

so that (12 9) holds. Hence we shall examine (12.16).

$$\begin{aligned} \frac{dF^i_{lk}}{dt} &= -\frac{\partial}{\partial a^k}(\delta^i_l - C^i_{pz} a^z f^p_l) + \frac{\partial}{\partial a^l}(\delta^i_k - C^i_{pz} a^z f^p_k) \\ &\quad - C^i_{pq} f^p_l (\delta^q_k - C^q_{xz} a^z f^x_k) - C^i_{pq} f^p_k (\delta^p_l - C^p_{xz} a^z f^x_l) \\ &= C^i_{pk} f^p_l - C^i_{pl} f^p_k + C^i_{pz} a^z \left(\frac{\partial f^p_l}{\partial a^k} - \frac{\partial f^p_k}{\partial a^l} \right) \\ &\quad - C^i_{pk} f^p_l - C^i_{lq} f^q_k + C^i_{pq} C^q_{xz} a^z f^p_l f^x_k + C^i_{pq} C^p_{xz} a^z f^q_k f^x_l. \end{aligned}$$

If we introduce

$$\frac{\partial f^p_l}{\partial a^k} - \frac{\partial f^p_k}{\partial a^l} = -F^p_{lk} - C^p_{xy} f^x_l f^y_k,$$

obtained from (12.15) into the last equation, then it follows that

$$\frac{dF^i_{lk}}{dt} = -C^i_{pz} a^z F^p_{lk} - C^i_{pz} C^p_{xy} f^x_l f^y_k a^z + C^i_{pq} C^q_{xz} f^p_l f^x_k a^z + C^i_{pq} C^p_{xz} f^q_k f^x_l a^z.$$

Replacing the indices (x, p, q) by (y, x, p), (x, p, y) respectively and utilizing (8.15) and (8.16), we obtain

$$\frac{dF^i_{lk}}{dt} = - C^i_{pz}a^z F^p_{lk} - (C^p_{xy}C^i_{pz} + C^p_{yz}C^i_{px} + C^p_{zx}C^i_{py}) f^x_l f^y_k a^z$$

$$= - C^i_{pz}a^z F^p_{lk} .$$

In a similar way, for

$$G^i(t) = f^i_j(a, t)a^j - ta^i, \tag{12.17}$$

we have $G^i(O) = 0$. For (12.17), we examine

$$\frac{dG^i}{dt} = C^i_{pj}a^p G^j . \tag{12.18}$$

We see that $G^i(t) = 0$ and in particular, $G^i(1) = 0$. Now

$$\frac{dG^i}{dt} = (\delta^i_j + C^i_{pq}a^p f^q_j)a^j - a^i .$$

$$= C^i_{pq}a^p a^j f^q_j = C^i_{pq}a^p (f^q_j a^j - ta^q),$$

since $C^i_{pp} = 0$, $C^i_{pq} = - C^i_{qp}$, $(p > q)$, so that (12.18) is regitimate (7). That (12.10) holds, follows from the fact that the solutions of (12.12) for $a^1 = \dots = a^r = 0$ becomes $f^i_l = \delta^i_l t$.

III. Lastly, we shall prove that when (12.11) holds, the (extended) LIE group (germ) obtained under I is of the canonical parameter.

By (12.6), for the $\bar{G}$ obtained under I the relation $b^i_j(a) = \beta^i_j(a)$ holds for the $\beta^i_j(a)$ in (8.5). Hence by (12.11) we have

$$\alpha^i_j(a)a^j = a^i \tag{12.19}$$

also.

Next we shall prove that

$$c^i = a^i_0(s + t), \qquad (i = 1, 2, \dots, r) \tag{12.20}$$

(7) The reason why we considered (12.12) consists in that when conversely (12.9) and (12.11) hold, it is easily seen that $f^i_j(t) = tb^i_j(t)$ satisfies (12.12). Cf. Pontrijagin, [16], p. 253.

for

(12.21) $$a^i = a_0^i s, \quad b^i = b_0^i t, \qquad (i = 1, 2, \ldots, r).$$

Consider

$$c^i = \varphi^i(as, at) = c^i(t)$$

fixing s for a while. Then for (12.21), we have

(12.22) $$\frac{dc^i}{dt} = \frac{\partial \varphi^i(a; b)}{\partial b^j} \frac{db^j}{dt} = \frac{\partial \varphi^i}{\partial b^j} a_0^j.$$

Now we introduce

(12.23) $$\alpha_j^i(a_0 t) a_0^j = \frac{1}{t} \alpha_j^i(a_0 t) a_0^j t = a_0^i$$

obained from (12.19) into (12.22), it results that

$$\frac{dc^i}{dt} = \frac{\partial \varphi^i}{\partial b^j} \alpha_k^j(b) a_0^k.$$

Utilizing (8.13) herein, we obtain

(12.24) $$\frac{dc^i}{dt} = \alpha_k^i(c) a_0^k.$$

The solution of (12.24) such that $c^i(O) = c_0^i s$ for $t = 0$ is, by (12.23) and (3.1):

$$c^i(t) = a_0^i(s + t).$$

Thus (12.20) is proved.

N. B. - It is easily seen that conversely the (12.19) holds for the canonical parameter.

13. - Reciprocal Isomorphism.

THEOREM - *If two r-dimensional (extended) Lie groups (germs) G and G^* be reciprocally isomorphic* (cf. [24], ... [31]), *then their structure constants c_{jk}^i and c_{jk}^{*i} are related to each other by*

(13.1) $$c_{jk}^i = - c_{jk}^{*i}, \qquad (i, j, k = 1, 2, \ldots, r).$$

PROOF. - Consider the Pfaffians

$$\omega^{*i}(a, da) = \beta^{i}_{j}(a) da^{j}, \tag{13.2}$$

where (cf. [24], ..., [31])

$$\alpha^{*i}_{j}(a) = \left(\frac{\partial \varphi^{i}(c; a)}{\partial c^{j}}\right)_{c=0}, \tag{13.3}$$

$$\alpha^{*i}_{k}(a)\beta^{*k}_{j}(a) = \delta^{i}_{j}, \qquad \alpha^{*k}_{j}(a)\beta^{*i}_{k}(a) = \delta^{i}_{j}. \tag{13.4}$$

Then as in the case of the Third Theorem of Art. 8, the transformation under which $\omega^{*1}, \ldots, \omega^{*r}$ are invariant, is

$$a^{i} \longrightarrow \bar{a}^{i} = \varphi^{i}(a; k), \qquad (i = 1, 2, \ldots, r). \tag{13.5}$$

$\mathbf{d}\omega^{*i}$ is expressible in the form

$$\mathbf{d}\omega^{*i} = c^{*i}_{jk}\omega^{*j} \wedge \omega^{*k}. \tag{13.6}$$

We consider the expansions analogous to those in Art. 8:

$$\begin{cases} \alpha^{*i}_{j}(b) = \delta^{i}_{j} + d^{i}_{jk}b^{k} + \varepsilon^{*2}, \\ \beta^{*i}_{j}(b) = \delta^{i}_{j} - d^{i}_{jk}b^{k} + \varepsilon^{*3}, \\ \omega^{*i}(b, db) = db^{i} - d^{i}_{jk}b^{k}db^{j} + \varepsilon^{*i}_{j}db^{j}, \end{cases}$$

whence we have

$$C^{*i}_{jk} = d^{i}_{kj} - d^{i}_{jk} = - C^{i}_{jk} \tag{13.7}$$

quite as in the case of (8.20).

Consider the totality G^{*} of

$$T_{x} \overset{\text{def}}{=} S_{x}^{-1}, \qquad (S_{x} \in G).$$

Then we have

$$T_{x}T_{y} = (S_{y}S_{x})^{-1},$$

so that G and G^{*} become reciprocally isomorphic. If we set

$$T_{x}T_{y} = T_{z}, \quad z^{i} = \varphi^{*i}(x; y), \qquad (i = 1, 2, \ldots, r), \tag{13.8}$$

then it follows that

$$\varphi^{*i}(x;y) = \varphi^i(y;x). \tag{13.9}$$

Hence the theorem.

§ 3. - Extended Lie Transformation Groups.

14. The Lie Transformation Group Germ. - Let G be an r-dimensional LIE group germ; Let D_0 be a vicinity of a point (x_0) of an n-dimensional Euclidean space E^n taken merely auxiliarily.

(i) Let

$$x'^i = f^i(x^1, \dots, x^n; a^1, a^2, \dots, a^r), \qquad (i = 1, 2, \dots, n), \tag{14.1}$$

be a one-to-one transformation T_a mapping a vicinity $D_1 \subset D_0$ of (x_0) into D_0:

$$x' \in D_0, \ f^i(x; a) \in C^3, \qquad (i = 1, 2, .., n).$$

(ii)
$$x'^i = f^i(x; a) = x^i, \qquad (i = 1, 2, \dots, n)$$

is the *unit transformation*. (It is convenient to write

(ii')
$$x'^i = f^i(x; a) = x^i, \qquad (i = 1, 2, \dots, n)$$

in place of (ii)

(iii) If $S_a \cdot S_b = S_c$ in G, we have

$$f^i(f^1(x;a), \dots, f^n(x;a); b^1, b^2, \dots, b^r) \equiv f^i(x^1, \dots, x^n; c^1, c^2, \dots, c^r), \tag{14.2}$$

where

$$c^k = \varphi^k(a^1, a^2, \dots, a^r; b^1, b^2, \dots, b^r), \qquad (k = 1, 2, \dots, r). \tag{14.3}$$

The G will be called thereby the *parameter group germ* of $T = (T_a)$.

When the function $f^i(x;a)$ of (14.1) are regular analytic functions of x and a for the analytic LIE group germ G, the T is called the *analytic Lie transformation group germ.*

15. Extended Lie Transformation Group in the Large. - The element

(15.1) $x = (x^1, x^2, \dots, x^n)$ | $a = (a^1, a^2, \dots, a^r)$

of the

base manifold M | LIE group germ G

admits of being made *global* by the principle stated in Art. 6, so that we have

(15.2) $x^i = \xi^i, \quad (i = 1, 2, \dots, n),$ | $a^l = \eta^l, \quad (l = 1, 2, \dots, r),$

where the

ξ^i | η^l

are II-geodesic parallel coordinates in the global

base manifold M. | LIE group space G.

Hereafter, we assume *the*

x^i | a^l

themselves to be the global ones:

ξ^i, | η^l,

and extend the LIE transformation group to the case that a^l *are functions of* x:

$$a^l = a^l(x). \tag{15.3}$$

Thus we obtain an *extended Lie transformation group G.*

A concrete example will be found in the case, where

$$a = (a^l_m(\xi^r)), \qquad (r = n^2)$$

in the sense of the right-hand side of Art. 6.

If we interpret

$$\frac{\partial f^i(x;\, a(x))}{\partial x^j} = \alpha^i_j(x) \quad \text{as} \quad a^i_j(x^p), \qquad (r = n^2),$$

then, *for the general* $a^l(x)$ *we obtain* $a^i_j(x^p)$ *correspondingly and the results for the right-hand side of Art.* 6 *applies to the case of general* $a^l(x)$.

In the following articles, the following Fundamental Theorem will be established.

FUNDAMENTAL THEOREM. - *For the extended Lie transformation groups, the theory* (Art. 16-17) *of the ordinary Lie transformation groups applies*

16. The Fundamental Theorems. - We set

$$\xi^i_j(x^1, x^2, \dots, x^n) = \left(\frac{\partial f^i(x;\ a(x))}{\partial a^j}\right)_{a=0}, \qquad (i = 1, 2, \dots, n;\ j = 1, 2, \dots, r) \tag{16.1}$$

and

$$\omega^l(a(x),\ da(x)) = \beta^l_j(a(x))da^j(x), \qquad (l = 1, 2, \dots, r) \tag{16.2}$$

as before. Further we set

$$\theta^i = dx^i + \omega^l(a(x), da(x))\xi^i_l(x), \qquad (i = 1, 2, \dots, n). \tag{16.3}$$

THEOREM 1°. - *The simultaneous total differential equations*

$$\theta^1 = 0,\ \ \theta^2 = 0,\ \dots,\ \theta^n = 0 \tag{16.4}$$

are completely integrable and

$$f^1(x; a(x)),\ f^2(x; a(x)),\ \dots,\ f^n(x; a(x)) \tag{16.5}$$

are n *independent first integrals of* (16.4) *such that*

$$f^i(x; O) = x^i, \qquad (i = 1, 2, \dots, n),$$

so that

$$(\theta^1, \theta^2, \dots, \theta^n) = (df^1(x; a(x)), \dots,\ df^n(x; a(x))) \tag{16.6}$$

for the ideals.

PROOF. - We differentiate (14.2):

$$f^i(f^1(x; b), \dots, f^n(x; b);\ a^1, \dots, a^r) \tag{16.7}$$

$$= f^i(x^1, \dots x^n;\ \varphi^1(a; b), \dots, \varphi^r(a; b)), \qquad (i = 1, 2, \dots, n)$$

with respect to b and set $b=O$. Then it follows that

$$\frac{\partial f^i}{\partial x^k}\xi_l^k=\frac{\partial f^i}{\partial a^j}\alpha_l^j. \tag{16.8}$$

From (8.7) and (16.3), we obtain

$$\begin{aligned} df^i(x;\ a) &= \frac{\partial f^i}{\partial x^k}dx^k+\frac{\partial f^i}{\partial a^j}da^j \\ &= \frac{\partial f^i}{\partial x^k}(\theta^k-\omega^l\xi_l^k)+\frac{\partial f^i}{\partial a^j}(\alpha_l^j\omega^l) \\ &\equiv \frac{\partial f^i}{\partial x^k}\theta^k-\left\{\frac{\partial f^i}{\partial x^k}\xi_j^k-\frac{\partial f^i}{\partial a^j}\alpha_l^j\right\}\omega^l \\ &= \frac{\partial f^i}{\partial x^k}\theta^k\in(\theta^1,\ \dots,\ \theta^n). \end{aligned}$$

Since $df^1(x;a)$, ..., $df^n(x;a)$ are linearly independent, the (16.6) holds. Q.E.D.

The converse of the Theorem 1° holds as follows.

THEOREM 2°. - *When we introduce*

$$\xi_j^i(x)\in C^2, \qquad (i=1,\ 2,\ \dots\ n;\ j=1,\ 2,\ \dots,\ r)$$

appropriately for the fundamental Pfaffians ω^1, ..., ω^r *of an* r*-dimensional extended Lie group (germ)* G *and the simultaneous equations*

$$\theta^1=0,\ \ \theta^2=0,\ \ \dots,\ \ \theta^n=0 \tag{16.9}$$

are completely integrable, the n *independent first integrals* f^1, ..., f^r *such that*

$$f^i(x;O)=x^i, \qquad (i=1,\ 2,\ \dots,\ r) \tag{16.10}$$

determine an n*-dimensional extended Lie transformation group (germ) and the given* $\xi_j^i(x)$ *satisfy* (16.10).

PROOF. - If (16.9) be completely integrable, then there exist n first integrals f^1, f^2, ..., f^n satisfying (16.10). It suffices to show that these satisfy (16.7). Since

$$\bar{a}^i(x)=\varphi^i(k(x);a(x)), \qquad (i=1,\ 2,\ \dots,\ r)$$

satisfies

$$\omega^i(\bar{a}(x), d\bar{a}(x)) = \omega^i(a(x), da(x)), \qquad (i = 1, 2, \dots, r),$$

the functions

$$f^i(x; \bar{a}(x)) = f^i(x; \varphi(k(x); a(x))), \qquad (i = 1, 2, \dots, n) \tag{16.11}$$

satisfy

$$\bar{\theta}^i = dx^i + \bar{\omega}^j \xi_j^i = dx^i + \omega^j \xi_j^i = \theta^i, \qquad (i = 1, 2, \dots, n), \tag{16.12}$$

i. e. (16.11) become the first integrals of (16.9). Since (16.11) implies

$$f^i(x; \varphi(k(x); O)) = f^i(x; k(x)),$$

they take values for $a = O$ with the integrals $f^i(f(x; a(x)); a(x))$ of (16.9) in common. Hence we must have

$$f^i(x; \varphi(kx); a(x))) = f^i(f(x; k(x)); a(x)), \qquad (i = 1, 2, \dots, n).$$

Since thereby $df^i \in (\theta^1, \dots, \theta^n)$, pursueing the process of proof for Theorem 1° reversedly, we see that (16.8) must hold. If we set $a = O$ in (16.8), then we obtain (16.1), since

$$\alpha_l^j = \delta_l^j, \quad \frac{\partial f^i}{\partial x^j} = \delta_j^i, \text{ Q. E. D.}$$

The first Fundamental Theorem of the extended LIE transformation group (germ) below makes a liaison between the property of the extended LIE transformation group germ and the fundamental differential operators. In order to prove it, we shall try to replace the above properties with those of the simultaneous linear partial differential equations of the first order by virtue of the following Lemma.

LEMMA. – That the simultaneous total differential equations

$$\omega^i = a_l^i(x) dx^l = 0, \qquad (i = 1, 2, \dots, n) \tag{16.10}$$

are completely integrable is equivalent to that the simultaneous linear partial differential equations of the first order

$$X_{r+1} f = 0, \ \dots, \ X_{r+s} f = 0, \qquad (n = r + s) \tag{16.11}$$

are completely integrable. The first integrals are thereby common to (16.10) and (16.11). Thereby we have put

$$X_i f = b_i^l(x) \frac{\partial f}{\partial x^l}, \qquad (i,\ l = 1,\ 2,\ \dots,\ n), \tag{16.12}$$

where $(b_i^l(x))$ is the inverse transformation of $(a_l^i(x))$.

THE FIRST FUNDAMENTAL THEOREM. - *In an extended Lie transformation group (germ) G having G as extended parameter group (germ), the functions*

$$f^k(x;\ a(x)), \qquad (k = 1,\ 2,\ \dots,\ n)$$

are n independent solutions of the completely integrable simultaneous linear partial differential equations

$$\frac{\partial f}{\partial a^l} = \xi_j^k(x)\beta_l^j(a(x)) \frac{\partial f}{\partial x^k}, \qquad (k = 1,\ 2,\ \dots,\ r) \tag{16.13}$$

such that

$$x^k = f^k(x; O). \tag{16.14}$$

Conversely, when an r-dimensional extended Lie group (germ) G is given, the (16.13) *are completely integrabte for certain*

$$\xi_j^i(x) \in C^2, \quad (i = 1,\ 2,\ \dots, n;\ j = 1,\ 2,\ \dots,\ r),$$

the solutions

$$f^1(x; a(x)),\ f^2(x; a(x)),\ \dots,\ f^n(x; a(x))$$

satisfying (16.14) *determine an extended Lie transformation group (germ) having G as extended parameter group (germ).*

PROOF. - We consider two r-dimensional square matrices A and B defined by

$$A = (\alpha_k^i(a(x))),\ \ B = (\beta_k^i(a(x))),\ \ AB = BA = (\delta_k^i),$$

having defined $\alpha_k^i(a(x))$ and $\beta_k^i(a(x))$ by (8.3) and (8.5). Then

$$\begin{cases} \theta^1 = dx^1 + \{ \beta_k^i(a(x))\xi_j^1(x) \}\, da^k(x), \\ \dots\dots\dots\dots \\ \theta^n = dx^n + \{ \beta_k^i(a(x))\xi_j^n(x) \}\, da^k(x), \end{cases}$$

and

$$\begin{cases} \theta^{n+1} = \beta^1_k(a(x))da^k(x), \\ \dots\dots\dots\dots \\ \theta^{n+r} = \beta^r_k(a(x))da^k(x) \end{cases}$$

are linearly independent and the determinant D of their coefficients may be expressed as follows:

$$D = \begin{pmatrix} E_n & \Xi' B \\ 0 & B \end{pmatrix},$$

where

$$E_n = \begin{vmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 1 \end{vmatrix}$$

is the unit determinant of *n-th* order and Ξ' the determinant obtained from $|\xi^i_j(x)|$ by interchanging the rows with columns. The reciprocal determinant of D is

$$D^{-1} = \begin{pmatrix} E_n & -\Xi' \\ 0 & A \end{pmatrix}.$$

We set

$$(16.15) \qquad A_l f = \left(\alpha^j_l(a(x)) \frac{\partial}{\partial a^j}\right) f, \qquad (l = 1, 2, \dots, r),$$

$$(16.16) \qquad X_j f = \left(\xi^k_j(x) \frac{\partial}{\partial x}\right) f, \qquad (j = 1, 2, \dots, r).$$

By the above Lemma, when the simultaneous total differential equations

$$(16.17) \qquad \theta^1 = 0, \quad \theta^2 = 0, \dots, \quad \theta^n = 0$$

are completely integrable, the simultaneous linear partial differential equations

$$(16.18) \qquad \bar{X}_1 f = 0, \ \dots, \ \bar{X}_r f = 0,$$

where

$$\overline{X}_j = -\xi_j^k(x)\frac{\partial}{\partial x^k} + \alpha_j^l(a(x))\frac{\partial}{\partial a^l} \tag{16.19}$$

$$= -X_j + A_j, \qquad (j = 1, 2, \dots, r),$$

are also completely integrable, the first integrals of (16·9) = (16·17) coincide with the solutions of (16.18).

Now (16.18) and the simultaneous linear partial differential equations

$$Y_1 f = 0, \dots, Y_r f = 0, \tag{16.20}$$

where

$$Y_j = \beta_j^l(a(x))\overline{X}_l = \frac{\partial}{\partial a^j} - \xi_l^i(x)\beta_j^l(a(x))\frac{\partial}{\partial x^i}, \quad (j, l = 1, 2, \dots, r), \tag{16.21}$$

are equivalent.

Hence the Theorems 1° and 2° may be restated in the form of our First Fundamental Theorem.

N. B. - Our Fundamental Theorem is often stated in the following forms Cor. 1° and Cor. 2°.

COR. 1°. - (An Extension of the Lie's First Fundamental Theorem.) *In the extended Lie transformation group (germ) having G as extended parameter group (germ), the* $f^k(x; a(x))$, $(k = 1, 2, \dots, n)$ *are n independent solutions of the completely integrable simultaneous linear partial differential equations*

$$\frac{\partial x'^i}{\partial b^l} = \xi_j^i(x)\beta_l^{*j}(b(x)), \quad (i = 1, 2, \dots, n; j, l = 1, 2, \dots, r) \tag{16.22}$$

such that

$$x^i = f^i(x; 0). \tag{16.23}$$

Conversely, when an r-dimensional extended Lie group (germ) G is given, the (16.22) *are completely integrable for certain*

$$\xi_j^i(x) \in C^2, \ (i = 1, 2, \dots, n; \ j = 1, 2, \dots, r),$$

their solutions

$$f^1(x; a(x)), \ f^2(x; a(x)), \dots, f^n(x; a(x))$$

satisfying (16.21) *determine an extended Lie transformation group (germ) having G as extended parameter group (germ).*

PROOF. - If we differentiate the two sides of (16.7) with respect to $a^j(x)$ and set $a = O$, then, for

$$x'^i = f^i(x; b(x)), \qquad (i = 1, 2, \dots, n), \tag{16.24}$$

we obtain

$$\xi^i_j(x') = \frac{\partial x'^i}{\partial b^l} \alpha^{*l}_j(b(x))$$

by (13.2), (13.3) and (13.4). If we solve this by virtue of (13.2), (13.3) and (13.4), it results that (16.22):

$$\frac{\partial x'^i}{\partial b^l} = \xi^i_j(x')\beta^{*j}_l(b(x)), \qquad (i = 1, 2, \dots, n; \quad l = 1, 2, \dots, r). \tag{16.25}$$

These are partial differential equations in the case, where in (16.24), the (x^i) are considered as parameters and (x'^i) are considered as functions of $b^1, \dots, b^r$. Hence our Cor. is proved by proceeding quite as in the case of our First Fundamental Theorem.

COR. 2°. - *In the extended Lie transformation group (germ) having G as extended parameter group (germ), the*

$$f^k(x; a(x)), \ (k = 1, 2, \dots, n)$$

are n independent solutions of the completely integrable simultaneous linear partial differential equations

$$\alpha^j_l \frac{\partial f}{\partial a^j} = \xi^i_l \frac{\partial f}{\partial x^i} \ (j, \ l = 1, 2, \dots, r; \ i, \ k = 1, 2, \dots, n) \tag{16.26}$$

such that

$$x^k = f^k(x; O). \tag{16.27}$$

Conversely, when an r-dimensional extended Lie group (germ) G is given, (16.26) *are completely integrable for certain*

$$\xi^i_j(x) \in C^2, \ (i = 1, 2, \dots, n; \ j = 1, 2, \dots, r),$$

their solutions

$$f^1(x; a(x)), \dots, f^n(x; a(x))$$

satisfying (16.23) = (16.27), *determine an extended Lie transformation group (germ) having G as extended parameter group (germ).*

PROOF. - Now it suffices to show that (16.1) $\rightleftarrows$ (16.26). For it, multiplying (16.13):

$$\frac{\partial f}{\partial a^l} = \xi^i_j(x)\beta^j_l(a(x))\frac{\partial f}{\partial x^i} \tag{16.28}$$

with $\alpha^l_h(a(x))$, we see that

$$\alpha^l_h(a(x))\frac{\partial f}{\partial a^l} = \xi^i_j(x)\alpha^l_h(a(x))\beta^j_l(a(x))\frac{\partial f}{\partial x^i}, \qquad (i=1, 2, \ldots, n; j, h, k = 1, 2, \ldots, r)$$

$$= \xi^i_j(x)\delta^j_h\frac{\partial f}{\partial x^i} = \xi^i_h(x)\frac{\partial f}{\partial x^i},$$

by (8.5) and conversely, multiplying the last relation with $\beta^h_l(a(x))$, we return to (16.28).

THE SECOND FUNDAMENTAL THEOREM. - (An Extension of the Lie's Fundamental Theorem.) *When a given r-dimensional extended Lie group (germ) G as an extended parameter gronp (germ) has the structure constants* C^k_{ij}, $(i, j, k = 1, 2, \ldots, r)$, *the necessary and sufficient condition for that* (16.13) *may be completely integrable, is that the relations*

$$(X_j, X_l) = C^h_{jl}X_h, \qquad (h, j, l = 1, 2, \ldots, r) \tag{16;29}$$

hold for the fundamental operators

$$X_j = \xi^i_j(x)\frac{\partial}{\partial x^i} \qquad (i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, r). \tag{16.30}$$

Hereby (X_i, X_j) *is the Jacobi's parenthesis.*

PROOF. - We have seen that that the (16.13) = (16.20) is completely integrable is equivalent to that (16.18) is completely integrable. Now it is known that the necessary and sufficient condition for that (16.18) is completely integrable is that $\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_r$ form a *complete system* i. e. that $\bar{X}_1, \ldots, \bar{X}_r$ satisfy

$$(\bar{X}_j, \bar{X}_l) = -C^h_{jl}(x; a)\bar{X}_h, \qquad (j, l, h = 1, 2, \ldots, r). \tag{16.31}$$

Now (16.19):

$$\bar{X}_h = -X_h + A_h \tag{16.32}$$

gives

(16.33) $$(\overline{X}_j, \overline{X}_l) = (X_j, X_l) + (A_j, A\ ),$$

and after setting

(16.34) $$\mathbf{d}\omega^l = \frac{1}{2} C_{jl}^h \omega^j \wedge \omega^l, \quad \omega^l = \beta_j^l(a(x)) da^j(x),$$

(16.35) $$C_{jl}^h = - C_{lj}^h,$$

apply the operator **d** to

$$\mathbf{d}f = \omega^l(A_l f):$$

$$O = \mathbf{d}(\mathbf{d}f) = (A_l f)\mathbf{d}\omega^l + \mathbf{d}(A_l f) \wedge \omega^l$$

$$= \frac{1}{2} C_{jl}^h (A_h f)\omega^j \wedge \omega^l + A_j (A_h f) \omega^j \wedge \omega^h$$

$$= \sum_{j<l} \{ C_{jl}^h (A_h f) + (A_j, A_l) f \} \omega^j \wedge \omega^l.$$

Thus we obtain

(16.36) $$(A_j, A_l) = - C_{jl}^h A_h, \qquad (j,\ l = 1,\ 2,\ \dots,\ r).$$

Owing to (16.32), (16.33) and (16.36), the (16.31) becomes

$$(X_j, X_l) - C_{jl}^h A_h = - C_{jl}^h(x; a)(- X_h + A_h),$$

so that

(17.37) $$C_{jl}^h(x; a) = C_{jl}^h$$

and thus finally we have

(16.38) $$(X_j, X_l) = C_{jl}^h X_h, \qquad (j,\ l,\ h = 1,\ 2,\ \dots,\ r).$$

THE THIRD FUNDAMENTAL THEOREM. – *When r linearly independent differential operators*

(17,37) $$X_j f = \xi_j^i(x) \frac{\partial f}{\partial x^i}, \qquad (i = 1,\ 2,\ \dots,\ n;\ j = 1,\ 2,\ \dots,\ r),\ (\xi_j^i(x) \in C^2)$$

are given, the necessary and sufficient condition for that they are the fundamental differential operators for an extended Lie transformation group (germ), *is that the relations*

(16.38) $$(X_j, X_l) = C_{jl}^h X_h, \qquad (h, j, l = 1, 2, ..., r)$$

hold for certain constants

(16.39) $$C_{jl}^k, \qquad (j, k, l = 1, 2, ..., r).$$

PROOF. - The necessity is implied in the last theorem. It is known that when $\xi_j^i(x) \in C^v$, $(v \geq 2)$, the Jacobi's parentheses satisfy the identities

(16.40) $$(X_i, X_j) = -(X_j, X_i),$$

(16.41) $$((X_i, X_j), X_k) + ((X_j, X_k), X_i) + ((X_k, X_i), X_j) = 0.$$

For the complete system accompanied by (16.38), the relations (9.4):

(16.42) $$C_{ij}^k = -C_{ji}^k,$$

(16.43) $$C_{ij}^h C_{hk}^l + C_{jk}^h C_{hi}^l + C_{ki}^h C_{hj}^l = 0, \qquad (i, j, k, l = 1, 2, ..., n)$$

hold. Hence, by Theorem 2° of Art. 12, there exists an r-dimensional extended LIE group (germ) G having C_{ij}^k as structure constants. If we adopt this G, we are led to the last Theorem for sufficiency.

THE FOURTH FUNDAMENTAL THEOREM. - (An Extension of S. Lie's Third Fundamental Theorem). *The necessary and sufficient condition for that the r^3 given constants C_{jl}^h, $(h, j, l = 1, 2, ..., r)$ may establish the relations*

$$(X_i, X_j) = C_{ij}^k X_k$$

for the fundamental differential operators $X_1, ... X_r$ of an extended Lie transformation group (germ), is that they satisfy the following two conditions (16.42), (16.43):

(16.44) $$C_{ij}^k = -C_{ji}^k,$$

(16.45) $$C_{ij}^h C_{hk}^l + C_{jk}^h C_{hi}^i + C_{ki}^h C_{hj}^l = 0, \qquad (i, j, k, l = 1, 2, ..., r).$$

17. The Lie Ring composed of the Fundamental Differential Operators. - We have represented the (extended) parameter group (germ) G by the extended transformation group (germ) T, so that the abstract (extended)

Lie ring $\mathbb{R}$ has become homeomorphic to the extended Lie ring $\mathbb{O}$ consisting of the totality of

$$X = \lambda_i X_i, \qquad (\lambda_i = \text{constants}).$$

Thus we obtain the following homeomorphic correspondence:

Abstract (extended) Lie ring $\mathbb{R}$	Extended Lie ring $\mathbb{O}$
(Extended) parameter group (germ) G	Extended transformation group (germ) T
Basis $E_1, E_2, \dots, E_r$	Fundamental differential operators $X_1, X_2, \dots, X_r$
$A = \lambda_i E_i \in \mathbb{R}$	$X = \lambda_i X_i \in \mathbb{O}$
$B = \mu_i E_i \in \mathbb{R}$	$Y = \mu_i X_i \in \mathbb{O}$
$\alpha A + \beta B$	$\alpha X + \beta Y$
(A, B)	(X, Y)

Concerning this correspondence, we get the following theorem.

Theorem 1°. – *In order that an extended Lie transformation group (germ) may be a faithful representation of its extended parameter group (germ) G, is that the*

extended Lie ring composed of the fundamental differential operators and the abstract (extended) Lie ring $\mathbb{R}$ may be isomorphic to each other.	*correspondence of the two sides of the above table is one-to-one.*

Proof. – We utilize the canonical parameter t of the extended Lie group (germ) G. Taking a point $(a^1, \dots, a^r)$ in the vicinity of the origin (unit element) and set

$$f^i(x^1, \dots, x^n;\ a^1 t, \dots, a^r t) = f^i(x^1, \dots, x^n;\ t), \qquad (i = 1, 2, \dots, n).$$

Then, by (16.13) and (12.11), we have

$$\frac{\partial f^i}{\partial t}=\frac{\partial f^i}{\partial a^l}a^l=(\beta^k_l(a)a^l)\xi^k_j(x)\frac{\partial f^i}{\partial x^j}$$ (17.1)

$$=a^k\left(\xi^j_k(x)\frac{\partial}{\partial x^j}\right)f^i=(a^kX_k)f^i.$$

Hence, in the case that the correspondence between the two sides of the above table is not one-to-one, we have

$$X=\lambda^1X_1+\dots+\lambda^rX_r=0,$$ (17.2)

where $\lambda^1,\dots,\lambda^r$ are sufficiently small values, which are not zero at the same time.

If we take them for $(a^1,\dots,a^r)=(\lambda^1,\dots,\lambda^r)$, from (17.1), we obtain

$$\frac{\partial f^i}{\partial t}=0, \qquad (i=1,\ 2,\ \dots,\ n)$$

i. e.

$$f^i(x^1,\ \dots,\ x^n;\ a^1t,\ \dots,\ a^rt)=x^i, \qquad (i=1,\ 2,\ \dots,\ n).$$

Thus G and T do not correspond one-to-one.

In the case, where (17.2) holds when and only when $\lambda^1=\lambda^2=\dots=\lambda^r$, take a hypersphere with sufficiently small radius ε and with the origin as center. Then, since $a^kX_k\neq 0$ in (17 1) for each point $(a^1,\ \dots,\ a^r)$ on it, we get

$$(f^i(x^1,\ \dots,\ x^n;\ a^1t,\ \dots,\ a^nt)\neq x^i,$$ (17.3)

$$(i=1,\ 2,\ \dots,\ r;\ |\,t\,|<\delta(a^1,\ \dots,\ a^r).$$

Since $\delta(a^1,\ \dots,\ a^r)$ is evidently a continuous function of $(a^1,\ \dots,\ a^r)$, for the least value δ_0 of it, we must have

$$T_a\neq T_0, \qquad (a^ia^i<\delta_0).$$ (17.4)

Since T makes an extended group (germ), from (17.4), we can conclude that G and T correspond one-to-one in a sufficiently small vicinity of the origin. Q.E.D.

Let us considsr now the case where $\mathfrak{R}$ and $\mathfrak{G}$ are not isomorphic to each other generally. Let s ($\leqq r$) out of the r fundamental differential operators $X_1, \dots, X_r$ be linearly independent with constant coefficients. Let

(17.5) $$Y_i = h_i^j X_j, \qquad (i = 1, 2, \dots, s)$$

be linearly independent and suppose that in terms of them we have

(17.6) $$X_j = g_j^i Y_i, \qquad (j = 1, 2, \dots, r).$$

Since $Y_1, \dots, Y_s$ are linearly independent, we have

(17.7) $$h_i^j g_j^k = \delta_i^k, \qquad (i, k = 1, 2, \dots, s).$$

Utilizing

$$(X_k, X_l) = C_{kl}^m X_m,$$

we obtain

$$(Y_i, Y_j) = h_i^k h_j^l (X_k, X_l) = h_i^k h_j^l C_{kl}^m X_m = h_i^k h_j^l C_{kl}^m g_m^p Y_p,$$

i. e.

(17.8) $$(Y_i Y_j) = \gamma_{ij}^p Y_p, \qquad (i, j = 1, 2, \dots, s),$$

where

(17.9) $$\gamma_{ij}^p = h_i^k h_j^l g_m^p C_{kl}^m.$$

Further we set

(17.10) $$\tau^i(a, da) = g_j^i \omega^j (a, da), \qquad (i = 1, 2, \dots, s).$$

From (12.4) and (16.38), it results that

(17.11) $$\mathbf{d}\omega^m(X_m f) - \frac{1}{2}\,\omega^p \wedge \omega^q (X_p, X_q) f = 0,$$

which becomes

(18.12) $$\mathbf{d}\tau^i(Y_i f) - \frac{1}{2}\,\tau^j \wedge \tau^k (Y_j, Y_k) f = 0$$

by virtue of (17.6) and (17.10).

Utilizing (17.8), thence we obtain

(17.13) $$(\mathbf{d}\tau^i - \frac{1}{2}\,\gamma_{jk}^i \tau^j \wedge \tau^k)(Y_i f) = 0.$$

Now since $Y_1, \dots, Y_r$ are linearly independent, their coefficients must vanish severally, i. e.

$$\mathbf{d}\tau^i(a, da) = \frac{1}{2}\gamma^i_{jk}\tau^j \wedge \tau^k, \qquad (i = 1, 2, \dots, s). \tag{17.14}$$

Consequently the simultaneous total differential equations

$$\tau^1(a, da) = 0, \dots, \tau^s(a, da) = 0 \tag{17.15}$$

are completely integrable. Since further $Y_1, \dots, Y_s$ are linearly independent, the rank of (g^i_j) is s. Hence $\tau^1, \dots, \tau^s$ are also linearly independent by virtue of (17.10). Thus there exist s independent first integrals of (17.15), which are 0 at the origin. Let them be

$$b^1(a^1, \dots, a^r), \dots, b^s(a^1, \dots, a^r) \in C^2,$$

where

$$b^i(0, \ ., 0) = 0, \qquad (i = 1, 2, \dots, s).$$

Taking $(r - s)$ adequate functions

$$b^{s+1}(a^1, \dots, a^r), \dots, b^r(a^1, \dots, a^r) \in C^2,$$

where

$$b^j(0, \dots, 0) = 0, \qquad (j = s+1, s+2, \dots, r),$$

in addition, we have one-to-one correspondence

$$(a^1, \dots, a \ ) \longrightarrow (b^1, \dots, b^r)$$

in the vicinity of the origin. Noticing this transformation of the variables, we write

$$\tau^i(a, da) = \pi^i(b, db), \qquad (i = 1, 2, \dots, s).$$

Since $b^1, \dots, b^s$ are s independent first integrals of (17.15), the relation

$$(\pi^1, \dots, \pi^s) = (db^1, \dots, db^s)$$

holds, so that we may write

$$\pi^i(b, db) = \psi^i_j(b^1, \dots, b^r)db^j, \qquad (i = 1, 2, \dots, s).$$

Now, by (17.14), we must have

$$\mathbf{d}\pi^i(b, db) = \frac{\partial \psi^i_l(b^1, \dots, b^r)}{\partial b^h} db^h \wedge db^l, \qquad (h, l = 1, 2, \dots, r)$$

$$= \frac{1}{2}\gamma^i_{jk}\pi^j \wedge \pi^k \qquad (j, k = 1, 2, \dots, s)$$

$$= \frac{1}{2}\gamma^i_{jk}\psi^j_h(b^1, \dots, b^r) db^h \wedge \psi^k_l(b^1, \dots, b^r) db^l$$

$$= \frac{1}{2}\gamma^i_{jk}\psi^j_h(b^1, \dots, b^r)\psi^k_l(b^1, \dots, b^r) db^h \wedge db^l,$$

so that

$$\frac{\partial \psi^i_j}{\partial b^k} = 0, \qquad (i, j = 1, 2, \dots, s;\ k = s+1, \dots, r).$$

Hence we have

$$\psi^i_j = \psi^i_j(b^1, \dots, b^s),$$

and consequently π^i must be expressible in terms of $b^1, \dots, b^s, db^1 \dots, db^s$ only.

We denote the s-dimensional (extended) LIE group (germ) defined uniquely by

$$\mathbf{d}\pi^i = \frac{1}{2}\pi^j \wedge \pi^k, \qquad (i = 1, 2, \dots, s) \tag{17.16}$$

in the s-dimensional neighborhood of the origin of $(b^1, \dots, b^r)$ by $\overline{G}$. Now, by Theorem 1° of Art 16, the $f^i(x; a)$ are the first integrals of

$$dx^i + \omega^j(a, da)\xi^i_j(x) = 0, \qquad (i = 1, 2, \dots, n)$$

such that $f^i(x; 0) = x^i$. Taking the last differential equations together with (17.5), (17.6) and (17.10), we can deduce

$$dx^i + \pi^j(b, db)\eta^i_j(x) = 0, \qquad (i = 1, 2, \dots, n), \tag{17.17}$$

$$Y_i = h^j_i X_j = \eta^j_i(x)\frac{\partial}{\partial x^j} \tag{17.18}$$

for

$$\eta^j_i(x) = h^j_k \xi^k_i(x).$$

Hence (17.17) are also completely integrable and its first integral is expressible as

$$f^i(x^1, \dots, x^n;\ a^1, \dots, a^r) = g^i(x^1, \dots, x^n;\ b^1(a), \dots, b^s(a)), \quad (i = 1, 2, \dots, n). \tag{17.19}$$

Thus we obtain the following theorem.

THEOREM 2°. – *When the rank of the fundamental* (*extended*) *Lie ring composed of the fundamental differential operators* $X_1, \dots, X_r$ *is* s $(\leqq r)$, *there exists an s-dimensional* (*extended*) *Lie group* (*germ*) $\bar{G}$ *as extended parameter group* (*germ*) *having linearly independent* (17.18) *as fundamental differential operators, for which we have* (17.19). *In this case, the given transformation group* (*germ*) *becomes faithful representation of* $\bar{G}$.

18. The Relation between the Π-Geodesic Curves in the Base Manifold *M* and the Extended Lie Transformation Group Manifold *G*. – We must not overlook that *we are considering both the* Π-*geodesic curves in the*

base manifold M.	*extended Lie transformation group manifold G.*

Now we will seek for how the Π-geodesic curves in the

base manifold M	extended LIE transformation group manifold G

behave in the

extended LIE transformation group manifold G.	base manifold M.

I. For a while, let x^λ denote the *local coordinates* in M and consider a matrix $\omega^p_\mu(x^\nu)$ in place of $a^l_m(x^\nu)$.

We seek for tensors

ω^λ_π	Ω^l_p

such that

$\omega^\lambda_\pi d\Omega^\pi_m = \Omega^\lambda_m \omega^m = \Omega^\lambda,$	$\Omega^l_p d\omega^p_\mu = \omega^l_\nu dx^\nu = \omega^l,$

i. e. that

(18.1) $dx^\lambda = \Omega^\lambda = \delta^\lambda_\varkappa \Omega^\varkappa = \varphi^\lambda_\pi \dfrac{\partial \Omega^\pi_m}{\partial x^\varkappa} dx^\varkappa,$	$d\xi^l = \omega^l = \delta^l_q \omega^q = \Omega^l_p \dfrac{\partial \omega^p_\mu}{\omega^q} \omega^q,$

for which it suffices to set

(18.2) $\omega^\lambda_\pi \dfrac{\partial \Omega^\pi_m}{\partial x^K} = \delta^\lambda_K.$	$\Omega^l_p \dfrac{\partial \omega^p_\mu}{\omega^q} = \delta^l_q.$

Thus the

(ω^λ_π)	(Ω^l_p)

is the inverse transformation of

$\left(\dfrac{\partial \Omega^\pi_m}{\partial x^\kappa}\right).$	$\left(\dfrac{\partial \omega^p_\mu}{\omega^q}\right).$

From (18.1), we have

(18.3)
$$\frac{d}{dt}\frac{\Omega^\lambda}{dt} = \Omega^\lambda_m\left(\frac{d}{dt}\frac{\omega^m}{dt} + \Lambda^m_{rs}\frac{\omega^r}{dt}\frac{\omega^s}{dt}\right) = \frac{d}{dt}\left(\omega^\lambda_\pi \frac{d\Omega^\pi_m}{dt}\right) = \omega^\lambda_\pi\left(\frac{d^2\Omega^\pi_m}{dt^2} + \Delta^\pi_{\rho\sigma}\frac{\Omega^\rho_m}{dt}\frac{\Omega^\sigma_m}{dt}\right) = 0$$

$$\frac{d}{dt}\frac{\omega^l}{dt} = \omega^l_\nu\left(\frac{d^2x^\nu}{dt^2} + \Lambda^\nu_{\rho\sigma}\frac{dx^\rho}{dt}\frac{dx^\sigma}{dt}\right) = \frac{d}{dt}\left(\Omega^l_p \frac{d\omega^p_\mu}{dt}\right) = \Omega^l_p\left(\frac{d^2\omega^p_\mu}{dt^2} + \Delta^p_{rs}\frac{d\omega^r_\mu}{dt}\frac{d\omega^s_\mu}{dt}\right) = 0$$

giving

(18.4)
$$\Omega^\lambda = \Omega^\lambda_m(\xi^p)\omega^m = dx^\lambda = a^\lambda dt = \omega^\lambda_\pi d\Omega^\pi_m, (m=1,\dots,n), \quad (a^\lambda = \text{const.}),$$

$$\omega^l = \omega^l_\mu(x^\nu)dx^\mu = d\xi^l = a^l dt = \Omega^l_p d\omega^p_\mu, \ (\mu = 1, \dots, n), \quad (a^l = \text{const.}),$$

where

(18.5) $\Lambda^m_{rs} = \omega^m_\lambda \dfrac{\partial \Omega^\lambda_r}{\omega^s} \equiv -\Omega^\lambda_r \dfrac{\partial \omega^m_\lambda}{\omega^s},$	$\Lambda^\nu_{\rho\sigma} = \Omega^\nu_l \dfrac{\partial \omega^l_\rho}{\partial x^\sigma} \equiv -\omega^l_\rho \dfrac{\partial \Omega^\nu_l}{\partial x^\sigma},$
(18.6) $\Delta^\pi_{\rho\sigma} = -\omega^\lambda_\rho \dfrac{\partial^2 \Omega^\pi_m}{\partial x^\sigma \partial x^\lambda} \equiv \dfrac{\partial \Omega^\pi_m}{\partial x^\lambda}\dfrac{\partial \omega^\lambda_\rho}{\partial x^\sigma}.$	$\Delta^p_{rs} = -\Omega^l_r \dfrac{\partial^2 \omega^p_\mu}{\omega^s\, \omega^l} \equiv \dfrac{\partial \omega^p_\mu}{\omega^l}\dfrac{\partial \Omega^l_r}{\omega^s}.$

Thus, by (18.4), *the* Π-*geodesic curves* (18.4):

(18.7) $dx^\lambda = \Omega^\lambda_m(\xi^p)d\xi^m = a^\lambda dt$ | $\omega^l = \omega^l_\mu(x^\nu)dx^\mu = a^l dt$

for the group manifold G, provided that

ω^λ_π | Ω^l_p

are defined by (18.2).

II. - Let us take now another view point. Let

$(\omega^l_\mu(x^\nu))$ | $(\Omega^\lambda_l(x^\nu))$

be the inverse transformation of

$(\Omega^\lambda_l(x^\nu))$ | $(\omega^l_\mu(x^\nu))$

as before. Then

(18.9)
$$\alpha^l_m \overset{\text{def}}{=} \omega^l_\lambda d\Omega^\lambda_m = \omega^l_\lambda \frac{\partial \Omega^\lambda_m}{\omega^r}\omega^r = \Lambda^l_{mr}\omega^r, \qquad \Big| \qquad \alpha^\lambda_\mu \overset{\text{def}}{=} \Omega^\lambda_l d\omega^l_\mu = \Omega^\lambda_l \frac{\partial \omega^l_\mu}{\partial x^\nu} dx^\nu = \Lambda^\lambda_{\mu\nu} dx^\nu,$$

(18.10)
$$\frac{d}{dt}\frac{\alpha^l_m}{dt} \equiv \omega^l_\lambda\left(\frac{d^2\Omega^\lambda_m}{dt^2} + \Lambda^\lambda_{K\rho}\frac{d\Omega^K_m}{dt}\frac{d\Omega^\rho_m}{dt}\right) = 0, \qquad \Big| \qquad \frac{d}{dt}\frac{\alpha^\lambda_\mu}{dt} \equiv \Omega^\lambda_l\left(\frac{d^2\omega_\mu}{dt^2} + \Lambda^l_{qr}\frac{d\omega^q_\mu}{dt}\frac{d\omega^r_\mu}{dt}\right) = 0,$$

where

(18.11)
$$\Lambda^\lambda_{\mu\nu} \overset{\text{def}}{=} \Omega^\lambda_l \frac{\partial \omega^l_\mu}{\partial x^\nu} \equiv -\omega^l_\mu \frac{\partial \Omega^\lambda_l}{\partial x^\nu}; \qquad \Big| \qquad \Lambda^l_{qr} \overset{\text{def}}{=} \omega^l_\lambda \frac{\partial \Omega^\lambda_q}{\partial \omega^r_\mu} \equiv -\Omega^\lambda_q \frac{\partial \omega^l_\lambda}{\partial \omega^r_\mu};$$

(18.12)
$$\Omega^\lambda_l \omega^l_\mu = \delta^\lambda_\mu. \qquad \Big| \qquad \omega^p_\lambda \Omega^\lambda_q = \delta^p_q.$$

Hence we have

(18.13)
$$\alpha^l_m \overset{\text{def}}{=} \omega^l_\lambda d\Omega^\lambda_m = a^l_m\, dt,\ (a^l_m = \text{const.}) = d\eta^l \qquad \Big| \qquad \alpha^\lambda_\mu \overset{\text{def}}{=} \Omega^\lambda_l\, d\omega^l_\mu = a^\lambda_\mu dt,\ (a^\lambda_\mu = \text{const}) = d\eta^\lambda$$

by (5.4) and (5.6). On the other hand, by (18.9), we have

(18.14) $\alpha_m^l = \Lambda_{mr}^l \omega^r = a_m^l dt = d\eta^l.$ | $\alpha_\mu^\lambda = \Lambda_{\mu\nu}^\lambda dx^\nu = a_\mu^\lambda dt = d\eta^\lambda.$

(18.15) $\frac{d}{dt}\left(\Lambda_{mr}^l \frac{\omega^r}{dt}\right) = \Lambda_{mr}^l\left(\frac{d}{dt}\frac{\omega^r}{dt} + \Delta_{st}^r \frac{\omega^s}{dt}\frac{\omega^t}{dt}\right) = 0,$ | $\frac{d}{dt}\left(\Lambda_{\mu\nu}\frac{dx^\nu}{dt}\right) = \Lambda_{\mu\nu}^\lambda\left(\frac{d^2x^\nu}{dt^2} + \Delta_{\sigma\tau}^\nu \frac{dx^\sigma}{dt}\frac{dx^\tau}{dt}\right) = 0,$

where

(18.16) $\Delta_{st}^r \overset{\text{def}}{=} \tilde{\Lambda}_{ml}^r \frac{\partial \Lambda_{ms}^l}{\omega^t} \equiv -\Lambda_{ms}^l \frac{\partial \tilde{\Lambda}_{ml}^r}{\omega t},$ (m: not summed). | $\Lambda_{\sigma\tau}^\nu \overset{\text{def}}{=} \tilde{\Lambda}_{\mu\lambda}^r \frac{\partial \Lambda_{\mu\sigma}^\lambda}{\partial x^\tau} \equiv -\Lambda_{\mu\sigma}^\lambda \frac{\partial \tilde{\Lambda}_{\mu\lambda}^\nu}{\partial x^\tau},$ (μ: not summed).

From (18.15), we obtain

(18.17) $\Lambda_{mr}^l \omega^r = a_m^l dt = d\eta^l.$ | $\Lambda_{\mu\nu}^\lambda dx^\nu = a_\mu^\lambda dt = d\eta^\lambda.$

This gives another system of Π-*geodesic curves in* M. *The corresponding* Π-*geodesic curves in* G *are given by* (18.13).

III. If we multiply (16.3) with $\bar{\xi}_i^l(x)$ defined by $\xi_i^j \bar{\xi}_l^i = \delta_l^j$ [(21.1)], then

$$\bar{\xi}_i^l(x)\theta^i = \bar{\xi}_i^l(x)dx^i + \beta_j^l(a(x))da^j(x), \tag{18.19}$$

so that the differential equations (16.14) $\theta^1 = 0, \ldots, \theta^n = 0$ give

$$\bar{\xi}_i^l(x)dx^i = -\beta_j^l(a(x))da^j(x) \tag{18.20}$$

i. e. (21.13) = (21.15):

$$d\xi^j = -d\alpha^j = e^j dt \tag{19.21}$$

by (5.7). Thus *to the* Π-*geodesic curves* $d\xi^j = e^j dt$ *in the base manifold* M, *there correspond the* Π-*geodesic curves* $d\alpha^j = -e^j dt$ *in the extended Lie transformation group manifold* G.

19. Two Systems of Equipollences of Vectors in the Extended Lie Transformation Group Space.

(i) Consider an extended LIE transformation group G with r extended parameters $a^1(x), a^2(x), \ldots, a^r(x)$. The coordinates $x = (x^1, x^2, \ldots, x^n)$, which undergo the extended LIE transformations $a(x)$ will play the quite an accessory rôle in the following lines. We will extend the E. CARTAN'S theory [15] of two kinds of parallelisms of the vectors in the group space to the case of our extended LIE transformation group space $\mathfrak{E}$.

Let us denote the elements of G corresponding to $a(x)$ as an operator by T_a and the product of T_a and T_b by $T_b T_a$, and the inverse of T_a by T_a^{-1}, so that $(T_b T_a)^{-1} = T_a^{-1} T_b^{-1}$.

We will call a pair of points $(a(x))$ and $(b(x))$ taken in this order a *vector* $\overrightarrow{ab}$ of $\mathfrak{E}$ and when $a(x) = b(x)$, we will call the vector a *nul vector*.

(ii) DEFINITION. - We will say that two vectors $\overrightarrow{ab}$ and $\overrightarrow{a'b'}$ are *equipollent of the*

first	*second*

kind, when

(19.1)	$T_b T_a^{-1} = T_{b'} T_{a'}^{-1}$.	$T_a^{-1} T_b = T_{a'}^{-1} T_{b'}$.

Considering the inverses, we may replace (19.1) by

$T_a T_b^{-1} = T_{a'} T_{b'}^{-1}$.	$T_b^{-1} T_a = T_{b'}^{-1} T_{a'}$.

The equipollences have the following properties.

1°. *Every vector, which is equipollent to a nul vector, is nul.*

2°. *Every vector is equipollent to itself.*

3°. *If a vector is equipollent to a second vector, then the second vector is equipollent to the first.*

4°. *If two vectors are equipollent, then their inverses are also equipollent.*

5°. *Every point of the group spacc* $\mathfrak{E}$ *may be considered as the origin of one and only one vector, which is equipollent to a given vector.*

6°. *Two vectors, which are equipollent to a third vector, are equipollent to each other.*

7°. *If* $\overrightarrow{ab}$ *is equipollent to* $\overrightarrow{a'b'}$ *and* $\overrightarrow{bc}$ *equipollent to* $\overrightarrow{b'c'}$, *then the vector* $\overrightarrow{ac}$ *is equipollent to* $\overrightarrow{a'c'}$.

The 7° may be proved as follows. From

$$T_b T_a^{-1} = T_{b'} T_{a'}^{-1}, \quad T_c T_b^{-1} = T_{c'} T_{b'}^{-1},$$

we obtain

$$(T_c T_b^{-1})(T_b T_a^{-1}) = (T_{c'} T_{b'}^{-1})(T_{b'} T_{a'}^{-1})$$

i. e.

$$T_c T_a^{-1} = T_{c'} T_{a'}^{-1}.$$

(iii) THEOREM. - *When* $\overrightarrow{ab}$ *is equipollent of the first kind to* $\overrightarrow{a'b'}$, *the vector* $\overrightarrow{aa'}$ *is equipollent of the second kind to* $\overrightarrow{bb'}$ *and vice versa.*

PROOF. - From (19.1), we have

$$T_b^{-1}T_b T_a^{-1}T_{a'} = T_b^{-1}T_{b'}T_{a'}^{-1}T_{a'},$$

i. e.

$$T_a^{-1}T_{a'} = T_b^{-1}T_{b'},$$

which is of the form (11.2) for $\overrightarrow{aa'}$ and $\overrightarrow{bb'}$.

THEOREM. - *When the first equipollence plays property* 7°, *the second equipollence plays the property* 6° *and vice versa.*

PROOF. - Suppose that an equipollence satisfying the properties 1° — 6° is defined in an r-dimensional space in a certain way. Thence we can dedude an equipollence of the second kind saying that $\overrightarrow{aa'}$ is equipollent of the second kind to $\overrightarrow{bb'}$ when $\overrightarrow{ab}$ is equipollent of the first kind to $\overrightarrow{a'b'}$. It is easy to see that the properties 1° — 5° are verified for this equipollence of the second kind. But as for the property 6°, it is not necessarily the case· Suppose $\overrightarrow{aa'}$ and $\overrightarrow{bb'}$ are equipollent of the second kind to $\overrightarrow{cc'}$. This means that $\overrightarrow{ac}$ is equipollent of the first kind to $\overrightarrow{a'c'}$ and that $\overrightarrow{bc}$ is equipollent of the first kind to $\overrightarrow{b'c'}$. In order that $\overrightarrow{aa'}$ and $\overrightarrow{bb'}$ may be equipollent of the first kind to $\overrightarrow{b'c'}$. In order that $\overrightarrow{aa'}$ and $\overrightarrow{bb'}$ may be equipollent of the second kind to each other, it is necessary and sufficient that $\overrightarrow{ab}$ is equipollent of the first kind to $\overrightarrow{a'b'}$; in other words, the equipollence of the second kind will verify 6° when the equipollence of the first kind verifies 7° and vice versa.

(iv) The two kinds of equipollence are in close relation to *two groups* of extended parameters of G. Indeed, let us consider the geometrical operation consisting of laying through a variable point $(\xi(x))$ a vector $\xi\xi'$, which is equipollent of the first kind to a fixed vector. Let $(a(x))$ be the extremity of the vector which is equipollent to the fixed vector and is drawn through the origin of $\mathfrak{E}$. The operation considered is expressed analytically by

$$T_{\xi'}T^{-1} = T_\xi$$

or by

$$T_{\xi'} = T_a T_\xi. \tag{19.3}$$

This is thus analytically indentical to one of the transformations of the first group of extended parameters [8].

---

[8] An extension of the analogous result in [12], p. 449.

Similarly the operation consisting in drawing a vector $\overrightarrow{\xi\xi'}$ through a variable point $(\xi(x))$, which is equipollent of the second kind to a fixed vector $\overrightarrow{Oa}$, may be expressed analytically by

$$T_{\xi'} = T_{\xi} T_a. \tag{19.4}$$

This is thus analytically identical to one of the transformations of the second group of extended parameters (9).

(v) The property explained by the Theorem under (ii) is a geometrical interpretation of the fact that the extended transformations of the two groups of extended parameters are interchanged among themselves.

The properties $1^\circ - 7^\circ$ are the characteristic properties of the equipollence attached to the groups. We shall prove that *when we have defined an equipollence of vectors in extended group space $\mathbb{E}$ playing the seven properties $1^\circ - 7^\circ$, the space $\mathbb{E}$ can be considerd as a space of group, the equipollence defined in $\mathbb{E}$ being the first equipollence attached to extended group.*

For this purpose, let us take au origin (O) in the space $\mathbb{E}$ quite arbitrarily. Let $(a(x))$ be any point of $\mathbb{E}$. Consider an operation S_a, by which we pass from a variable point $(\xi(x))$ to the extremity $(\xi'(x))$ of the vector $\overrightarrow{\xi\xi'}$, which is equipollent to $\overrightarrow{Oa}$ (a vector which exists by 5°). We will prove first that *these operations constitute a group.*

To prove this, we proceed as follows. Those operations contain evidently the identical operation (by 1°). Let S_a abd S_b be two such operators. Let $c(x))$ be the transform of $(a(x))$ by S_b. Executing the operation S_a and S_b successively, we pass from the point $(\xi(x))$ to the point $(\xi'(x))$ and then to $(\xi''(x))$ by virtne of

$$(S_a) \qquad \overrightarrow{\xi\xi'} = \overrightarrow{Oa},$$

$$(S_b) \qquad \overrightarrow{\xi'\xi''} = \overrightarrow{Ob}.$$

Now, by the hypothesis, $\overrightarrow{ac}$ is equipollent to $\overrightarrow{Ob}$. Hence $\overrightarrow{\xi'\xi''}$ is equipollent to $\overrightarrow{ac}$ (by 6°). From the equipollences

$$\overrightarrow{\xi\xi'} = \overrightarrow{Oa}, \qquad \overrightarrow{\xi'\xi''} = \overrightarrow{ac},$$

follows thus (7°) that

$$\overrightarrow{\xi\xi''} = \overrightarrow{Oc},$$

whence we obtain

$$S_b S_a = S_c. \qquad \text{Q. E. D.}$$

(9) [12], p. 633.

Next, let G be the group composed of the operations S_a. *This group is simply transitive.* This means that it contains one and only one transformation, which maps a given point $(\xi(x))$ to another given point $(\xi'(x))$, obtaining the transformation S_a corresponding to the extremity of the vector $\overrightarrow{Oa}$, which is equipollent to $\overrightarrow{\xi\xi'}$. Consider next two arbitrary equipollent vectors ab and $\overrightarrow{a'b'}$. The vector $\overrightarrow{Oc}$, which is equipollent to $\overrightarrow{ab}$ is also equipollent to $a'b'$ (property 6°). Hence the transformation S_e maps $(a(x))$ to $(b(x))$ and $(a'(x))$ to $(b'(v))$ simultaneously. Now the transformation $S_bS_a^{-1}$ also maps $(a(x))$ to $(b(x))$ (by the mediation of the origin (O)), and transformation $S_{b'}S_{a'}^{-1}$ maps likewise $(a'(x))$ to $(b'(x))$.
Hence we have

$$S_c = S_bS_a^{-1} = S_{b'}S_{a'}^{-1},$$

what shows us that the equipollence defined in $\mathfrak{E}$ is idenitcal with the equipollence of the first kind attached to the group G.

(vi) The results of the last theorem that the equipollence of the second kind of the space of group may be considered as equipollence of the first kind attached to another group admitting the same representative space. It is easy to see that *the second group of extended parameters* will admit the second equipollence of the group G for the first equipollence.

Now we encounter another important remark. Consider a set of transformation T_a depending on r extended parameters, *not forming a group,* but playing the property that *the transformations* $T_bT_a^{-1}$ *do not depend on not more than* r *extended parameters* (when $a(x)$ and $b(x)$ take all possible values). We can define an equipollence of vectors in the space of this set of transformations by the equality.

$$T_bT_a^{-1} = T_{b'}T_{a'}^{-1}, \tag{19.5}$$

and this equipollence plays the seven properties $1^\circ - 7^\circ$ as we can easily verify. Choose *an arbitrary origin transformation* T_0. The transformation S_a defined above may be expressed as follows:

$$T_{\xi'}T_{\xi}^{-1} = T_aT_0^{-1}$$

i,e.

$$(S_a) \qquad T_{\xi'} = T_aT_0^{-1}T_{\xi}.$$

Execute the transformations S_a and S_b successively and set

$$S_bS_a = S_c.$$

We shall obtain

$$T_{\xi'} = T_aT_0^{-1}T_{\xi},$$

$$T_{\xi''} = T_bT_0^{-1}T_{\xi'} = T_bT_0^{-1}T_aT_0^{-1}T_{\xi} = T_cT_0^{-1}T_{\xi}$$

by S_a and S_b successively. Hence the equality

$$(T_b T_a^{-1})(T_a T_0^{-1}) = T_c T_0^{-1}$$

results, so that *the transformations* $T_a T_0^{-1}$ *form a group.*

This theorem, which is of purely analytical nature, may be proved else directly. Consider a set of transformations $T_b T_a^{-1}$ of r extended parameters. From the product

$$(T_{b'} T_{a'}^{-1})(T_b T_a^{-1})$$

of such transformations, we see that there exists a transformation T_c such that

$$T_{a'}^{-1} T_b = T_{b'}^{-1} T_c. \tag{19.6}$$

For, the transformations $T_b T_\xi^{-1}$, where we let the extended parameters ξ vary, must have all the transformations of set $T_\xi T_\eta^{-1}$, so, in particular, the transformation $T_{a'} T_{b'}^{-1}$. Therefore there exists a point $(c(x))$ such that

$$T_b T^{-1} = T_{a'} T_{b'}^{-1}.$$

This equality is equivalent to the equality (19.6). Thus from (19.6), we deduce

$$(T_{b'} T_{a'}^{-1})(T_b T_c^{-1}) = T_{b'} T_{b'}^{-1} T_c T_a^{-1} = T_c T_a^{-1},$$

which shows us that the transformations $T_b T_a^{-1}$ form a group. Moreover all the transformations of this group are obtainable by letting $(a(x))$ fix and letting $(b(x))$ vary.

(vii) We know that two groups G and G' of the same order are said to be ***isomorph*** (holoedrique), when we can establish among their transformations a correspondence such that to the product of two arbitrary tranformations of the first group there corresponds the product of two corresponding transformations of the second group. In the correspondence, which realize the isomorphism, the identity transformations correspond to each other. Moreover, to the inverse of transformation of the first group there corresponds the inverse of the corresponding transformation of the second group.

Let G and G' be two isomorphic groups and $\mathbb{E}$ and $\mathbb{E}'$ their spaces.

All correspondence by isomorphism of two groups may be interpreted by the point-correspondence of two $\mathbb{E}$ *spaces and* $\mathbb{E}'$, *such that to two vectors of* $\mathbb{E}$, *which are equipollent of the first (second) kind to each other, there correspond two vectors of* $\mathbb{E}'$, *which are equipollent of the first (second) kind to each other.*

Indeed, we can choose the extended parameters of two groups in such a way that in the correspondence by isomorphism under consideration, the extended parameters of two corresponding transformations are the same. We denote the transformations of the two groups by T and Θ. Then the equality

$$T_b T_a^{-1} = T_{b'} T_{a'}^{-1}$$

signifies that there exists a transformation T_c such that we have

$$T_b = T_c T_a, \qquad T_{b'} = T_c T_{a'},$$

whence follows:

$$\Theta_b = \Theta_c \Theta_a, \qquad \Theta_{b'} = \Theta_c \Theta_{a'},$$

so that

$$\Theta_b \Theta_a^{-1} = \Theta_{b'} \Theta_{a'}^{-1}. \qquad \text{Q. E. D.}$$

The demonstration will be the same for the parallelism of the second kind

(viii) Conversely, *suppose that we can establish a point correspondence between the spaces* $\mathfrak{E}$ *and* $\mathfrak{E}'$ *of two groups* G *and* G' *of the same order* r *such that to two vectors of* $\mathfrak{E}$, *which are equipollent of the first kind, there correspond two vectors of* $\mathfrak{E}'$ *which are equipollent of the first kind, then the two groups* G *and* G' *are isomorphic.*

To prove this, let (ω) be the point of $\mathfrak{E}'$ corresponding to the origin (O) of $\mathfrak{E}$, and let $(a(x))$, $(b(x))$ and $(c(x))$ be three arbitrary points of $\mathfrak{E}$ and $(\alpha(x))$, $(\beta(x))$, $(\gamma(x))$ the corresponding points of $\mathfrak{E}'$. From the equality

$$T_b T_a^{-1} = T_c = T_c T_0^{-1}$$

follows:

$$\Theta_\beta \Theta_\alpha^{-1} = \Theta_\gamma \Theta_\omega^{-1}$$

by hypothesis. In other words, from the equality

$$T_b = T_c T_a$$

follows:

$$\Theta_\beta \Theta_\omega^{-1} = (\Theta_\gamma \Theta_\omega^{-1})(\Theta_\alpha \Theta_\omega^{-1}).$$

Then we let the transformation $\Theta_\alpha \Theta_\omega^{-1}$ of G' correspond to the transformation T_a of G. This correspondence shows the isomorphism of the two groups by the last equality.

We can make the remark that it is very easy to establish a correspondence with a given group by interchanging the two kinds of equipollence attached to the group: *it suffices to make the transformation* $T_\alpha = T_a^{-1}$ *correspond to the transformation* T_a. Then the equality

$$T_b T_a^{-1} = T_{b'} T_{a'}^{-1},$$

which defines the equipollence of the first kind, becomes changed into the equality

$$T_\beta^{-1} T_\alpha = T_{\beta'}^{-1} T_{\alpha'},$$

which defines the equipollence of the second kind.

It results from this remark, that *in order that two groups of the same order may be isomorphic, it is necessary and sufficient that we can establish a point correspondence between the spaces of these two groups transforming one of the spaces into a certain of the spaces by an equipollence of the second kind.*

(ix) The preceding consideration proposes the question of determination of all the point transformations of a space of group into itself, which play the property to conserve the two kinds of equipollence of the space.

It is firstly evident that a point transformation, which conserves the equipollence of the first kind, conserves the equipollence of the second kind and vice versa. Let $(\alpha(x))$, $(\beta(x))$, etc. be the points transformed from $(a(x))$, $(b(x))$, etc. From the equipollence of the first kind of $\overrightarrow{ab}$ and $\overrightarrow{a'b'}$ follows that of $\overrightarrow{\alpha\beta}$ and $\overrightarrow{\alpha'\beta'}$ by hypothesis, whence follows that from the equipollence of the second kind of $\overrightarrow{aa'}$ and $\overrightarrow{bb'}$ follows that of $\overrightarrow{\alpha\alpha'}$ and $\overrightarrow{\beta\beta'}$.

Let us commence with determination of the point transformations, which conserve the equipollence of the first kind and *let the point origin be invariant.* The equality

$$T_c = T_b T_a^{-1}$$

expresses simply the equipollence of the first kind of vectors $\overrightarrow{Ob}$ and $\overrightarrow{ac}$, whence follows the equipollence of $\overrightarrow{O\beta}$ and $\overrightarrow{\alpha\gamma}$, and consequently the equality

$$T_\gamma = T_\beta T_\alpha^{-1}.$$

Hence the transformations sought for are *autoisomorphisms* of group G. Among the automorphisms, there exist in particular the transformations of the *adjoint group*

$$T_{\bar\xi} = T_a^{-1} T_\xi T_a,$$

where $(a(x))$ is a fixed point.

If the group G is semi-simpls, the adjoint group is the largest maximum continuous group of automorphisms of G. ([10]).

In order to obtain all the point transformations conserving the equipollence of the first kind, it will suffice to combine the preceding point transformations with the transformations

$$T_{\xi'} = T_{\xi} T_a \quad \text{or} \quad T_{\xi'} = T_a T_{\xi},$$

or further with the transformations

$$T_{\xi'} = T_a T_{\xi} T_b, \tag{19.7}$$

$(a(x))$ and $(b(x)$ denoting two fixed points. The point transformations (19.7) transform the equality

$$T_{\eta} T_{\xi}^{-1} = T_{\bar{\eta}} T_{\bar{\xi}}^{-1}.$$

into the equality:

$$T_{\eta'} T_{\xi'}^{-1} = T_{\bar{\eta}'} T_{\bar{\xi}'}^{-1}.$$

Evidently the transformations (19.7) form a group Γ_0, which is a subgroup of the total group Γ of transformations, which conserve the equipollence of the first kind. It is likewise easy to see that Γ_0 *is an invariant subgroup* of Γ. It suffices to prove that all the transformations of Γ_0 are changed into other transformations of Γ_0 by an automorphism of the group G. If the points $(a'(x))$, $(b'(x))$, $(\xi'(x))$, $(\eta'(x))$ correspond to $(a(x))$, $(b(x))$, $(\xi(x))$, $(\eta(x))$ by this automorphism, the relation

$$T_{\eta} = T_a T_{\xi} T_b$$

is changed into

$$T_{\eta'} = T_{a'} T_{\xi'} T_{b'},$$

the transformation of Γ_0 corresponding to points $(a(x))$ and $(b(x))$ is changed into another transformation of Γ_0, that which correspond to points $(a'(x))$ and $(b'(x))$.

We will give the name *group of isomorphism* of $\mathfrak{E}$, to Γ.

The group of point transformations of $\mathfrak{E}$, which conserve the set of two *equipollences* will easily be deduced from Γ by combining it with

([10]) E. CARTAN, Le principe de dualité et la théorie des groupes simples et semi-simples. Bull. Sc. math. 2e série, t. 49 (1925), 363-364.

the transformation

$$T_{\xi} = T_{\xi'}.$$

It may be remarked that the group Γ_0 defined by the equations (18.7) is at most of $2r$ extended parameters. Precisely, it is of $2r-\rho$ extended parameters, where ρ denotes the order of the subgroup formod of those transformations of G, which are interchangeable with all the other transformations of G. The group Γ_0 contains evidently the adjoint group (19.6), which is itself of $r-\rho$ extended parameters.

20. - Extension of E. Cartan's Geodesics, His Two Kinds of Parallelisms and His Transformations.

(i) In case of the ordinary equipollence of two vectors, the straight lines play the following characteristic property:

If we take three arbitrary points (a), (b), (c) on a straight line, the vector $\overrightarrow{cd}$, which is equipollent to $\overrightarrow{ab}$ has its extremity (d) on the straight line.

E. CARTAN [15] has generalized this notion in his space of group. Now we will *generalize* his notion further to the case of *the groups of extended parameters* as follows.

DEFINITION. - A curve (C) traced in a space of group of extended parameters will be called a II'_1-*geodesic* (read: the first geodesic of the second kind), when three arbitary points $(a(x))$, $(b(x))$ and $(c(x))$ are taken on this curve, the extremity $(d(x))$ of the vector $\overrightarrow{cd}$, which is equipollent of the first kind to $\overrightarrow{ab}$, lie also on this curve. The II'_2-geodesics may be defined similarly with respect to the equipollence of the second kind.

But we have to make the following important remark.

All the

II'_1-*geodesics*	II_2'-*geodesics*
are	
II_2'-*geodesics.*	II_1'-*geodesics.*

For, if $\overrightarrow{cd}$ be equipollent of the first kind to $\overrightarrow{ab}$, then this implies that $\overrightarrow{bd}$ is equipollent of the second kind to $\overrightarrow{ac}$ and vice versa.

Thus *there exist really only* II-*geodesics.*

(ii) The primary question arising is that of the existence of the II-geodesics. Now it is easy to find a priori an infinity of II'-geodesics in the spaces of groups with extended parameters. For this purpose, take of $(a(x))$ a fixed point (a). Let us consider a one-parametric subgroup g of G. Denote its general transformation by Θ_u. The point $(\xi(x))$ defined by

$$T_\xi = \Theta_u T_a \tag{20.1}$$

describes a II′-geodesic. For, if u_1, u_2 and u_3 be three arbitrary particular values of the parameter u, and $(\xi_1(x))$, $(\xi_2(x))$, $(\xi_3(x))$ the three corresponding points and if $(\xi_4(x))$ be the extremity of the vector $\overrightarrow{\xi_3\xi_4}$, which is equipollent of the first kind to $\xi_1\xi_2$, then we have

$$T_{\xi_4}T_{\xi_3}^{-1} = T_{\xi_2}T_{\xi_1}^{-1}$$

i. e.

$$T_{\xi_4} = T_{\xi_2}T_{\xi_1}^{-1}T_{\xi_3} = \Theta_{u_2}\Theta_{u_1}^{-1}\Theta_{u_3}T_a = \Theta_{u_4}T_a. \qquad \text{Q. E. D.}$$

Conversely, *we can obtain all the* II′-*geodesics in this manner.*

For, if $(\xi(x))$ and $(\eta(x))$ be two variable points and $(a(x))$ a fixed point on a II′-geodesic, then there exists on this II′-geodesic a point $(\zeta(x))$ such that

$$T_\eta T_\xi^{-1} = T_\zeta T_a^{-1}$$

and consequently the transformations $T_\eta T_\xi^{-1}$ depend only on a single parameter, whence follows that these transformations and especially the transformations $T_\xi T_a^{-1}$ form a one-parametric subgroup g of G. Denoting its general transformation by Θ_u, we obtain

$$T_\xi = \Theta_u T_a. \qquad \text{Q. E. D.}$$

It should be remarked that any II-geodesic may be defined also by

$$T_\xi = T_a\Theta_u, \tag{20.2}$$

the Θ_u forming a one-parametric group, or more generally by

$$T_\xi = T_a\Theta_u T_b. \tag{20.3}$$

Moreover the (20.10) may be rewritten as follows:

$$T_\xi = (T_a\Theta_a T_a^{-1})\ (T_a T_b),$$

and the transformation $T_a\Theta_u T_a^{-1}$ constitute a group being led to the transformation group of g by T_a. Thus we fall on the expression (20.1) again.

(iii) Hitherto we have considered a vector $\overrightarrow{ab}$ to be defined uniquely by its origin $(a(x))$ and its extremity $(b(x))$. When the parameters of $(b(x))$ do not differ much from those of $(a(x))$, the transformation $T_b T_a^{-1}$ belongs to one and only one-parametric subgroup g of G as in the case of the theory of continuous groups of S. LIE; consequently the two points $(a(x))$ and $(b(x))$ belong to one and only one II′-geodesic, which is the locus of the point $(\xi(x))$ defined by

$$T_\xi = \Theta_u T_a ,$$

where Θ_u is the general transformation of g. Thus the vector assimilates to the II′-geodesic segment limited by $(a(x))$ and $(b(x))$.

We can then state as follows:

All vectors lying on a II′-*geodesic is equipollent of the first and the second kind to a determined vector lying on the* II-*geodesic and having for the origin a given point of this* II′-*geodesic.*

If we define the equality of two segments by the equipollence of corresponding vectors, we can measure the segment of one and the same II′-geodesic as soon as we choose a unit segment on this II′-geodesic segment.

If, in particular, we have taken *our parameter* t (the *affine length*: a generalization of the canonic parameter of S. LIE) introduced in (5 5) for the parameter u of the general transformation g such that

$$\Theta_u \Theta_{u'} = \Theta_{u+u'} , \qquad (u = t, \ u' = t')$$

the measure of the segment $\overrightarrow{\xi_1 \xi_2}$ with

$$T_{\xi_1} = \Theta_{u_1} T_a , \qquad T_{\xi_2} = \Theta_{u_2} T_a ,$$

will be $| u_2 - u_1 | = | t_2 - t_1 |$. The change of u into ku means a change of the unit of length. The algebraic ratio of two vectors $\overrightarrow{\xi_1 \xi_2}$ and $\overrightarrow{\xi_3 \xi_4}$ taken on one and the same II′-geodesic has the determinate value

$$\frac{u_4 - u_3}{u_2 - u_1} = \frac{t_4 - t_3}{t_2 - t_1} .$$

Thus we may now drop the dashes (primes) from II′-geodesics and write down merely II-*geodesics* in place of II′-geodesics.

THEOREM. - *The* II′-*geodesics in this section are the* II-*geodesics in the sense of our* Art. 5.

(iv) PARALLELISMS. - If we draw through a point $(b(x))$ outside of a II-geodesic (C) passing through $(a(x))$ vectors, which are equipollent of the first kind to several vectors lying on (C), we obtain the vector $\overrightarrow{b\eta}$, which is equipollent of the first kind to the vector $\overrightarrow{a\xi}$ whose extremity $(\xi(x))$ describes (C). Hence the point (η) describes a curve (C') and this curve is a II-geodesic. If we have

$$T_{\xi} = T_u T_a ,$$

then we deduce

$$T_{\eta} = T_u T_b$$

thence.

We say that (C') is *parallell of the first kind* to (C) and any vector lying on (C') is equipollent of the first kind to a vector lying on (C).

Two II-*geodesics, which are parallel of the first kind to a third, are parallel of the first kind to each other.*

We can define similarly II-*geodesics, which are parallel of the second kind to each other.* When this is defined by

$$T_{\xi} = T_a \Theta_u ,$$

we obtain II-geodesics defined by

$$T_{\eta} = T_b \Theta_u ,$$

where $(b(x))$ is an arbitrary fixed point.

Thus we have defined two kinds of parallelisms for the II-geodesics and for *each of these kinds,* we have the following properties:

1°. *Each* II-*geodesic is parallel to itself.*

2°. *Two* II-*geodesics, which are parallel to a third, are parallel to each other.*

3° *Through any point taken outside of a* II-*geodesic, there exists one and only one* II-*geodesic, which is parallel to the former.*

It should be remarked that the two parallelisms permit us easily to construct the vector $\overrightarrow{\xi\eta}$ equipollent of the

first | second

kind to a given vector $\overrightarrow{ab}$ and having a given origin $(\xi(x))$; for this it suffices to draw through $(\xi(x))$ the II-geodesic, which is parallel of the

first | second

kind to $\overrightarrow{ab}$ and then through $(b(x))$ the II-geodesic, which is parallel of the

second | first

kind to $\overrightarrow{a\xi}$; these two II-geodesics meet in the point $(\eta(x))$ sought for.

(v) It is convenient to say that two II-geodesics, which are parallel of the

first | second

kind, have *the same direction of the*

first | *second*

kind.

If we draw through the origin the parallel of the

first | second

kind to a given II-geodesic, then several points of this parallel represent the trasformations of a one-parametric group g. Hence we can say that *any direction of the*

first | *second*

kind is defined by a one-parametric subgroup of G.

If a one-parametric subgroup g of G together with a point $(a(x))$ of the space is given, starting from the point $(a(x))$ we can make a displacement in the direction of the

first | second

kind defined by g, and thus we obtain *two distinct* II-*geodesics* starting from $(a(x))$.

(vi) The equipollences of the first and second kinds permit us, as we have done in (iii) to define the equality and then the *ratio* of two segments lying on two geodesics, which are parallel of the first or second kind. If on a given II-geodesic, we choose a unit of length, we can thus measure the segment on all the geodesics, which are parallel of the first kind to given II-geodesic and then on any II-geodesic, which is parallel of the second kind to one of those latter and so on. Suppose that the given II-geodesic starting from the point of origin and defined by a subgroup g of transformations Θ_u, the u being the affine length (canonical parameter) The II-geodesics which thus arise by the indicated process are the loci of the points $(\xi(x))$ given by

$$T_\xi = T_a \Theta_u T_b,$$

the $(a(x))$ and the $(b(x))$ denoting two arbitrary fixed points, in particular, those among such II-geodesics, which pass through the point of origin, are given by

$$T_\xi = T_a \Theta_u T_a^{-1};$$

their directions are defined by the various homologous (gleichberechtigte ([11])) subgroups of g in the total group G. It is only in the set of these directions, that the space admits of an intrinsic metric.

(vii) Any point transformation of the group of isomorphism of the space $\mathfrak{E}$ transforms evidently a II-geodesic into a II-geodesic, the ratio of segments being conserved. It transforms further two parallel II-geodesic into two parallel II-geodesics.

Consider, in particular, the transformation

$$T_{\xi'} = T_a T_\xi.$$

By this transformation, the points of the space describe the vectors, which are equipollent of the first kind to one another. Moreover any vector is transformed into another vector, which is equipollent of the second kind to the former, and any II-geodesic into another II-geodesic, which is parallel of the second kind. We may give to such a transformation the name *"the translation of the first kind"*. These translations are the transformations of the first group of extended parameters ((ii) of Art. 18).

The equation

$$T_{\xi'} = T_\xi T_a$$

defines similarly *a translation of the second kind.*

The continuous translation of the first kind

$$T_{\xi'} = \Theta_u T_\xi,$$

([11]) Cf. [12], p. 474.

where Θ_u denote an arbitrary transformation of the one-parametric group $g(u$ playing the rôle of the time), plays the property, that respective points of the space describe the II-geodesics, which are parallel of the first kind to one another, while respective II-geodesics displace remaining parallel of the second kind to one another. We will call this continuous translation the II-*geodesic translation of the first kind.* We define similarly the II-*geodesic translation of the second kind.*

§4. - Simplification of the Fundamental Theorems on the Extended Lie Transformation Groups by Means of the II-Geodesic Parallel Coordinates.

21. **II-Geodesic Parallel Coordinates in the Base Manifold and the Group Space.** - In (6.6), we have already introduced II-*geodesic parallel coordinates* $\overline{\eta}^\lambda$ *in the extended Lie group manifolds.* Now we shall introduce II-*geodesic parallel coordinates* ξ *in the base manifold.* For this purpose we introduce a matrix

$$\overline{\xi}_i^j(x) \in C^2, \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, r)$$

corresponding to the matrix $\xi_j^i(x)$ introduced by (16.1) by the conditions:

$$\overline{\xi}_i^j \xi_l^i = \delta_l^j, \quad (i, k, p, q, \ldots = 1, 2, \ldots, n;\ h, j, l, \ldots = 1, 2, \ldots, r). \tag{21.1}$$

Multiplying

$$\overline{\Xi}_k^l \xi_l^i = \delta_k^i,$$

where $\overline{\Xi}_k^l$ are unknowns, by $\overline{\xi}_i^j$, we obtain $\overline{\Xi}_k^j = \overline{\xi}_k^j$ by virtue of (21.1), so that it results that

$$\overline{\xi}_k^l \xi_l^i = \delta_k^i \tag{21.2}$$

and multiplying $\overline{\Xi}_k^l \xi_l^i = \delta_k^i$ by $\overline{\xi}_i^j$, we obtain $\overline{\Xi}_k^j = \overline{\xi}_k^j$ arriving at (21.1). Thus we see that

$$(21.1) \rightleftharpoons (21.2).$$

For,

$$|\bar{\xi}^l_i\xi^i_l| = \|\bar{\xi}^l_i\|\cdot\|\xi^i_l\|$$

$$= \begin{vmatrix} \eta^1_1\eta^1_2 & \cdots & \eta^1_n & 0 & \cdots & 0 \\ \eta^2_1\eta^2_2 & \cdots & \eta^2_n & 0 & \cdots & 0 \\ \cdots & & & & & \cdots \\ \cdots & & & & & \cdots \\ \eta^r_1\eta^r_2 & \cdots & \eta^r_n & 0 & \cdots & 0 \end{vmatrix} \cdot \begin{vmatrix} \xi^1_1\xi^1_2 & \cdots & \xi^1_r \\ \xi^2_1\xi^2_2 & \cdots & \xi^2_r \\ \cdots & & \cdots \\ \cdots & & \cdots \\ \xi^n_1\xi^n_2 & \cdots & \xi^n_r \\ 0\ 0 & \cdots & 0 \\ \cdot\ \cdot & & \cdot \\ \cdot\ \cdot & & \cdot \\ 0\ 0 & \cdots & 0 \end{vmatrix}\Bigg\}{\scriptstyle r-n},$$

$$|\bar{\xi}^l_k\xi^i_l| = \|\bar{\xi}^l_k\|\cdot\|\xi^i_l\|$$

$$= {\scriptstyle r-n}\Bigg\{\begin{vmatrix} \xi^1_1\xi^1_2 & \cdots & \xi^1_r \\ \xi^2_1\xi^2_2 & \cdots & \xi^2_r \\ \cdots & & \cdots \\ \cdots & & \cdots \\ \xi^n_1\xi^n_2 & \cdots & \xi^n_r \\ 0\ 0 & \cdots & 0 \\ \cdot\ \cdot & & \cdot \\ \cdot\ \cdot & & \cdot \\ 0\ 0 & \cdots & 0 \end{vmatrix} \cdot \begin{vmatrix} \eta^1_1\eta^1_2 & \cdots & \eta^1_n & 0 \cdots 0 \\ \eta^2_1\eta^2_2 & \cdots & \eta^2_n & 0 \cdots 0 \\ \cdots & & & \cdots \\ \cdots & & & \cdots \\ \eta^r_1\eta^r_2 & \cdots & \eta^r_n & \underbrace{0 \cdots 0}_{r-n} \end{vmatrix},$$

we have

$$(21.3) \qquad |\bar{\xi}^j_i\xi^i_l| = |\delta^j_l| = 1. \qquad\qquad |\bar{\xi}^j_i\xi^i_l| = |\delta^i_l| = 1.$$

Replace $\underset{\mu}{\omega^l}(x^\nu)$ of Art. 5 by

$$\bar{\xi}^l_n, (l = 1, 2, \ldots, r; m, p, q = 1, 2, \ldots, n; \ r \geqq n)$$

and consider the Pfaffians

$$(21.4) \qquad \omega^l = \bar{\xi}^l_m(x)dx^m, \quad \left(\bar{\xi}^l_m(x^1, \ldots, x^n) = \left(\frac{\partial f^i(x; a(x))}{\partial a^j}\right)_{a^j = a^j_0}\right.,$$

which are assumed to be anholonomic in general and to be of rank r, so that the condition

$$(21.5) \qquad \|\bar{\xi}^l_m(x)\|^2 \neq 0 \quad \text{in } M$$

is satisfied.

We define the connection parameter Λ^l_{pq} by

$$(21.6) \qquad \Lambda^i_{pq} \overset{\text{def}}{=} \xi^i_l \frac{\partial\bar{\xi}^l_p}{\partial x^q} \equiv -\bar{\xi}^l_p \frac{\partial\xi^i_l}{\partial x^q},$$

the last identity arising from (21.2).

Consider a parametrized curve

$$x^i = x^i(t), \qquad (i = 1, 2, \ldots, n).$$

We can easily prove the identity

$$\frac{d}{dt}\frac{\omega^l}{dt} = \bar{\xi}^l_i(x)\left\{\frac{d^2x^i}{dt^2} + \Lambda^i_{pq}\frac{dx^p}{dt}\frac{dx^q}{dt}\right\}. \tag{21.7}$$

We consider the combined manifold:

$$\{x^i\} + \{\bar{\xi}^l_i(x)\}$$

forming a principal fibre bundle, the

$$\{\bar{\xi}^i_l(x)\} = \{\xi^i_l(x)\}$$

making the *structure group.* Although the group elements $\bar{\xi}^l_i(x)$ can contain the local coordinates (x^i), the *function forms* make the group elements (in a certain sense) independent of the local coordinates (x^i).

From (21.7), we have

$$\frac{d}{dt}\frac{\omega^l}{dt} = 0 \quad \rightleftharpoons \quad \frac{d^2x^i}{dt^2} + \Lambda^i_{pq}\frac{dx^p}{dt}\frac{dx^q}{dt} = 0. \tag{21.8}$$

Indeed, we can convert (21.7) into

$$\xi^i_l\frac{d}{dt}\frac{\omega^l}{dt} = \frac{d^2x^i}{dt^2} + \Lambda^i_{pq}\frac{dx^p}{dt}\frac{dx^q}{dt}. \tag{21.9}$$

The differential equations on the right-hand side of (21.8) define the autoparallel curves of the teleparallelism. *The left-hand side is convenient for the study of the global properties and is integrated readily.*

$$\omega^l = c^l dt, \qquad (c^l = \text{const.}), \tag{21.10}$$

$$\int\frac{\omega^l}{dt}dt = c^l t + d^l, \qquad (d^l = \text{const.}), \tag{21.11}$$

the (21.11) being guided by the simple character of the right-hand side of (21.10). Noticing again the simple character of the right-hand side of (21.11), we set

$$\xi^l = c^l t + d^l,$$

so that

$$\xi^l = \int \frac{\omega^l}{dt} dt = c^l t + d^l. \tag{21.12}$$

This means that *we adopt such curves as* ξ^l — *axes in the r-dimensional space cantaining subspace* $\{x^i\}$.

From (21.12), we see that *the curves represented by* (21.12) *behave as for meet and join like straight lines in the large.* We will call such curves II-*geodesic curves.*

Although the ω^l are anholonomic in general, we may write it in the form of differentials:

$$d\xi^l = \omega^l = \bar{\xi}^l_i(x(t))dx^i(t) \tag{21.13}$$

for the II-*geodesic line-elements, where*

$$\| \bar{\xi}^l_i(x) \| \neq 0 \quad \text{in} \quad M. \tag{21.14}$$

The expressions (21.12) tells us that, for the given $\bar{\xi}^l_i(x)dx^i$, there exists a curve $x^i(t)$, whose line-elements $\{dx^i\}$ with directions $\{c^l\}$ is given by the differential $d\xi^l$. This is the case for all the directions $\{c^l\}$. Thus in (21.13), we may omit t and write down as follows:

$$d\xi^l = \bar{\xi}^l_i(x)dx^i \tag{21.15}$$

notwithstanding the right-hand side is anholonomic in general.

The first differential equation of (21.8) may be rewritten as follows:

$$\frac{d^2\xi^l}{dt^2} = 0.$$

Multiplying (21.10) with $\xi^i(x)$ and taking (21.1) into account, we see that *the relations*

$$\frac{dx^i}{dt} = c^l \xi^i_l \tag{21.16}$$

hold along the II-*geodesic line-elements.*

We will call $\{\xi^l\}$ the II-*geodesic parallel coordinates corresponding to* $\bar{\xi}^l_i$ referred to the ξ^l-axes. The $\{\xi^l\}$ are *global* in the large.

From (21.15), we obtain

$$\xi^l = \int \bar{\xi}^l_i(x)dx^i = \bar{\xi}^l_i(x)x^i - \int x^i d\bar{\xi}^l_i(x),$$

$$\xi^l = \bar{\xi}^l_i(x)x^i + \bar{\xi}^l_0. \qquad (\bar{\xi}^l_0 = \text{const.}) \tag{21.17}$$

as in the case of (6.8), the differential equations to the II-geodesic curves being

$$d\bar{\xi}^l_i(x)dx^i = 0 \tag{21.18}$$

or

$$x^i d\bar{\xi}^l_i(x) = 0 \tag{21.19}$$

as in the case of (5.14) and (5.16).

22. To prove

$$\frac{\partial}{\partial \xi^l} = \xi^i_l \frac{\partial}{\partial x^i} \quad \textbf{and} \quad \frac{\partial}{\partial \alpha^l} = \alpha^j_l \frac{\partial}{\partial a^j}.$$

$$\tau^l = \bar{\xi}^l_k dx^k,$$

$$\frac{\partial \psi(x;a)}{\partial \xi^l} = \lim_{dx^i \to 0} \frac{\dfrac{\partial \psi(x;a)}{\partial x^i} dx^i}{\bar{\xi}^l_k dx^k}$$

$$= \lim_{dx^i \to 0} \frac{\dfrac{\partial \psi(x;a)}{\partial x^i} \xi^i_h \tau^h}{\bar{\xi}^l_k \xi^k_j \tau^j}$$

$$= \lim_{dx^i \to 0} \frac{\tau}{\delta^l_j \tau^j} \xi^i_h \frac{\partial \psi(x;a)}{\partial x^i}$$

$$= \lim_{dx^i \to 0} \frac{\tau^h}{\tau^l} \xi^i_h \frac{\partial \psi(x;a)}{\partial x^i}$$

$$= \delta^h_l \xi^i_h \frac{\partial \psi(x;a)}{\partial x^i} = \xi^i_l \frac{\partial \psi(x;a)}{\partial x^i}.$$

$$\alpha^j = \beta^j_l da^l,$$

$$\frac{\partial \psi(x;a)}{\partial \alpha^l} = \lim_{da^j \to a} \frac{\dfrac{\partial \psi(x;a)}{\partial a^j} da^j}{\beta^l_j da^j}$$

$$= \lim_{da^j \to 0} \frac{\dfrac{\partial \psi(x;a)}{\partial a^j} \alpha^j_k d\alpha^k}{\beta^l_j \alpha^j_h \alpha^h}$$

$$= \lim_{da^j \to 0} \frac{\alpha^k}{\delta^l_h \alpha^h} \alpha^i_k \frac{\partial \psi(x;a)}{\partial a^j}$$

$$= \lim_{da^j \to 0} \frac{\alpha^k}{\alpha^l} \alpha^j_k \frac{\partial \psi(x;a)}{\partial a^j}$$

$$= \delta^k_l \alpha^j_k \frac{\partial \psi(x;a)}{\partial a^j} = \alpha^j_l \frac{\partial \psi(x;a)}{\partial a^j}.$$

Hence

$$\frac{\partial}{\partial \xi^l} = \xi^i_l \frac{\partial}{\partial x^i}. \qquad \frac{\partial}{\partial \alpha^l} = \alpha^j_l \frac{\partial}{\partial a^j}. \tag{22.1}$$

23. Simplification of the First Fundamental Theorem on the Extended Lie Transformation Group by means of the II-Geodesic Parallel Coordinates. The First Fundamental Theorem of the Theory of the Extended Lie Transformation Groups has been stated in the form of Cor. 2° of Art. 16. Now by virtue of the last article, it may be simplified and made *global* as follows.

THE FIRST FUNDAMENTAL THEOREM (the simplified form). *In the extended Lie transformation group G as extended parameter group, the* $f^k(\xi; a(\xi))$, $(k=1, 2, \dots, n)$ *are n independent solutions of the completely integrable simultaneous linear partial differential equations*

$$\frac{\partial f}{\partial \alpha^l} = \frac{\partial f}{\partial \xi^l}, \quad (j,\ l=1,\ 2, \dots ,\ r;\ i,\ k=1,\ 2, \dots ,\ n) \tag{23.1}$$

such that

$$\xi^i = f^i(\xi;\ 0), \qquad (i=1,\ 2,\ \dots,\ n). \tag{23.2}$$

Conversely, when an r-dimensional extended Lie group G is given, the (23.1) *is completely integrable, their solutions* $f^l(\xi;\ a(x))$, $(l=1,\ 2,\ \dots,\ r)$ *satisfyling* (23.2), *determine an axtended Lie transformation group having G as extended parameter group.*

SOLUTION OF (23.1) The Lagrange's auxiliary differential equations of (23.1) are

$$d\xi^l = -d\alpha^l,\ df(x; a(x)) = 0. \qquad [(16.3), (16.9)] \tag{23.3}$$

The (16.19) becomes in this case:

$$\overline{X} = e^j\xi^i_j \frac{\partial}{\partial x^i} - e^i\alpha^l_j \frac{\partial}{\partial a^l} = e^j\left(-\frac{\partial}{\partial \xi^j} + \frac{\partial}{\partial x^j}\right). \tag{23.4}$$

Consider

$$-\overline{X}f = 0. \tag{23.5}$$

The Lagrange's auxiliary differential equations become

$$\begin{aligned} \frac{dx^i}{e^j\xi^i_j} &= \frac{da^k}{-e^j\alpha^k_j} = dt \\ &= \frac{\bar{\xi}^l_i dx^i}{e^j\bar{\xi}^l_i\xi^i_j} = \frac{\alpha^l_k da^k}{-e^j\alpha^l_k\alpha^k_j} \\ &= \frac{d\xi^l}{e^j\delta^l_j} = \frac{d\alpha^l}{-e^j\delta^l_j}, \end{aligned}$$

so that

$$d\xi^l = -d\alpha^l = e^l dt \tag{23.6}$$

in conformity with (23.3), whence follows:

$$\xi^l = \alpha_0^l - \alpha^l(x) = e^l(t - t_0), \qquad (\alpha_0^l,\ t_0 = \text{const.}), \tag{23.7}$$

which represents a II-*geodesic curve corresponding to*

ξ_j^i	α_j^l
in the differentiable base manifold.	*group manifold.*

The *complete integral* consists of (23.7) and the *general integral* is

$$\chi(\xi^1 + \alpha^1(\xi),\ \xi^2 + \alpha^2(\xi),\ \dots,\ \xi^r + \alpha^r(\xi)), \tag{23.8}$$

where χ is an arbitrary function.

Comparing (23.7) with

$$\xi^l = \bar{\xi}_i^l(x)x^i + \bar{\xi}_0^l, \qquad (\bar{\xi}_0^l = \text{const.},\ i = 1, 2, \dots, n;\ l = 1, 2, \dots, r), \tag{23.9}$$

we see that

$$\bar{\xi}_i^l(x)x^i + \bar{\xi}_0^l = \alpha_0^l - \alpha^l(x), \tag{23.10}$$

so that

$$\alpha^l(x) = \alpha_0^l - \bar{\xi}_0^l(x)x^i - \bar{\xi}_0^l = \alpha_0^l - \xi^l. \tag{23.11}$$

The inverse transformation of (23.9) was

$$x^i = \xi_j^i(\xi)\xi^j + \xi_0^i. \qquad (\xi_0^j = \text{const.}). \tag{23.12}$$

N.B. (i) The differential equations (16.13) reduce to (23.1). (ii) The differential equations (16.22) reduce to $d\xi'^l = d\alpha^{*l}(a(\xi))$.

24. **Simplificaton of the Second Fundamental Theorem.** – *When a given r-dimensional extended Lie group G as an extended parameter group has the structure constants*

$$C_{ij}^k, \quad (i, j, k = 1, 2, \dots, r),$$

the necessary and and sufficient condition for that (23.1) *may be completely integrable, is that the relations*

$$C_{jl}^h = 0, \qquad (h,\ j,\ l = 1,\ 1,\ \dots,\ r) \tag{24.1}$$

holds.

PROOF. - In (16.29), we have

$$(X_j, X_l) = \xi_j^h(x)\frac{\partial}{\partial x^h}\xi_l^k(x)\frac{\partial}{\partial x^k} - \xi_l^k(x)\frac{\partial}{\partial x^k}\xi_j^h(x)\frac{\partial}{\partial x^h}$$

$$= \frac{\partial^2}{\partial\xi^j\partial\xi^l} - \frac{\partial^2}{\partial\xi^l\partial\xi^j} = 0$$

and X_j are linearly independent.

25. **Simplification of the Third Fundamental Theorem.** - *When r linearly independent differential operators*

$$X_j f = \xi_j^i(x)\frac{\partial f}{\partial x^i} = \frac{\partial f}{\partial \xi^j}, \qquad (i = 1,\ 2,\ \dots,\ n;\ j = 1,\ 2,\ \dots,\ r), \tag{25.1}$$

$$(\xi_j^i(x) \in C^2)$$

are given, the necessary and sufficient condition for that they are the fundamental differential operators for an extended Lie transformation group, is that the relations

$$C_{jk}^h = 0, \qquad (h,\ j,\ k = 1,\ 2,\ \dots,\ r) \tag{25.2}$$

hold.

26. **Simplification of the Fourth Fundamental Theorem.** - *The* r^3 *constants*

$$C_{jl}^h = 0, \qquad (h,\ j,\ l = 1,\ 2,\ \dots,\ r)$$

for the fundamental differential operators

$$\frac{\partial}{\partial\xi^1},\ \frac{\partial}{\partial\xi^2},\ \dots,\ \frac{\partial}{\partial\xi^r}$$

of an extended Lie transformation group make the following three conditions:

$$(16.29)\qquad (X_j, X_l) = C^h_{jl}X_h, \qquad (h,\ k,\ j,\ l = 1,\ 2,\ \dots,\ r),$$

$$(16.44)\qquad C^h_{jl} = -\ C^h_{lj},$$

$$(16.45)\qquad C^h_{ij}C^l_{hk} + C^h_{jk}C^l_{hi} + C^h_{ki}C^l_{hj} = 0, \qquad (i,\ j,\ k,\ l = 1,\ 2,\ \dots,\ r)$$

identities, so that the Fourth Fondamental Theorem of Art. 16 *holds.*

§ 5. Adjoint Extended Lie Transformation Groups.

27. **Adjoint Extended Group of Extended Lie Transformation Groups.** - In Art. 11, we have extended the concept of adjoint group ([12], p. 450) of a LIE transformation group to the case of the *adjoint extended group of an extended Lie transformation group G.*

I. We shall first study the adjoint extended transformations

$$\tilde{e}^i = \xi^i_l(c^1,\ c^2,\ \dots,\ c^r)e^l,$$

where the e^l are those, which we have considered in (23.4).

Since

$$(27.1)\qquad x^i(t) = \tilde{e}^i t + \tilde{c}^i_0, \qquad (\tilde{c}^i_0 = \text{const.})$$

for the II-geodesic curves in the manifold, the (23.6) and (23.7) may be rewritten as follows:

$$(27.2)\qquad d\xi^l = \bar{\xi}^l_i(x)\tilde{e}^i dt = e^l dt, \qquad \Big|\qquad \xi^l = \bar{\xi}^l_i(x)(\tilde{e}^i t + \tilde{c}^i_0) + \bar{c}^l_0$$

$$= e^l + c^l,\ \ (c^l = \text{const.},\ \bar{c}^l = \bar{\xi}^l_0),$$

so that

$$(27.3)\qquad \bar{\xi}^l_i(x)\tilde{e}^i = e^l, \qquad \Big|\qquad \bar{\xi}^l_0(x)\tilde{c}^i_0 + \bar{c}^l_0 = c^l,$$

whose inverse transformation is

$$(27.4)\qquad \tilde{e}^i = \xi^i_l(x)e^l. \qquad \Big|\qquad \tilde{c}^i_0 = \xi^i_l(x)c^l + \xi^i_0.$$

Thus

e^l	$\tilde{e}^t$	c^l	$\tilde{c}_0^i$

undergo the extended affine transformations

(27.3).	(27.5).	(27.3).	(27.4).

II. Next we will consider the general case. Let us denote the operator corresponding to

$$x'^i = f^i(x;\ a) \tag{27.5}$$

by $X'_l f$. Then we shall have

$$e^h X_h f = e'^h X'_h f, \tag{27.6}$$

where e'^h are certain functions of

$$a^1,\ a^2,\ \dots,\ a^r,\ e^1,\ e^2,\ \dots,\ e^r$$

by virtue of (27.4).

If we set $f = x^i$ in (27.6), then it results that

$$e^l \xi_l^i(x) = e'^l X'_l x^i, \qquad (i = 1,\ 2,\ \dots,\ n). \tag{27.7}$$

If we give r determinate values $\overset{(p)}{x}{}^i$, $(p = 1,\ 2,\ \dots,\ r)$ to x^i, then x'^i becomes j functions of $a^1,\ a^2,\ \dots,\ a^r$. Thus we obtain

$$e^l \xi_l^i(\overset{(p)}{x}) = e'^l \xi'^k_l(\overset{(p)}{x'}) \left[\frac{\partial x^i}{\partial x'^k}\right]_{x^i = \overset{(p)}{x^i}} . \tag{27.8}$$

Thereby we assume that r values $(p = 1,\ 2,\ \dots,\ r)$ of $\overset{(p)}{x}{}^i$ have been so chosen that

$$|\ \xi_l^i(\overset{(p)}{x})\ | \neq 0, \qquad (i = 1,\ 2,\ \dots,\ n). \tag{27.9}$$

Let $\bar{\xi}_i^j(\overset{(p)}{x})$ be a matrix such that

$$\xi_l^i(\overset{(p)}{x}) \bar{\xi}_i^j(\overset{(p)}{x}) = \delta_l^j, \qquad (p\text{: summed};\ i\text{: not summed}), \tag{27.10}$$

and multiply (27.8) with $\bar{\xi}_i^j(\overset{(p)}{x})$ and sum the result with respect to p.

Then we obtain

$$e^j = e^l\delta^j_l = e'^l\xi'^k_l(x')\overline{\xi}^{j(p)}_i(x)\left[\frac{\partial x^i}{\partial x'^k}\right]_{x^i=\overset{(p)}{x}{}^i}, \qquad (p: \text{ summed})$$

i.e.

$$(27.11) \qquad e^j = \rho^j_l(a(x))e'^l, \qquad (\mid \rho^j_l(a(x)) \mid \neq 0),$$

where

$$(27.12) \qquad \rho^j_l(a(x)) = \xi'^k_l(x)\overline{\xi}^{j(p)}_i(x)\left[\frac{\partial x^i}{\partial x'^k}\right]_{x^i=\overset{(p)}{x}{}^i}, \qquad (p: \text{ summed}).$$

If we denote the inverse transformation of (27.12) by $\overline{\rho}^l_j(a(x))$, we have

$$(27.13) \qquad \rho^h_l(a(x))\overline{\rho}^l_k(a(x)) = \delta^h_k, \quad \rho^l_k(a(x))\overline{\rho}^h_l(a(x)) = \delta^h_k,$$

and

$$(27.14) \qquad e'^l = \rho^l_j(a(x))e^j.$$

That (27.11) *forms a group* may be proved as in the case of [12], p. 452.

28. **The Adjoint Extended Transformation Group in terms of the II-Geodesic Parallel Coordinates.** The (27.5) becomes

$$(28.1) \qquad e^l X_l f \equiv e^l\frac{\partial f}{\partial \xi^l} = e'^l X'_l f \equiv e'^l\frac{\partial f}{\partial \xi'^l},$$

when ξ^l and ξ'^l are respective II-geodesic parallel coordinates, such that

$$(28.2) \qquad \xi'^l = \overline{\xi}^l_j(\xi)\xi^j, \quad \xi^l = \xi^l_j(\xi')\xi'^j.$$

If we set $f = \xi^l$, we obtain

$$e^l = e^j\delta^l_j = e'^j\frac{\partial \xi^l}{\partial \xi'^j} = e'^j\xi^l_j(\xi')$$

i.e.

$$(28.3) \qquad e^l = e'^j\xi^l_j(\xi'), \qquad e'^l = \overline{\xi}^l_k(\xi)e^k.$$

Thus $\xi^l_j(\xi')$ *and* $\overline{\xi}^l_j(\xi)$ *themselves play the rôles of* $\rho^l_j(a(\xi'))$ *and* $\overline{\rho}^l_j(a(\xi))$ in (27.11) *and* (27.14) *respectively.*

29. The Canonical Equations of an r-Dimensional Extended Lie Transformation Group. – The following theorem is an extension of a theorem ([12], p. 454, Theorem 32) of Sophus Lie:

THEOREM. – *If*

$$x'^i = x^i + e^l X_l x^i + \frac{1}{2!} e^j e^l X_j X_l x^i + \dots \tag{29.1}$$

be the canonical equations of an r-dimensional extended Lie transformation gronp $X_1 f$, $X_2 f$, ..., $X_r f$ in n variables x^1, x^2, ..., x^n and if we apply the transformation (27.4), *then the transformations* (e^1, e^2, ..., e^r) *are transformed into* (e'^1, e'^2, ..., e'^r) *by the transformations*

(27.14),	(28.3),

where

$\mid \rho^l_j(a(x)) \mid \neq 0.$	$\mid \bar{\xi}^l_k(\xi) \mid \neq 0.$

The transformations

(27.4)	(28.3)

form a group and the relation

(27.6)	(28.1)

holds.

The part concerning (29.1) may be proved quite as in the case of ([12], p. 454, Theorem 32).

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