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Approximating the Ball by a Minkowski Sum of Segments with Equal Length*

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Abstract. It is proved that for $n \geq 2$ the Euclidean ball B_n can be approximated up to ε (in the Hausdorff distance) by a zonotope having N summands of equal length with $N \leq c(n)(\varepsilon^{-2}|\log \varepsilon|)^{(n-1)/(n+2)}$.

1. Introduction

A zonotope in R^n is defined to be the Minkowski sum of segments I_j in R^n:

$$Z = \sum_{j=1}^{N} I_j = \left\{ \sum_{j=1}^{N} x_j;\ x_j \in I_j,\ 1 \leq j \leq N \right\}. \tag{1.1}$$

We deal here with the problem of approximating the Euclidean ball B_n by zonotopes with as few as possible summands N. In [BLM] it was proved that if a zonotope Z has a Hausdorff distance $\leq \varepsilon$ from B_n, then the number N appearing in (1.1) has to satisfy

$$N \geq c_1(n)\varepsilon^{-2(n-1)/(n+2)}. \tag{1.2}$$

On the other hand, it was proved in [BL] that there is a zonotope approximating

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B_n to within ε in the Hausdorff metric with N satisfying

$$N = [c_2(n)(\varepsilon^{-2}|\log \varepsilon|)^{(n-1)/(n+2)}]. \tag{1.3}$$

Thus up to a possible logarithmic factor (1.2) gives the best result.

In the paper [BM] where the same problem is also considered it was asked what can be said of N if we require that the segments appearing in (1.1) all have the same length. In [W] the late G. Wagner showed that if $n \leq 6$, then with the same estimate on N as that given in (1.3) B_n can be approximated up to ε by a zonotope having N summands of equal length. Wagner based his proof on a formula for numeric integration with constant weights which he proved in an earlier paper (the close relation between the topic of numerical integration with equal weights and approximation by zonotopes with summands of the same length will become clear below).

Our aim here is to show that Wagner's result is true without the restriction $n \leq 6$. In other words, we prove here the following:

Theorem. *For every n, there is a constant $c_2(n)$ so that the Euclidean ball can be approximated up to $\varepsilon > 0$ in the Hausdorff metric by a zonotope having N summands, all of the same length with N satisfying* (1.3).

Our proof differs from that of Wagner (for $n \leq 6$). It is an adaptation of the method we used in [BL]. In the rest of this introduction we review the approach in [BL] and point out the place where the argument here has to differ from that of [BL]. Details of the proof of the theorem are given in Section 2. The most technical part of the proof (the proof of Lemma 4) deals with a topic very close to numerical integration with equal weights of functions defined on an arc. The proof of this part is given in Section 3. It should be pointed out that the proofs given in Sections 2 and 3 are self-contained and can be read without reference to [BL] or [W].

We first give the analytic expression for approximation by zonotopes. A zonotope with N summands can approximate B_n up to ε if and only if there are $\{z_j\}_{j=1}^N$ on the sphere S^{n-1} and positive scalars $\{\alpha_j\}_{j=1}^N$ so that

$$\left| \sum_{j=1}^{N} \alpha_j |\langle x, z_j \rangle| - 1 \right| \leq \varepsilon, \qquad x \in S^{n-1}. \tag{1.4}$$

The length of the segment I_j is $2\alpha_j$. Instead of (1.4) it is more convenient to deal with a similar expression with 1 replaced by

$$\beta_n = \int_{S^{n-1}} |\langle x, y \rangle| \, d\mu(y), \tag{1.5}$$

where μ is the normalized rotation invariant measure on S^{n-1} (the passage from 1 to β_n has effect only on the value of the constants $c_1(n)$ and $c_2(n)$ in (1.2) and

(1.3) which are of no interest to us here). The question of approximating B_n by zonotopes with sides of equal length thus becomes the question of finding N and $\{z_j\}_{j=1}^N$ on S^{n-1} so that

$$\left| N^{-1} \sum_{j=1}^{N} |\langle x, z_j \rangle| - \beta_n \right| < \varepsilon, \qquad x \in S^{n-1}. \tag{1.6}$$

Let us now explain our approach in [BL]. Let N be given by (1.3). We partition S^{n-1} into N parts $\{Q_j\}_{j=1}^N$ all having the same μ measure and having a diameter of the order of magnitude $\eta = N^{-1/(n-1)}$. On each such Q_j we consider the set Σ_j of probability measures on Q_j so that $\sigma \in \Sigma_j$ has the form $\sum_{i=1}^{n+2} \lambda_i(\sigma)\delta_{y_i(\sigma)}$, $y_i(\sigma) \in Q_j$ (i.e., the support of σ consists of at most $n+2$ points), and so that the barycenter of σ agrees with that of $N\mu_{|Q_j}$. By using Caratheodory's theorem (in R^{n+1}) and the separation theorem it is not hard to show that Σ_j is rich enough to contain in the weak* closure of its convex hull the measure $N\mu_{|Q_j}$ (the w^* topology on measures is the one induced by continuous functions on Q_j). Since Σ_j is w^*-compact it follows from the Krein–Milman theorem that there is a probability measure v_j on Σ_j so that

$$N\mu_{|Q_j} = \int_{\Sigma_j} \sigma \, dv_j(\sigma), \qquad 1 \le j \le N.$$

By choosing on each Q_j a measure σ_j in Σ_j according to the distribution v_j, doing it independently in each j, and using a standard inequality from probability theory (Lemma 3 below) it follows that, with a positive probability,

$$\left| N^{-1} \sum_{j=1}^{N} \sum_{i=1}^{n+2} \lambda_i(\sigma_j) |\langle x, y_i(\sigma_j) \rangle| - \beta_n \right| < \varepsilon, \qquad x \in S^{n-1}. \tag{1.7}$$

The fact that N is replaced by $N(n+2)$ does of course not matter in view of the form of (1.3). The reason that (1.7) is not of the desired form (1.6) is because of the presence of the weights $\lambda_i(\sigma_j)$. They enter via the use made of Caratheodory's theorem. A point in the convex hull of a set A in R^{n+1} can be expressed as a convex combination of $n+2$ points out of A but of course in general not as an arithmetic mean of such $n+2$ points.

This difficulty is overcome below by proving (using a lemma on numerical integration with equal weights) that, if Q is a "nice" subset of S^{n-1}, the normalized surface measure $\tilde{\mu}$ on Q is in the weak* closed convex hull of arithmetic means of $L(n)$ dirac measures which (i.e., the arithmetic means) have the same centroid as $\tilde{\mu}$. Moreover, S^{n-1} may be partitioned into subsets so that the large majority of them are "nice" in the sense above.

Let us point out that this proof strongly depends on the specific geometric structure of S^{n-1}. It does not generalize in an obvious way to the setting of approximating other zonoids (= Hausdorff limits of zonotopes) by zontopes with a small number of summands of equal length. This is in contrast to the argument in [BL], where general zonoids were treated, or to the arguments in Wagner's paper [W] which, as stated there, apply also to zonoids sufficiently close to balls.

2. Proof of the Theorem

The proof of the theorem is based on three lemmas. The first lemma is geometric in nature and for its formulation we need the notion of a good spherical parallelopiped (GSP in short) in S^{n-1}. We introduce in the usual manner spherical coordinates in S^{n-1}:

$$-\pi \le \theta_1 < \pi, \ -\frac{\pi}{2} \le \theta_2 \le \frac{\pi}{2}, \ldots, -\frac{\pi}{2} \le \theta_{n-1} \le \frac{\pi}{2}.$$

A GSP set in S^{n-1} is a set of all the points so that $\theta_1 \in [\alpha_1, \beta_1]$ for some $\alpha_1 < \beta_1$ and for $j \ge 2$, $\theta_j \in [\alpha_j, \beta_j]$ for some $\alpha_j < \beta_j$ with

$$0 \le \alpha_j < \beta_j < \pi/2 \ (\text{resp. } -\pi/2 < \alpha_j < \beta_j \le 0),$$

and moreover

$$\frac{\pi/2 - \alpha_j}{\pi/2 - \beta_j} \le 1 + \rho_j \qquad \left(\text{resp. } \frac{\beta_j + \pi/2}{\alpha_j + \pi/2} \le 1 + \rho_j\right), \tag{2.1}$$

where the $\{\rho_j\}_{j=1}^n$ are specific positive numbers to be determined in the proof of Lemma 2 below.

Lemma 1. *Let N be an even integer, and put $\eta = N^{-1/(n-1)}$. It is possible to find N compact subsets $\{Q_j\}_{j=1}^N$ of S^{n-1} so that:*

$$\mu(Q_j) = N^{-1} \quad \textit{for every } j, \qquad \mu(Q_{j_1} \cap Q_{j_2}) = 0 \quad \textit{for} \quad j_1 \ne j_2. \tag{2.2}$$

$$\textit{The diameter of each } Q_j \textit{ is at most } c_3(n)\eta. \tag{2.3}$$

$$\textit{The } Q_j \textit{ are GSPs except at most } c_4(n)N\eta^2 \textit{ of them.} \tag{2.4}$$

The constants $c_3(n)$ and $c_4(n)$ are, as usual, constants depending only on the dimension.

The next lemma deals with a subject closely related to integration formulas with equal weights.

Lemma 2. *There is an integer $L(n)$ so that if Q is a GSP on S^{n-1}, then the probability measure $\mu(Q)^{-1}\mu_{|Q}$ is in the w^* closed convex hull of the set Σ of all* *probability measures* σ *satisfying*

$$\sigma = L(n)^{-1} \sum_{i=1}^{L(n)} \delta_{y_i(\sigma)}, \qquad y_i(\sigma) \in Q, \quad 1 \le i \le L(n), \tag{2.5}$$

$$\int_Q f\, d\sigma = \mu(Q)^{-1} \int_Q f\, d\mu \qquad \textit{for all linear functions } f \textit{ on } R^n. \tag{2.6}$$

The third lemma is a standard inequality in probability theory (Bernstein's inequality).

Lemma 3. *Let* $\{g_j\}_{j=1}^J$ *be independent random variables with mean* 0 *and uniformly bounded by* 1 *on some probability space. Then for* $0 < \delta < 1$ *we have*

$$\mathrm{Prob}\left\{ \left| \sum_{j=1}^{J} g_j \right| > \delta J \right\} \le 2 \exp\left(-\frac{J\delta^2}{2} \right). \tag{2.7}$$

We prove Lemmas 1 and 2 below and show next how to derive the theorem from the three lemmas above.

Proof of the Theorem. Let $\varepsilon > 0$ and let N be an even integer. We partition S^{n-1} into sets $\{Q_j\}_{j=1}^N$ as in Lemma 1. We assume that for $j \le \tilde{N}$ the sets Q_j are GSPs. Recall that by (2.4)

$$N - \tilde{N} \le c_4(n) N \eta^2. \tag{2.8}$$

Let Σ_j be the set of probability measures on Q_j, $1 \le j \le \tilde{N}$, given by Lemma 2. Since Σ_j is w^*-compact it follows from Lemma 2 and the Krein–Milman theorem that there is a probability measure ν_j on Σ_j so that

$$N\mu_{|Q_j} = \int_{\Sigma_j} \sigma\, d\nu_j(\sigma), \qquad 1 \le j \le \tilde{N}. \tag{2.9}$$

Let $f \in C(S^{n-1})$ and consider the following N independent random variables (on the obvious product space):

$$h_{j,f}(\sigma) = L(n)^{-1} \sum_{i=1}^{L(n)} f(y_i(\sigma)) - N \int_{Q_j} f\, d\mu, \qquad \sigma \in \Sigma_j, \quad j \le \tilde{N}, \tag{2.10}$$

$$h_{j,f}(y) = f(y) - N \int_{Q_j} f\, d\mu, \qquad y \in Q_j, \quad \tilde{N} < j \le N. \tag{2.11}$$

All these variables have mean 0. Observe next that by (2.3) if f satisfies

$$|f(u) - f(v)| \le \|u - v\|_2, \qquad u, v \in S^{n-1}, \tag{2.12}$$

then

$$\|h_{j,f}\|_\infty \le c_3(n)\eta, \qquad 1 \le j \le N. \tag{2.13}$$

Also, in view of (2.6), if the restriction of f to Q_j is linear for some $j \le \tilde{N}$, then

$$h_{j,f} \equiv 0. \tag{2.14}$$

Now fix $x \in S^{n-1}$ and let $f(y) = f_x(y) = |\langle x, y\rangle|$ for $y \in S^{n-1}$. By (2.14) the set of indices j for which h_{j,f_x} does not vanish identically consists of at most the $\tilde{N} \le j \le N$ and those $j \le \tilde{N}$ for which Q_j intersects $\{y; \langle x, y\rangle = 0\}$. Hence by (2.3) and (2.8) the cardinality J of this set is at most $c_5(n)N\eta$. Also f_x clearly satisfies (2.12); thus (2.13) holds for every j and $f = f_x$. By applying Lemma 3 to the J nonvanishing $h_{j,f_x}/c_3(n)\eta$ we get for $0 < \delta < 1$

$$\operatorname{Prob}\left\{\left|\sum_{j=1}^{N} h_{j,f_x}\right| > c_3(n)c_5(n)\delta\eta^2 N\right\} \le 2\exp\left(-\frac{c_5(n)\eta N\delta^2}{2}\right). \tag{2.15}$$

We now take δ so that

$$\frac{\varepsilon}{2} = c_3(n)c_5(n)\delta\eta^2 N$$

(for N satisfying (1.3) we have $\delta < 1$ for $\varepsilon < \varepsilon_0(n)$). By using (1.5), (2.10), and (2.11), formula (2.15) becomes

$$\operatorname{Prob}\left\{\left|N^{-1}\left(\sum_{j=1}^{\tilde{N}} L(n)^{-1}\sum_{i=1}^{L(n)} |\langle x, y_i(\sigma_j)\rangle| + \sum_{j=\tilde{N}+1}^{N} |\langle x, y_j\rangle|\right) - \beta_n\right| > \frac{\varepsilon}{2}\right\}$$
$$\le 2\exp\left(-\frac{N^{(n+2)/(n-1)}\varepsilon^2}{c_6(n)}\right). \tag{2.16}$$

We now let x vary on an $\varepsilon/4$ net in S^{n-1}. The number of points in such a net is $\le (c_7(n)/\varepsilon)^{n-1}$. Hence if

$$2\left(\frac{c_7(n)}{\varepsilon}\right)^{n-1}\exp\left(-\frac{N^{(n+2)/(n-1)}\varepsilon^2}{c_6(n)}\right) < 1$$

(and this is the case if N is of the form (1.3) for suitable $c_2(n)$), then there is a choice of $\sigma_j \in \Sigma_j$, $1 \le j \le \tilde{N}$, and of $y_j \in Q_j$, $\tilde{N} < j \le N$, so that

$$\left|N^{-1}L(n)^{-1}\left(\sum_{j=1}^{\tilde{N}}\sum_{i=1}^{L(n)} |\langle x, y_i(\sigma_j)\rangle| + \sum_{j=\tilde{N}+1}^{N} L(n)|\langle x, y_j\rangle|\right) - \beta_n\right| \le \frac{\varepsilon}{2} \tag{2.17}$$

for every x in an $\varepsilon/4$ net on S^{n-1}. This clearly implies that (2.17) holds for every

$x \in S^{n-1}$ if $\varepsilon/2$ is replaced by ε. Thus we get that a formula of type (1.6) holds with $NL(n)$ where N is given by (1.3). This concludes the proof of the theorem. □

Proof of Lemma 1. The proof is by induction on n (starting with $n = 3$). Let us first consider the case $n = 3$. Let N be an even integer and let $\eta = N^{-1/2}$. We divide the equator ($\theta_2 = 0$) of S^2 into arcs A_j^1 all of an equal length between $\eta/2$ to η. We next find β_1 so that the spherical parallelograms $\theta_1 \in A_j^1$, $0 \le \theta_2 \le \beta_1$, all have area N^{-1}. We next divide the circle $\theta_2 = \beta_1$ into arcs A_j^2 all of an equal length between $\eta/2$ and η and find a β_2 so that the parallelograms $\theta_1 \in A_j^2$, $\beta_1 \le \theta_2 \le \beta_2$, all have area N^{-1}. We continue in an obvious manner and find $\beta_3, \ldots, \beta_k$. We do this as long as the parallelograms we get are "good," i.e., according to (2.1), as long as

$$\frac{\pi/2 - \beta_{k-1}}{\pi/2 - \beta_k} < 1 + \rho_2.$$

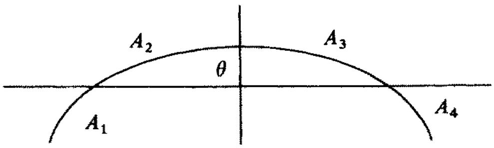
It follows that if we have to stop with β_k, then $\pi/2 - \beta_k \le d_2\eta$, where d_2 is a constant determined by ρ_2 (which in turn will be determined in the proof of Lemma 2). The cap around the north pole which remains uncovered has an area of the order of magnitude η^2 and thus can be divided into sets Q_j satisfying (2.2) and (2.3) whose number is of the order $\eta^2 N$ (in this case, i.e., $n = 3$, actually an absolute constant). In the lower hemisphere the partition is the reflection of what we did in the upper hemisphere.

We pass now from $n = 3$ to $n = 4$ (the general induction step is the same). We first partition the 2-sphere $\theta_3 = 0$ as above into parts Q_j^1 of equal two-dimensional measure between $\eta^2/2$ and η^2. We find γ_1 so that the sets $\theta_1, \theta_2 \in Q_j^1$, $0 \le \theta_3 \le \gamma_1$, all have μ measure N^{-1}. We continue in the same way finding $\gamma_2, \gamma_3, \ldots, \gamma_h$ as long as the sets

$$\theta_1, \theta_2 \in Q_j^{h-1}, \qquad \gamma_{h-1} \le \theta_3 \le \gamma_h,$$

are GSPs whenever Q_j^{h-1} is a GSP of one lower dimension. In other words, we continue till $\pi/2 - \gamma_h \le d_3\eta$ for a certain d_3 which is determined by ρ_3. The remaining cap around the point $\theta_3 = \pi/2$ with radius $\le d_2\eta$ we divide into sets Q_j satisfying (2.2) and (2.3). By reflection we deal with the part $\theta_3 \le 0$ of S^{n-1}. It is clear that the number of sets obtained in this way which fail to be GSPs is at most $c(3)N\eta^2$. □

The main part of the proof of Lemma 2 is the proof of a one-dimensional result which we state here as Lemma 4. First we need some notation. Let A be an arc of length 1 of a circle (typically with a large radius) in the plane. We assume that the origin in the plane is the center of gravity of A (we equip A with the homogeneous arc length measure ds). We assume also that A is symmetric with respect to the y axis. The parts into which A is cut by the coordinate axes are called A_1, A_2, A_3, and A_4, respectively. The y coordinate of the top of the circle is denoted by θ. We prefer to use θ as a parameter rather than the radius of the circle. The picture is thus:



With this notation in mind we state

Lemma 4. *There are absolute constants $\delta > 0$ and $c > 0$ and an interger L so that the following holds. Let $f \in L_1(A)$ satisfy*

$$\int_{A_1 \cup A_2} f\, ds = \int_{A_2 \cup A_3} f\, ds = \int_{A_3 \cup A_4} f\, ds = \int_{A_4 \cup A_1} f\, ds = 0 \tag{2.18}$$

and let $u_0 = (x_0, y_0) \in R^2$ satisfy

$$|x_0| \le \delta, \qquad |y_0| \le \theta\delta. \tag{2.19}$$

Then there are L points $\{u_i\}_{i=1}^L$ on A with

$$u_0 = L^{-1} \sum_{i=1}^{L} u_i, \tag{2.20}$$

$$L^{-1} \sum_{i=1}^{L} f(u_i) \ge c\|f\|_1 = c \int_A |f|\, ds. \tag{2.21}$$

The point in the lemma is that δ, c, and L are independent of the parameter θ of A and also of f and that no smoothness conditions are imposed on f.

The proof of Lemma 4 is the most technical part of this paper and is presented in Section 3. Here we assume Lemma 4 and deduce from it Lemma 2.

Corollary 5. *Let A be a circular arc of length 1 in the plane. Let c, δ, and L be the constants given in Lemma 4 and put $\rho = \min(c, \delta)$. Then any probability measure ν on A which is absolutely continuous with respect to ds and satisfies $\|d\nu/ds - 1\|_\infty \le \rho$ is in the weak* closed convex hull of the set Ω of measures σ on A of the form $\sigma = L^{-1} \sum_{i=1}^L \delta_{u_i}$, $u_i = (x_i, y_i) \in A$, for which*

$$L^{-1} \sum_{i=1}^{L} x_i = \int_A x(s)\, d\nu(s), \qquad L^{-1} \sum_{i=1}^{L} y_i = \int_A y(s)\, d\nu(s). \tag{2.22}$$

Proof. Assume that ν does not belong to the w^* closed convex hull of Ω. By the separation theorem there is a continuous function f on A so that

$$\int_A f\,dv > \sup_{\sigma\in\Omega} \int_A f\,d\sigma. \tag{2.23}$$

We may clearly choose the coordinate system in R^2 as in the setting for Lemma 4. Because of (2.22) we may add to f any restriction to A of a linear function on R^2 without affecting (2.23). Hence there is not loss of generality to assume that the f in (2.23) also satisfies (2.18). By our assumption on dv/ds we have

$$\left|\int_A x(s)\,dv(s)\right| = \left|\int_A x\,ds + \int_A x\left(\frac{dv}{ds} - 1\right) ds\right| = \left|\int_A x\left(\frac{dv}{ds} - 1\right) ds\right| \le \delta.$$

Similarly $|\int_A y\,dv(s)| \le \theta\delta$. Hence by Lemma 4 there is a $\sigma \in \Omega$ so that

$$\int_A f\,d\sigma \ge \rho \int_A |f|\,ds.$$

On the other hand,

$$\left|\int_A f\,dv\right| = \left|\int_A f\left(\frac{dv}{ds} - 1\right) ds\right| \le \rho \int_A |f|\,ds$$

and this contradicts (2.23). □

Remark. Corollary 5 is evidently valid for an arbitrary circular arc A in the plane provided we take as ds the normalized arc measure on A (which gives to A mass 1).

Proof of Lemma 2. The proof is by induction on n. The number $L(n)$ we get is L^{n-1} where L is the integer appearing in Lemma 4.

For $n = 2$ Lemma 2 is just the special case $dv = ds$ of Corollary 5.

We pass from $n \ge 2$ to $n + 1$ and determine at the same time the constant ρ_n appearing in the definition of GSP. Let $Q_n \subset S^n$ be the spherical parallelopiped $\alpha_i \le \theta_i \le \beta_i$, $1 \le i \le n$. We may assume that $\alpha_i \ge 0$ for all $i \ge 2$. We assume also that

$$\frac{\pi/2 - \alpha_i}{\pi/2 - \beta_i} \le 1 + \rho_i, \qquad 2 \le i \le n - 1.$$

For each fixed θ_n in $[\alpha_n, \beta_n]$ the spherical parallelopiped Q_{n-1,θ_n} (=those points in Q_n whose last coordinate is θ_n) is a GSP in S^{n-1}. By the induction hypothesis the normalized $(n - 1)$-dimensional measure of Q_{n-1,θ_n} is in the w^* closed convex hull of measures of the form

$$L(n)^{-1}\sum_{i=1}^{L(n)} \delta_{y_i}, \qquad y_i \in Q_{n-1,\theta_n},$$

whose barycenter agrees with the barycenter of Q_{n-1,θ_n}. The barycenters of Q_{n-1,θ_n}, $\alpha_n \le \theta_n \le \beta_n$, form an arc A of a circle (natually parametrized again by θ_n). Let ds be the normalized arclength measure on A and let ν be the measure induced on A by the normalized n-dimensional surface measure μ on Q_n (i.e., for $\alpha_n < \gamma' < \gamma'' < \beta_n$):

$$\nu[\gamma', \gamma''] = \mu\{y \in Q_n, \theta_n(y) \in [\gamma', \gamma'']\}.$$

We have

$$\left[\frac{\max_{\theta\in A}(d\nu/ds)(\theta)}{\min_{\theta\in A}(d\nu/ds)(\theta)}\right] - 1 \le c(n)\left[\left(\frac{\pi/2-\alpha_n}{\pi/2-\beta_n}\right)^{n-1} - 1\right]$$

for a suitable constant $c(n)$. Hence if ρ_n is defined by $c(n)\ ((1+\rho_n)^{n-1} - 1) = \rho$, where ρ is the constant appearing in Corollary 5, then by this corollary ν is in the w^* closed convex hull of measures of the form $L^{-1}\sum_{j=1}^{L} \delta_{z_j}$ with $z_j \in A$ which have as their barycenter the barycenter of Q_n. By using these facts (and Fubini's theorem) it follows that the normalized surface measure on Q_n is in the weak* closed convex hull of measures of the form

$$L(n)^{-1}L^{-1}\sum_{j=1}^{L}\sum_{i=1}^{L(n)} \delta_{y(i,j)}, \qquad y_{(i,j)} \in Q_{n-1,\theta_j},$$

which have the same barycenter as Q_n. □

3. Proof of Lemma 4

We now pass to the proof of Lemma 4. We use the notion of equivalence $\sim$ to denote two positive quantities whose ratio is bounded from above and below by absolute positive numbers. The symbol $\prec$ has a similar obvious meaning.

We start by pointing out some absolute constants which can be obtained from the setting of Lemma 4 and which are used later in determining δ, c, and L.

(i) Let Φ be the map from $A \times A$ into R^2 defined by $\Phi\{v, w\} = (v + w)/2$. On the subset $\tilde{A}$ of A, consisting of those pairs for which the x coordinate of v is smaller than that of w, the map Φ is one to one. The range of Φ contains a rectangle $[-\beta, \beta] \times [-\theta\beta, \theta\beta]$ for some absolute $\beta > 0$. The Jacobian of Φ on the intersection of $\tilde{A}$ with the inverse image of this rectangle is equivalent to θ.

(ii) Let $K = \{h \in L_1(A);\ \exists a,\ b,\ h(x, y) \le ax + by,\ \forall (x, y) \in A\}$. Then for every f satisfying (2.18) we have

$$d(f, K) \ge c_1 \|f\|_1 \tag{3.1}$$

for some absolute constant c_1.

The verification of (i) is obvious. Let us check (ii). Assume that $h \in K$ then $h(x, y) = ax + by + k(x, y)$ with $k(x, y) \le 0$ for all $(x, y) \in A$. By (2.18)

$$\|f - h\|_1 \ge \int_A (f - h)\, ds = -\int_A k\, ds = \|k\|_1.$$

Again, by (2.18),

$$\|f - h + k\|_1 \ge \left| \int_{A_2 \cup A_3} (f - (ax + by))\, ds \right| = |b| \int_{A_2 \cup A_3} y\, ds$$

and hence $\|by\|_1 \prec \|f - h + k\|_1$. Similarly, $\|ax\|_1 \prec \|f - h + k\|_1$. Inequality (3.1) is a direct consequence of these inequalities.

We now divide each of the subarcs $\{A_i\}_{i=1}^4$ of A into consecutive arcs of length between δ and $\delta/2$ (δ is determined later). A typical arc obtained this way is denoted by W. Each such W is partitioned into two subsets W' and W'' so that

$$|W'| \sim \delta \sim |W''| \tag{3.2}$$

($|W'|$ means the ds measure of W') and

$$\left| \int_{W'} f\, ds \right| \sim \int_{W'} |f|\, ds, \qquad \left| \int_{W''} f\, ds \right| \sim \int_{W''} |f|\, ds. \tag{3.3}$$

The collection of all subsets of A obtained in this way is denoted by $\{T_i\}_{i \in I}$. The conditional expectation of f with respect to this partition is denoted by g:

$$g = \sum_{i \in I} |T_i|^{-1} \cdot \int_{T_i} f\, ds \cdot \chi_{T_i}. \tag{3.4}$$

From (3.3) it follows that $\|f\|_1 \sim \|g\|_1$. It is clear from the construction that g like f satisfies (2.18) and hence by (3.2) there is an absolute positive c_2 so that

$$d(g, K) \ge c_2 \|f\|_1.$$

By the separation theorem we deduce that there is a $\Delta \in L_\infty(A)$ with $\Delta \geq 0$ and $\|\Delta\|_\infty = 1$ so that

$$\int_A g\Delta \, ds = \sum_{i \in I} |T_i|^{-1} \cdot \int_{T_i} f \, ds \cdot \int_{T_i} \Delta \, ds \geq c_2 \|f\|_1, \tag{3.5}$$

$$\int_A \Delta(s)x(s) \, ds = \int_A \Delta(s)y(s) \, ds = 0. \tag{3.6}$$

By replacing Δ with $\Delta' = \Delta/\|\Delta\|_1$ and noting that $\|\Delta\|_1 \leq 1$ we see that we can ensure that (3.5) and (3.6) hold with a Δ satisfying $\Delta \geq 0$ and $\|\Delta\|_1 = 1$.

From Caratheodory's theorem (in R^3) it follows that there exist four indices $\{i_j\}_{j=1}^4$ in I and scalars $\{\lambda_j\}_{j=1}^4$ with $\lambda_j \geq 0$ and $\sum \lambda_j = 1$ so that if we put $\tau_j = T_{i_j}$, then $\int_{\tau_j} \Delta \, ds \neq 0$ for all j and

$$\sum_{j=1}^4 \lambda_j |\tau_j|^{-1} \int_{\tau_j} f \, ds \geq c_2 \|f\|_1, \tag{3.7}$$

$$\sum_{j=1}^4 \lambda_j \frac{\int_{\tau_j} x\Delta \, ds}{\int_{\tau_j} \Delta \, ds} = \sum_{j=1}^4 \lambda_j \frac{\int_{\tau_j} y\Delta \, ds}{\int_{\tau_j} \Delta \, ds} = 0. \tag{3.8}$$

From (3.8) it follows that if u_j is any point in τ_j, then

$$\sum_{j=1}^4 \lambda_j u_j \in [-\delta, \delta] \times [-\theta\delta, \theta\delta]. \tag{3.9}$$

To continue we point out another fact which follows easily from the geometry of A. There are at least two indices out of the four (which we choose to take as 1 and 2) so that

$$\lambda_1 \geq c_3, \qquad \lambda_2 \geq c_3, \qquad d(\tau_1, \tau_2) \geq c_3, \tag{3.10}$$

where c_3 is an absolute positive constant. From (3.10) it follows that if W_1 (resp. W_2) are the arcs from which τ_1 and τ_2 were formed then for every fixed $p \in R^2$ the map from $W_1 \times W_2 \to R^2$ defined by

$$\{u_1, u_2\} \to \lambda_1 u_1 + \lambda_2 u_2 + p \tag{3.11}$$

has a Jacobian (obviously independent of p) which is $\sim \theta$.

Let $\tilde{M}$ be an even integer and let $\{\lambda_j'\}_{j=1}^4$ be nonnegative numbers which sum to 1, which are close to $\{\lambda_j\}_{j=1}^4$, and are rational numbers with $\tilde{M}$ as the denominator so that

$$\sum_{j=1}^4 \lambda_j' |\tau_j|^{-1} \int_{\tau_j} f \, ds \geq \frac{c_2 \|f\|_1}{2} \tag{3.12}$$

and for every choice of $u_j \in \tau_j$, $1 \le j \le 4$,

$$\sum_{j=1}^{4} \lambda'_j u_j \in [-2\delta, 2\delta] \times [-2\delta\theta, 2\delta\theta]. \tag{3.13}$$

In view of (3.7) and (3.9) we can ensure all this provided that

$$\tilde{M}\delta \ge c_4 \tag{3.14}$$

for some abosolute c_4. Let $M \ge 2$ be another integer so that

$$16M\delta \le \beta, \tag{3.15}$$

where β is the constant appearing in (i) at the beginning of this section. By our assumption on u_0 (see (2.19)) and (3.13)

$$Mu_0 - (M-1)\sum_{j=1}^{4} \lambda'_j u_j \in [-\beta, \beta] \times [-\theta\beta, \theta\beta]$$

for all $u_j \in \tau_j$, $1 \le j \le 4$. Hence, by observation (i), we can define maps $V(u_1, u_2, u_3, u_4)$ and $W(u_1, u_2, u_3, u_4)$ from $W_1 \times W_2 \times \tau_3 \times \tau_4$ into A so that

$$Mu_0 - (M-1)\sum_{j=1}^{4} \lambda'_j u_j = \tfrac{1}{2}(V(u_1, \ldots, u_4) + W(u_1, \ldots, u_4)). \tag{3.16}$$

Moreover, by (i) and (3.14), for every fixed u_3 and u_4 the Jacobian of the map from $W_1 \times W_2$ to $A \times A$ defined by

$$\{u_1, u_2\} \to \{V(u_1, u_2, u_3, u_4), W(u_1, u_2, u_3, u_4)\}$$

is equivalent to M^2 (with the equivalence constant independent of u_3 or u_4). The representation

$$u_0 = \frac{M-1}{M}\sum_{j=1}^{4} \lambda'_j u_j + \frac{1}{2M}(V(u_1, u_2, u_3, u_4) + W(u_1, u_2, u_3, u_4)) \tag{3.17}$$

is of the form (2.20) with $L = M\tilde{M}$. We now use an averaging argument to find $u_j \in \tau_j$, $1 \le j \le 4$, so that (2.21) holds (with a universal c) for the representation (3.17). By (3.12)

$$\frac{M-1}{M}\sum_{j=1}^{4} \lambda'_j |\tau_j|^{-1} \int_{\tau_j} f(u(s))\, ds \ge \frac{c_2\|f\|_1}{4}.$$

We next show that on average $|f(V)|$ is not too big. More precisely, we estimate from above the expression

$$\left(2M \cdot \prod_{j=1}^{4} |\tau_j|\right)^{-1} \int_{\tau_4}\int_{\tau_3}\int_{\tau_2}\int_{\tau_1} |f(V(u(s_1), u(s_2), u(s_3), u(s_4)))|\, ds_1\, ds_2\, ds_3\, ds_4. \tag{3.18}$$

For a fixed s_3 and s_4 we get by the remark made above on the Jacobian that

$$\int_{\tau_2}\int_{\tau_1} |f(V(u(s_1), u(s_2), u(s_3), u(s_4)))|\, ds_1\, ds_2 \le c_5 M^{-2}\|f\|_1.$$

Hence, in view of (3.2), expression (3.18) is bounded from above by $c_6\delta^{-2}M^{-3}\|f\|_1$. The same estimate holds for $|f(W)|$. Hence if

$$\frac{c_2}{8} \ge 2c_6\delta^{-2}M^{-3} \tag{3.19}$$

we can ensure that there are $\{s_j\}_{j=1}^4$ so that for $u_j = u(s_j) \in \tau_j$ estimate (2.21) holds with $c = c_2/8$ for the representation (3.17). Clearly, it is possible to find absolute M and δ so that (3.15) and (3.19) hold. Once δ is determined we determine $\tilde{M}$ so that (3.14) holds. The lemma holds thus with $c = c_2/8$, the δ we determined, and $L = M\tilde{M}$.

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