Nonlinear Characterization of Sand-asphalt Concrete by Means of Permanent-memory Norms

Experimental and theoretical investigations demonstrate that extended nonlinear, homogeneous constitutive equations using P^{th} order Lebesgue norms are readily applicable to defining the mechanical behavior of sand-asphalt concrete

by J. E. Fitzgerald and J. Vakili

ABSTRACT—Employing a constitutive equation developed by Farris and Fitzgerald which accounts for the maximum strain ever imposed upon a material as well as a weighted average of the strain history, the family of P^{th} order Lebesgue norms, the applicability to a sand-asphalt concrete is demonstrated. The inadequacy of linear viscoelasticity theory under repeated or decreasing loadings for these materials is also demonstrated. Practical laboratory determination of the material parameters is described.

Introduction

The theory of linear viscoelasticity is often used for determining the stresses and strains in asphalt pavements due to wheel loads, the rationale being that experimental relaxation and creep curves fit the theoretical results fairly well. This fit occurs because materials with asphalt binder satisfy one of the requirements for linearity, namely, homogeneity wherein the stress, σ, strain, ϵ, relation is

$$\sigma(a_1 e) = a_1 \sigma(e) \quad (1)$$

with a_1 constant. However, the assumption of additivity which is also a linearity requirement is not valid for these materials wherein

$$\sigma(e_1 + e_2) = \sigma(e_1) + \sigma(e_2) \quad (2)$$

This observation has been previously pointed out for solid propellants,[9] and can be shown by conducting other tests such as interrupted ramp-strain or reverse ramp-strain tests.

Other theories have previously been developed to characterize nonlinear viscoelastic materials. Volterra,[10] by using the Frechet expansion for nonlinear functions of one variable, suggested an approximation to a continuous function by a polynomial. Green and Rivlin[11] have developed continuous tensor-valued functionals in a power series of homogeneous tensor-valued functionals. Hermann[12] suggested an energy method for nonlinear viscoelasticity. His assumption was that a viscoelastic material is capable of instantaneous deformation but not of instantaneous dissipation of energy. By this assumption he found a constitutive equation which is essentially the same as defined by Green and Rivlin. Coleman and Noll[13] have also contributed to the theory of nonlinear viscoelasticity in solids as well as liquids.

The work of all the above mentioned authors is based on the fading-memory assumption which means that the distant past memory of, say, strain is not as effective as the near-present value of strain in determining the present stress state of a material. Fitzgerald[14] proposed a constitutive equation wherein the stress is a function(al) of the present value of deformation gradient and its P^{th} order Lebesgue norm. The Lebesgue norm, $||f(t)||_2$, of a function of time, $f(t)$, is defined as

$$||f(t)||_P = \left[\int_{\tau=0}^{1} |f(\tau)|^P d\tau \right]^{1/P} ; \quad 1 \leq P \leq \infty . \quad (3)$$

It will be noted that for $P = 2$ the above expression is the usual R.M.S., root-mean-square, expression from elementary statistics, i.e.,

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$$||f(t)||_2 = \left[\int_{\tau=0}^{1} |f(\tau)|^2 d\tau \right]^{1/2} . \quad (4)$$

Fitzgerald and Farris[15] then enlarged the norm into a family of norms by changing the upper limit of the integral from $\tau = 1$ to $\tau = t$. Coleman and Noll have also suggested these norms as a possible approximation for the functionals. Fitzgerald and Hufferd[2] have formulated the general case of nonlinear thermoviscoelasticity by using the permanent-memory assumption.

Fitzgerald and Farris[1,15] developed a specific kind of Lebesgue-norm constitutive equation for nonlinear viscoelastic materials. They used the nonlinear homogeneous constitutive equations of degree one. They satisfy the homogeneity condition which is one of the requirements for linearity and appears to be the simplest kind of constitutive equation for nonlinear viscoelastic materials. They observed that the mechanical response of filled polymeric materials were time dependent, even though the filler particles and the polymeric binder were elastic. They have modeled the irreversible time-dependent Mullins' effect and, by using the permanent-memory hypothesis, they have formulated constitutive equations and subsequently developed a one-dimensional constitutive theory and analysis method valid for nonlinear viscoelastic materials. To obtain the experimental results, they used solid propellants which are filled polymeric materials.

The present work is an extension of their work showing that these homogeneous constitutive equations are also a valid theory for characterizing asphalt-binder materials such as sand asphalt.

We assume a constant temperature in our equations herein and all the experimental tests were conducted at room temperature (70°F).

Theoretical Background

The family of norms is, in itself, homogeneous since it can readily be seen from eq (4) that

$$||a_1 f(t)||_P = a_1 ||f(t)||_P . \quad (5)$$

Thus, the ratio of two norms, say the Qth and Pth norm, is independent of the multiplier which cancels out. The norm ratio then characterizes the normalized weighted averages of the past history. It will thus distinguish, by its numerical value, between different sequences of loading or straining.

A second ratio, that of the absolute present value of the function divided by the, say, Pth norm, compares the present value with the integral of the past weighted average.

Letting $P \to \infty$ produces the Chebychev or maximum norm wherein for continuous functions

$$||f(t)||_{P=\infty} = \max |f(t)| . \quad (6)$$

Thus, ratios involving the above norm bring into play the concept of the maximum strain or load on the material. A detailed discussion may be found in Ref. 15.

Asphalt-binder materials have generally a relaxation-modulus behavior which obeys an inverse power law:

$$E(t) = \sum_{i=1}^{N} \beta_i t^{-r_i} \quad (7)$$

Thus, a feasible uniaxial form for the stress functional $S(t)$, which verifies the relaxation behavior of the material is:

$$S(t) = \sum_{i=0}^{N} \sum_{j=0}^{N} A_{ij} \left(\frac{||e||_{Q_i}}{||e||_{P_i}} \right)^{n_i} \int_0^t (t-\tau)^{-m_j} \dot{e}(\tau) d\tau \quad (8)$$

For material characterization, one restricts the number of these parameters and determines the coefficients by curve fitting. For example, if one takes only the first term of the expansion, $i = j = 0$, and assumes $Q_o = \infty$, we find

$$S(t) = A_{00} \left[\frac{||e(t)||_\infty}{||e(t)||_{P_o}} \right]^{n_o} e(t) \quad (9)$$

where $||e||_\infty$ is the Chebychev norm.

This constitutive equation is valid for our materials only when the strain is a nondecreasing function of time. For weakly monotonically increasing strains, the maximum strain equals the present strain and eq (9) becomes

$$S(t) = A_{00} \left[\frac{|e(t)|}{||e(t)||_{P_o}} \right]^{n_o} e(t) \quad (10)$$

For example, assume that the strain is applied as a step function of magnitude $e_o U(t)$. Then the P_oth norm is

$$||e_o U(t)||_{P_o} = e_o \left[\int_{t=0}^{t} d\tau \right]^{1/P_o} = e_o t^{1/P_o} \quad (11)$$

and the stress from eq (13) is

$$S(t) = A_{00} e_0 t^{-n_o/P_o} \quad (12)$$

confirming the usually observed inverse-power-law stress-relaxation behavior.

When the strain is a decreasing function or a combination of increasing and decreasing portions, the foregoing equation is no longer valid, because the Chebychev norm of the strain is not equal to the current value of strain. Then we must add some additional terms to correct the results. A three-term expansion initially yields:

$$S(t) = A_{00} \left(\frac{|e(t)|}{||e(t)||_{P_o}} \right)^{n_o} e(t) + A_{01} \int_{\tau=0}^{t} (t-\tau)^{-m_1} \dot{e}(\tau) d\tau + A_{11} \left(\frac{|e(t)|}{||e(t)||_{P_1}} \right)^{n_1} \int_{\tau=0}^{t} (t-\tau)^{-m_1} \dot{e}(\tau) d\tau \quad (13)$$

Equation (13) is a more general case of eq (9) and, if the strain function is nondecreasing, eq (13) should be reducible to eq (9), which requires that:

$$A_{01} = -A_{11}$$

and

$$P_1 = \infty . \quad (14)$$

TABLE 1—GRADATION FOR THE AGGREGATE SAND

Weight Percentage	Passing through Sieve of Size:	Remaining on Sieve of Size:
60%	No. 30	No. 50
20%	No. 40	No. 100
10%	No. 30	No. 40
10%	No. 50	No. 200

TABLE 2—CHARACTERISTICS OF SPECIMEN

No. of Specimen	Height (in.)	Diameter (in.)	Age at Test Time (days)
1	3.282	2.000	4
2	3.190	2.030	4
3	3.195	2.030	4
4	3.130	2.030	4
5	3.095	2.030	5
6	3.065	2.030	5
7	3.005	2.030	6
8	3.000	2.030	6
9	2.935	2.030	7
10	2.920	2.030	7
11	2.700	2.060	7

Hence, the above equation becomes with, for convenience in writing, $A_1 = A_{00}$ and $A_2 = A_{01}$

$$S(t) = A_1 \left[\frac{|e(t)|}{||e(t)||_{P_o}} \right]^{n_o} e(t) + A_2 \left[1 - \left(\frac{|e(t)|}{||e(t)||_\infty} \right)^{n_1} \right] \int_o^t (t-\tau)^{-m_1} e_{11}(\tau)\,d\tau . \quad (15)$$

Now, if $e(t)$ is a weakly monotone increasing function, $|e| = ||e||_\infty$, and the second term of eq (13) vanishes and reduces to eq (9).

The development of eq (13) follows the specifics as originally carried out by the senior author's graduate student, R. Farris, in his doctoral dissertation, the essence of which may be found in Ref. 1.

Determination of Material Constants

It was shown in eq (12) that the stress response to a step strain input was

$$S(t) = A_1 e_0 t^{-n_o/P_o} \quad (16)$$

so that, in logarithmic form,

$$\log \frac{S(t)}{e_o} = \log A_1 - \frac{n_o}{P_o} \log t$$

$$\triangleq \log \text{ (relaxation modulus)} \quad (17)$$

Thus, the ratio n_o/P_o is determined experimentally from the slope of the relaxation curve and A_1 is determined as the value of the relaxation modulus at $t = 1$ when a log-log plot is used.

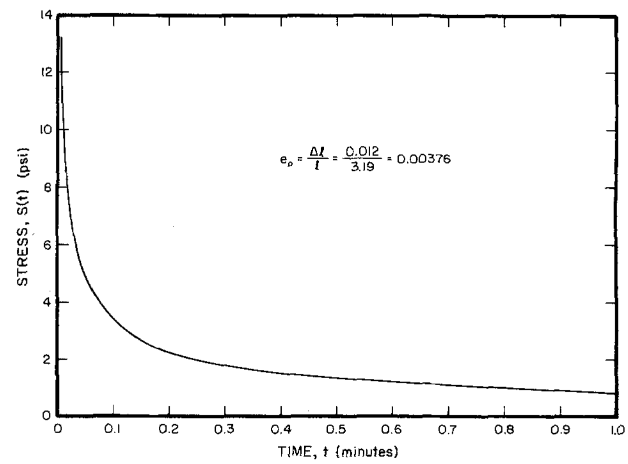


Fig. 1—Stress output for a relaxation test on a sand asphalt

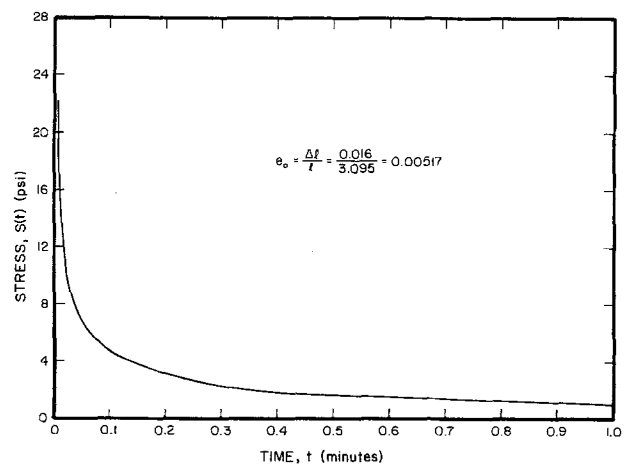


Fig. 2—Stress output for a relaxation test on a sand asphalt

A constant-strain-rate test with $e(t) = Rt$ yields from eq (15)

$$S(t) = A_1 R\ (P_o + 1)^{n_o/P_o}\ \ t^{(1-n_o/P_o)} . \quad (18)$$

Thus, the value of P_o may be found from the intercept of log $S(t)$ and therefore, the value of n_o found since n_o/P_o has been determined.

The values of A_2, n_1, and $-m_1$ can only be determined by conducting interrupted-ramp strain or cyclic-ramp strain tests.

Experimental Results

Description of Samples

The experiments were conducted using 2-in. ϕ × 3-in. cylindrical specimens of sand asphalt. The specimens were made with an 88-percent-by-weight aggregate and (80-100) asphalt. The gradation for the

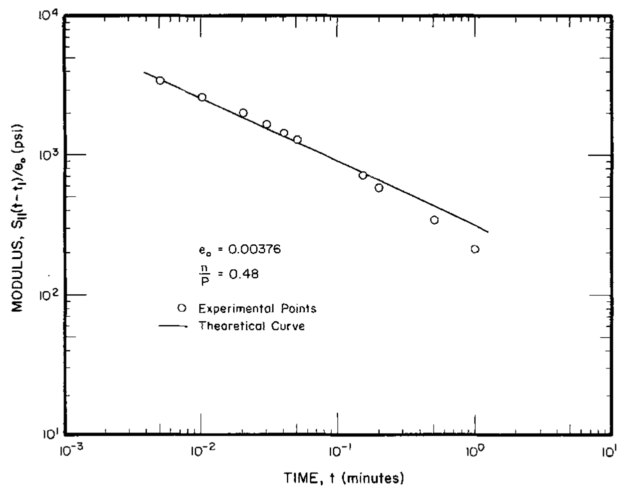


Fig. 3—Relaxation modulus of sand asphalt for first test

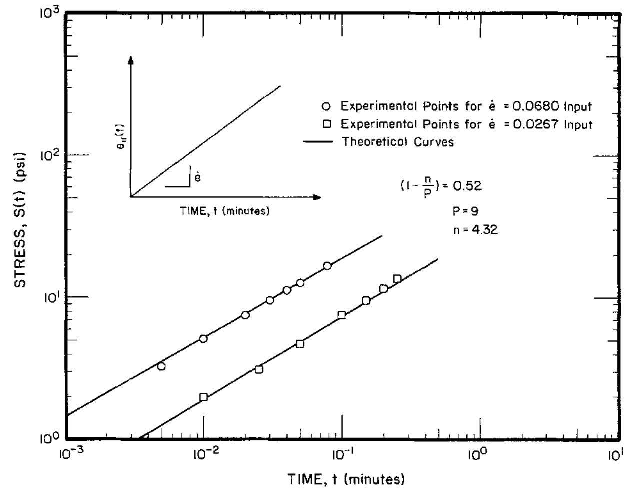


Fig. 5—Stress output for two different constant-strain-rate tests

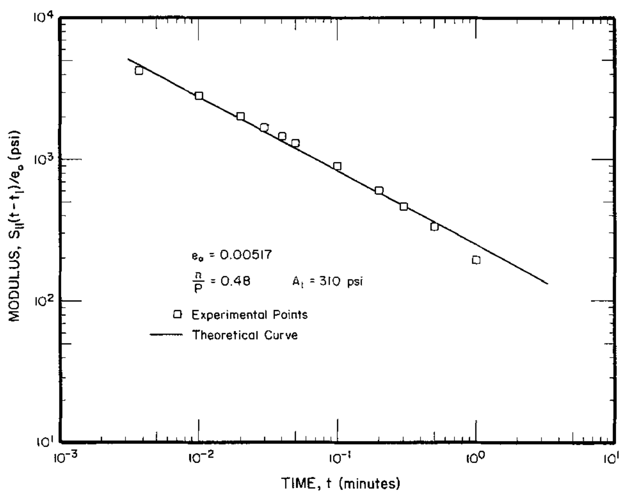


Fig. 4—Relaxation modulus of sand asphalt for second test

aggregate sand is given in Table 1.

The temperature of the asphalt before mixing was 275°F and that of the aggregate 350°F. The mixing time was about 2-3 min and all the specimens were compacted by 30 blows of a hammer. The total amount of compaction energy for each specimen was 500 ft-lb. The characteristics of the specimens are given in Table 2.

Relaxation Tests

Relaxation tests with various strain magnitudes were conducted using an Instron tester. A fast initial cross-head speed (about 5 in./min) was applied and, when the desired strain was reached, the cross head was stopped. Comparison of values of $\frac{S(t)}{e_o}$ for various relaxation tests indicates that the material obeys a homogeneous constitutive equation of degree one. However, the other tests demonstrated that the second requirement for linearity, additivity, is not valid for the sand asphalt. The results plotted logarithmically have the straight-line form of eq (17). The tangent of this line is the value of $\frac{n}{P}$ and one can also find the value of the coefficient A from the relaxation curves. Analyzing Figs. 3 and 4, the values of $\frac{n}{P}$ and A are:

$$n/P = .48 \qquad A = 310 \text{ psi}$$

Constant-strain-rate Tests

This test was conducted for various values of strain rate. The slope of $S(t)$ on a logarithmic scale is the value of $(1 - n/P)$. Looking at Fig. 5, one observes that the value of $n/P = .48$ obtained from the relaxation test is also justified by the constant-strain-rate test. The value of P, and hence n, may also be found from the experimental stress curve for the constant-strain-rate input and is from eq (18)

$$P = 9 \qquad n = 4.32$$

Interrupted-ramp Strain Tests

In this test, an arbitrary constant strain rate, R, was applied to the specimen for a time T_1 and the cross head stopped. After a time T_2, the same rate of strain was again applied for the same increment of time T_1 and the procedure repeated K times.

Introducing the reduced time t' wherein

$$t' = t - (K-1)(T_1 + T_2) \qquad (19)$$

and defining the parameters

$$\alpha = \frac{t'}{T_1}, \quad h = \frac{T_2}{T_1} \qquad (20)$$

one introduces the evaluated norms into eq (15) which yields

$$S(t) = A_1 R T_1^{(1-n/P)} K \left[\alpha + \frac{K}{P+1} + \frac{h}{K^P} \sum_{j=1}^{K-1} j^P \right]^{-n/P} \quad (21)$$

One may write the expression (21) in the form:

$$\text{Log}\, S(t) = \text{Log}\, C - \frac{n}{P} \text{Log}\, (\alpha + D)$$

where

$$C = KA\, RT_1^{(1-n/P)}$$

and

$$D = \frac{K}{P+1} + \frac{h}{K^P} \sum_{J=1}^{K-1} J^P \quad (22)$$

Then the slope of the decreasing parts of the experimental curve of $S(t)$ on a logarithmic scale should be the value of n/P.

Figure 6 shows that the value of $n/P = .48$ is again justified. In Figs. 7 and 8, experimental curves of interrupted-ramp strain are compared with the theoretical points. It is obvious that the type of constitutive equation used in this study is a good approximation to the constitutive equation for a sand-asphalt material.

In Fig. 9, the inadequacy of the theory of linear viscoelasticity for this material is shown. It is observed that the assumption of additivity is not valid for the sand asphalt.

Reverse-ramp Strain Tests

In this test, we effectively cycle the stress which is raised to an arbitrary value with a constant strain rate and is decreased with the same but negative strain rate. When the stress reaches zero, the strain is increased again at the same rate and so on. Figure 10 shows one cycle of the curve of stress vs. strain; we observe that the second term in eq (15) must be of importance. As the stress reduces to zero, the strain has decreased to only 80 percent of its maximum value. By curve fitting, Figs. 10 and 11, we find

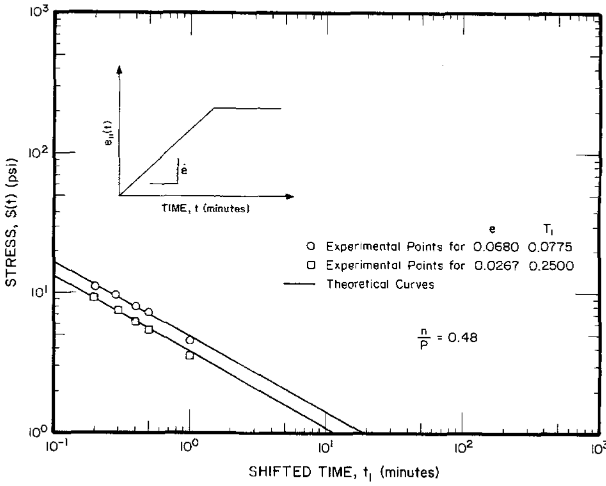


Fig. 6—Stress output for two different ramp relaxation tests

convenient values for the parameter, m, and the coefficient, A_2;

$$m = .8 \qquad A_2 = A_1 = 310 \text{ psi}$$

In Fig. 11 we observe that the value found for m is also applicable for the second cycle of the curve of stress vs. time.

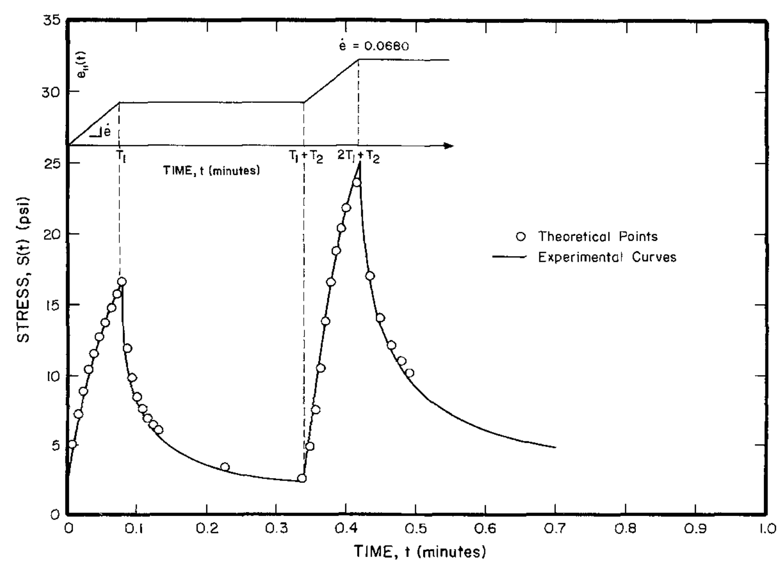
Combining these results, the constitutive equation for the sand asphalt becomes:

$$S(t) = 310 \left\{ \left(\frac{|e|}{||e||_9} \right)^{4.32} e(t) + \left[1 - \frac{|e|}{||e||_\infty} \right] \int_o^t (t-\tau)^{-.8} e(\tau)\, d\tau \right\} \quad (23)$$

Conclusion

Based on the limited theoretical and experimental

Fig. 7—Stress output for interrupted-ramp strain input



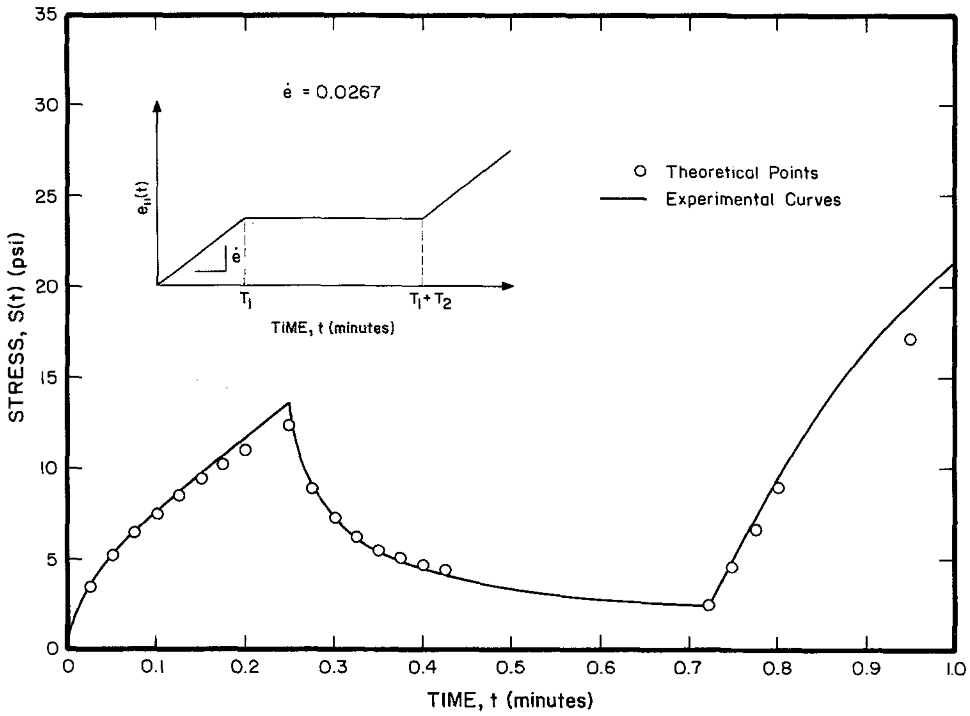


Fig. 8—Stress output for interrupted-ramp strain input

study presented herein, the following conclusions can be drawn:

1. Linear viscoelasticity does not seem to be an applicable theory for characterizing materials with asphalt binder under repeated loads.
2. An analysis of the experimental results and a comparison with theoretical data demonstrate that extended nonlinear, homogeneous constitutive equations using P^{th} order Lebesgue norms become a powerful theory to define the mechanical behavior of sand-asphalt concrete.
3. The simple three-term expression used herein may be rather obviously extended by utilizing the expansion of eq (8) when such an extension is required.

Acknowledgment

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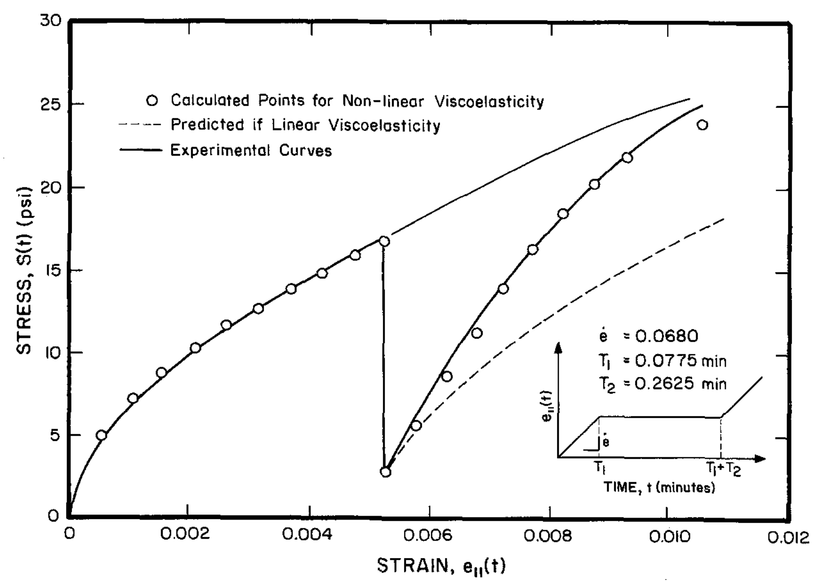

Fig. 9—Comparison of calculated and observed stress–strain output for an interrupted-ramp strain input

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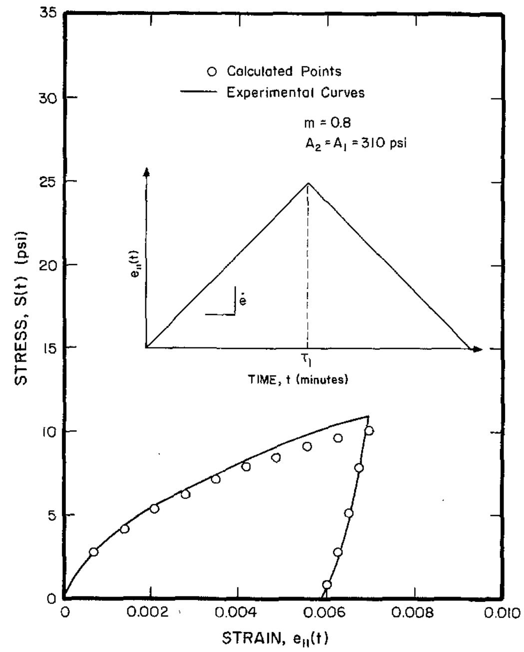

Fig. 10—Comparison of calculated and observed stress–strain behavior of sand asphalt

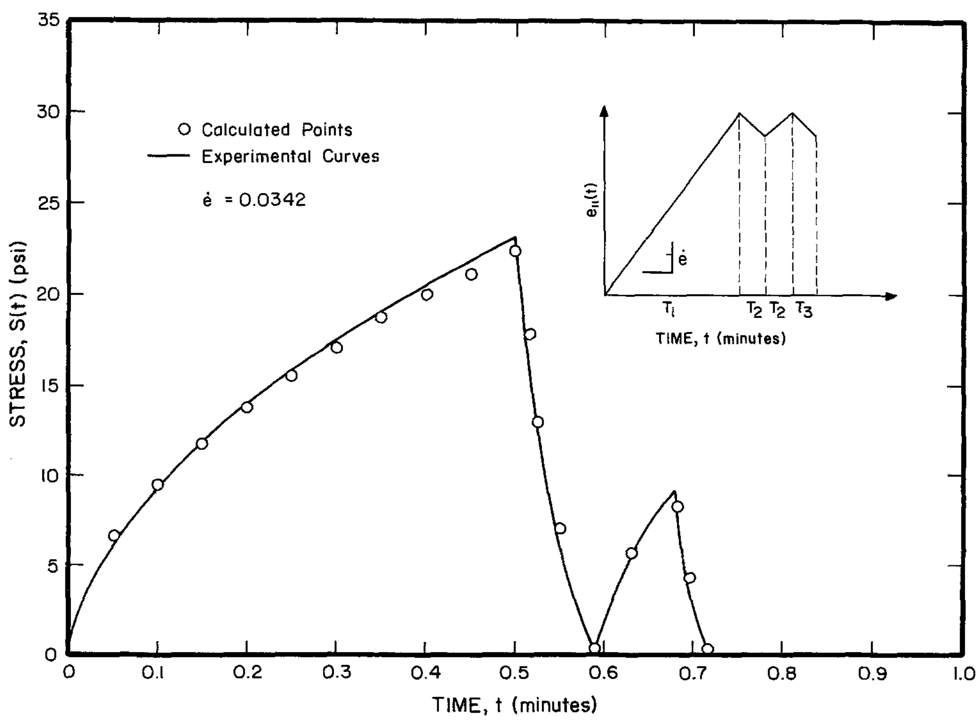

Fig. 11—Comparison of calculated and observed stress output for a reversed-ramp strain test