Discrete Comput Geom 7:281–293 (1992)

Discrete & Computational
Geometry

X-Rays of Polygons*

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Abstract. Various results are given concerning X-rays of polygons in $\mathbb{R}^2$. It is shown that no finite set of X-rays determines every star-shaped polygon, partially answering a question of S. Skiena. For any n, there are simple polygons which cannot be verified by any set of n X-rays. Convex polygons are uniquely determined by X-rays at any two points. Finally, it is proved that given a convex polygon, certain sets of three X-rays will distinguish it from other Lebesgue measurable sets.

1. Introduction

The determination of sets (or, more generally, of density distributions) in $\mathbb{R}^d$ by X-rays is of interest in various fields. There is the obvious geometrical aspect, since in this context X-rays are essentially symmetrals, and there are the rapidly growing areas of computerized tomography in medicine and of tactile sensing in robotics. The present paper, which concerns only polygons, was in part stimulated by work done on "geometric probing," which has connections with computer science. In particular, one of our results (Theorem 1) is a partial answer to a question of Skiena in his Ph.D thesis with this title (Question 4.3 of [12], see also Question 12 of [13]). The paper also continues work done on X-rays of convex bodies over the last 25 years or so.

In order to put our results in context, we need to clarify several different concepts. If $\mathscr{E}$ is a class of sets, we say that $\mathscr{E}$ is *determined* by n directions if there is a set of n directions such that whenever $E, E' \in \mathscr{E}$ have the same X-rays in these directions, then $E' = E$. (We neglect sets of measure zero, although in fact this is not necessary in this paper except in Section 6.) To illustrate, we mention the

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results of [7], which imply that convex bodies in $\mathbb{R}^d$ are determined by four directions, but not by three directions. To put it another way, there are four directions such that X-rays in these directions determine the shape of a convex hole in a uniform solid, among all convex shapes.

A different idea is that of verification. A class of sets $\mathscr{E}$ can be *verified* by n directions if, given $E \in \mathscr{E}$, a set of n directions can be chosen (depending on E) so that if $E' \in \mathscr{E}$ and E' has the same X-rays as E in these directions, then $E' = E$. (We might imagine being asked to check if a hole in a uniform solid has a certain shape.) In this language a theorem of Giering [8] (or see [5] and [14] for shorter proofs) says that convex bodies in $\mathbb{R}^2$ can be verified by three directions.

An intermediate notion worth mentioning, though we will not need it here, has been considered in [3] and [13]. We say that $E \in \mathscr{E}$ can be *successively determined* by n directions if we can inductively choose directions θ_k, $1 \leq k \leq n$, the choice of θ_k depending on the X-rays of E in the directions θ_i for $1 \leq i \leq k-1$, such that if $E' \in \mathscr{E}$ and E' has the same X-rays as E in the directions θ_k, $1 \leq k \leq n$, then $E' = E$. The literature presently contains only Theorem 7 of [3] as an example of this very natural scheme; this says that convex polygons can be successively determined by three directions. (Further results of this type appear in [6].) Note that an example in [5] shows that two directions are not sufficient to verify, and therefore are also not sufficient to successively determine, convex polygons.

We can now state our results. In contrast to the theorem of [7], Theorem 1 says that polygons star-shaped at the origin cannot be determined by n directions for any n. (Note that only finite sets of directions need be considered, since it is known [11, Theorem 3.5] that bounded density distributions are determined by any infinite set of directions.) Theorem 2 is a contribution to a suggestion on p. 881 of [3] to try to extend known results for convex polygons to the case of simple polygons. However, the situation here is even worse, since we prove that there is no n such that simple polygons can be verified by n directions (so that a Giering-type theorem cannot be proved for this class of sets).

X-rays issuing from points rather than from directions (i.e., from infinity) have also been widely studied, and though they are much more difficult to work with, they are recognized as being generally more efficient, even in the practical context of computerized tomography (see [11]). We use the same terms as above when directions are replaced by points. Then Volčič's theorem [15] shows that convex bodies in $\mathbb{R}^2$ are determined by four points in general position (with our definition of point X-ray, see Section 2). It is unknown if fewer points suffice for convex bodies, but we show here (Theorem 4) that two suffice for convex polygons. In the course of proving this we obtain (Theorem 3) a necessary and sufficient condition for two general polygons to have equal X-rays at a given point.

Perhaps it should be stressed that all the above results concern distinguishing a set belonging to some class of sets from others in the same class. Our last result, however, is of a new type. Theorem 5 states that given a convex polygon P, three directions may be chosen so that the X-rays in these directions verify P not only among convex polygons, but among measurable sets (indeed, as Volčič has pointed out, among density distributions bounded by 0 and 1).

2. Definitions and Notation

If A is a set, we denote by $|A|$, cl A, int A, ∂A, and cA the *cardinality*, *closure*, *interior*, *boundary*, and *complement* of A, respectively. Further, if $r \in \mathbb{R}$, $rA = \{rx: x \in A\}$. If A and B are sets, $A + B$ denotes the vector sum $A + B = \{x + y: x \in A, y \in B\}$. The *convex hull* of a family of sets A_i is denoted by $\text{conv}\{A_i\}$.

By a *polygon* (*polytope*) we mean a finite union of triangles (simplices) (with their interiors). A *simple polygon* is a polygon whose boundary is simple and closed.

By *measure* we always mean planar Lebesgue measure, denoted by λ_2.

Let θ be a direction in $\mathbb{R}^d$ (which we identify with a unit vector in S^{d-1}), and let $\theta^\perp$ be the hyperplane containing the origin $\mathbf{o}$ and orthogonal to θ. If f is a bounded integrable function defined on a bounded set in $\mathbb{R}^d$, the *X-ray of f in the direction θ* is the function

$$P_\theta f(x) = \int_{-\infty}^{\infty} f(x + t\theta)\, dt$$

for $x \in \theta^\perp$. The X-ray of a set E in the direction θ is $P_\theta 1_E$, where 1_E is the characteristic function of E. We find it convenient to identify $P_\theta f$ with the set of points between its graph and $\theta^\perp$, so that if K is convex, so is $P_\theta 1_K$ (an alternative approach is to work with the Steiner symmetral in the direction θ; indeed, in the formulation of our theorems the terms X-ray and Steiner symmetral are interchangeable).

Suppose $p \in \mathbb{R}^d$ and f is a bounded integrable function on a bounded set in $\mathbb{R}^d$. The *X-ray of f at p* is the function

$$L_p f(\theta) = \int_{-\infty}^{\infty} f(p + t\theta)\, dt$$

for $\theta \in S^{d-1}$, and the X-ray of a set E at p is $L_p 1_E$.

The functions $P_\theta f$ and $L_p f$ are also called the X-ray transform and line transform, respectively, of f. All we need here is that the X-ray of a polygon P in the direction θ gives the total length of the intersection of P with each line parallel to θ, and the X-ray of P at a point p gives the total length of the intersection of P with each straight line through p.

3. Switching Components

The concept of a switching component is essential for some of our results. Let $\Theta = \{\theta_1, \ldots, \theta_m\}$ be a finite set of directions, not necessarily distinct, in $\mathbb{R}^2$. Suppose A and B are two disjoint nonempty finite sets of points such that if l is any line parallel to θ_i, $1 \le i \le m$, then

$$|l \cap A| = |l \cap B|.$$

Then we call $A \cup B$ a *Θ-switching component*. If E is any set, $A \cup B$ is called a *Θ-switching component for* E if in addition $A \subset \text{int } E$ and $B \subset \text{int}(cE)$.

The following proposition, due to Lorentz [10] (see also [1]), guarantees the existence of switching components. We give the proof for completeness.

Proposition 1. *If* $\Theta = \{\theta_1, \ldots, \theta_m\}$ *is any set of directions in* $\mathbb{R}^2$, *there is a Θ-switching component.*

Proof. If $m = 1$, let $A_1 = \{\mathbf{o}\}$, let v_1 be any vector parallel to θ_1, and $B_1 = (A_1 + v_1)$. Then $A_1 \cup B_1$ is a $\{\theta_1\}$-switching component.

If the proposition is true for $m = k$, let $A_k \cup B_k$ be the corresponding switching component. Choose a vector v_{k+1} parallel to θ_{k+1}, such that the sets A_k, B_k, $(A_k + v_{k+1})$, and $(B_k + v_{k+1})$ are all disjoint. Let $A_{k+1} = A_k \cup (B_k + v_{k+1})$ and $B_{k+1} = B_k \cup (A_k + v_{k+1})$. Then $A_{k+1} \cup B_{k+1}$ is a Θ-switching component for $m = k + 1$. □

Note that the switching component constructed by the method of Proposition 1 is simply a projection of the vertices of an m-dimensional parallelepiped onto $\mathbb{R}^2$.

4. Parallel X-Rays of Polygons

Theorem 1. *If Θ is any finite set of directions in* $\mathbb{R}^2$, *there exist two distinct simple polygons P and Q, star-shaped at a common point, with equal X-rays in the directions in Θ.*

Proof. Let $A \cup B$ be any Θ-switching component (see Proposition 1), and let p be a point not on any line containing two points of $A \cup B$ (in particular, $p \notin A \cup B$). Let T be a triangle containing the origin, such that no side of T is parallel to a line joining p to some $x \in A \cup B$. Next we choose $\varepsilon > 0$ small enough so that the set $\varepsilon T + (A \cup B)$ has the following properties:

(1) If $x \in A \cup B$ and l is a straight line meeting both the triangles $\varepsilon T + \{p\}$ and $\varepsilon T + \{x\}$, then l does not meet $\varepsilon T + \{y\}$ for any $y \in A \cup B$ with $y \neq x$.

(2) If e is an edge of a triangle $\varepsilon T + \{x\}$, $x \in A \cup B$, visible from a vertex of the triangle $\varepsilon T + \{p\}$, then e is also visible from p.

Let

$$C = \bigcup\{\text{conv}\{\varepsilon T + \{p\}, \varepsilon T + \{x\}\}: x \in A \cup B\}$$

and

$$P = \text{cl}(C - \cup\{\varepsilon T + \{x\}: x \in A\}),$$

$$Q = \text{cl}(C - \cup\{\varepsilon T + \{x\}: x \in B\}).$$

Then P and Q are the required simple polygons; both are star-shaped at the point p (see Fig. 1). □

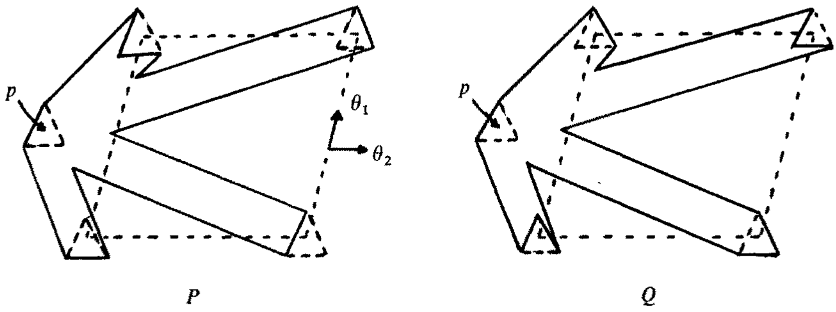

Fig. 1. The polygons P and Q of Theorem 1 (two directions)

Theorem 2. *For each $n \in \mathbb{N}$, there is a simple polygon P_n such that if Θ is any set of n directions in $\mathbb{R}^2$, there is a different simple polygon $Q(\Theta)$ with the same X-rays as P_n in the directions in Θ.*

Proof. If S^1 is the unit circle, we identify ordered sets Θ of n (not necessarily distinct) directions with points in the compact metric space $(S^1)^n$ with the product topology. By a neighborhood of Θ we mean a neighborhood in this space.

Let $\Theta = \{\theta_1, \ldots, \theta_n\}$ be an ordered set of n (not necessarily distinct) directions. Following the proof of Proposition 1, we inductively construct $\{\theta_1, \ldots, \theta_k\}$-switching components $A_k \cup B_k$, using vectors v_i parallel to θ_i, $1 \le i \le k$, and with $A_k \cup B_k \subset A_{k+1} \cup B_{k+1}$ for $k = 1, \ldots, n-1$. Let s be a closed line segment, containing $\mathbf{o}$ in its interior, not parallel to any $\theta \in \Theta$ and short enough so that the copies $\{s + x : x \in A_n \cup B_n\}$ of s are all disjoint. Put

$$S = \bigcup \{s + x : x \in A_n \cup B_n\}.$$

We claim that there is a neighborhood U of Θ such that if $\Theta' \in U$, then S also contains a Θ'-switching component.

The claim can be proved by induction on n. Specifically, we prove the following. Suppose t is a closed line segment with $\mathbf{o}$ in its interior and such that t is strictly contained in s. Then we can find another such segment $u_n \subset t$ and a neighborhood U_n of Θ, such that, if $\Theta' \in \mathrm{U}_n$ and $z \in u_n$, there is a Θ'-switching component $A'_n \cup B'_n$, constructed as in Proposition 1 starting with the point z instead of $\mathbf{o}$, such that

$$A'_n \cup B'_n \subset \bigcup \{t + x : x \in A_n \cup B_n\}.$$

Let $n = 1$. Given a segment t as above, there is a segment $u_1 \subset t$ and a neighborhood U_1 of θ_1, such that if $\theta'_1 \in U_1$ and $z \in u_1$, there is a vector v'_1 parallel to θ'_1 with $z + v'_1 \in (t + v_1)$. Then $\{z\} \cup \{z + v'_1\}$ is the required $\{\theta'_1\}$-switching component.

Suppose we have proved the above for $n \le k$. Apply the argument for $n = 1$ to the case where $\Theta = \{\theta_{k+1}\}$. We obtain a segment $u' \subset t$ and a neighborhood U' of θ_{k+1} such that if $\theta'_{k+1} \in U'$ and $z \in u'$, there is a vector v'_{k+1} parallel to θ'_{k+1} with

$z + v'_{k+1} \in (t + v_{k+1})$. Now applying the case $n = k$ with $\Theta = \{\theta_1, \ldots, \theta_k\}$ and $t = u'$, we get a segment $u_{k+1} \subset u'$ and a neighborhood U'' of $\{\theta_1, \ldots, \theta_k\}$, such that if $\{\theta'_1, \ldots, \theta'_k\} \in U''$ and $z \in u_{k+1}$, there is a Θ'-switching component $A'_k \cup B'_k$, constructed as in Proposition 1 starting with z, such that $A'_k \cup B'_k \subset \bigcup \{u' + x: x \in A_k \cup B_k\}$.

Now let $U_{k+1} = U'' \times U'$; then U_{k+1} is a neighborhood of $\{\theta_1, \ldots, \theta_{k+1}\}$. If $z \in u_{k+1}$ and $\{\theta'_1, \ldots, \theta'_{k+1}\} \in U_{k+1}$, then for each $y \in A'_k \cup B'_k$ we have $y \in (u' + x)$ for some $x \in A \cup B$, so $(y - x) \in u'$ and $(y + v'_{k+1}) \in (t + v_{k+1} + x)$. Therefore if we define $A'_{k+1} = A'_k \cup (B'_k + v'_{k+1})$ and $B'_{k+1} = B'_k \cup (A'_k + v'_{k+1})$, then $A'_{k+1} \cup B'_{k+1}$ is a $\{\theta'_1, \ldots, \theta'_{k+1}\}$-switching component contained in $\bigcup\{t + x: x \in A_{k+1} \cup B_{k+1}\}$. This completes the proof of the claim.

With n still fixed we choose, for each $\Theta \in (S^1)^n$, a neighborhood $U(\Theta)$ as above. By compactness there is a finite set $\Theta_1, \ldots, \Theta_m$, such that the associated neighborhoods cover $(S^1)^n$. For each i, $1 \le i \le m$, let S_i be the finite set of parallel line segments corresponding to Θ_i as above, where we assume that these sets S_i are translated so that they are all disjoint.

By the construction above, we may partition each S_i into two disjoint unions C_i, D_i of parallel line segments, so that if $\Theta \in U(\Theta_i)$, there is a Θ-switching component $A \cup B$ with $A \subset C_i$ and $B \subset D_i$. Let $S(n) = \bigcup_{i=1}^m S_i$, $C(n) = \bigcup_{i=1}^m C_i$, and $D(n) = \bigcup_{i=1}^m D_i$. Then $S(n)$ is a finite disjoint union of line segments such that, for each $\Theta \in (S^1)^n$, there is a Θ-switching component $A \cup B \subset S(n)$, with $A \subset C(n)$ and $B \subset D(n)$. We note that each of the switching components we are considering meets each line segment in $S(n)$ in at most one point, and that we may assume that all the line segments in $S(n)$ are parallel.

Let S_ε be the closed square with center $\mathbf{o}$ and one side of length ε and parallel to a line segment in $S(n)$. It is easy to see that we can find an $\varepsilon > 0$ and a simple polygon P_n with the following property: If l is one of the line segments which form $C(n)$ (or $D(n)$) and $x \in l$, then $S_\varepsilon + x \subset P_n$ and $\mathrm{cl}(P_n \backslash S_\varepsilon)$ is a simple polygon (or $\mathrm{int}(S_\varepsilon + x) \cap \mathrm{int}\, P_n = \varnothing$ and $P_n \cup S_\varepsilon$ is a simple polygon, respectively).

Suppose that Θ is any set of n directions, and let $A \cup B$ be the corresponding Θ-switching component with $A \subset C(n)$ and $B \subset D(n)$. If $x \in A$, x belongs to one of the line segments which form $C(n)$. We delete from P_n the square $S_\varepsilon + x$. If $x \in B$, we adjoin $S_\varepsilon + x$ to P_n. In this way we construct a different simple polygon Q with the same X-rays as P_n in the directions in Θ. □

Corollary. *There is a compact set E (a countable union of simple polygons) which cannot be verified by any finite set of X-rays.*

Proof. Any translation or dilation of the simple polygon P_n from Theorem 2 has the same property. For each n, there is a disk C_n containing P_n such that, for each set Θ of n directions, the polygon $Q(\Theta)$ of Theorem 2 is contained in C_n. Choose constants α_n and vectors w_n so that the sets $(\alpha_n C_n + w_n)$ are disjoint and converge to a single point. Then $E = \mathrm{cl}(\bigcup_n(\alpha_n P_n + w_n))$ is the required compact set. □

In Question 12 of [13] Skiena asks if star-shaped polygons can be determined (by which he means successively determined) by finitely many directions. If Theorem 2 remains true for star-shaped polygons, this would provide a strong negative answer to Skiena's question. However, we do not know if this is the case.

Higher-dimensional analogues of Theorem 2 can be proved in the same way. In particular, there are (nonconvex) polyhedra in $\mathbb{R}^3$ which cannot be verified by any pair of directions. At first sight this seems to contradict the result of Golubjatnikov [9], that, given any polyhedron P in $\mathbb{R}^3$, two directions can be found so that P can be reconstructed from its X-rays in these directions. However, this reconstruction can only be made using information about how the directions were chosen, and this involves additional knowledge about the relative position of vertices and of edges. In short, the theorem of [9] is not of the type considered in this paper (but see [6] for results on the successive determination of convex polytopes).

5. X-Rays from Points

In our next theorem we state a necessary and sufficient condition for X-rays of two polygons to be equal at $\mathbf{o}$. If P and Q are polygons, we can partition $\mathbb{R}^2$ into a finite set of double cones with vertex at $\mathbf{o}$, such that neither P nor Q has any of its vertices in the interiors of these cones. By subdividing the cones if necessary, we may assume that for each one we can choose an axis for polar coordinates at $\mathbf{o}$ so that it can be represented as

$$C(\alpha, \beta) = \{(r, \theta): r \in \mathbb{R}, 0 \leq \alpha \leq \theta \leq \beta < \pi\}.$$

Each edge of P or Q meeting int $C(\alpha, \beta)$ also meets its bounding lines $\{\theta = \alpha\}$ and $\{\theta = \beta\}$.

We need the following lemma, whose simple proof we omit.

Lemma. *Let T be the triangle with vertices at $\mathbf{o}$, (s, α), and (t, β), $s > 0$, $t > 0$, $0 \leq \alpha < \beta < \pi$, in polar coordinates centered at $\mathbf{o}$. The X-ray of T at $\mathbf{o}$ is given by*

$$\rho(\theta) = \frac{st \sin(\beta - \alpha)}{t \sin(\beta - \theta) + s \sin(\theta - \alpha)}$$

for $\alpha \leq \theta \leq \beta$.

Suppose $C(\alpha, \beta)$ is a cone as above, and let $\{e_i\}$ be the set of edges of P or Q which meet int $C(\alpha, \beta)$. If φ is a direction, $0 \leq \varphi < \pi$, let $\{e_i, i \in I(\varphi)\}$ be the set of those e_i which are parallel to φ. Suppose e_i intersects $\{\theta = \alpha\}$ at (s_i, α). If $e_i \subset P$, define $\varepsilon_i = +1$ or -1 according as a line $\{\theta = \gamma, \alpha < \gamma < \beta\}$ leaves P as r increases across e_i, or enters P as r increases across e_i; and vice versa for edges $e_i \subset Q$.

Theorem 3. *The following is a necessary and sufficient condition for two polygons P, Q to have equal X-rays at* $\mathbf{o}$: *in the above notation, for each appropriate cone $C(\alpha, \beta)$ and direction φ, the family $\{e_i, i \in I(\varphi)\}$ of edges must satisfy $\sum\{\varepsilon_i s_i \colon i \in I(\varphi)\} = 0$.*

Proof. Using the lemma and the notation above, we see that P and Q have the same X-rays for $\alpha \le \theta \le \beta$ if and only if

$$\sum_i \frac{\varepsilon_i s_i t_i \sin(\beta - \alpha)}{t_i \sin(\beta - \theta) + s_i \sin(\theta - \alpha)} = 0 \tag{1}$$

for $\alpha \le \theta \le \beta$, where the sum runs over all edges e_i of P or Q, containing (s_i, α) and (t_i, β), and with appropriate weights $\varepsilon_i = \pm 1$.

If φ is a direction, and e_i is an edge of P or Q parallel to φ and containing (s_i, α) and (t_i, β), then

$$t_i \sin(\beta - \varphi) + s_i \sin(\varphi - \alpha) = 0. \tag{2}$$

Solving for t_i, and substituting into the corresponding term in (1), or by direct calculation, we get

$$\frac{\varepsilon_i s_i t_i \sin(\beta - \alpha)}{t_i \sin(\beta - \theta) + s_i \sin(\theta - \alpha)} = \varepsilon_i s_i \cdot \frac{\sin(\alpha - \varphi)}{\sin(\theta - \varphi)}. \tag{3}$$

If e_i, $i \in I(\varphi)$, are all the relevant edges of P and Q parallel to φ, then by (3) we may replace the corresponding terms in (1) by

$$\sum_{i \in I(\varphi)} \varepsilon_i s_i \cdot \frac{\sin(\alpha - \varphi)}{\sin(\theta - \varphi)} = \left(\sum_{i \in I(\varphi)} \varepsilon_i s_i \right) \cdot \frac{\sin(\alpha - \varphi)}{\sin(\theta - \varphi)}.$$

This implies that polygons satisfying the condition of the theorem will have equal X-rays.

A similar calculation can be made with t_i instead of s_i. This means that the left-hand side of (1) is unchanged if we replace the set of edges $\{e_i \colon i \in I(\varphi)\}$ by a single segment $e'(\varphi)$ joining the points $(\sum\{\varepsilon_i s_i \colon i \in I(\varphi)\}, \alpha)$ and $(\sum\{\varepsilon_i t_i \colon i \in I(\varphi)\}, \beta)$.

Let us make this replacement for each direction φ_j parallel to an edge of P or Q. Cancelling the constant $\sin(\beta - \alpha)$, we obtain

$$\sum_j \frac{p_j q_j}{q_j \sin(\beta - \theta) + p_j \sin(\theta - \alpha)} = 0 \tag{4}$$

for $\alpha \le \theta \le \beta$, where the jth term now corresponds to a segment e'_j joining (p_j, α) and (q_j, β), no two of the segments e'_j are parallel, and each p_j is a sum of the form $p_j = \sum\{\varepsilon_i s_i \colon i \in I(\varphi_j)\}$.

Simplifying (4) gives

$$\sum_j \left[p_j q_j \prod_{k \neq j} (q_k \sin(\beta - \theta) + p_k \sin(\theta - \alpha)) \right] = 0 \tag{5}$$

for $\alpha \leq \theta \leq \beta$. Dividing by $\cos \theta$ we see that the left-hand side of (5) is a polynomial in $\tan \theta$, so it must vanish for all θ.

For each j we choose $\theta = \theta_j$, where

$$q_j \sin(\beta - \theta_j) + p_j \sin(\theta_j - \alpha) = 0. \tag{6}$$

Substituting into (5) we get

$$p_j q_j \prod_{k \neq j} (q_k \sin(\beta - \theta_j) + p_k \sin(\theta_j - \alpha)) = 0.$$

If, for some $k \neq j$,

$$q_k \sin(\beta - \theta_j) + p_k \sin(\theta_j - \alpha) = 0, \tag{7}$$

then (6) and (7) imply (compare (2)) that the segment e'_j joining (p_j, α) and (q_j, β) is parallel to e'_k, joining (p_k, α) and (q_k, β), which is not the case. Consequently, $p_j = 0$ (or, equivalently, $q_j = 0$). This means that for the corresponding sum

$$\sum \{\varepsilon_i s_i : i \in I(\varphi_j)\} = 0,$$

completing the proof. □

Despite Theorem 3, uniqueness results for general polygons may be difficult to obtain. Brehm [2] has found two different sets of ten homothetic triangles whose unions have the same X-rays from two points. (The centers of these triangles form a special switching component suitable for point X-rays.) It is then easy to construct, as in Theorem 1, two different star-shaped polygons with the same X-rays at two points. In contrast, our next theorem shows that two points suffice for convex polygons. It is very similar to Theorem 6.3 of [12], but the latter is stated only for X-rays from points interior to the polygon.

Theorem 4. *X-rays from two points determine a convex polygon uniquely.*

Proof. Suppose P and Q are two convex polygons with equal X-rays at $\mathbf{o}$. Choose a polar axis and double cone $C(\alpha, \beta)$ as in Theorem 3, so that both P and Q meet int $C(\alpha, \beta)$ but the latter set contains none of their vertices. Then by convexity there are exactly two edges e_i, $i = 1, 2$, of P, and two edges e_i, $i = 3, 4$, of Q, with e_i containing (s_i, α) and (t_i, β), and $s_1 < s_2$, $s_3 < s_4$, say.

If $P \cap C(\alpha, \beta) \neq Q \cap C(\alpha, \beta)$, Theorem 3 implies that there are only two possibilities. The first is that the edges e_1 and e_4 are parallel and $s_1 + s_4 = 0$, and the edges

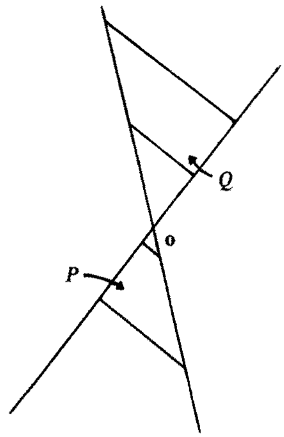

Fig. 2. Polygons P and Q with equal X-rays at $\mathbf{o}$.

e_2 and e_3 are parallel and $s_2 + s_3 = 0$. This means that $Q \cap C(\alpha, \beta)$ is the reflection of $P \cap C(\alpha, \beta)$ in the origin. Now suppose that this is not the case. Then the second possibility is that all the edges e_i are parallel and $s_4 - s_3 = s_2 - s_1$. The intersections of P and Q with $C(\alpha, \beta)$ are then formed by congruent parallel strips intersecting $C(\alpha, \beta)$. Reducing α and increasing β until we meet a vertex of P or Q, we obtain a maximal double cone $C(\alpha, \beta)$ for which this holds. Using convexity, it is easy to see that $P \cap cC(\alpha, \beta) = \varnothing$.

We conclude that if two unequal convex polygons have equal X-rays at $\mathbf{o}$, then either one is the reflection of the other in $\mathbf{o}$, or else they must be as in Fig. 2, the intersection of congruent parallel strips and a cone with vertex at $\mathbf{o}$. However, such polygons must have different X-rays at any point other than $\mathbf{o}$, unless they are equal. □

Corollary. *X-rays from two points determine a convex polytope in* $\mathbb{R}^d$, $d \geq 2$, *uniquely.*

Proof. By Theorem 4, each slice by a two-dimensional plane containing the two points is determined uniquely. □

6. Uniqueness Among Measurable Sets

Can we determine or verify a convex polygon or body if we do not know *a priori* that it is such a set, but merely a measurable set? Here we present a result of this new type.

A theorem of Falconer [4] says that if the X-ray of a compact set in $\mathbb{R}^2$ in every direction is convex, then the set itself must be convex. Therefore it is pertinent to remark here that, for any finite set of directions, there are nonconvex sets whose

X-rays in these directions are convex. We now describe a construction which is based on an idea of Volčič.

Let $\theta_i, i = 1, \ldots, m, m \geq 2$, be given directions in $\mathbb{R}^2$, and let s be a line segment not parallel to any θ_i. For each i, let S_i be the closed infinite strip whose two bounding lines are parallel to θ_i and each contain one endpoint of s. Next construct a convex polygon P, containing s in its interior and with $4n$ sides, such that for each i there are two sides of P which have the same orthogonal projection onto $\theta_i^{\perp}$ as s, and each of the remaining $2n$ sides is not contained in any S_i. Find a triangle T contained in S_i for each i (and therefore in int P), one of whose sides is s, and which is sufficiently thin to satisfy the following property: For each i there is a triangle T_i contained in S_i, such that T_i and T have equal X-rays in the direction θ_i, int $T_i \cap$ int $P = \emptyset$, and $P \cup \bigcup_{i=1}^{m} T_i$ is a convex polygon. Now

$$(P - \text{int } T) \cup \bigcup_{i=1}^{m} T_i$$

is a nonconvex set whose X-rays in the directions θ_i are convex polygons.

In proving our theorem the main tool is a lemma due to Giering [8, Satz 8] as generalized by Volčič [16, Theorem 2.3]. To assist the reader we give the simple but clever proof.

Lemma. *Let K be a convex body in $\mathbb{R}^2$, and let Θ be a finite set of directions. Suppose Q is a convex polygon with its vertices in ∂K, such that each edge of Q is parallel to some $\theta \in \Theta$. If E is any measurable set such that E has the same X-rays as K in the directions in Θ, then $Q \subset E$ except for a set of measure zero.*

Proof. If e_i is an edge of Q and S is any set, let S_i be the part of S in the open half-space bounded by the line containing e_i, which does not contain Q. Since Q is inscribed in ∂K we have

$$\lambda_2(K) = \lambda_2(Q) + \sum_i \lambda_2(K_i).$$

If E has the same X-rays as K in the directions in Θ, then $\lambda_2(E) = \lambda_2(K)$ and $\lambda_2(E_i) = \lambda_2(K_i)$ for each i. Now $(E - \bigcup_i E_i) \subset Q$ and

$$\lambda_2\left(E - \bigcup_i E_i\right) \geq \lambda_2(E) - \sum_i \lambda_2(E_i) = \lambda_2(K) - \sum_i \lambda_2(K_i) = \lambda_2(Q).$$

This means that Q and $(E - \bigcup_i E_i)$ differ by at most a set of measure zero, so $Q \subset E$ except for such a set. □

Theorem 5. *If P is a convex polygon, we can choose three directions such that if E is a measurable set with the same X-rays as P in these directions, then $E = P$ up to a set of measure zero.*

Proof. The directions we choose are those parallel to three adjacent edges of P. Let these edges be e_1, e_2, and e_3, in order around ∂P, and let θ_i be the corresponding directions, $1 \le i \le 3$. We may assume that P lies in the upper half-plane with its base, the edge e_2, on the x-axis.

An easy continuity argument (also employed in [16]) shows that there is a triangle T_1 with one vertex in e_2, the others in $(\partial P - e_2)$, and with sides parallel to the directions θ_i, $1 \le i \le 3$ (so that the uppermost side, e_2' say, is parallel to e_2, i.e., horizontal). Each point $x \in \partial P$ below e_2' is a vertex of a convex quadrilateral inscribed in ∂P, all of those edges are parallel to one of the directions θ_i, $1 \le i \le 3$. Thus, according to the lemma, the polygonal part P' of P which lies between e_2 and e_2' is contained (modulo a set of measure zero) in any measurable set E which has the same X-rays as P in the directions θ_i, $1 \le i \le 3$.

Note that this argument works when we take one direction parallel to an edge of P and the two others parallel to lines through each of the endpoints of this edge which do not intersect int P.

By subtracting the X-rays of P' from those of P we obtain the X-rays of the convex polygon $P_1 = \text{cl}(P - P')$. Note that P_1 has its base on the horizontal edge e_2', and that we can repeat the above argument with P replaced by P_1, obtaining another triangle T_2, and so on.

The process may not terminate after finitely many steps, but it is easy to see that otherwise the similar triangles T_n, $n = 1, 2, \ldots$, will converge to a unique vertex of P with largest y-coordinate. So after ω steps we have shown that $P \subset E$, except for a set of measure zero; and, since $\lambda_2(P) = \lambda_2(E)$, we are finished. □

We conclude with some remarks concerning Theorem 5. Firstly, it can be shown that the known proofs of Giering's theorem cannot work in the context of Theorem 5. It is not difficult to construct a convex polygon P such that when the three directions are selected according to any of the published proofs of Giering's theorem, there may be a switching component $A \cup B$ for P (see Section 2) in these directions. Then, removing small congruent disks centered at points in $A \subset \text{int}\, P$ and adding disks of the same size centered at points in $B \subset cP$, we obtain a measurable set E with the same X-rays as P in the three directions.

Volčič has noticed that both the lemma and Theorem 5 can be modified to apply to density distributions bounded by 0 and 1 instead of measurable sets.

As mentioned in the Introduction, Example 2.5 of [5] shows that in general two directions do not suffice.

Finally, we note that the proof of Theorem 5 is to some extent constructive; in [16] it is proved that the triangle T_1 (and hence T_n, $n \ge 2$) can actually be computed from the X-rays.

Acknowledgment

I thank A. Volčič for several helpful discussions and permission to include a construction of his (third paragraph of Section 6).

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Received March 19, 1990, and in revised form August 15, 1990.