A Gallai-Type Transversal Problem in the Plane

Jürgen Eckhoff

Fachbereich Mathematik, Universität Dortmund,
W-4600 Dortmund 50, Federal Republic of Germany

Abstract. Let $\mathcal{K}$ be a family of compact convex sets in the plane. We show that if every three members of $\mathcal{K}$ admit a common line transversal, then there exist four lines which together meet all the members of $\mathcal{K}$.

1. Introduction

Let $\mathcal{K}$ be a family of compact convex sets in the plane. We say that $\mathcal{K}$ has property $T(r)$ if every r or fewer members of $\mathcal{K}$ admit a *line transversal*, that is, a straight line which intersects these members (and possibly other members of $\mathcal{K}$ as well). We say that $\mathcal{K}$ has property T if there exists a line transversal for all the members of $\mathcal{K}$. Such a line is also called a *common 1-transversal*, or a *stabbing line* for $\mathcal{K}$.

Over the years, considerable effort has been devoted to finding general conditions on the family $\mathcal{K}$ which assure that if $\mathcal{K}$ has property $T(r)$, for some fixed integer r, then $\mathcal{K}$ has property T. Such a result, if true, is said to be of *Helly-type*. We mention two prominent examples below, one going back to 1940, the other one dating from 1989. (For historical comments and a survey of the literature on transversals up to 1963, the reader is referred to [2]. Transversal problems are by no means restricted to the plane. A wealth of more recent results is documented in a new report by Goodman *et al.* [5].)

Our first example is a special case of a theorem of Santaló [13]. It asserts that, for any family of rectangles in the plane with sides parallel to the coordinate axes, $T(6)$ implies T. This is best possible in the sense that $T(6)$ cannot be replaced by $T(5)$ and that the conclusion fails for more general families of convex sets. The higher-dimensional version of Santaló's result deals with line (resp. hyperplane) transversals for families of parallelotopes in R^d. (For an extension in the latter case, see [8].)

Our second example is supplied by the recent proof—due to Tverberg [14]—of a long-standing conjecture of Grünbaum [7]. This conjecture (now a theorem) claimed that, for any family of pairwise disjoint translates of a given compact convex set in the plane, $T(5)$ implies T. Again, simple examples show that the weaker assumption $T(4)$ would not suffice for the conclusion and that neither the disjointness hypothesis nor the requirement that the members be translates of one another can be dropped. We point out that Grünbaum's conjecture is far more difficult to prove than Santaló's result and, for that matter, most results in this area. (Compare the discussion in Section 4 of [5].) The conjecture had been verified earlier for disks (hence ellipses) by Danzer [1] and for squares (hence parallelograms) by Grünbaum [7]. A weaker form of the conjecture which assumed $T(128)$ instead of $T(5)$ had been established by Katchalski [11].

The above examples only serve to confirm the general rule that rather stringent conditions on the shape and the mutual position of the sets in $\mathcal{K}$ must be imposed in order to obtain a Helly-type transversal theorem. In this paper, therefore, we are concerned with transversal problems of a different nature (to be explained in a moment). We thereby continue our research on line transversals in the plane begun in [3] and [4].

As before, let $\mathcal{K}$ be a family of compact convex sets in the plane, and let n be a positive integer. We say that $\mathcal{K}$ has property T^n if $\mathcal{K}$ can be partitioned into n or fewer subfamilies, each of which have a line transversal. In other words, $\mathcal{K}$ has property T^n if n lines in the plane whose union meets every member of $\mathcal{K}$ can be found. Occasionally, we say that $\mathcal{K}$ can be "stabbed" by those lines. Notice that T^1 is the same as T. The idea here is to make the "stabbing number" n as small as possible while allowing the members of $\mathcal{K}$ to be quite general. This leads to a class of transversal problems called *Gallai-type* problems (see [2]–[5] and [12]). A variant has been considered by Wegner [15].

For each $r \geq 3$, the problem is to determine the smallest possible integer n such that, for any family of compact convex sets in the plane, $T(r)$ implies T^n. It will be seen shortly that such an integer always exists but that its value is only interesting for $r = 3$ and $r = 4$. The problem was first studied by Hadwiger and Debrunner (see Proposition 26 of [10]) who dealt with the case $r = 4$. To be precise, these authors restricted their attention to families of positive homothets (resp. translates) of a given compact convex set K. They proved that $T(4)$ implies T^4 for families of homothets of K, whereas it implies T^3 for families of translates of K. Later it turned out that even T^2 can be achieved and, moreover, the special assumptions on the shape of the sets can be dropped altogether. This shows

$$T(4) \Rightarrow T^2. \tag{1.1}$$

Assertion (1.1) was proved by the author in 1969 (see [3] and [4]). As there are no Helly-type transversal theorems for general convex sets (i.e., $T(r) \not\Rightarrow T$ for $r \geq 3$), this settles the problem except for the case $r = 3$ which to this day is still open. A family of 14 convex sets having property $T(3)$, but not T^2, is exhibited in [4]. It simplifies an earlier example of an infinite family with the same properties

described in [3]. Hence we have

$$T(3) \nRightarrow T^2. \tag{1.2}$$

For some time, even the existence of a finite stabbing number for $T(3)$ families was in doubt but in 1975 Kramer [12] was able to show that, in fact, $T(3)$ implies T^5. This result appeared in Kramer's doctoral dissertation (see also p. 257 of [6]) and has not been published elsewhere. Our goal in this paper is to improve on Kramer's upper bound (while employing his main ideas) by showing that

$$T(3) \Rightarrow T^4. \tag{1.3}$$

In other words, we establish the following:

Theorem. *If a family of compact convex sets in the plane is such that every three of its members admit a common line transversal, then the family can be stabbed by four lines.*

We do not know whether the stabbing number "four" in this statement may be replaced by "three." [1] For additional remarks, see the final section of this paper.

2. Preliminaries

Before proceeding to the proof of our theorem, we need to establish some geometric terminology and describe certain reductions of the problem which will facilitate the ensuing discussion.

In what follows, $\mathcal{X}$ is always a family of compact convex sets in the plane, and n is a positive integer. The first reduction uses a compactness argument to show that it is sufficient to deal with *finite* families $\mathcal{X}$.

If every finite subfamily of $\mathcal{X}$ has property T^n, then so has $\mathcal{X}$. (2.1)

This was proved in a more general context in [3]. The proof is quite standard to anyone familiar with geometric transversals and is not given here.

The second reduction allows us to work with convex sets which have a *nonempty interior*. Strictly speaking, this reduction is not essential for the proof but it saves us some extra considerations. For K a set in the plane and ε a positive real number, let $K(\varepsilon)$ denote the Minkowski sum $K + B_\varepsilon$, where B_ε is the closed disk of radius ε centered at the origin. It is well known that if K is compact and convex, then $K(\varepsilon)$ is compact, convex, and smooth; in particular, $K(\varepsilon)$ has interior points. Set $\mathcal{X}(\varepsilon) := \{K(\varepsilon) \mid K \in \mathcal{X}\}$. Then we have

If $\mathcal{X}(\varepsilon)$ has property T^n for every $\varepsilon > 0$, then so has $\mathcal{X}$. (2.2)

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[1] In the *Proceedings of the Second Colloquium on Discrete Geometry*, Salzburg (1980), pp. 45–49, we claimed to have proved that $T(3) \Rightarrow T^3$. Unfortunately, our proof was in error.
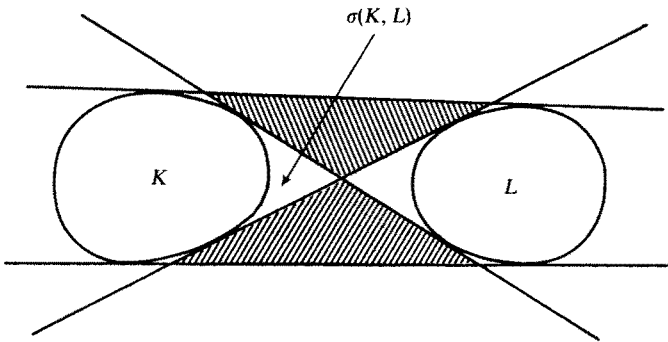
Fig. 1.   The inner and outer common tangents of K and L.

This is an easy consequence of assertion (2.1). Again, the proof is omitted.

Our last reduction is almost trivial. It states that in proving the main theorem we may as well assume that $\mathscr{K}$ contains pairs of *disjoint* sets. For, otherwise, $\mathscr{K}$ would have property T. To see this, project the members of $\mathscr{K}$ orthogonally onto some fixed line. If every two members intersect, then the same is true of the resulting segments. Thus, by Helly's theorem on the line, there is a point common to all these segments and so the projecting line corresponding to this point is a stabbing line for $\mathscr{K}$.

The following special terminology is used in this paper. Suppose K and L are two disjoint compact convex sets, both with interior points. There are exactly four lines in the plane which simultaneously support K and L (see Fig. 1). These lines are called the *inner* or *outer common tangents* of K and L, depending on whether they separate the sets or not.[2] The angle between the inner common tangents is denoted by $\sigma(K, L)$. To be precise, this is the interior angle of the closed double cone which contains K and L and is bounded by the inner tangents. This double cone is said to be *associated* with the pair $\{K, L\}$. The outer common tangents of K and L form two triangles (shaded in Fig. 1) with the inner common tangents. If these triangles are added to the double cone just mentioned, the resulting (closed) region is clearly the set of all points in the plane through which there passes a common line transversal of K and L.

This observation leads at once to the following useful criterion. Suppose K, L, and M are three pairwise disjoint compact convex sets (not necessarily with interior points). If there exist three lines a, b, and c in the plane such that each of the sets is separated from the other two by one of these lines, while none of the lines intersects all three sets, then there is no line transversal for K, L, and M. (Incidentally, the converse is also true.) We refer to this condition by saying that K, L, and M are *separated* by a, b, and c. The lines must then be distinct but they are allowed to meet in a point.

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[2] In this paper, separation by lines is always understood to be weak separation.

3. Proof of the Theorem

We are now ready to prove our main result.

Let $\mathscr{K}$ be a family of compact convex sets in the plane, and suppose $\mathscr{K}$ has property $T(3)$. We must show that $\mathscr{K}$ has property T^4. In view of the reductions described in the preceding section, we may assume that $\mathscr{K}$ is finite and that every member of $\mathscr{K}$ has a nonempty interior. Furthermore, we may assume that $\mathscr{K}$ contains at least one pair of disjoint sets. Among all such pairs choose one, say $\{K_0, L_0\}$, for which the interior angle $\sigma(K_0, L_0)$ defined in Section 2 is minimal. These assumptions imply that $\sigma(K_0, L_0) > 0$ and that any member of $\mathscr{K}$ which lies in the double cone associated with $\{K_0, L_0\}$ meets both the inner common tangents of K_0 and L_0. In particular, no member of $\mathscr{K}$ is contained in the interior of that cone.

Denote the inner common tangents of K_0 and L_0 by b_0 and c_0 and their intersection point by A_0. The complement of the double cone above consists of two opposite open cones R_1 and R_2, say, bounded by b_0 and c_0 and separated by the common apex A_0. Let $\mathscr{K}_1$ and $\mathscr{K}_2$ be the subfamilies of $\mathscr{K}$ whose members lie entirely in R_1 and R_2, respectively. Of course, $\mathscr{K}_1$ or $\mathscr{K}_2$ may be empty. Each set in $\mathscr{K}\backslash(\mathscr{K}_1 \cup \mathscr{K}_2)$ meets one of the lines b_0 and c_0 (or both). The following observation was already made by Kramer [12].

> Let g be a line through A_0. If g separates two members of $\mathscr{K}_1$, say, then g is a stabbing line for $\mathscr{K}_2$.   (3.1)

Suppose this is not true. Then, without loss of generality, g separates two members K_1 and L_1 of $\mathscr{K}_1$ and misses some member K_2 of $\mathscr{K}_2$, where K_0, K_1, and K_2 lie in one of the closed half-planes bounded by g and L_0 and L_1 lie in the other (see Fig. 2). It follows at once that K_1, K_2, and L_0 are separated by b_0, c_0, and g, even though it may happen that L_0 touches g at the point A_0. This violates the hypothesis of a common line transversal for K_1, K_2, and L_0, and so the assertion is proved. $\square$
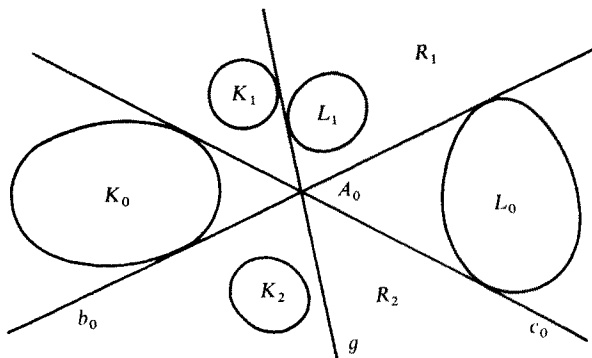

Fig. 2. Two members K_1 and L_1 of $\mathscr{K}_1$ separated by g.

Assume, for the moment, that every two members of $\mathcal{K}_1$ admit a line transversal passing through A_0. Then $\mathcal{K}_1$ has a stabbing line through A_0. To see this, map the sets in $\mathcal{K}_1$ onto a suitable line which cuts through the cone R_1, using central projection with center A_0. By Helly's theorem on that line, the resulting segments have a common point and so the line connecting that point to A_0 is the desired stabbing line for $\mathcal{K}_1$. If, in addition, every two members of $\mathcal{K}_2$ admit a line transversal through A_0, then $\mathcal{K}_2$ also has a stabbing line passing through A_0. Consequently, each set in $\mathcal{K}$ meets b_0 or c_0 or one of the stabbing lines for $\mathcal{K}_1$ and $\mathcal{K}_2$. This would conclude the proof of our theorem.

For the remainder of the proof it therefore suffices to assume that one of the families $\mathcal{K}_1$ and $\mathcal{K}_2$ contains two members without a common transversal through A_0. In fact, it is enough to require that there is some line through A_0 which separates the two sets. Without loss of generality, suppose it separates two sets in $\mathcal{K}_1$.

Let a_0 denote the outer common tangent of K_0 and L_0 which cuts through the cone R_1, and let B_0 and C_0 be its intersection points with c_0 and b_0, respectively. To simplify the following discussion, we assume that the ordered triple (A_0, B_0, C_0) is clockwise oriented and that K_0 touches c_0 between the points A_0 and B_0 (see Fig. 3). This can always be achieved by relabeling the pairs b_0, c_0 and K_0, L_0, if necessary. Denote the closed triangle with vertices A_0, B_0, and C_0 by Δ_0. Every member of $\mathcal{K} \backslash \mathcal{K}_2$ not contained in the interior of Δ_0 meets one of the lines a_0, b_0, and c_0.

We now proceed as follows. We rotate the line c_0 clockwise around A_0 until it is about to leave—for the first time—some member of $\mathcal{K}_1$. Let c_0' denote the rotated line at this point, and let K' be such a member. Since $\mathcal{K}_1$ is assumed to be nonempty, K' and c_0' clearly exist. By construction, no member of $\mathcal{K}_1$ lies in the open wedge formed by c_0 and c_0' and contained in R_1. This means that c_0'
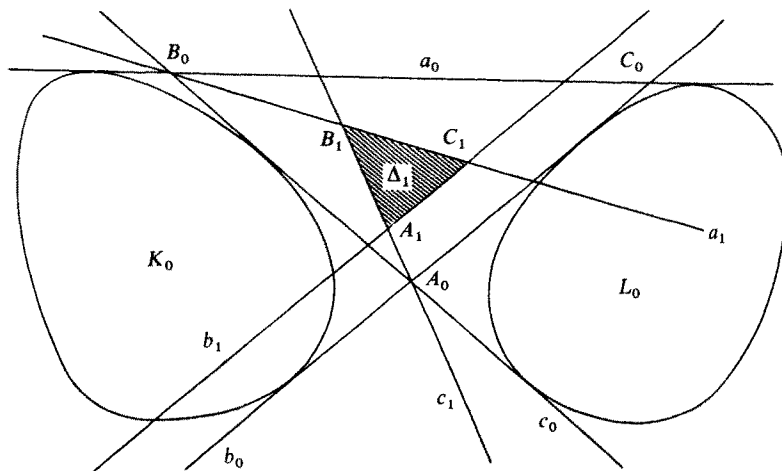

Fig. 3. The ordered triple (A_0, B_0, C_0).

must separate K' from some member of $\mathcal{K}_1$; otherwise, no two sets in $\mathcal{K}_1$ could be separated by a line through A_0, contrary to the assumption on $\mathcal{K}_1$ made earlier in the proof. Thus, by (3.1), c_0' is a stabbing line for $\mathcal{K}_2$. With Δ_0' denoting the closed triangle formed by a_0, b_0, and c_0', the four lines a_0, b_0, c_0, and c_0' clearly intersect all the members of $\mathcal{K}$ except those lying in the interior of Δ_0' (if such members exist at all).

We next apply a technique which was introduced by Kramer [12] and played a central role in this proof that $T(3)$ implies T^5. (A brief description of Kramer's approach is given in Section 4.) Since our goal here is to improve on Kramer's result by replacing T^5 with T^4, we find it necessary to modify and at the same time extend his method. The general idea is to move the lines a_0, b_0, and c_0' from their original position to a more tractable "extremal" position, making sure that the property stated in the last sentence of the preceding paragraph is preserved during the transition.

To be specific, we rotate a_0 about B_0 and c_0' about A_0, both in a clockwise direction, while translating b_0 toward B_0. We continue to do so as long as the equivalent of the above condition holds, that is as long as c_0 and the three moving lines intersect all the members of $\mathcal{K}$ except those lying in the interior of the triangle formed by the moving lines. For convenience, and only in the present context, the latter condition is called the "stabbing condition."

Two cases can occur.

In the first (and easy) case, the lines in question can be moved in such a way that the triangle formed by them degenerates, i.e., shrinks to a point, with the stabbing condition being satisfied all the time. Clearly, every member of $\mathcal{K}$ is then intersected by c_0 or one of the three versions of the moving lines passing through that point. Hence $\mathcal{K}$ can be stabbed by four lines, and we are done.

In the contrary case, no matter how the three lines are moved, they will always reach a position in which a nondegenerate triangle is formed, the stabbing condition is satisfied but fails if any of the lines is moved any further. This case is more difficult to analyze and occupies us for the rest of the proof.

Let a_1, b_1, and c_1 be versions of the original lines a_0, b_0, and c_0' which form a "limit triangle" as described above. (It may happen, of course, that $a_1 = a_0$, $b_1 = b_0$, or $c_1 = c_0'$.) Obviously, each of these lines must be "fixed" by some member of $\mathcal{K}$, or else we would be back to the first case. That is to say, there exist sets K_a, K_b, and K_c in $\mathcal{K}$ such that a_1, b_1, and c_1 are tangent to K_a, K_b, and K_c, respectively, while after a (small) further rotation of a_1 or c_1 or a further translation of b_1, the contact is lost; moreover, each of the sets K_a, K_b, and K_c misses c_0 and intersects only the line having the same label.

It follows immediately that K_c belongs to $\mathcal{K}_1$ or to $\mathcal{K}_2$ and that K_a and K_0 lie on different sides of c_0. Indeed, if K_a and K_0 were on the same side of c_0, then K_a would have to intersect b_0 since otherwise the interior angle $\sigma(K_a, L_0)$ would be smaller than $\sigma(K_0, L_0)$, a contradiction. *A fortiori*, K_a would have to intersect b_1; however, this is excluded by the choice of K_a.

Let Δ_1 denote the closed triangle formed by a_1, b_1, and c_1, and let A_1, B_1, and C_1 be its vertices, where the labeling is such that A_1 lies across from a_1, and so on. We claim that Δ_1 does not contain any member of $\mathcal{K}$ in its interior. This

will conclude the proof, as is evident from the construction of Δ_1. So assume, contrary to the claimed result, that there do exist members of $\mathscr{K}$ which lie in the interior of Δ_1. Let K_1 be such a member. We then consider the six unbounded (closed) polygonal regions in the plane generated by the arrangement of lines a_1, b_1, and c_1 (see Fig. 3). For brevity, these regions are called *pointed* or *blunt*, depending on whether they intersect the triangle Δ_1 in a single vertex or a full side. Each of the sets K_a, K_b, and K_c lies entirely in one of these regions but of course a given region may contain more than one of the sets. We now show that of all the combinatorially distinct ways of assigning the sets to the regions, none can actually occur.

To this end, consider first K_a. It is clear that K_a cannot lie in the pointed region with vertex B_1. For, otherwise, sets K_a, K_1, and K_0 would be separated by lines a_1, c_1, and c_0 and so would not admit a line transversal. (Here and in similar cases below we invite the reader to make the necessary verifications.) This is impossible. On the other hand, K_a cannot lie in the pointed region with vertex C_1 either because this would mean that $\sigma(K_1, K_a) < \sigma(K_0, L_0)$, contrary to the assumption that the latter angle is minimal. Since K_a and K_0 are separated by c_0, we conclude that K_a is contained in the blunt region with vertices B_1 and C_1.

Next consider K_c. Recall that K_c is a member of $\mathscr{K}_1$ or $\mathscr{K}_2$. If $K_c \in \mathscr{K}_1$, then, for the same reason as above, K_c cannot lie in the pointed region with vertex B_1. Furthermore, it cannot lie in the pointed region with vertex A_1 since in that case K_c, K_1, and L_0 would be separated by b_0, c_1 and the line connecting A_1 and C_0, again a contradiction. If, on the other hand, $K_c \in \mathscr{K}_2$, then K_0, K_1, and K_c are separated by the concurrent lines b_0, c_0, and c_1 (compare the proof of (3.1) above). Hence K_c belongs to $\mathscr{K}_1$ and must be included in the blunt region with vertices A_1 and B_1.

Finally, what can be said about K_b? On the one hand, K_b cannot lie in the pointed region with vertex C_1. This follows exactly as in the case of K_a above. On the other hand, K_b cannot lie in the pointed region with vertex A_1 either. Here the argument is a little different. Since K_b is not intersected by c_0 we know that K_b is either contained in the triangle formed by b_1, c_1, and c_0, or is separated from that triangle by c_0. In the first case, K_b, K_1, and L_0 would have no common line transversal, by the same reasoning as was used above for K_c. In the second case, K_b, K_c, and L_0 would fail to have a line transversal because they are separated by b_1, c_1, and c_0. Therefore, both cases cannot occur and we conclude that K_b is contained in the blunt region with vertices C_1 and A_1.

To summarize the above discussion, it is found that the sets K_a, K_b, and K_c lie one in each of the three blunt regions surrounding the triangle Δ_1. By construction, none of the sets touches the two infinite rays on the boundary of its region. Hence K_a, K_b, and K_c are separated by a_1, b_1, and c_1 and so do not admit a common line transversal. However, this is forbidden by the basic assumption that $\mathscr{K}$ has property $T(3)$. It turns out, therefore, that the geometric situation described above cannot arise after all. That is, K_1 does not exist; in other words, the triangle Δ_1 does not contain any member of $\mathscr{K}$ in its interior. This means that every member of $\mathscr{K}$ intersects a_1, b_1, c_1, or c_0, whence $\mathscr{K}$ can be stabbed by four lines. This completes the proof of our theorem. $\qquad\square$

4. Remarks

1. We like to comment on the way in which the stabbing lines a_1, b_1, and c_1 are constructed in the foregoing proof. The reader has certainly noticed that our definition of these lines is not symmetric, in the sense that a_1 and c_1 result from a clockwise rotation of a_0 about B_0 and c_0' about A_0, respectively, while b_1 is obtained by translating b_0 toward B_0. Why cannot b_1 be a clockwise rotation of b_0 around C_0? The reason is that we want the set K_b which fixes b_1 to be included in the blunt region with vertices C_1 and A_1. If b_1 were obtained from b_0 by rotating the latter instead of translating it, then it could happen that K_b and L_0 lie on different sides of b_1. In that case, K_b would be contained in the blunt region with vertices B_1 and C_1 and so K_a, K_b, and K_c would not be separated by a_1, b_1, and c_1 any more. Hence the ensuing argument which leads to the nonexistence of K_1 would break down.

In principle, the same kind of problem could arise with the set K_a which fixes a_1. However, here the situation is different, due to the fact that we chose c_0 to be one of the stabbing lines for $\mathcal{K}$. It was shown in the proof that the presence of c_0 forces K_a and K_0 to be separated by c_0 and thus prevents K_a from lying in the blunt region with vertices A_1 and B_1.

This seems a good place to point out the main differences between Kramer's method (as demonstrated in [12]) and the approach taken in this paper.

Kramer starts out with a pair $\{K_0, L_0\}$ having smallest interior angle $\sigma(K_0, L_0)$ and defines a_0, b_0, c_0, A_0, B_0, C_0, $\mathcal{K}_1$, and $\mathcal{K}_2$ just as we do here. He assumes, as he may, that there is some line through A_0 which separates two members of $\mathcal{K}_1$. He then goes on to show that $\mathcal{K}_1$ can be stabbed by three suitable lines a_1, b_1, and c_1. These lines are obtained in a symmetric fashion by rotating a_0, b_0, and c_0 clockwise about B_0, C_0, and A_0, respectively. Except for the fact that b_0 is rotated and not translated, the procedure is completely analogous to the one used in defining the corresponding lines in this paper. However, the necessary case analysis is much easier because all that has to be cared about is the sets in $\mathcal{K}_1$. It follows that every member of $\mathcal{K} \setminus \mathcal{K}_2$ intersects b_0, c_0, a_1, b_1, or c_1.

The last step in Kramer's proof consists in rotating the line c_1 "backwards," i.e., counterclockwise around A_0, until it is about to leave some member of $\mathcal{K}_1$ for the first time. Such a member must exist. Let c_1' denote the rotated line at this point, where of course $c_1' = c_1$ is possible. As is readily seen, c_1' separates two members of $\mathcal{K}_1$ since otherwise no two members of $\mathcal{K}_1$ could be separated by a line through A_0; the latter was excluded above. Hence by (3.1), c_1' is a line transversal for $\mathcal{K}_2$. On the other hand, it is obvious that a_1, b_1, and c_1' still meet all the sets in $\mathcal{K}_1$. Consequently, every member of $\mathcal{K}$ is intersected by one of the lines b_0, c_0, a_1, b_1, and c_1'. This completes Kramer's reasoning.

We gratefully acknowledge that without Kramer's pioneering work, this paper would not have been written.

2. As pointed out in the Introduction, the proof of our theorem (satisfying as it may be) leaves open the possibility that $T(3)$ might imply not only T^4, but even T^3. In fact, we believe that the latter is true although we cannot offer any supporting evidence for this except our inability to find a counterexample.

Part of the difficulty seems to lie in the fact that $T(3)$ is a far weaker property than, say, $T(4)$. To justify this statement, let us briefly recall (from [3] or [4]) how we proved that $T(4)$ implies T^2. Consider a family $\mathcal{K}$ of compact convex sets in the plane. A result of Hadwiger and Debrunner (see p. 62 of [10]) asserts that if $\mathcal{K}$ satisfies $T(4)$, then there are two directions in the plane such that any two members of $\mathcal{K}$ admit a line transversal in one or the other direction. More generally, let us say that $\mathcal{K}$ has property $D_m(r)$, with D representing direction, provided m directions in the plane can be found such that any r or fewer sets in $\mathcal{K}$ have a line transversal in one of these directions. (Transversal properties of this and a closely related type have been studied by Wegner [15].) Thus $T(4)$ implies $D_2(2)$. In turn, $D_2(2)$ can be shown to be strong enough to imply T^2.

It might be expected that similar (albeit weaker) conclusions can be established in the case when $\mathcal{K}$ has property $T(3)$. For example, it is plausible that $T(3)$ should imply $D_3(2)$. Since it is easily verified that properties $T(3)$ and $D_3(2)$ combined imply T^3, this would settle the problem.[3] Unfortunately, $D_3(2)$ is not in general a consequence of $T(3)$. Wegner (personal communication) has constructed examples which show that $T(3)$ does not even imply $D_m(2)$, no matter how large an m is chosen. For $m = 3$, Wegner's example is illustrated in Fig. 4.

We must conclude, therefore, that the problem of finding the desired stabbing number cannot be tackled in the manner suggested by these (and similar) considerations. It seems safe to say that new tools and ideas are required to solve the problem.

3. Gallai-type problems for line transversals in the plane can, of course, be studied in any sufficiently large "ground family" of compact convex sets. Apart from the family of all such sets considered in this paper, the most interesting and natural examples are supplied by the family of all (positive) homothets and the family of all translates of a given compact convex set K. As already mentioned, the stabbing numbers for these families were first investigated by Hadwiger and Debrunner [10].

Again, in view of (1.1) and the fact that there is no Helly-type transversal theorem for translates, the only unresolved problem in this context arises for $r = 3$. The following is known (see [4]):

$$T(3) \Rightarrow T^4 \qquad \text{for families of homothets of } K,$$

$$T(3) \Rightarrow T^2 \qquad \text{for families of translates of } K.$$

Moreover, any family of translates of K satisfying $T(3)$ can be stabbed by two *parallel* lines. Of course, the result for homothets is now superseded by the corresponding result for general convex sets, i.e., assertion (1.3). However, a direct proof is still of interest and, in fact, very easy. It proceeds as follows:

Let $\mathcal{K}$ be a family of homothets of K having property $T(3)$. It may be assumed that K has nonempty interior and that $\mathcal{K}$ is finite, hence contains a smallest

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[3] Whether $D_3(2)$ alone is sufficient for T^3 is not known. It follows from a result of Gyárfás and Lehel [9] (see the note at the end of [15]) that $D_3(2)$ implies T^4.
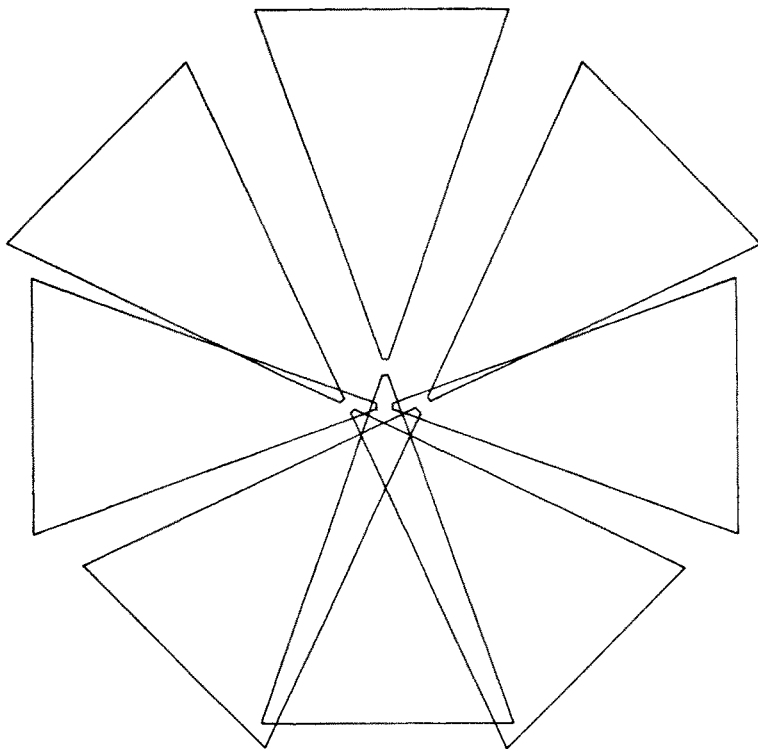
Fig. 4. Wegner's example.

homothet K_0, say. If every other member of $\mathscr{K}$ intersects K_0, then any two parallel supporting lines of K_0 suffice to stab $\mathscr{K}$. Otherwise, let L_0 be disjoint from K_0 and such that the interior angle $\sigma(K_0, L_0)$ defined in Section 2 is minimal. Then it is immediate that each set in $\mathscr{K}$ intersects one of the four supporting lines of K_0 which are laid out in the directions of the inner common tangents of K_0 and L_0. Thus $\mathscr{K}$ satisfies T^4, as claimed.

This straightforward argument is due to Wegner and was reported in [4]. It contrasts sharply with the rather complicated proof for general families of convex sets given in this paper. We believe that for any family of homothets of K, just as in the case of translates, $T(3)$ implies T^2.

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