Transformations of relations between numerical functions.

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1. On several occasions Cesàro [1] considered arithmetical identities between functions

$$f[\varepsilon_\alpha(x)], \quad g[\varepsilon_\beta(x)], \ldots \qquad (\alpha, \beta, \ldots = 1, 2, 3, \ldots),$$

in which $\varepsilon_\alpha(x)$, $\varepsilon_\beta(x)$,... are single valued functions of x, and $f(y)$, $g(y)$,... single valued functions of y, and where, moreover, for every pair of integers α, $\beta > 0$,

$$\text{(1)} \qquad \varepsilon_\alpha[\varepsilon_\beta(x)] = \varepsilon_{\alpha\beta}(x) = \varepsilon_\beta[\varepsilon_\alpha(x)].$$

If in (1) we put $\beta = 1$, we see that $\varepsilon_1(x) = x$, and evidently

$$\text{(2)} \qquad \varepsilon_\alpha(x), \qquad \alpha = 1, 2, 3, \ldots,$$

form an abelian semi-group [2], whose order may be finite or infinite.

It is noted by Cesàro that obviously it is sufficient, if the $\varepsilon_\alpha(x)$ are to form an abelian semi-group, to take for α those integers > 0 which are prime to a given integer n. The case above is that in which $n = 1$. He considered also [3] *enumerative functions* $\Omega(x)$, such that $\Omega(x) = 1$ or 0 according as the integer $x > 0$ is or is not a member of a given class, and in particular he discusses the case for which

$$\text{(3)} \qquad \Omega(x)\Omega(y) = \Omega(xy),$$

viz., that in which the product of two integers of a given class is contained in that class, and the product of two integers not both in the class is not a

(1) *Annali di Matematica pura ed applicata*, serie 2ª, vol. 13, pp. 235-351; reprinted in *Excursions arithmétiques à l'infini*, pp. 104-117, Paris, 1885; also elsewhere, cf. references in Dickson's: *History of the Theory of Numbers*, vol. I (Washington, D. C., 1919).

(2) It is inexact to say that the functions (2) form an abelian *group*. Thus, for example, if α is prime, $\varepsilon_\alpha(x)$ has no inverse in the set (2). Cesàro, (*Annali*, vol. 14, p. 155), calls (2) a *groupe fermé*; the nomenclature here employed is in accordance with modern usage. Likewise for Cesàro's Ω.

(3) Mémoires de l'Académie de Belgique, 6 février 1886; *Annali*, vol. 14, pp. 141 et seq..

member of the class. Putting $x = 1$ we see that $\Omega(1) = 1$. From these considerations CESÀRO obtains a remarkable inversion of series [1] « le plus général que l'on connaisse », and the points out « la possibilité de remplacer le système des nombres entiers par un groupe fermé quelconque, détaché du même système. Cette possibilité a été implicitement reconnue par MÖBIUS, qui s'est occupé, il y a longtemps, (*Crelle's Journal*, vol. 9, p. 105), de l'inversion des séries dans un cas particulier ».

I will show that the inversions of CESÀRO and MÖBIUS are abstractly identical, so that, in particular either can be immediately inferred from the other. Incidentally in § 2 there is obtained and stated an extremely general theorem of reciprocity, of which the CESÀRO-MÖBIUS theorem is the simplest instance.

2. Let n denote an integer > 0. Since $\varepsilon_n(x)$ is a single valued function of x for $n = 1, 2, 3, \ldots$, and since $f(y)$ is a single valued function of y, it follows that $f[\varepsilon_n(x)]$ takes a single definite value for each n. A function $\psi(y)$ is called *numerical* if $\psi(y)$ takes a single definite value for each integer value of $y > 0$. We can therefore consider $f[\varepsilon_n(x)]$ as a numerical function of n, and to emphasize that n rather than x is the variable under attention, we shall write

$$f[\varepsilon_n(x)] \equiv f_n(x) \equiv f_x(n).$$

I have fully discussed the consequences of this point of view elsewhere [2].

Here it is sufficient to note the following: $f_x(n)$ is a *particular instance* of $\psi(n)$. Hence, if we have established a specific relation between arbitrary numerical functions $f(n)$, $g(n)$, $h(n), \ldots$, it follows that the relation remains valid when $f, g, h, \ldots$ are replaced by $f_x, g_x, h_x, \ldots$ respectively.

Conversely, if in a specific relation between $f_x(n)$, $g_x(n)$, $h_x(n), \ldots$ we take the special $\varepsilon_n(x)$ such that $\varepsilon_n(x) \equiv nx$, as obviously is permissible by (1), and if in the result we replace x by 1, we obtain a relation between the numerical functions $f(n)$, $g(n)$, $h(n), \ldots$, and this relation can be written down at once from the original by suppressing the suffix x in $f_x, g_x, h_x, \ldots$.

Hence either of the relations between $f, g, h, \ldots$ or $f_x, g_x, h_x, \ldots$ implies the other, and the second can be written down from the first by supplying the suffix x, the first from the second by dropping the suffix x.

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(1) *Annali*, loc. cit., p. 155, (12), (13).

(2) *Bulletin of the American Mathematical Society*, *Transactions* (of the same) vol. 25, pp. 145-154.

From the definitions it is evident that $\Omega(y)$ is a particular instance of a numerical function of y. Hence if $f(y)$ is a numerical function of y, so also is $\Omega(y)f(y)$. Note that $\Omega(y)f(y)$ is the ordinary algebraic product of $\Omega(y)$ and $f(y)$. For n an arbitrary integer >0 it is sometimes convenient to write symbolically

$$\Omega(n)f(n) \equiv (\Omega f)(n) \equiv (\Omega f),$$

the parenthesis enclosing (Ωf) being used to distinguish this type of product from Ωf in § 3 which does *not* symbolize the ordinary algebraic product. As before we write

$$\Omega[\varepsilon_n(x)]\, f[\varepsilon_n(x)] \equiv \Omega_x(n)\, f_x(n) \equiv (\Omega_x f_x),$$

and note that, if $f(n)$, $g(n)$ are arbitrary numerical functions of n, then a relation involving $f(n)$ remains valid when for f is substituted $(\Omega_x g_x)$, and conversely.

From the foregoing we now have the following general statement: *A relation between arbitrary numerical functions*

(3) $$f(n),\quad g(n),\quad h(n),\ldots,$$

remains valid when for these functions are substituted respectively

(4) $$(\Omega_x^{(a)} f_x)(n),\quad (\Omega_y^{(b)} g_y)(n),\quad (\Omega_z^{(c)} h_z)(n),\ldots,$$

where the superscripts (a), (b), (c),... *refer to any classes* A, B, C,... *of integers respectively, and where the variables* x, y, z,... *are not necessarily independent, nor the classes* A, B, C,... *necessarily mutually exclusive, nor the abelian semi-groups*

$$e_\alpha(x),\quad e_\beta(y),\quad e_\gamma(z),\ldots \qquad (\alpha, \beta, \gamma,\ldots = 1, 2, 3,\ldots)$$

necessarily distinct.

To pass from the relation for (3) to that for (4) we replace in the former f by $\Omega_x^{(a)} f_x$, g by $\Omega_y^{(b)} g_y$, h by $\Omega_z^{(c)} h_z,\ldots$; to pass from the relation for (4) to that for (3) we take

$$e_\alpha(x) = \alpha x,\quad e_\beta(y) = \beta y,\ldots \qquad (\alpha, \beta,\ldots = 1, 2, 3,\ldots),$$

$A = B = C = \ldots =$ the class of all integers >0, and in the result so obtained put

$$x = y = z = \ldots = 1.$$

3. Let $\eta(n) = 1$ or 0 according as the integer $n \overline{\overline{>}} 1$, and write symbolically

$$\Sigma_n f(d) g(\delta) = fg,$$

the summation extending to all pairs (d, δ) of integers > 0 such that $n = d\delta$. Then if $f(n)$ is an arbitrary numerical function, there exists a unique numerical function $f'(n)$, called the reciprocal of $f(n)$, such that $ff' = \eta$, or in full,

$$\Sigma_n f(d) f'(\delta) = \eta(n).$$

Elsewhere I have given the explicit form of f' in terms of f, and have developed a simple algebra for determining the reduced form of f' when f is a specific function (1).

Note that if g is any numerical function, $\eta g = g$, so that in this symbolic calculus of numerical functions, η has the multiplicative properties of unity in common algebra or arithmetic, and hence it is called the unit function.

4. Assuming with CESÀRO that all of the infinite processes concerned are significant, we shall now examine his most general inversion in the light of § 2. For j an arbitrary integer > 0, write

$$F(j) \equiv \sum_{i=1} h(i) f(ij), \tag{5}$$

and let f, g, h denote numerical functions. Note that (5) is the definition merely of $F(j)$. Immediately by § 3 we have

$$\sum_{j=1} g(j) F(j) = \sum_{j=1} hg(j) f(j),$$

and therefore, if in particular $hg = \eta$, we get $g \equiv h'$, the reciprocal of h, and obtain

$$f(1) = \sum_{j=1} h'(j) F(j), \tag{6}$$

which can be considered as an inversion of (5). We have shown that (5) implies (6), and will prove next that this proposition implies CESÀRO'S most general inversion.

(1) *Tohoku Mathematical Journal*, vol. 17, pp. 221-231; *Trans. American Math. Soc.*, loc. cit..

In (5) take $f \equiv f_x$, and *define* F_x by

$$F_x(j) \equiv \sum_{i=1} h(i) f_x(ij), \tag{7}$$

which is a *particular instance* of (5), and which, therefore, by the proposition just proved, implies

$$f_x(1) = \sum_{j=1} h'(j) F(j), \tag{8}$$

so that (7), (8) *constitute a particular instance of the inversion* (5), (6). We note that (7) *is implied by* its apparently special case $j = 1$, for

$$F_x(1) \equiv \sum_{i=1} h(i) f_x(i), \tag{9}$$

on transforming to the other notation, is

$$F(x) \equiv \sum_{i=1} h(i) f[e_i(x)],$$

and if in this we replace x by $e_j(x)$ and transform back to the first notation, we obtain (7). Hence (9) implies (8).

By a further *specialization* we reach CESÀRO'S inversion. Let $\Omega^{(u)}$ refer to u_i $(i = 1, 2, 3, \ldots)$, and be such that $\Omega^{(u)}(m)\Omega^{(u)}(n) = \Omega^{(u)}(mn)$. Take in (7) for $h(i)$ the *particular* function $\Omega^{(u)}(i)h(i)$. Then the reciprocal function is $\Omega^{(u)}(i)h'(i)$, where, as before, h, h' are reciprocals. For we have

$$\Sigma_n \Omega^{(u)}(d) h(d) \Omega^{(u)}(\delta) h'(\delta) = \Sigma_n \Omega^{(u)}(d\delta) h(d) h'(\delta) = \Omega^{(u)}(n)\eta(n),$$

and hence, since $\eta(n)$ is the unit function, the value becomes $\Omega^{(u)}(1) = 1$, or 0, according as $n = 1$, or $n > 1$. As a special case, then, of the result just proved that (9) implies (8), we have now shown that

$$F_x(1) = \sum_{i=1} \Omega^{(u)}(i) h(i) f_x(i) \quad \textit{implies} \quad f_x(1) = \sum_{j=1} \Omega^{(u)}(j) h'(j) F_x(j), \tag{10}$$

which, transformed to the other notation,

$$F_x(1) \equiv F(x), \quad f_x(i) \equiv f[e_i(x)], \quad f_x(1) \equiv f(x), \quad F_x(j) \equiv F[e_j(x)],$$

is CESÀRO'S most general inversion, with $hh' = \eta$ and $\Omega^{(u)}(m)\Omega^{(u)}(n) = \Omega^{(u)}(mn)$.

We have shown therefore that this most general inversion is implied by its apparently special case (5), (6).

To show next that the inversion (5), (6) is implied by (10), we replace x by $e_j(x)$ in the first of (10), take $u_i = i$ $(i = 1, 2, 3, \ldots)$, $\Omega(i) = i$, $e_i(x) = ix$, and in the result put $x = 1$.

We have proved then that each pair of inversions (5), (6) and (10) implies the other; that is, the pairs are equivalent.

5. The purpose of such transformations as we have discussed in § 2 and illustrated in § 4, is to throw arithmetical identities into more suggestive forms. For example, taking $u_i = u$ $(i = 1, 2, \ldots)$ where the u_i are prime to a given integer n, and choosing for $f_x(i)$ the function $g_x(i)\lambda(i)$, where g_x is an arbitrary numerical function, and $\lambda(i) = 1$ or 0 according as $i \leqq n$ or $i > n$, we transform a specific identity involving n into another involving only the integers not greater than n and prime to n.