

Many Triangulated Spheres

Gil Kalai*

Institute of Mathematics, Hebrew University, Jerusalem, Israel

Abstract. Let $s(d, n)$ be the number of triangulations with n labeled vertices of S^{d-1} , the $(d-1)$ -dimensional sphere. We extend a construction of Billera and Lee to obtain a large family of triangulated spheres. Our construction shows that $\log s(d, n) \geq C_1(d)n^{\lfloor (d-1)/2 \rfloor}$, while the known upper bound is $\log s(d, n) \leq C_2(d)n^{\lfloor d/2 \rfloor} \log n$.

Let $c(d, n)$ be the number of combinatorial types of simplicial d -polytopes with n labeled vertices. (Clearly, $c(d, n) \leq s(d, n)$.) Goodman and Pollack have recently proved the upper bound: $\log c(d, n) \leq d(d+1)n \log n$. Combining this upper bound for $c(d, n)$ with our lower bounds for $s(d, n)$, we obtain, for every $d \geq 5$, that $\lim_{n \rightarrow \infty} (c(d, n)/s(d, n)) = 0$. The case $d = 4$ is left open. (Steinitz's fundamental theorem asserts that $s(3, n) = c(3, n)$, for every n .) We also prove that, for every $b \geq 4$, $\lim_{d \rightarrow \infty} (c(d, d+b)/s(d, d+b)) = 0$. (Mani proved that $s(d, d+3) = c(d, d+3)$, for every d .)

Let $s(n)$ be the number of triangulated spheres with n labeled vertices. We prove that $\log s(n) = 2^{0.69424} n^{(1+o(1))}$. The same asymptotic formula describes the number of triangulated manifolds with n labeled vertices.

1. Introduction

How many triangulations with n (labeled) vertices of S^d (the d -dimensional sphere) are there? Let $s(d+1, n)$ denote the number of such triangulations.

Theorem 1.1.

$$3 \log n \binom{n - \lfloor (d+1)/2 \rfloor}{\lfloor d/2 \rfloor} \geq \log s(d, n) \geq (1/(n-d)(d+1)) \binom{n - \lfloor (d+2)/2 \rfloor}{\lfloor (d+1)/2 \rfloor} \tag{1.1}$$

* Research done, in part, while the author visited the mathematics research center at AT&T Bell Laboratories.

The upper bound for $s(d, n)$ follows from Stanley's upper bound theorem [41] for the numbers of faces of triangulated d -spheres with n vertices. (Compare, Klee [26] and McMullen [34].)

Our main purpose is to construct a large family of triangulated spheres which demonstrates the lower bound.

We indicate two special cases of Theorem 1.1. For fixed $d \geq 3$,

$$C_1(d)n^{\lfloor d/2 \rfloor} \log n \geq s(d, n) \geq C_2(d)n^{\lfloor (d-1)/2 \rfloor}. \quad (1.2)$$

Here, $C_1(d)$ and $C_2(d)$ are constants depending on d .

Define $b(z) = (1-z)^{1-z}/z^z(1-2z)^{1-2z}$. Let x be a real number, $0 < x < 1$. Theorem 1.1 implies that $\log s(\lfloor xn \rfloor, n) = (b(x/2))^{n(1+o(1))}$. Note that $b(x/2) > 0$ for every $0 < x < 1$. $b(x/2)$ is maximal and is equal to 1.61803... for $x = (1 - 5^{-1/2}) = 0.5528...$.

Let $s(n)$ be the total number of triangulated spheres (of all dimensions) on n labeled vertices. The most accurate estimate we can give for $s(n)$ is (C_3, C_4 are constants)

$$C_3 \log nn^{-1/2} 1.61803...^n \geq \log s(n) \geq \log s(0.5528... n, n) \geq C_4 n^{-2} 1.61803...^n. \quad (1.3)$$

Note that the number $\text{simp}(n)$ of simplicial complexes on n labeled vertices satisfies $\log \text{simp}(n) = 2^{n(1+o(1))}$. More precisely, Kleitman [30] proved

$$\log \text{simp}(n) = \binom{n}{\lfloor n/2 \rfloor} (1 + o(1)).$$

Recently, Korshunov [31] completely determined the asymptotic behavior of $\text{simp}(n)!$

The boundary complex of a simplicial d -polytope is a triangulated $(d-1)$ -sphere. A triangulated sphere is *polytopal* if it is isomorphic to the boundary complex of a simplicial polytope.

Let $c(d, n)$ be the number of polytopal $(d-1)$ -spheres with n labeled vertices. Thus $c(d, n) \leq s(d, n)$. Denote $b = n - d$.

A fundamental theorem of Steinitz [44], [21, Chapter 13] asserts that every triangulated 2-sphere is polytopal. (Steinitz's theorem in its full generality asserts that every polyhedral 2-sphere is isomorphic to the boundary complex of a 3-polytope.) Only a few decades after Steinitz's theorem was established, mathematicians have realized that nonpolytopal triangulated spheres do exist in higher dimensions. Simple examples of nonpolytopal triangulated 3-spheres with eight vertices were given by Grünbaum [20] and by Barnette [8]. The "Steinitz Problem"—the determination of polytopal spheres among all triangulated spheres—is one of the main problems in convex polytope theory. For a recent work on this subject, see Bokowski and Sturmfels [14].

Steinitz's theorem asserts that $s(3, n) = c(3, n)$. Works of Tutte [45], [46], Brown [15], and Richmond and Wormald [39] give good asymptotic estimates for $s(3, n)$ and exact formulas for related enumeration problems.

An important result of Mani [33] asserts that every triangulated $(d - 1)$ -sphere with at most $(d + 3)$ -vertices is polytopal. (Kleinschmidt [28] extended Mani's result and proved that every polyhedral $(d - 1)$ -sphere with at most $d + 3$ vertices is polytopal.) There are exactly $\lfloor d/2 \rfloor$ simplicial d -polytopes with $d + 2$ vertices and the value of $c(d, d + 3) (= s(d, d + 3))$ was determined by Perles. (See Chapter 6 of [21].)

In the last 20 years extensive work has been done on the enumeration of triangulated 3-spheres and simplicial 4-polytopes with few vertices. It was proved that $c(4, 8) = 37$ (Grünbaum and Sreedharan [22], correcting Brückner [16]), $s(4, 8) = 39$ [9], $c(4, 9) = 1142$, and $s(4, 9) = 1296$ [3], [6]. For further results of this type, see [2], [4], and [5].

It was conjectured by several people that for every fixed $d \geq 4$, "most" triangulated $(d - 1)$ -spheres are *not* polytopal, i.e., that $\lim_{n \rightarrow \infty} (c(d, n) / s(d, n)) = 0$. Similarly, it was believed that for every $b, b \geq 4$, $\lim_{d \rightarrow \infty} (c(d, d + b) / s(d, d + b)) = 0$.

Goodman and Pollack [18], [19] have recently proved that $\log c(d, n) \leq d(d + 1)n \log n$. This remarkable result is proved using a theorem of Milnor which gives bounds on the sum of the Betti numbers of real algebraic varieties. Alon [1] extended their result to arbitrary polytopes.

Goodman and Pollack's upper bound on $c(d, n)$ combined with our lower bound on $s(d, n)$ imply (in a very strong sense) that for every $d \geq 5$, $\lim_{n \rightarrow \infty} (c(d, n) / s(d, n)) = 0$. A proof that most triangulations of the 3-sphere are not polytopal is still unknown.

Our construction gives the bound $\log s(d, d + b) > D(b) \cdot d^{b-2}$, while the method of proof of Goodman and Pollack's theorem gives $\log c(d, d + b) \leq b(b - 1)d \log d$ (see [1]). This shows that for every $b \geq 4$,

$$\lim_{d \rightarrow \infty} (c(d, d + b) / s(d, d + b)) = 0.$$

By Goodman and Pollack's bound the total number of polytopal triangulated spheres with n labeled vertices is bounded by $2^{(n+1)^3}$. (Alon's result gives the same bound for the nonsimplicial case.) Thus, the gap between the numbers of triangulated spheres and polytopal spheres is most striking when $d = \lfloor xn \rfloor$ for some constant x .

For example, the number of triangulated spheres with 1000 labeled vertices is more than $2^{2^{50}}$, but less than $2^{2^{30}}$ out of them are polytopal. The number of triangulated spheres with 1 000 000 vertices is between $2^{2^{694,200}}$ and $2^{2^{694,250}}$, but less than $2^{2^{62}}$ out of them are polytopal.

Our construction is a modification of a construction of Billera and Lee [10], [11]. For a simplicial complex C , $f_k(C)$ denotes the number of k -dimensional faces of C . The vector $f(C) = (f_0(C), f_1(C), \dots)$ is called the f -vector of C . The same definition applies for polytopes. McMullen [35] proposed a complete characterization of f -vectors of simplicial d -polytopes. Billera and Lee built, for every vector f of nonnegative integers which satisfies McMullen's conditions, a shellable $(d - 1)$ -sphere with f as its f -vector. They further realized this sphere as a boundary complex of a simplicial d -polytope, thus proving the sufficiency part of McMullen's conjecture. Stanley [43] proved the necessity part of McMullen's conjecture using deep results from algebraic-geometry.

Billera and Lee's $(d-1)$ -spheres as well as our more general class of spheres are obtained as the boundaries of shellable balls spanned by some of the facets ($=d$ -faces) of a cyclic $(d+1)$ -polytope. Specifically, let $C(d+1, n)$ denote the cyclic $(d+1)$ -polytope with n vertices. Billera and Lee considered the (reverse) lexicographic order on the facets of $C(d+1, n)$, and studied simplicial complexes spanned by initial sets of facets. They proved that simplicial complexes obtained in this way are actually shellable d -balls, and their boundaries are polytopal $(d-1)$ -spheres. We consider a certain partial order of the facets of $C(d+1, n)$, and study simplicial complexes which are spanned by initial sets of facets. All these complexes are shellable d -balls and their boundaries form a large class of triangulated $(d-1)$ -spheres, which we call *squeezed spheres*. Squeezed spheres demonstrate the lower bounds in (1.1).

McMullen's conditions are conjectured to hold for arbitrary triangulated spheres. They hold almost trivially for squeezed spheres. Results by Lee [32] on the diameter of the Billera–Lee polytopes (related to the ‘‘Hirsch conjecture’’) also extend to arbitrary squeezed spheres.

We hope that squeezed spheres will play a role among triangulated spheres similar to that which shifted complexes play among simplicial complexes. (See [24] and [13].) In particular, a *squeezing* operation for spheres is now desirable.

Problems which remains open are to construct many triangulations (say, 2^{Cn^2}) for the three-dimensional sphere, and to give an explicit example of a nonpolytopal (say, four-dimensional) squeezed sphere.

2. Preliminaries

2.1. Orderings

For an integer n , $[n]$ denotes the set $\{1, 2, \dots, n\}$. \mathbb{N} will denote the set of positive integers. For a set A and a nonnegative integer d , $A^{(d)}$ is the set of d -subsets of A . The *partial order* $<_p$ on $\mathbb{N}^{(d)}$ is defined as follows: if $S, T \in \mathbb{N}^{(d)}$, $S = \{i_1, \dots, i_d\}$, $i_1 < \dots < i_d$, and $T = \{j_1, \dots, j_d\}$, $j_1 < \dots < j_d$, then $S \leq_p T$ iff for every $1 \leq k \leq d$, $i_k \leq j_k$.

Define the lexicographic order $<_L$ on $\mathbb{N}^{(d)}$ as follows: for $S, T \in \mathbb{N}^{(d)}$, $S <_L T$ if $\min(S \Delta T) \in S$. The reverse lexicographic order $<_{RL}$ is defined by $S <_{RL} T$ if $\max(S \Delta T) \in T$.

2.2. Simplicial Complexes

We follow the definitions and notation of Billera and Lee [11]. We repeat here some basic definitions.

Let C be a simplicial complex, and let ν be a vertex not in C . The *cone* over C with *apex* ν is defined by

$$\text{cone}(C, \nu) = C \cup \{S \cup \{\nu\}: S \in C\}.$$

A simplicial complex is *pure* if all its maximal faces have the same size. Maximal faces of a pure simplicial complex are called *facets*. Two facets S and T of a pure simplicial complex are *adjacent* if they intersect in a maximal proper face of each.

Let C be a pure simplicial complex and let S and T be two facets of C . The *distance* between S and T , $d(S, T)$, is the minimal integer k such that there exists a sequence of facets $S = F_0, F_1, \dots, F_k = T$ with the property that F_i is adjacent to F_{i+1} for every $1 \leq i < k$. (If no such sequence exists $d(S, T) = \infty$.) The *diameter* of C , $\Delta(C)$, is the maximal value of $d(S, T)$ taken over all pairs of facets of C . A pure simplicial complex C is *strongly connected* if it has a finite diameter.

A *d-pseudomanifold* C is a strongly connected d -dimensional simplicial complex, such that every $(d-1)$ -face is included in at most two facets. The *boundary* of C , ∂C , is the $(d-1)$ -dimensional simplicial complex spanned by those $(d-1)$ -faces of C which are included in exactly one facet of C .

2.3. Shelling

A pure $(d-1)$ -dimensional simplicial complex C is *shellable* if its maximal faces can be ordered F_1, F_2, \dots, F_t , so that for every i and k , $1 \leq i < k \leq t$, there exists j , $1 \leq j \leq k$, such that $F_i \cap F_k \subseteq F_j \cap F_k$ and F_j is adjacent to F_k . Such an ordering is called a *shelling order* of C .

2.4. Vertex Decomposability and the Hirsch Condition (needed for Section 5)

Let C be a pure d -dimensional simplicial complex. C satisfies the *Hirsch condition* if $\Delta(C) \leq f_0(C) - d$. (Recall that $f_0(C)$ is the number of vertices of C .) The Hirsch conjecture asserts that every polytopal sphere satisfies the Hirsch condition. (See Klee and Kleinschmidt [27] for a recent survey on this conjecture.)

Provan and Billera [38] related the Hirsch condition to a strong form of shellability called *vertex-decomposability*.

Vertex-decomposable simplicial complexes are defined recursively as follows: $C = \{\emptyset\}$ is vertex-decomposable. A pure simplicial complex C is vertex-decomposable if there exists a vertex $v \in C$ (called a *shedding vertex*) such that $lk(v, C)$ and $ast(v, C)$ are vertex-decomposable. (Here, $lk(v, C) = \{S \setminus \{v\} : S \in C, v \in S\}$ and $ast(v, C) = \{S \in C : v \notin S\}$.)

A pure simplicial $(d-1)$ -dimensional complex C is *weakly vertex-decomposable* if C is a single $(d-1)$ -simplex together with its faces, or else there is a vertex v of C (a weak shedding vertex) such that $ast(v, C)$ is a weakly vertex-decomposable $(d-1)$ -dimensional complex.

Theorem 2.1 (Provan and Billera [38]).

- (i) *Every vertex-decomposable simplicial complex satisfies the Hirsch condition.*
- (ii) *The diameter of a weakly vertex-decomposable $(d-1)$ -dimensional simplicial complex with n vertices is at most $2(n-d)$, twice the Hirsch bound.*

3. Squeezed Balls and Spheres

Let $d > 0$ be a fixed odd integer. Define a collection F_d of $(d+1)$ -subsets of \mathbb{N} as follows: put $e = (d+1)/2$. For $F \in \mathbb{N}^{(d+1)}$, $F \in F_d$ iff $F = \{i_1, i_1+1\} \cup \{i_2, i_2+1\} \cup \cdots \cup \{i_e, i_e+1\}$, where $i_1 \geq 1$ and for every j , $e > j \geq 1$, $i_{j+1} > i_j + 1$. Define $F_d(n) = F_d \cap [n]^{(d+1)}$. Let I be an initial set of F_d with respect to the partial order $<_p$ on $\mathbb{N}^{(d+1)}$ and let $B(I)$ be the simplicial complex spanned by I . We will prove that $B(I)$ is a shellable $(d+1)$ -ball.

For d even put $F_d = \{\{0\} \cup F : F \in F_{d-1}\}$. For an initial set I of F_d with respect to the partial ordering, let $B(I)$ be the simplicial complex spanned by I . We will show that in this case $B(I)$ is also a shellable ball. In fact, for d even and I an initial set of F_d , $B(I)$ is easily seen to be a cone over some $B(J)$, where J is an initial set of F_{d-1} . We call a simplicial complex $B(I)$ of the form described above a *squeezed ball*. The boundary of a squeezed ball is called a *squeezed sphere*.

Remark. $F_d(n)$ can be regarded as a subcollection of the facets of the cyclic $(d+1)$ -polytope with n vertices.

Theorem 3.1. *Let $d \geq 1$ be a fixed integer. Let I be an initial set of F_d with respect to the partial order $<_p$. Let $B(I)$ be the simplicial complex spanned by I . Then any ordering F_1, F_2, \dots, F_t of I which extends the partial order $<_p$, is a shelling order for $B(I)$.*

Proof. The proof is similar to the proof of Lemma 4 in Billera and Lee [11]. We will consider the case of odd d . (The proof for even d is the same.) Let F_1, F_2, \dots, F_t be an ordering of I which extends the partial order $<_p$. Let $F_k = [1, d+1-2p] \cup \{i_1, i_1+1\} \cup \cdots \cup \{i_p, i_p+1\}$ where $i_1 > d+2-2p$ and $i_{j+1} > i_j + 1$ for $p > j \geq 1$. Define, for $1 \leq j \leq p$, $G_k^j = F_k \setminus \{i_j+1\}$. G_k^j is a $(d-1)$ -face of F_k . If u_j is the maximal element in $[i_j] \setminus F_k$ then $G_k^j \cup \{u_j\}$ is a d -face in F_d which precedes F_k in the partial order.

It remains to show that if $S = F_k \cap F_i$ for $i < k$ then $S \subset G_k^j$ for some j , $p \geq j \geq 1$. If $S \not\subset G_k^j$ for every j , $p \geq j \geq 1$ then $S \supset R = \{i_1+1, \dots, i_p+1\}$. But it is easy to see that every $F \in F_d$ which includes R satisfies $F \geq_p F_k$. Thus, $F_i \geq_p F_k$. A contradiction. \square

Corollary 3.2. *$B(I)$ is a shellable ball.*

Proof. It is easy to see that every $(d-1)$ -face of $B(I)$ is included in at most two d -faces. Thus, $B(I)$ is a shellable pseudomanifold and as is well known (see Danaraj and Klee [17]), $B(I)$ is either a shellable sphere or a shellable ball. It is easy to see that there is a $(d-1)$ -face of $B(I)$ which is included in exactly one d -face. Hence $B(I)$ is a shellable ball. \square

Since squeezed spheres are boundaries of shellable balls, they are PL-spheres. We do not know if they are, in general, shellable. (A recent result of Pachner [37] asserts that every PL-sphere is the boundary of some shellable ball.)

The construction of Billera and Lee is the special case where I is an initial set of F_d , with respect to the reverse lexicographic order. (They also consider only a subset of F_d for odd d .) We will refer to the triangulated spheres considered by Billera and Lee as *compressed spheres*. Billera and Lee proved that compressed spheres are polytopal. We refer to the corresponding polytopes as Billera-Lee polytopes.

Proposition 3.3. *Let B_1, B_2 be two squeezed d -balls. If $\partial B_1 = \partial B_2$ then $B_1 = B_2$.*

Proof. Let B be a squeezed d -ball and let S be a facet of ∂B . S is included in exactly one facet of B . Note that S is included in at most two members of F_d . Note also that if S is included in two members of F_d , say F_1 and F_2 , then they are comparable with respect to the partial order $<_p$. Define $F(S)$ to be the minimal set in F_d with respect to $<_p$ which contains S . From the discussion above it follows that $F(S)$ is defined and belongs to B . Define now $\hat{B} = \{F(S) : |S| = d - 1, S \in \partial B\} \subset B$. Let F be a facet of B which is maximal in B with respect to $<_p$ and let $u = \min F$. Then $S = (F \setminus \{u\}) \in \partial B$ and $F = F(S) \in \hat{B}$. Therefore, $B = \{F \in F_d : F \leq_p \hat{F}, \hat{F} \in \hat{B}\}$. \square

Proposition 3.4. *Let $S(I) = \partial B(I)$ be a squeezed sphere. Then if i is a vertex of $S(I)$ and $j \leq i$ then j is a vertex of $S(I)$.*

Proof. Since I contains a set which contains i , I also contains a set which contains j . Consider $F \in I$, which is maximal (with respect to the partial order) among sets in I which contain j . It is easy to see that for some $k \in S$, $k \neq j$, $F \setminus \{k\} \in S(I)$. \square

Remark. Let P be a ranked poset. A subset Q of P is a *ranked subposet* of P if Q forms a ranked poset whose rank function is the restriction from P to Q of the rank function of P . Clearly every initial set in P is a ranked subposet.

Let I be a ranked subposet of F_d and let $B(I)$ be the simplicial complex spanned by I . The proof of Theorem 3.1 can be directly extended to show that $B(I)$ is a shellable ball. This construction does not improve substantially the lower bounds for $s(d, n)$.

4. The Number of Triangulated Balls and Spheres

In this section we will estimate the number $s(d, n)$ of triangulated $(d - 1)$ -spheres with n vertices. We will also consider the number $m(d, n)$ of triangulated $(d - 1)$ -manifolds with n vertices.

We need the following lemma. The proof is an easy application of Stirling's estimation of $n!$

Lemma 4.1.

- (i) Define $b(x) = (1-x)^{1-x}/x^x(1-2x)^{1-2x}$. Then for a fixed real number x , $0 < x < \frac{1}{2}$, $\binom{n-\lfloor xn \rfloor}{\lfloor xn \rfloor} = c(x)n^{-(1/2)}(b(x))^n(1+o(1))$. (Here, $c(x) = (1-x)^{1/2}(2\pi x(1-2x))^{-1/2}$.)
- (ii) $b(x)$ attains its maximum at $x_{\max} = (1-5^{-1/2})/2$.

4.1. The Lower Bounds

We now give the lower bounds obtained by the family of squeezed spheres.

Let $f(d, n)$ be the number of initial sets in $F_d(n)$. It is clear that for even d , $f(d, n) = f(d-1, n-1)$. We will therefore assume that d is odd. As before let $e = (d+1)/2$. Let $sq(d, n)$ be the number of squeezed $(d-1)$ -spheres with n vertices.

F_d is a ranked poset, for $F = \{i_1, i_1+1\} \cup \dots \cup \{i_e, i_{e+1}\} \in F_d$. $r(F) = \sum_{j=1}^e i_j - e^2$. Thus, for $F \in F_d(n)$, $0 \leq r(F) \leq (n-2e)e$.

Let $a(n, k, r)$ be the number of ways to represent r as a sum $r = a_1 + a_2 + \dots + a_k$ where $0 \leq a_1 \leq a_2 \leq \dots \leq a_k \leq n$. Define $a(n, k) = \max\{a(n, k, r) : r \geq 0\}$. Clearly, $a(n, k, r) = 0$ for $r > kn$. Also $\sum_{r=0} \binom{n+k}{k} = \frac{1}{(1+kn)} \binom{n+k+1}{k}$. Therefore $a(n, k) \geq$

Remark. It is well known (see, for example, Chapter 3 of Andrews [7] that $a(n, k, r)$ is the coefficient of q^r in $\binom{n+k}{k}(q)$. It is also known that the sequence $a(n, k, 0), a(n, k, 1), \dots, a(n, k, kn)$ is symmetric and unimodal, hence, $a(n, k) = a(n, k, \lfloor kn/2 \rfloor)$.

Let $b(n, d, r) = |\{S \in F_d(n) : r(S) = r\}|$ and $b(n, d) = \max\{b(n, d, r) : r \geq 0\}$. A simple inspection shows that $b(n, d, r) = a(n-1-d, e, r)$. Thus, $b(n, d) = a(n-1-d, e)$.

Theorem 4.2. Put $e = \lfloor (d+1)/2 \rfloor$ and $e' = \lfloor (d+2)/2 \rfloor$.

$$\log s(d, n) \geq \log sq(d, n) \geq (1/(n-d)e) \binom{n-e'}{e} - 1. \quad (4.1)$$

In particular:

- (i) Let $d \geq 2$ be a fixed integer. Then $\log sq(d, n) \geq C_2(d)n^{\lfloor (d+1)/2 \rfloor - 1}$.
- (ii) Let $b \geq 2$ be a fixed integer. Then $\log sq(d, d+b) \geq C_4(b)d^{b-2}$.
- (iii) $\log sq(\lfloor xn \rfloor, n) \geq C_6(x)n^{-(5/2)}(b(x/2))^n$.

Proof. To prove (4.1) it is enough to consider the case that d is odd. The number of squeezed d -balls with at most n vertices is exactly $f(d, n)$. By Proposition 3.3, this is also the number of squeezed $(d-1)$ -spheres with at most n vertices. By Proposition 3.4, the number $sq(d, n)$ of squeezed $(d-1)$ -spheres with exactly n

vertices is equal to the number of squeezed d -balls with exactly n vertices, namely $f(d, n) - f(d, n - 1)$. Now, $f(d, n) \geq 2^{b(n,d)} = 2^{a(n-1-d,e)}$. Since, for $n > k > 1$, $a(n, k) > a(n - 1, k)$ we obtain, for odd $d \geq 3$,

$$\log s(d, n) \geq \log sq(d, n) > a(n - 1 - d, e) - 1 \geq (1/(n - d)e) \binom{n - e}{e} - 1. \quad \square$$

Remark. With some more effort it can be shown that

$$\log sq([xn], n) \geq a(n - 1 - d, e) - 1 \geq C_8(x)n^{-2}(b(x/2))^n(1 + o(1)). \quad (4.2)$$

4.2. The Upper Bounds

Here are the upper bounds on $s(d, n)$ and $m(d, n)$ which follow from the upper bound for the number of facets (= maximal faces) of triangulated spheres. The upper bound theorem for triangulated spheres (proved by Stanley [41]) asserts that a triangulated $(d - 1)$ -sphere C with n vertices has at most $\binom{n - [(d + 1)/2]}{n - d} + \binom{n - [(d + 2)/2]}{n - d}$ $(d - 1)$ -faces. This expression is smaller than $2 \binom{n - [(d + 1)/2]}{[d/2]}$.

The number of pure $(d - 1)$ -dimensional simplicial complexes with n labeled vertices and u facets is clearly $\binom{\binom{n}{d}}{u}$. This gives $\log s(d, n) \leq 2d \log n \binom{n - [(d + 1)/2]}{[d/2]} + 1$. We can gain a factor of d in this expression by using the following:

Proposition 4.3. *Let $pm(d, n, u)$ be the number of $(d - 1)$ -pseudomanifolds with n vertices and u facets. Then $pm(d, n, u) \leq n^{u+d}$.*

Proof. Put $t = d + u - 1$. Let C be a $(d - 1)$ -pseudomanifold on the vertex set $[n]$. Associate to C an integral vector $x(C) = (x_1, \dots, x_t)$ which satisfies $1 \leq x_i \leq n$ for every $1 \leq i \leq t$ and a chain of subcomplexes C_1, \dots, C_u as follows: x_1, \dots, x_d are the vertices of the lexicographically first facet F of C , and C_1 is the complex spanned by F . Assume that $x_i, i \leq k$, and $C_j, j \leq k - d + 1$, are defined. Let R be the lexicographically minimal $(d - 2)$ -face in the boundary of C_{k-d+1} and let S be the facet of C , not in C_{k-d+1} which contains R . Define x_{k+1} by $\{x_{k+1}\} = S \setminus R$ and define C_{k-d+2} to be the complex spanned by $C_{k-d+1} \cup \{S\}$. As easily seen, $x(C)$ determines C . \square

Stanley's theorem implies that a triangulated $(d - 1)$ -manifold with n vertices has at most $2nd^{-1} \binom{n - [d/2] - 1}{[(d - 1)/2] - 1}$ facets.

We summarize the asymptotic upper bounds for $s(d, n)$ and $m(d, n)$ which are obtained from the upper bound theorems by Proposition 4.3:

Theorem 4.4.

$$\log s(d, n) \leq 2 \log n \binom{n - \lfloor (d+1)/2 \rfloor}{\lfloor d/2 \rfloor} + d. \quad (4.3)$$

$$\log m(d, n) \leq 2nd^{-1} \log n \binom{n - \lfloor d/2 \rfloor - 1}{\lfloor (d-1)/2 \rfloor} + d. \quad (4.4)$$

In particular;

- (i) Let $d \geq 2$ be a fixed integer. Then $\log s(d, n) \leq C_1(d) \log nn^{\lfloor d/2 \rfloor}$ and $\log m(d, n) \leq C_3(d) \log nn^{\lfloor (d+1)/2 \rfloor}$.
- (ii) Let $b \geq 2$ be fixed. Then $\log s(d, d+b) \leq \log m(d, d+b) \leq C_5(b) \log dd^{b+1}$.
- (iii) Let x be a fixed real number $0 < x < 1$. Then, $\log s(\lfloor xn \rfloor, n) \leq \log m(\lfloor xn \rfloor, n) \leq C_7(x) n^{(-1/2)} (b(x/2))^n (1 + o(1))$.

Theorem 1.1 is a combination (in a weaker but more elegant form) of (4.1) and (4.3).

5. Further Properties of Squeezed Balls and Spheres

5.1. Squeezed Spheres and the ‘‘Hirsch Conjecture’’

Proposition 5.1 (Compare Lee [32]).

- (i) Squeezed balls satisfy the Hirsch condition.
- (ii) Let d be even. For a squeezed $(d-1)$ -sphere S , $\Delta(S) \leq f_0(S) - d + 1$.
- (iii) Let d be odd. For a squeezed $(d-1)$ -sphere S , $\Delta(S) \leq 2(f_0(S) - d)$.
- (iv) Squeezed 3-spheres satisfy the Hirsch condition.

Proof. The proofs of Lee [32] of the special case of compressed spheres extend directly. By the Provan–Billera’s theorem (Theorem 2.1(i)) in order to prove part (i) it is enough to show that squeezed balls are vertex-decomposable. Indeed, if B is a squeezed d -ball then $lk(1, B)$ and $ast(1, B)$ are combinatorially isomorphic to squeezed balls (of dimensions $d-1$ and d , respectively).

For the proof of (ii) see the argument in Lee [32]. (The argument applies to boundaries of those squeezed balls which are cones.)

To prove (iii) it is left by Theorem 2.1(ii) to show that squeezed spheres are weakly vertex-decomposable. The proof is identical to the following unpublished proof by Lee for compressed spheres, which preceded his stronger published result. Let I be an initial set of F_d and let n be the maximal vertex of I . We prove by induction on n . The case $n = d + 1$ is easy, so assume $n > d + 1$. Let \hat{I} be an initial set in F_d consisting of those F in I that do not contain n . Compare

the facets of $\partial B(I)$ and $\partial B(\hat{I})$. It is easy to see from the construction that each facet of $\partial B(\hat{I})$ that is not in $\partial B(I)$ contains $n-1$, and that each facet of $\partial B(I)$ that is not in $\partial B(\hat{I})$ contains n . Hence the facets of $ast(n-1, ast(n, \partial B(I)))$ are precisely those of $ast(n-1, \partial B(\hat{I}))$, the latter being weakly vertex-decomposable by induction. So $ast(n, \partial B(I))$ is weakly vertex-decomposable with a weak shedding vertex n .

Squeezed 3-spheres are obtained from squeezed 3-balls by forming a cone over the boundary. Part (iv) is therefore a consequence of (i) and the fact that every triangulated 2-sphere satisfies the Hirsch condition. \square

5.2. Squeezed Spheres and the “g-Conjecture”

Our next results are of a technical nature. We point out that f -vectors of squeezed spheres satisfy McMullen’s conditions (which are conjectured to hold for *all* triangulated spheres.) For the definitions of the h -vector, an M -vector, and the assertion of the “g-conjecture” we refer the reader to Billera and Lee [11] or Stanley [43].

Let d be fixed. Let $e = \lfloor (d+1)/2 \rfloor$, and let $\Phi^{(e)}$ be the set of all monomials of degree at most e in the variables Y_1, Y_2, \dots . Put $Y_0 = 1$.

A bijection $\alpha: F_d \rightarrow \Phi^{(e)}$ is defined as in Billera and Lee [11]: for $F \in F(n)$, $F = V_d \cup \{i_1, i_1+1\} \cup \dots \cup \{i_e, i_e+1\}$. (Here, $V_d = \{\emptyset\}$ for odd d and $V_d = \{0\}$ for d even.) $\alpha(F) = Y_{u_1} \cdot Y_{u_2} \cdot \dots \cdot Y_{u_e}$ where $u_j = i_j - 2j + 1$. Note that if $F = V_d \cup \{1, 2e-2p\} \cup \{j_1, j_1+1\} \cup \dots \cup \{j_p, j_p+1\}$ where $j_1 > 2e-2p+1$ then $\deg \alpha(F) = p$.

Proposition 5.2. *The f -vector of squeezed spheres satisfies McMullen’s conditions.*

Proof. Let I be an initial set in F . Let $M(I) = \alpha(I) \subset \Phi^{(n)}$. $M(I)$ is an order ideal of monomials. Let $k_i = |\{m \in M(I) : \deg m = i\}|$. k_1, k_2, \dots is an M -vector. The shelling of $B(I)$ demonstrates that $h_i(B(I)) = k_i$ for $0 \leq i \leq e$ and $h_i(B(I)) = 0$ for $i > e$. The h -vector of $S(I) = \partial B(I)$ is determined by the h -vector of $B(I)$ by $h_i(S(I)) - h_{i-1}(S(I)) = h_i(B(I)) - h_{d+1-i}(B(I)) = h_i(B(I))$. (See [11].) This proves that $(1, h_1(S(I)) - h_0(S(I)), h_2(S(I)) - h_1(S(I)), \dots, h_{e-1}(S(I)) - h_e(S(I)))$ is an M -vector. \square

A triangulated $(d-1)$ -sphere is k -stacked if it is the boundary of a triangulated d -ball with the same $(d-k-1)$ -skeleton.

Proposition 5.3. *Let S be a squeezed $(d-1)$ -sphere, then (i) S is e -stacked, and (ii) if $h_k(S) = h_{k+1}(S)$ for some $0 \leq k \leq \lfloor d/2 \rfloor - 1$, then S is k -stacked.*

The proofs are the same as the proofs for the compressed case by Billera and Lee [11] (for (i)) and by Kleinschmidt and Lee [29] (for (ii)).

Remarks. 1. Stanley [42] proved that *every* e -stacked $(d-1)$ -sphere satisfies McMullen’s conditions.

2. A d -polytope P is k -stacked if it can be triangulated without introducing new j -faces for $j \leq d - k - 1$. McMullen and Walkup [36] conjectured that every simplicial d -polytope P that satisfies $h_k(P) = h_{k+1}(P)$ for some k , $i \leq k \leq \lfloor d/2 \rfloor - 1$, is a k -stacked polytope. Kleinschmidt and Lee [29] proved the assertion of this conjecture for the Billera-Lee polytopes.

6. Open Problems

6.1. Properties of Squeezed Spheres

1. Which squeezed d -spheres are polytopal? For $d \geq 4$ very few of them are. Yet, I do not know, at present, any explicit example of a nonpolytopal squeezed sphere.

Perhaps all squeezed 3-spheres are polytopal (in analogy with Proposition 9.2 of [23].)

2. Are squeezed spheres shellable?

6.2. More Precision

3. For fixed d and $n \rightarrow \infty$ there is still a substantial gap between the lower and upper bounds of $\log s(d, n)$, especially for d even. I believe that the value of $s(d, n)$ is closer to the upper bounds in (1.2). It can be shown that our lower bounds cannot be substantially improved by considering balls spanned by subsets of facets of cyclic polytopes.

We left a large gap between the lower and upper bounds for $m(d, n)$. I believe that in this case also the truth is closer to the upper bound.

Most urgently, we would like to estimate the number of triangulated 3-spheres with n vertices and the number of triangulated 2-manifolds with n vertices. I expect that 2^{n^2} is a lower bound for both these questions.

4. How many *topologically distinct* triangulated d -manifolds on n vertices are there? For $d = 2$ the answer is Cn^2 . For $d = 3$ the answer is somewhere between n^2 and 2^{n^2} .

6.3. Greater Generality

5. Let $f = (f_0, \dots, f_{d-1})$ be a vector of nonnegative integers. Let $s(f)$ be the number of triangulated $(d-1)$ -spheres with f as their f -vector. Estimate $s(f)$.

Note that for d even there is only one neighborly squeezed $(d-1)$ -sphere, namely, that corresponding to the cyclic d -polytope. We expect, however, that the number $sn(d, n)$ of neighborly $(d-1)$ -spheres with n vertices is very large. Perhaps for fixed even d , $\lim_{n \rightarrow \infty} (\log sn(d, n) / \log s(d, n)) = 1$. A large number of neighborly polytopes was constructed by Shemer [40].

In some cases, estimating $s(f)$ is related to structural properties of triangulated spheres with f as their f -vector [25].

6. It is possible that the upper bounds for $s(n)$ and $s(d, n)$ apply to arbitrary ranked Eulerian posets (of rank $d + 1$ and n atoms). This requires a far-reaching extension of the upper bound theorem which is still unknown. By Klee's upper bound theorem [26], the upper bound in (1.2) applies to arbitrary Eulerian $(d - 1)$ -dimensional simplicial complexes.

Acknowledgments

I am thankful to N. Alon, L. J. Billera, C. W. Lee, N. Linial, and M. A. Perles for helpful comments.

References

1. N. Alon, The number of polytopes, configurations and real matroids, *Mathematika* **33** (1986), 62-71.
2. A. Altshuler, Neighborly 4-polytopes and neighborly combinatorial manifolds with ten vertices, *Canad. J. Math.* **29** (1977), 400-420.
3. A. Altshuler and L. Steinberg, An enumeration of combinatorial 3-manifolds with nine-vertices, *Discrete Math.* **16** (1976), 113-137.
4. A. Altshuler and L. Steinberg, Enumeration of the quasi-simplicial 3-spheres and 4-polytopes with eight vertices, *Pacific J. Math.* **113** (1984), 269-288.
5. A. Altshuler and L. Steinberg, The complete enumeration of the 4-polytopes and 3-spheres with eight vertices, *Pacific J. Math.* **118** (1985), 1-16.
6. A. Altshuler, J. Bokowski, and L. Steinberg, The classification of simplicial 3-spheres with nine vertices into polytopes and non-polytopes, *Discrete Math.* **31** (1980), 115-125.
7. G. Andrews, *The Theory of Partitions*, Addison-Wesley, Reading, MA, 1976.
8. D. Barnette, Diagrams and Schlegel diagrams, in *Combinatorial Structures and Their Applications* (R. Guy, H. Hanani, N. Sauer, and J. Schonheim, eds.), 1-5, Gordon and Breach, New York, 1970.
9. D. Barnette, The triangulations of the 3-sphere with up to 8 vertices, *J. Combin. Theory Ser. A* **14** (1973), 37-52.
10. L. Billera and C. Lee, Sufficiency of McMullen's conditions for f -vectors of simplicial polytopes, *Bull. Amer. Math. Soc.* **2** (1980), 181-185.
11. L. Billera and C. Lee, A proof of the sufficiency of McMullen's conditions for f -vectors of simplicial convex polytopes, *J. Combin. Theory Ser. A* **31** (1981), 237-255.
12. A. Björner, Homology of matroids, in *Combinatorial Geometries*, Vol. II (N. White, ed.), Cambridge University Press, Cambridge, in press.
13. A. Björner and G. Kalai, On f -vectors and homology, *Proc. NY Acad. Sci.*, to appear.
14. J. Bokowski and B. Sturmfels, Polytopal and non-polytopal spheres—an algorithmic approach, preprint, TH, Darmstadt, 1985.
15. W. Brown, Enumeration of non-separable planar maps, *Canad. J. Math.* **15** (1963), 526-545.
16. M. Brückner, Ueber die Ableitung der allgemeinen Polytope and und die nach Isomorphismus verschiedenen Typen der allgemeinen Achtzelle (Oktatope), *Verh. Konnik. Konink. Akad. Wetensch. België*, **10** (1909).
17. G. Danaraj and V. Klee, Which spheres are shellable? *Ann. Discrete Math.* **2** (1978), 33-52.
18. J. Goodman and R. Pollack, There are asymptotically far fewer polytopes than we thought, *Bull. Amer. Math. Soc.* **14** (1986), 127-129.
19. J. Goodman and R. Pollack, Upper bound for configurations and polytopes in R^d , *Discrete Comput. Geom.*, to appear.
20. B. Grünbaum, Diagrams and Schlegel diagrams. Abstract 625-112, *Notices Amer. Math. Soc.* **12** (1965), 578.

21. B. Grünbaum, *Convex Polytopes*, Wiley-Interscience, London–New York–Sidney, 1967.
22. B. Grünbaum and V. Sreedharan, An enumeration of simplicial 4 polytopes with 8 vertices, *J. Combin. Theory* **2** (1967), 437–465.
23. G. Kalai, A characterization of f -vectors of families of convex sets in R^d , part II: Sufficiency of Eckhoff's conditions, *J. Combin. Theory Ser. A* **41** (1986), 167–188.
24. G. Kalai, Algebraic shifting operations and iterated homology groups, to appear.
25. G. Kalai, Rigidity and the lower bound theorem, III: Triangulated manifolds with “few edges”, in preparation.
26. V. Klee, The number of vertices of a convex polytope, *Canad. J. Math.* **16** (1964), 701–720.
27. V. Klee and P. Kleinschmidt, The d -step conjecture and its relatives, *Math. Oper. Res.*, to appear.
28. P. Kleinschmidt, Sphären mit wenigen Ecken, *Geom. Dedicata* **5** (1976), 307–320.
29. P. Kleinschmidt and C. Lee, On k -stacked polytopes, *Discrete Math.* **48** (1984), 125–127.
30. D. J. Kleitman, On Dedekind's problem: the number of monotone Boolean functions, *Proc. Amer. Math. Soc.* **21** (1969), 677–682.
31. A. D. Korshunov, On the number of monotone Boolean functions, *Problemy Kibernet.* **38** (1981), 5–108.
32. C. Lee, Two combinatorial properties of a class of simplicial polytopes, *Israel J. Math.* **47** (1984), 261–269.
33. P. Mani, Spheres with few vertices, *J. Combin. Theory Ser. A* **13** (1972), 346–352.
34. P. McMullen, The maximum number of faces of a convex polytope, *Mathematika* **17** (1970), 179–185.
35. P. McMullen, The number of faces of simplicial polytopes, *Israel J. Math.* **9** (1971), 559–570.
36. P. McMullen, and D. Walkup, A generalized lower bound conjecture for simplicial polytopes, *Mathematika* **18** (1971), 264–273.
37. U. Pachner, Konstruktionsmethoden und das kombinatorische Homoomorphieproblem für Triangulationen kompakter semilinearer Mannigfaltigkeiten, preprint.
38. J. Provan and L. Billera, Decompositions of simplicial complexes related to diameters of convex polyhedra, *Math. Oper. Res.* **5** (1980), 576–595.
39. L. B. Richmond and N. C. Wormald, The asymptotic number of convex polyhedra, *Trans. Amer. Math. Soc.* **273** (1982), 721–735.
40. I. Shemer, Neighborly polytopes, *Israel J. Math.* **43** (1982), 291–314.
41. R. Stanley, The upper-bound conjecture and Cohen–Macaulay rings, *Stud. Appl. Math.* **54** (1975), 135–142.
42. R. Stanley, Cohen–Macaulay complexes, in *Higher Combinatorics* (M. Aigner, ed.), 51–62, Reidel, Dordrecht, 1977.
43. R. Stanley, The number of faces of a simplicial convex polytope, *Adv. in Math.* **35** (1980), 236–238.
44. E. Steinitz and H. Radenmacher *Vorlesungen über die Theorie der Polyeder*, Springer-Verlag, Berlin, 1934.
45. W. Tutte, A census of planar triangulations, *Canad. J. Math.* **14** (1962), 21–38.
46. W. Tutte, On the enumeration of convex polyhedra, *J. Combin. Theory Ser. A* **28** (1980), 105–126.

Received December 3, 1985, and in revised form October 9, 1986.