An Elementary Derivation of an Inequality Involving R-Sequences.

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Dedicated to Professor B. SEGRE on his seventieth birthday

Summary. - *A basic inequality is derived for the grade of a finitely generated ideal on a general module. The methods used are both simple and elementary. No Noetherian conditions are needed.*

Introduction.

Let R be a commutative ring with an identity element and let E be an R-module. A sequence $\beta_1, \beta_2, \ldots, \beta_s$ of elements of R is called an *R-sequence on E* if for each i $(1 \leqslant i \leqslant s)$ β_i is a non-zerodivisor on $E/(\beta_1, \ldots, \beta_{i-1})E$, that is if

$$(\beta_1, \ldots, \beta_{i-1})E :_E \beta_i = (\beta_1, \ldots, \beta_{i-1})E$$

for $1 \leqslant i \leqslant s$. The inequality referred to in the title is described in the following

THEOREM. - *Let A be an ideal of R which can be generated by n $(n \geqslant 0)$ elements and suppose that $E \neq AE$. If now $\beta_1, \beta_2, \ldots, \beta_s$ is an R-sequence on E contained in A, then $s \leqslant n$.*

One can approach this result in various ways. First suppose that E is a *Noetherian* R-module and let I be the ideal formed by the annihilators of E so that $I = 0 :_R E$. Then R/I is a Noetherian ring and one can show [(1) Proposition 4, p. 247] that s does not exceed the rank (= height) of the ideal $(I, A)/I$ of this ring. However this is a proper ideal generated by n elements. Consequently, by Krull's *Principal Ideal Theorem*, the rank of the ideal in question does not exceed n and so we arrive at the desired inequality.

Apart from the fact that this method uses some rather powerful results from the theory of commutative Noetherian rings, it is unsatisfactory because it introduces an unnecessary condition namely that the module E is Noetherian. One way in which to avoid this complication is to use the theory of the Koszul complex (**). But although

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(**) See, for example, [(1) Exercise 9, p. 375].

this achieves the extra generality, the procedure is still open to the criticism that it uses techniques that are more elaborate than the situation appears to demand. It will now be shown how these difficulties may be circumvented.

PROOF OF THE THEOREM. – The argument depends on two lemmas neither of which is new. However for the sake of completeness we shall give proofs. The notation remains as before.

LEMMA 1. – *Let $\beta_1, \beta_2, \ldots, \beta_s$ be an R-sequence on E. Suppose $1 \leqslant j < s$ and that*

$$(\beta_1, \ldots, \beta_{j-1})E :_E \beta_{j+1} = (\beta_1, \ldots, \beta_{j-1})E\,.$$

Then $\beta_1, \ldots, \beta_{j-1}, \beta_{j+1}, \beta_j, \beta_{j+2}, \ldots, \beta_s$ is also an R-sequence on E.

PROOF. – It is enough to show that

$$(\beta_1, \ldots, \beta_{j-1}, \beta_{j+1})E :_E \beta_j = (\beta_1, \ldots, \beta_{j-1}, \beta_{j+1})E\,.$$

To this end suppose that $\beta_j e = \beta_1 e_1 + \ldots + \beta_{j-1}e_{j-1} + \beta_{j+1}e_{j+1}$, where e, e_1, e_2 etc. denote elements of E. Then $\beta_{j+1}e_{j+1} \in (\beta_1, \ldots, \beta_{j-1}, \beta_j)E$. Accordingly

$$e_{j+1} \in (\beta_1, \ldots, \beta_{j-1}, \beta_j)E$$

and therefore $e_{j+1} - \beta_j\varepsilon$ is in $(\beta_1, \ldots, \beta_{j-1})E$ for a suitable ε in E. It follows that $\beta_j(e - \beta_{j+1}\varepsilon) \in (\beta_1, \ldots, \beta_{j-1})E$ whence $e - \beta_{j+1}\varepsilon$ is in $(\beta_1, \ldots, \beta_{j-1})E$. Thus $e \in (\beta_1, \ldots, \beta_{j-1}, \beta_{j+1})E$ and now the lemma follows.

Let x be an indeterminate. Then besides the polynomial ring $R[x]$ we can also form the $R[x]$-module $E[x]$ which consists of polynomials in x with coefficients in E.

LEMMA 2. – *Let $A = (\alpha_0, \alpha_1, \ldots, \alpha_m)$ be an ideal of R and put $\varphi = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_m x^m$. Then the following statements are equivalent:*

(1) $0 :_E A = 0$;

(2) *φ is a non-zerodivisor on $E[x]$.*

PROOF. – We assume that (1) is true and (2) is false and from this we derive a contradiction. This will show that (1) implies (2). The converse is trivial.

We can find $\omega = e_0 + e_1 x + \ldots + e_q x^q$ in $E[x]$ such that $\omega \neq 0$ but $\varphi\omega = 0$, and we arrange that q is minimal. Then $e_q \neq 0$ and, since $\varphi\omega = 0$, we have $\alpha_m e_q = 0$. Now $\alpha_m\omega$ has smaller degree than ω and it is annihilated by φ. Consequently, by the minimality of q, $\alpha_m\omega = 0$ and therefore

$$(\alpha_0 + \alpha_1 x + \ldots + \alpha_{m-1}x^{m-1})\,\omega = (\varphi - \alpha_m x^m)\,\omega = 0\,.$$

Accordingly $\alpha_{m-1} e_q = 0$, whence $\alpha_{m-1}\omega$ has smaller degree than ω and is annihilated by φ. Thus $\alpha_{m-1}\omega = 0$ and now we see that

$$(\alpha_0 + \alpha_1 x + \ldots + \alpha_{m-2}x^{m-2})\,\omega = (\varphi - \alpha_m x^m - \alpha_{m-1}x^{m-1})\,\omega = 0\ .$$

Proceeding in this way we find that $\alpha_i e_q = 0$ for $i = 0, 1, \ldots, m$. Thus $Ae_q = 0$ but $e_q \neq 0$. This is the required contradiction.

We turn our attention to the theorem stated in the introduction and use the notation described there. The argument uses induction on n. If $n = 0$ the assertion is trivial. Suppose therefore that the result in question has been proved for $n = m$ and now let $A = (\alpha_0, \alpha_1, \ldots, \alpha_m)$. We have an R-sequence $\beta_1, \beta_2, \ldots, \beta_s$ on E contained in A. Since we wish to show that $s \leqslant m + 1$, we may suppose that $s \geqslant 1$. Now $\beta_1, \beta_2, \ldots, \beta_{s-1}$ is an R-sequence on E in A and a simple verification shows that it is also an $R[x]$-sequence on $E[x]$ in $AR[x]$. (Here x is an indeterminate.) Next β_s is not a zerodivisor on the R-module $K = E/(\beta_1, \ldots, \beta_{s-1})E$ and therefore $0 :_K A = 0$. Consequently, by Lemma 2, $\varphi = \alpha_0 + \alpha_1 x + \ldots + \alpha_m x^m$ is not a zerodivisor on the $R[x]$-module $K[x]$. But we may identify $K[x]$ with $E[x]/(\beta_1, \ldots, \beta_{s-1})E[x]$. It follows that *$\beta_1, \beta_2, \ldots, \beta_{s-1}, \varphi$ is an $R[x]$-sequence on $E[x]$ in $AR[x]$*.

Suppose that $0 \leqslant j < s$. We can apply the conclusion of the last paragraph to $\beta_1, \ldots, \beta_j, \beta_{j+1}$ and so deduce that $\beta_1, \ldots, \beta_j, \varphi$ is an $R[x]$-sequence on $E[x]$ for $j = 0, 1, \ldots, s-1$. It is now possible to make repeated applications of Lemma 1 and thus conclude that *$\varphi, \beta_1, \ldots, \beta_{s-1}$ is an $R[x]$-sequence on $E[x]$*.

Next we note that

$$AR[x] = (\alpha_0, \alpha_1, \ldots, \alpha_m)R[x] = (\varphi, \alpha_1, \ldots, \alpha_m)R[x]\ .$$

Put $R^* = R[x]/\varphi R[x]$, $A^* = AR[x]/\varphi R[x]$, $E^* = E[x]/\varphi E[x]$ and denote by α_i^*, β_j^* the images of α_i, β_j in R^*. Then $\beta_1^*, \beta_2^*, \ldots, \beta_{s-1}^*$ is an R^*-sequence on the R^*-module E^* and it is contained in $A^* = (\alpha_1^*, \alpha_2^*, \ldots, \alpha_m^*)$. Further, because $AE \neq E$, we must have $(AR[x])E[x] \neq E[x]$ and therefore $E^* \neq A^*E^*$. At this point we may apply the inductive hypothesis to deduce that $s - 1 \leqslant m$, that is $s \leqslant m + 1$. This completes the proof.

REFERENCES

[1] D. G. NORTHCOTT, *Lesson on rings, modules and multiplicities*, Cambridge University Press, 1968.