Discrete Comput Geom 3:49–54 (1988)

Discrete & Computational
Geometry

On the Classification of Toric Fano Varieties

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Abstract. Toric Fano varieties are algebraic varieties associated with a special class of convex polytopes in R^n. We extend results of V. E. Voskresenskij and A. A. Klyachko about the classification of such varieties using a purely combinatorial method of proof.

Let P be a simplical convex polytope in $\mathbb{R}^n$ whose vertices are primitive lattice vectors ($\in\mathbb{Z}^n$), and which contains 0 in its interior. If $a_1, \ldots, a_n$ are the vertices of a facet of P we suppose $\det(a_1, \ldots, a_n) = \pm 1$, for all facets of P. Then we call P a *Fano polytope.*

Let $\Sigma = \Sigma(P)$ be a system of cones each of which joins 0 to all points of a face of P, so that the toric or toroidal variety X_Σ associated with the fan Σ (see, for example, [1], [2], [3]) is projective. In case P is a Fano polytope, $X_{\Sigma(P)}$ is said to be a *toric* (or *toroidal*) *Fano variety.* (It is, equivalently, a complete smooth toric variety whose anticanonical divisor is ample.)

Let $e_1, \ldots, e_n$ be the canonical basis vectors of $\mathbb{R}^n$. If $n = 2k$ is even >0; then for $Q := \operatorname{conv}\{\pm e_1, \ldots, \pm e_n, e_1 - e_2 + \cdots + e_{n-1} - e_n, -e_1 + e_2 + \cdots + e_{n-1} - e_n\}$ we obtain an example of a Fano variety $X_{\Sigma(Q)}$, called a *del Pezzo variety* V^{2k}. It can be obtained from $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ (n times) by blowing-up twice in regular points. Voskresenskij and Klyachko [4] have classified all *symmetric* toroidal Fano varieties, i.e., Fano varieties $X_{\Sigma(P)}$ with centrally symmetric P, hence possessing a torus-invariant symmetry:

Any symmetric toroidal Fano variety splits into a product of projective lines and del Pezzo varieties. (1)

In terms of convexity a split $X_{\Sigma(P)} = X_1 \times X_2$ of $X_{\Sigma(P)}$ is given by $P = \operatorname{conv}(P_1 \cup P_2)$ where P_1, P_2 are polytopes whose linear spans have only 0 in common. So $X_{\Sigma(P)} = X_{\Sigma(P_1)} \times X_{\Sigma(P_2)}$. A dual polytope P^* of P then decomposes into the

Minkowski sum $P^* = P_1^* + P_2^*$ where P_1^*, P_2^* are dual lattice polytopes of P_1, P_2 relative to aff P_1, aff P_2 (affine hulls), respectively, such that $\dim(\text{aff } P_1 \cap P_2) = 0$. This again implies that the invertible sheaf associated with P^* splits into the tensor product of the invertible sheafs associated with P_1^*, P_2^*. If $X_{\Sigma(P)}$ does not split we call P *irreducible.*

In their proof of (1) the authors make use of an interesting relationship between symmetric Fano varieties and Dynkin diagrams of root systems. Nevertheless, in this note we present a direct and short proof of (1). Furthermore, we extend the result as follows:

We call a Fano polytope *pseudo-symmetric* if it has two facets F, F' centrally symmetric to each other with respect to 0. Let, for example, n be even and

$$Q := \text{conv}\{\pm e_1, \ldots, \pm e_n, e_1 - e_2 + \cdots + e_{n-1} - e_n\}.$$

In this case we call $X_{\Sigma(Q)}$ a *pseudo-del Pezzo variety.* We shall prove:

Theorem. *Any pseudo-symmetric toroidal Fano variety splits into a product of projective lines, del Pezzo varieties, and pseudo-del Pezzo varieties.*

Some consequences may be noted:

Any pseudo symmetric Fano variety

$$X_\Sigma = \mathbb{P}_1^{(1)} \times \cdots \times \mathbb{P}_1^{(P)} \times V^{2k_1} \times \cdots \times V^{2k_r} \times \tilde{V}^{2m_1} \times \cdots \times \tilde{V}^{2m_s} \tag{2}$$

($\mathbb{P}_1^{(q)}$ projective lines, V^{2k_i} del Pezzo varieties, $\tilde{V}^{2m_j}$ pseudo-del Pezzo varieties) can be blown-down

(a) into a product $\mathbb{P}_1^{(1)} \times \cdots \times \mathbb{P}^{(n)}$ of projective lines; $2r + s$ $(\leq n)$ blow-downs are hereby needed;
(b) into a product $\mathbb{P}_1^{(1)} \times \cdots \times \mathbb{P}_1^{(p)} \times \mathbb{P}_{2k_1} \times \cdots \times \mathbb{P}_{2k_r} \times \mathbb{P}_{2m_1} \times \cdots \times \mathbb{P}_{2m_s}$ of projective spaces; $r + k_1 + \cdots + k_r + m_1 + \cdots + m_s$ $(\leq n + r)$ blow-downs are hereby needed.

We say, a polytope P is *inscribed* in a polytope $\hat{P}$ if $\dim P = \dim \hat{P}$, and if any vertex of P is also a vertex of $\hat{P}$. We call a Fano polytope *maximal* if it is not inscribed in any Fano polytope $\hat{P} \neq P$.

Each maximal pseudo-symmetric Fano polytope is centrally symmetric. (3)

Further Fano polytopes can be obtained from the pseudo-symmetric ones by omitting vertices and considering the convex hull of the remaining polytopes (provided, it contains 0 in its interior). The following polytope $P \subset \mathbb{R}^3$, however, is not of this type:

$$P = \text{conv}\{e_1, e_2, e_3, -e_2, -e_3, -e_1 - e_2, -e_2 + e_3\}.$$

A construction method ("suspension") for Fano polytopes is also presented by Voskresenskij and Klyachko [4] using Gale diagrams. Since this construction assigns to any lattice polytope P_0 a Fano polytope P the authors doubt whether it might make sense to try a classification of all Fano polytopes. It should be noted, however, that the suspension (in [2] also used for the construction of nonprojective toroidal varieties) destroys the original structure of P. Also, the authors show that the number of vertices of any Fano polytope in $\mathbb{R}^n$ is $\leq n^2+1$. So we think a classification is possible. As intermediate results we conjecture:

Conjecture 1. *There are at most $n-1$ types of maximal irreducible n-dimensional Fano polytopes. Each of them possesses at most $2n+2$ vertices.*

Conjecture 2. *Up to a unimodular transformation, all vertices of a Fano polytope have coordinates* 1, −1, 0 *only.*

The proof of our theorem is achieved by proving several lemmas.

Lemma 1. *Let the coordinates be chosen such that $F=\operatorname{conv}\{e_1, e_2, \dots, e_n\}$ and $F'=\operatorname{conv}\{-e_1, \dots, -e_n\}$ are facets of a pseudo-symmetric Fano polytope P. Then any further vertex $a=(a_1, \dots, a_n)$ of P satisfies*

$$-1 \leq a_j \leq 1, \qquad j=1, \dots, n. \tag{4}$$

Proof. Let (for row vectors)

$$\delta=\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 1 & 0 & \cdots & 1 \end{pmatrix}$$

be a unimodular transformation. Then

$$F^{\delta}=\operatorname{conv}\{e_1, e_1+e_2, \dots, e_1+e_n\},$$
$$F'^{\delta}=\operatorname{conv}\{-e_1, -e_1-e_2, \dots, -e_1-e_n\}.$$

Let

$$F_j := \operatorname{conv}\{e_1, e_1+e_2, \dots, e_1+e_{j-1}, e_1+e_{j+1}, \dots, e_1+e_n\}$$

be a face of F^{δ}, and let $F_a := \operatorname{conv}(\{a\} \cup F_j)$ be a facet of P^{δ} adjacent to F^{δ}. Clearly, $|a_1|<1$, hence $a_1=0$. From the regularity of $X_{\Sigma(P)}$ we obtain

$$a_j=\det(e_1, e_1+e_2, \dots, e_1+e_{j-1}, a, e_1+e_{j+1}, \dots, e_1+e_n)=-1.$$

The supporting hyperplane $H_j := \{x_1-x_j=1\}$ of P^{δ} carried by F_a intersects the x_j-axis in $(0, \dots, 0, t_j, 0, \dots, 0)$ where $-1 \leq t_j < 0$. Furthermore, $H_j \cap \{x_1=0\}$ is a supporting hyperplane of $P^\delta \cap \{x_1 = 0\}$ relative to $\{x_1 = 0\}$. Since all vertices $\notin F^\delta \cup F'^\delta$ of P^δ lie in $\{x_1 = 0\}$, $\{x_j = -1\}$ is a supporting hyperplane of P^δ, $j = 2, \ldots, n$. Similarly, we see that $\{x_j = +1\}$ is a supporting hyperplane of P^δ, $j = 2, \ldots, n$.

If $\hat{F} := \operatorname{conv}\{b, e_1 + e_2, \ldots, e_1 + e_n\}$ is a facet adjacent to F we obtain, from the regularity of X_Σ,

$$-b_2 - \cdots - b_n = \det(b, e_1 + e_2, \ldots, e_1 + e_n) = -1.$$

The supporting hyperplane carried by $\hat{F}$ has the equation $x_2 + \cdots + x_n = 1$. Also $x_2 + \cdots + x_n = -1$ is a supporting hyperplane of P^δ. Therefore,

$$-1 \le x_2 + \cdots + x_n \le 1 \quad \text{for all vertices} \quad x = (x_1, \ldots, x_n) \text{ of } P^\delta.$$

Now $(0, x_2, \ldots, x_n)^{\delta^{-1}} = (-x_2 - \cdots - x_n, x_2, \ldots, x_n)$; furthermore, the hyperplanes $\{x_j = 1\}$ and $\{x_j = -1\}$, $j = 2, \ldots, n$, are invariant under δ^{-1}. So all vertices of P satisfy (4). □

Lemma 2. *Let $a = (a_1, \ldots, a_n)$, $a' = (a'_1, \ldots, a'_n)$ be two vertices $\neq \pm e_1, \ldots, \pm e_n$ of P. Then for any $j = 1, \ldots, n$, neither $a_j = a'_j = 1$ nor $a_j = a'_j - 1$ is true.*

Proof. If $j > 1$ we can, according to the proof of Lemma 1, consider P^δ instead of P. Then $a_1 = a'_1 = 0$ and we deduce from $a_j = a'_j = -1$ that both, a and a', lie in the supporting hyperplane H_j of P. This implies that $e_1^\delta, \ldots, e_{j-1}^\delta, e_{j+1}^\delta, \ldots, e_n^\delta$, a, a' are affinely dependent, a contradiction, since P^δ is simplicial.

If $j = 1$ we may assume $n > 1$ (otherwise everything is trivial), and interchange the roles of 1 and a $j_0 > 1$.

We call vertices $\neq \pm e_1, \ldots, \pm e_n$ of P *extra* vertices. □

Lemma 3. *Any extra vertex a of P lies in the hyperplane $x_1 + \cdots + x_n = 0$ and hence can be represented (up to renumbering of coordinates) in the form*

$$a = (1, -1, \ldots, 1, -1, 0, \ldots, 0) \quad (\textit{possibly no } 0).$$

Proof. This follows from the fact that F, F' carry the supporting hyperplanes $x_1 + \cdots + x_n = 1$, $x_1 + \cdots + x_n = -1$, respectively. □

Proof of (1). If P is centrally symmetric with respect to 0, and if $a = (1, -1, \ldots, 1, -1, 0, \ldots, 0)$ is an extra vertex, then also $-a$ is an extra vertex. By Lemma 3 the matrix having all extra vertices as rows splits as follows:

$$\left(\begin{array}{ccccc|c} 1 & -1 & \cdots & 1 & -1 & 0 \\ -1 & 1 & \cdots & -1 & 1 & \\ \hline & & 0 & & & * \end{array}\right).$$

This implies P itself splits unless a and $-a$ have no zero coordinates. □

Lemma 4. *If P does not split, $n=\dim P>1$, then n is even and either $a=(1,-1,\ldots,1,-1)$ is the only extra vertex or a, $-a$ are the only extra vertices.*

Proof. Suppose, Lemma 4 is false and we have two extra vertices a, b. Then we can assume the matrix of extra vertices to have the form (m always even)

$$\begin{pmatrix} a \\ b \\ \vdots \end{pmatrix}=\begin{pmatrix} 1 & -1 & \cdots & -1 & 1 & -1 & \cdots & 1 & -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -1 & 1 & \cdots & -1 & 1 & -1 & \cdots & -1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & & & & \vdots & & & & \vdots & & & \vdots \\ & & & \uparrow & & & & & \uparrow & & & & \uparrow & & & \\ & & & k\,(\text{even}) & & & & & m & & & & r & & & \end{pmatrix}, \tag{5}$$

$$\begin{pmatrix} a \\ b \\ \vdots \end{pmatrix}=\begin{pmatrix} 1 & -1 & \cdots & 1 & -1 & 1 & \cdots & 1 & -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & -1 & \cdots & -1 & 1 & -1 & \cdots & 1 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & & & & \vdots & & & & \vdots & & & \vdots \\ & & & \uparrow & & & & & \uparrow & & & & \uparrow & & & \\ & & & k\,(\text{odd}) & & & & & m & & & & r & & & \end{pmatrix}. \tag{6}$$

In both cases at least one 0 occurs in the first row; furthermore, $k<m-1$. Setting $e'_{r-1}:=-e_{r-1}$ we obtain in case (5)

$$a+b-e_1+e_2-\cdots-e_{k-1}+e_k+e_{m+1}-e_{m+2}-\cdots-e_{r-2}-e'_{r-1}-e_r=0,$$

which expresses an affine dependence of $e_1,\ldots,e_k$, $e_{m+1},\ldots,e_{r-2}$, e'_{r-1}, e_r, a, b. These points lie in the hyperplane

$$L(x):=x_1+\cdots+x_{k-1}+x_k+x_{k+1}+x_{m+1}+\cdots+x_{r-2}-x_{r-1}+x_r=1.$$

For all vertices $\pm e_j$ we have $L(\pm e_j)\leq 1$. If c is a further extra vertex we can write the matrix of extra vertices as

$$\begin{pmatrix} a \\ b \\ c \\ \vdots \end{pmatrix}=\left(\begin{array}{ccccccccccc} 1 & -1 & \cdots & 1 & -1 & 1 & \cdots & -1 & 1 & -1 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -1 & 1 & \cdots \\ -1 & 1 & \cdots & -1 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & & & & & & \vdots & & & \\ & & & & & & & \uparrow & & & \\ & & & & & & & k & & & \end{array}\right.$$

$$\left.\begin{array}{ccccccccccccc} 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ -1 & 1 & -1 & 1 & \cdots & -1 & 1 & -1 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -1 & \cdots & 1 & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ & \vdots & & & & & & & & \vdots & & & \vdots \\ & \uparrow & & & & & & & & \uparrow & & & \\ & m & & & & & & & & r & & & \end{array}\right)$$

from which we see $L(c) \le 1$. Hence $L(x)=1$ is a supporting hyperplane of P. this implies $e_1, \dots, e_k, e_{m+1}, \dots, e_{r-2}, e'_{r-1}, e_r, a, b$ to be vertices of a nonsimplicial face of P, a contradiction.

Suppose now case (6). We obtain (for $e'_r := -e_r$)

$$a+b-e_1+e_2-\cdots-e_k+e_{m+1}-e_{m+2}+\cdots+e_{r-2}-e_{r-1}-e'_r=0,$$

which is an affine dependence of the points involved. These points lie on the hyperplane

$$L(x) := x_1+x_2+\cdots+x_k+x_{m+1}+\cdots+x_{r-1}-x_r=1.$$

Again it is seen that $L(\pm e_j) \le 1, j=1, \dots, n$, also $L(c) \le 1$ for any other vertex c of P. This implies again a contradiction to P being simplicial.

Therefore, no 0 occurs in a and b, from which the lemma follows. □

From Lemmas 1-4 the theorem is now readily obtained.

References

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Received January 7, 1986