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Separating Convex Sets in the Plane

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Abstract. Given a set A in R^2 and a collection S of plane sets, we say that a line L separates A from S if A is contained in one of the closed half-planes defined by L, while every set in S is contained in the complementary closed half-plane.

We prove that, for any collection F of n disjoint disks in R^2, there is a line L that separates a disk in F from a subcollection of F with at least $\lceil (n-7)/4 \rceil$ disks. We produce configurations H_n and G_n, with n and $2n$ disks, respectively, such that no pair of disks in H_n can be simultaneously separated from any set with more than one disk of H_n, and no disk in G_n can be separated from any subset of G_n with more than n disks.

We also present a set J_m with $3m$ line segments in R^2, such that no segment in J_m can be separated from a subset of J_m with more than $m + 1$ elements. This disproves a conjecture by N. Alon *et al.* Finally we show that if F is a set of n disjoint line segments in the plane such that they can be extended to be disjoint semilines, then there is a line L that separates one of the segments from at least $\lceil n/3 \rceil + 1$ elements of F.

1. Introduction

Given a collection F of disjoint compact convex sets in the plane, an element $A \in F$, and a subcollection S of F, we say that a line L separates A from S if A is contained in one of the closed half-planes defined by L, while every set in S is contained in the complementary closed half-plane.

In [3] Tverberg proves that, for any positive integer k, there is a minimum integer $N(k)$ such that in any family F of $N(k)$ or more disjoint compact convex sets in the plane there is one that can be separated from a subfamily with at least k sets. In [2] Hope and Katchalski prove that $3k - 1 \leq N(k) \leq 12(k - 1)$.

In this article we show that, for any collection F of n disjoint disks in R^2, there is a line L that separates a disk in F from a subcollection of F with at least $\lceil (n-7)/4 \rceil$ disks. We produce configurations H_n and G_n with n and $2n$ disks, respectively, such that no pair of disks in H_n can be simultaneously separated from any set with more than one disk of H_n; and no disk in G_n can be separated from any subset of G_n with more than n disks.

In Section 3 we present a configuration J_m with $3m$ line segments in R^2, such that no segment in J_m can be separated from a subset of J_m with more than $m + 1$ elements. This disproves a conjecture by Alon *et al.* presented in [1]. Finally, we show that if F is a collection of n line segments such that they can be extended to be disjoint semilines, then there is a line L that separates one of the segments from a subcollection of F with at least $\lceil (n+3)/3 \rceil$ elements.

The results in this article remain valid for corresponding collections of convex sets with pairwise disjoint relative interiors. This allows us to present, as examples, the configurations H_n, G_n, and J_m that contain sets with common boundaries but pairwise disjoint relative interiors.

2. Separating Disks

In [1], Alon *et al.* proved that there is a constant $c > 0$ such that, for any family F with n disjoint congruent disks, there is a line L that leaves at least $k/2 - c\sqrt{k}\sqrt{\log k}$ disks on each closed half-plane defined by L. When the disks are allowed to have arbitrary radii the situation is entirely different as the following example illustrates.

We describe a configuration H_n of n disks in which no pair C_i, C_j of disks in H_n can be simultaneously separated by one line L from any other pair C_k, C_l in H_n.

Let $S_1 > S_2 > \cdots > S_n$ be n different slopes such that $0 \leq S_i \leq \varepsilon$, with ε small enough. Let H_n consist of n disks defined recursively as follows:

(a) C_1 is any disk in R^2.
(b) C_{i+1} is a disk tangent to C_i such that the slope of the line that separates them is S_i.
(c) C_{i+1} is large enough such that any line L separating C_j from C_{i+1}, $1 \leq j < i + 1$, has slope $s(L)$ contained in the interval $(S_i - \delta, S_i + \delta)$, $\delta > 0$, δ much smaller than ε. Observe that $s(L)$ is contained in the interval $(-\delta, \varepsilon + \delta)$ since $0 \leq S_i \leq \varepsilon$.

Moreover, if δ is small enough, C_{i+1} can be chosen such that:

(d) Any line separating C_j from C_i, $1 \leq j < i$, intersects C_{i+1}.

It follows that there are no different pairs of disks $\{C_i, C_j\}$ and $\{C_k, C_l\}$ in H_n, such that there is a line separating $\{C_i, C_j\}$ from $\{C_k, C_l\}$. For let us assume that

i is the smallest of i, j, k, and l, and that $k < l$. It now follows from (d) that any line separating C_i from C_k must intersect C_l.

Notice that in H_n, C_i can be separated from $C_1, \ldots, C_{i-1}$, $i = 1, \ldots, k$, and that C_i cannot be separated from any pair C_k, C_l, $i < k < l$.

For any family of disjoint disks we have the following theorem:

Theorem 1. *In any family F of n disjoint disks, there is one disk that can be separated from a subfamily of F with at least $\lceil (n-7)/4 \rceil$ disks.*

The following lemma will be used in the proof; the reader may wish to verify it.

Lemma 2. *Let H be a family of m disjoint disks, all of which are intersected by two orthogonal lines. There is a disk in H that can be separated from a subfamily of H with at least $\lceil (m-5)/2 \rceil$ disks.*

Proof of Theorem 1. Sweep a vertical line L_1, from left to right, until one disk is left to the left of L_1. Then sweep a horizontal line L_2, from bottom to top, until a disk is left below L_2. Let n_1 and n_2 denote the number of disks to the right of L_1 and above L_2, respectively. Also let H be the set of disks in F, intersected by both L_1 and L_2 and denote by n_3 the number of disks in H. Clearly, $n_3 \geq n - n_1 - n_2 - 2$.

By Lemma 2, there is a disk in H that can be separated from a subfamily with at least $\lceil (n_3 - 5)/2 \rceil$. If $n_1 < \lceil (n-7)/4 \rceil$ and $n_2 < \lceil (n-7)/4 \rceil$, then $\lceil (n_3 - 5)/2 \rceil \geq \lceil (n-7)/4 \rceil$ and the result follows. □

The following example shows that, occasionally, we cannot separate any disk of a family of m disks from any subfamily with more than $m/2$ disks.

To construct the family G_n let us take a copy $H'_n = \{C'_1, C'_2, \ldots, C'_n\}$ of the configuration H_n as follows: reflect H_n along the x-axis and translate it in the northwest direction until all the lines separating C_i from C_j intersect only C'_n in H'_n and all lines separating C'_i from C'_j intersect only C_n in H_n (see Fig. 1).

Any line separating two elements C_i, C_j in H_n leaves at most $C_1, \ldots, C_i$ on one side and $C'_1, \ldots, C'_{n-1}$ on the other; similarly, for any line separating two elements in H'_n. Then G_n is a configuration with $2n$ disks and none of them can be separated from any set of disks in G_n with more than n disks.

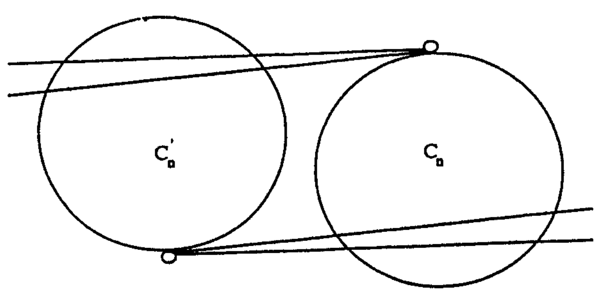

Fig. 1. $C_1, C_2, \ldots, C_{n-1}$ are contained in a small circle above C_n.

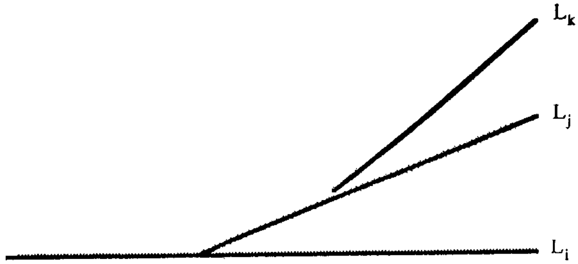

Fig. 2

3. Separating Line Segments and Semilines

In [1] the following conjecture is presented: for any collection F of n disjoint line segments on the plane, there is an element S of F that can be separated from close to $n/2$ elements of F. In this section we disprove the conjecture by producing a family J_m of $3m$ line segments such that no element of J_m can be separated from more than $m + 1$ elements of J_m.

To describe J_m we use a configuration due to K. P. Villanger, see [3]. He constructs a family T of m line segments $L_1, L_2, \ldots, L_m$ with the property that, for each $k = 3, \ldots, m$, L_k intersects the convex closure of $L_i \cup L_j$, $1 \leq i < j < k$, and therefore L_k cannot be separated by a line from $\{L_i, L_j\}$ (see Fig. 2).

His construction may be reproduced in such a way that $L_1, L_2, \ldots, L_m$ have slopes $0 = S(L_1) < S(L_2) < \cdots < S(L_m) = \delta < \pi/2$, respectively; and such that, for $i = 1, 2, \ldots, m$, the left endpoint of L_{i+1} lies in an interior point of L_i within distance ε of the left endpoint of L_1 (see Fig. 3).

Our example is a set J_m of $3m$ line segments consisting of three copies $T_0 = \{L_{0,1}, \ldots, L_{0,k}\}$, $T_1 = \{L_{1,1}, \ldots, L_{1,k}\}$, and $T_2 = \{L_{2,1}, \ldots, L_{2,k}\}$ of T placed around a triangle Q with vertices v_0, v_1, v_2 (see Fig. 4). The values of ε and δ are chosen in such a way that the line supporting any element of T_i, intersects all the elements of T_{i+1}; addition taken mod 2.

Let us consider the case where the segments in F can be extended to semilines so that they remain pairwise disjoint.

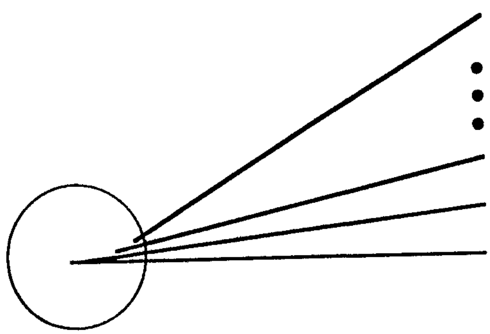

Fig. 3

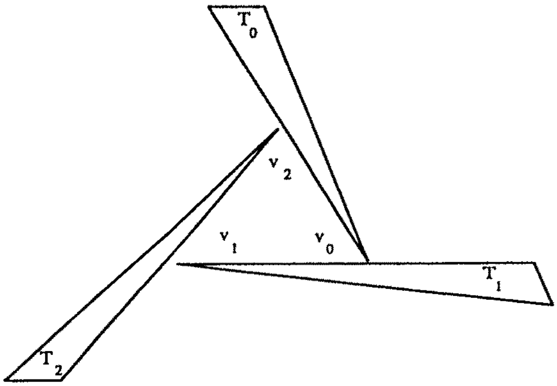

Fig. 4

Theorem 3. *Let $F = \{L_1, \ldots, L_n\}$ be a family of n disjoint line segments, $n \geq 4$. If they can be extended to form a collection of disjoint semilines, then there is a line L that separates an element L_i of F from a subset of F with at least $\lfloor n/3 \rfloor + 1$ elements.*

Proof. If there is an element L_i of F that can be extended to a whole line without intersecting any other element of F, then L_i can be separated from a subfamily of F with at least $\lceil (n-1)/2 \rceil$ elements of F. Suppose then that the line supporting each L_i intersects at least another element L_i of F. Extend the elements of F as much as possible until a family $F' = \{L'_1, \ldots, L'_n\}$ of semilines is obtained such that;

1. The endpoint of every element of F' lies on an interior point of another element of F'.
2. No two elements of F' cross each other (see Fig. 5).

We say that L'_i hits L'_j if the endpoint of L'_i lies on L'_j. For example, in Fig. 5 L'_1 hits L'_4. It is easy to see that in F' there is a cyclic sequence of elements, say $L'_1, \ldots, L'_j$, $j \leq n$, such that L'_{i+1} hits L'_i, $i = 1, \ldots, j-1$, and L'_1 hits L'_j.

For the case when $j = n$ we can easily show that there is an element of F separable from a set with at least $\lceil n/2 \rceil$ elements of F; in the remainder of this section we assume that $j < n$.

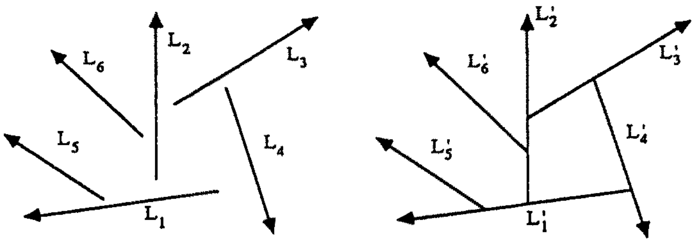

Fig. 5

For every $i = 2, \dots, j$ let S_i be the subset of F' consisting of L'_i together with all the elements of F' contained in the open region bounded by L'_i and L'_{i-1} and let S_1 be the subset of L' consisting of L'_1 and all elements of F' contained in the open region bounded by L'_1 and L'_j.

Let i be the smallest index such that the line L supporting L'_1 intersects L'_i. Then it is easy to see that the set $A = S_2 \cup \cdots \cup S_{i-1}$ is separable from L_1. It is also easy to see that $B = S_i$ is separable from L'_{i-1} and that

$$C = S_{i+1} \cup \cdots \cup S_j \cup S_1$$

may be separated from L'_i (see Fig. 6).

However, since $A \cup B \cup C = F'$, at least one of them has $\lfloor n/3 \rfloor$ elements; moreover, if not all their cardinalities are the same, then at least one of them has $\lfloor n/3 \rfloor + 1$ elements and the result is proved. Assume then that A, B, and C have the same cardinality. Since $j < n$, then at least one of the sets S_i, without loss of generality say S_1, contains more than one element $L'_a \in S_1$, $L'_a \neq L'_1$. It is now easy to see that L_a is separable from $A \cup \{L'_1\}$. □

4. Conclusions

The segments in the example J_m may be extended to semilines in such a way that they remain pairwise disjoint. This shows that the bound in Theorem 3 is tight. We think that the $\lceil (n-7)/4 \rceil$ lower bound given in Theorem 1 should be improved to something close to $n/2$. Like Alon *et al.* some of us believe that in any family F of n disjoint line segments there is one that can be separated from considerably more than $\lceil (n-1)/4 \rceil$; perhaps from close to $n/3$ segments. Unlike them, some of us think that the $\lceil (n-1)/4 \rceil$ bound cannot be substantially improved.

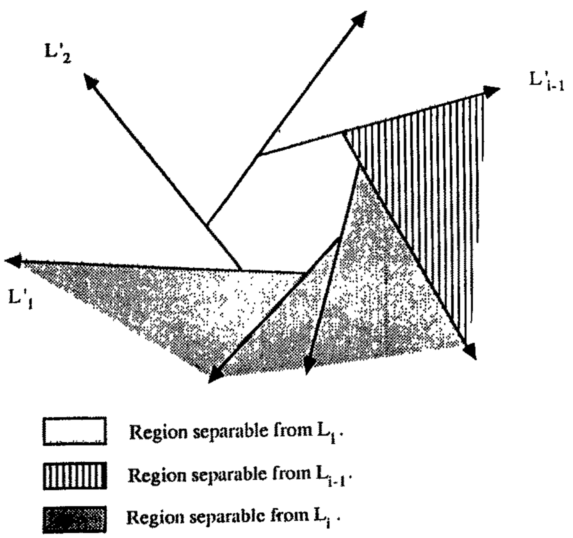

Fig. 6

References

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2. K. Hope and M. Katchalski, Separating plane convex sets, *Math. Scand.*, **66** (1990), 44–46.
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