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# The Euler Characteristic is the Unique Locally Determined Numerical Homotopy Invariant of Finite Complexes

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**Abstract.** If a numerical homotopy invariant of finite simplicial complexes has a local formula, then, up to multiplication by an obvious constant, the invariant is the Euler characteristic. Moreover, the Euler characteristic itself has a unique local formula.

## 1. Introduction

The Euler characteristic $\chi$ is the best known as well as the most ancient topological invariant. For a finite simplicial complex $K$ (or, more generally, a C-W complex) there is the familiar definition

$$\chi(K) = \sum_{i=0}^{\dim K} (-1)^i c_i,$$

where $c_i$ = number of $i$-simplices (or $i$-cells if $K$ is a C-W complex.) That $\chi(K)$ is an invariant of homotopy type follows from the alternative definition

$$\chi(K) = \sum_{i=0}^{\dim K} (-1)^i \text{ rank } H_i(K; \mathbb{Z}).$$

It is well known and easily verified that $\chi(K)$ is *locally determined* in the sense that given $K$, we may assign to each vertex $v \in K$ a rational number $e_1(v)$ such that $\chi(V) = \sum_v e_1(v)$. Here, $e_1(v)$ depends only on the simplicial structure of star

$v = \bigcup_{v\in\sigma} \sigma$ ($\sigma$ a simplex of $K$) and is given by

$$e_1(v) = \sum_i \frac{(-1)^i}{i+1} \cdot s_i(v),$$

where $s_i(v)$ is the number of $i$-simplices of $K$ containing $v$. Since star $v$ is, simplicially, the cone $c$(link $v$), we may think of $e_1(v)$ as an invariant of the simplicial isomorphism type of link $v = \bigcup_{v\notin\sigma\subset \text{star}\, v} \sigma$, i.e.,

$$e_1(v) = e(\text{link } v) = 1 + \sum_{i=0}^{\dim \text{link } v} \frac{(-1)^{i+1}}{i+2} \cdot (\text{number of } i\text{-simplices of link } v).$$

Of course, there are countless other $\mathbb{Z}$-valued (or $\mathbb{R}$-valued) invariants of finite complexes of finite complexes. It seems natural to ask whether any of these, other than the Euler characteristic, is locally determined in this sense. Specifically, let $\rho$ denote any $\mathbb{R}$-valued homotopy invariant of finite complexes. We always assume, by way of normalization, that $\rho(\emptyset) = 0$. Consider a real-valued function $d(L)$ defined on the set of finite simplicial complexes and depending only on the simplicial isomorphism type of $L$. We say that $\rho$ is *locally determined* by $d$ if and only if given any finite simplicial complex $K$ we have $\rho(K) = \sum_{v\in K} d(\text{link } v)$, where the sum is taken over the vertices $v$ of $K$. Clearly, the example we have in mind is the Euler characteristic $\chi$, locally determined by $e$ as above, and our question is whether there are any other numerical homotopy invariants (in a nontrivial sense) which are locally determined. The answer turns out to be negative.

**Theorem A.** *Let $\rho$ be any $\mathbb{R}$-valued homotopy invariant of finite complexes locally determined by some function $d$ on simplicial-isomorphism classes of finite complexes. Then $\rho = \rho(\text{pt.}) \cdot \chi$.*

In other words, up to multiplication by a constant, $\chi$ is the unique locally determined homotopy invariant.

We prove Theorem A in the following form:

**Theorem A′.** *If $\rho$ is an $\mathbb{R}$-valued homotopy invariant of finite complexes locally determined by $d$ and such that $\rho(\text{pt.}) = 1$, then $\rho \equiv \chi$.*

Theorem A obviously implies Theorem A′ and is, in turn, implied by it for the following reason: Let $\rho$ be as in the statement of Theorem A. If $\rho(\text{pt.}) \neq 0$, replace $\rho$ by $\rho' = \rho/\rho(\text{pt.})$ and apply Theorem A′ to conclude $\rho' = \chi$, hence $\rho = \rho(\text{pt.}) \cdot \chi$. If, however, $\rho(\text{pt.}) = 0$ let $\rho' = \rho + \chi$. Applying Theorem A′ to $\rho'$, we have $\rho' = \chi$ hence $\rho = 0 = \rho(\text{pt.}) \cdot \chi$.

The author is indebted to the referee for pointing out that the techniques below will, in fact, lead to a somewhat stronger result.

Consider compact PL $n$-manifolds (not necessarily closed). Let $\rho$ now denote a real-valued PL-homeomorphism invariant of such manifolds. Let $d$ be a real-valued function defined on triangulations of $S^{n-1}$ and $D^{n-1}$. Then the notion of $\rho$ being locally determined by $d$ transcribes, in an obvious way, to this context from the definition given above. Corresponding to Theorem A′ we have

**Theorem A″.** *If $\rho$ is an $\mathbb{R}$-valued invariant of compact PL n-manifolds with $\rho(\varnothing) = 0$, $\rho(D^n) = 1$ and $\rho$ is locally determined by some function d, then $\rho = \chi$.*

Note that Theorem A″ does, in fact, imply Theorem A′. Let $\rho$ be a numerical homotopy invariant of finite complexes with $\rho(\varnothing) = 0$, $\rho(\text{pt.}) = 1$. Then, for any $n$, $\rho$ is, *a fortiori*, a PL-homeomorphism invariant of compact PL $n$-manifolds with $\rho(D^n) = 1$. If $\rho$ is locally determined, then Theorem A″ tells us that $\rho(M^n) = \chi(M^n)$ for compact PL manifolds $M^n$ ($n$ arbitrary). However, given a finite complex $K$, there exists a compact manifold $M^n$ with $K$ homotopically equivalent to $M^n$ (see, e.g., [W1]). Hence $\rho(K) = \rho(M^n) = \chi(M^n) = \chi(K)$.

The observation above notwithstanding, we shall, in the interest of simplicity of exposition, prove Theorem A′ directly first and then show how Theorem A″ follows by a straightforward modification of the proof.

If we now go on to ask how many functions $d$, in addition to the $e$ given above, locally determine $\chi$, we find, in fact, that an even greater degree of rigidity prevails than is asserted by Theorem A. Not only is $\chi$ the only locally determined homotopy invariant which evaluates to 1 on a point but, as well, there is only one function, namely $e(L)$, which determines it. We rephrase this:

**Theorem B.** *If $\chi$ is locally determined by d, then $d = e$.*

Some remarks before we proceed to the proofs: If we examine more restricted classes of finite complexes, Theorem A no longer holds. For instance, if we look at the class of triangulated, oriented closed $4k$-manifolds $M$, then the signature of $M$, certainly an invariant of orientation-preserving homotopy type within this class of spaces, is locally determined by a function defined on simplicial-isomorphism classes of triangulated, oriented $(4k-1)$-spheres. This is a special case of the fact that rational Pontrjagin classes (as well as other PL characteristic classes) are locally determined. See [C], [L], [LR], and [GM] for details.

Moreover, as the referee has astutely pointed out, Theorem B fails as well in the context of closed PL $n$-manifolds. Given a combinational triangulation of such a manifold $M$, let $c_i$ denote the number of $i$-simplices. Klee [K] discovered a family of algebraic relations among the $c_i$ valid for all $M$ and Wall [W2] showed this list to be exhaustive. In consequence, there are nonstandard formulae for $\chi(M)$ in terms of $c_i$ differing from the classical formula cited in the first paragraph. For example (and, once more, the author is indebted to the referee for the observation),

$$\chi(M^n) = c_0 + \sum_{j=1}^{[(n+1)/2]} (-1)^j 2B_{2j} c_{2j-1},$$

where $B_{2j}$ is the $2j$th Bernoulli number. From this it immediately follows that if we set

$$d(L) = 1 + \sum_{j=0}^{[(n-1)/2]} \frac{(-1)^{j+1}}{j+1} B_{2j+2} \text{ (number of } 2j \text{ simplices of } L),$$

then $d$ locally determines $\chi$ on closed $n$-manifolds.

A word concerning notation: We abbreviate star $v$ by st $v$ and link $v$ by lk $v$. If there is any ambiguity as to which ambient complex $K$ is under consideration, we resolve this by recourse to subscripts, e.g. $\mathrm{lk}_K\, v$, $\mathrm{st}_K\, v$, etc.

Finally, we note that our proof applies to $\mathbb{C}$-valued or $\mathbb{Q}$-valued invariants as well.

## 2. Proof of Theorem A′

Theorem A′ follows immediately from:

**Lemma 1.** *Let $\rho$ be a numerical homotopy invariant of finite complexes locally determined by some function $d$. Let $K$ be a finite simplicial complex with $K = K_0 \cup K_1$, $K_0 \cap K_1 = K_2$ where $K_0$, $K_1$, $K_2$ are subcomplexes of $K$. Then $\rho(K) = \rho(K_0) + \rho(K_1) - \rho(K_2)$.*

The derivation of Theorem A′ from Lemma 1 comes via a straightforward induction. Given Lemma 1 and the hypothesis that $\rho(\text{pt.}) = 1$, it is immediate that for a 0-complex (i.e., discrete finite set) $K$, $\rho(K) =$ number of points of $K = \chi(K)$. So assume, inductively, that $\rho = \chi$ holds for complexes of dimension $\leq k$ and for $(k+1)$-complexes having $\leq j$ $(k+1)$-simplices. Let $K$ be a $(k+1)$-dimensional complex with exactly $(j+1)$ $(k+1)$-simplices. Choose a $(k+1)$-simplex $\sigma$. Let $K_0 = \sigma$, $K_1 = K\text{-int}\,\sigma$, so that $K_2 = K_0 \cap K_1$ is a $k$-sphere. Then

$$\rho(K) = \rho(K_0) + \rho(K_1) - \rho(K_2) = 1 + \chi(K_1) - \chi(S^k) = \chi(K_1) + (-1)^{k+1} = \chi(K).$$

To prove Lemma 1, in turn, it is technically convenient to consider finite *regular cell complexes* in addition to the more special category of finite simplicial complexes. (See [SCF] for definitions and basic properties of regular cell complexes.) Let $d_1$ be a function defined on pairs $(J, p)$ where $J$ is a regular cell complex which is the union of cells all containing the vertex $p$. It is understood that $d_1$ depends only on the isomorphism class of $(J, p)$ as a regular cell–complex pair. Thus in the case when $J$ happens to be simplicial, we see that $d_1$ depends only on the simplicial isomorphism class of $(J, p)$ and thus, since $J$ will in this instance be $c(\mathrm{lk}\, p)$, only on the simplicial isomorphism class of lk $p$. Consequently, $d_1$ may be viewed as an extension of a function $d(L)$, defined on simplicial complexes of the sort we have heretofore been considering (that is, $d(L) = d_1(cL, *)$). We say that $d_1$ determines an $\mathbb{R}$-valued homotopy invariant $\rho$ of finite regular cell-complexes if and only if $\rho(K) = \sum_v d_1(\text{st } v, v)$ where the sum is taken over the vertices $v$ of $K$ and where $\text{st}(v)$ is now understood to mean the union of all those cells of $K$ containing $v$.

**Lemma 2.** *If $d$ (defined on simplicial complexes) locally determines $\rho$ on simplicial complexes, then there is an extension of $d$ to regular cell-complex pairs $(J, p)$ as above which locally determines $\rho$ on regular cell-complexes.*

(Of course, it is understood that since regular cell-complexes are triangulable—in fact the first barycentric subdivision is a simplicial complex—the invariant $\rho$ automatically extends to regular cell-complexes.)

The proof of Lemma 2 is quite straightforward. Given $(J, p)$ let $K$ be the first barycentric subdivision of $J$, and hence a simplicial complex. Define $d_1(J, p)$ as follows: let $e$ be a cell of $J$, $b_e$ its barycenter, hence a vertex of $K$. Let $V(e)$ denote the number of vertices of the regular cell $e$. Then set $d_1(J, p) = \sum_e (1/v(e))\, d(\text{lk}_K\, b_e)$ where the sum is taken over all cells of $J$. The assertion that $d_1$ must locally determine $\rho$ on regular cell complexes is an immediate consequence of this definition.

We now proceed to the proof of Lemma 1. Assume that $d$, which locally determines $\rho$ on simplicial complexes, has been extended to $d_1$, which locally determines $\rho$ on regular cell-complexes. Let $M$ be a finite regular cell-complex and let $I$ denote, as usual, the unit interval as a simplicial complex with one 1-simplex $[0, 1]$ and two vertices 0 and 1. $M \times I$ is then well defined as a regular cell–complex without need of further subdivision. If $M$ has vertices $v_1, \ldots, v_k$, then $M \times I$ has vertices $u_1, \ldots, u_k, w_1, \ldots, w_k$ where $u_i = (v_i, 0)$, $w_i = (v_i, 1)$:

**Lemma 3.**

$$\sum_{i=1}^{k} d_1(\text{st}_{M\times I}\, u_i, u_i) = \sum_{i=1}^{k} d_1(\text{st}_{M\times I}\, w_i, w_i)$$
$$= \tfrac{1}{2} \sum_{i=1}^{k} d_1(\text{st}_M\, v_i, v_i) = \tfrac{1}{2}\rho(M).$$

*Proof.* $(\text{st}_{M\times I}\, u_i, u_i)$ is isomorphic as a regular cell-complex pair to $(\text{st}_{M\times I}\, w_i, w_i)$, thus it is immediate that

$$\sum_{i=1}^{k} d_1(\text{st}_{M\times I}\, u_i, u_i) = \sum_{i=1}^{k} d_1(\text{st}_{M\times I}\, w_i, w_i).$$

But

$$\sum_{i=1}^{k} d_1(\text{st}_{M\times I}\, u_i, u_i) + \sum_{i=1}^{k} d_1(\text{st}_{M\times I}\, w_i, w_i) = \rho(M \times I) = \rho(M) = \sum_{i=1}^{k} d_1(\text{st}_M\, v_i, v_i),$$

which yields the remainder of the lemma. □

Now let $I'$ denote the first subdivision of $I$ with two 1-simplices $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ and three vertices $0, \frac{1}{2}, 1$. With $M$, $v_i$ as above, $M \times I'$ is a regular cell-complex with vertices $u_i = (v_i, 0)$, $w_i = (v_i, 1)$, $x_i = (v_i, \frac{1}{2})$. Clearly, $(\mathrm{st}_{M \times I'} u_i, u_i)$ is isomorphic as a regular cell-complex pair to $(\mathrm{st}_{M \times I} u_i, u_i)$ and similarly for $w_i$. This observation leads to:

**Lemma 4.** $\sum_{i=1}^{k} d_1(\mathrm{st}_{M \times I'} x_i, x_i) = 0.$

*Proof.*

$$\rho(M \times I') = \sum_{i=1}^{k} d_1(\mathrm{st}_{M \times I'} u_i, u_i) + \sum_{i=1}^{k} d_1(\mathrm{st}_{M \times I'} w_i, w_i) + \sum_{i=1}^{k} d_1(\mathrm{st}_{M \times I'} x_i, x_i).$$

Also

$$\rho(M \times I') = \rho(M \times I) = \sum_{i=1}^{k} d_1(\mathrm{st}_{M \times I} u_i, u_i) + \sum_{i=1}^{k} d_1(\mathrm{st}_{M \times I} w_i, w_i).$$

By the remarks immediately preceding the statement of the lemma, the two summands on the right-hand side of the second equation are respectively equal to the first two summands in the right-hand side of the second. Hence the remaining summand, namely $\sum_{i=1}^{k} d_1(\mathrm{st}_{M \times I'} x_i, x_i)$, must vanish. □

Now we complete the proof of Lemma 1. Let $K = K_0 \cup K_1$ be a simplicial complex with $K_0 \cap K_1 = K_2$. We construct a homotopy-equivalent regular cell-complex $K_0 \cup (K_2 \times I') \cup K_1 = B$ where $K_0$, $K_1$ are now disjoint and $K_2 \times 0$ is identified with the copy of $K_2$ in $K_0$ and $K_2 \times 1$ with the copy of $K_2$ in $K_1$. Let $B_0 = K_0 \cup (K_2 \times [0, \frac{1}{2}])$, $B_1 = K_2 \times [\frac{1}{2}, 1] \cup K_1$. Thus $B_0 \cap B_1 = K_2 \times \frac{1}{2}$. We denote the vertices of $K_2$ by $v_1, \ldots, v_k$. Thus $K_2 \times \frac{1}{2}$ has vertices $x_i = (v_i, \frac{1}{2})$, $i = 1, \ldots, k$. Two elementary observations:

(i) $(\mathrm{st}_{B_1} x_i, x_i)$ is isomorphic (as a regular cell-complex pair) to $(\mathrm{st}_{K_2 \times I}(v_i, 0), (v_i, 0))$ and likewise for $(\mathrm{st}_{B_2} x_i, x_i)$.

(ii) $(\mathrm{st}_B x_i, x_i)$ is identical with $(\mathrm{st}_{K_2 \times I'} x_i, x_i)$.

Now, since $d_1$ locally determines $\rho$, we have

$$\rho(K_0) = \rho(B_0) = \sum_{i=1}^{k} d_1(\mathrm{st}_{B_0} x_i, x_i) + Y,$$

where $Y$ involves only the stars of vertices of $B_0$ not in $K_2 \times \frac{1}{2}$. Likewise,

$$\rho(K_1) = \sum_{i=1}^{k} d_1(\mathrm{st}_{B_1} x_i, x_i) + Z,$$

where $Z$ involves only the stars of vertices of $B_1$ not in $K_2 \times \frac{1}{2}$. Thus, by Lemma 3 and observation (i) above, we see immediately that

$$\sum_{i=1}^{k} d_1(\mathrm{st}_{B_0} x_i, x_i) = \sum_{i=1}^{k} d_1(\mathrm{st}_{B_1} x_i, x_i) = \tfrac{1}{2}\rho(K_2).$$

On the other hand, it is directly seen that $\rho(K) = \rho(B) = \sum_{i=1}^{k} d_1(\mathrm{st}_{B_1} x_i, x_i) + Y + Z$. However, $\sum_{i=1}^{k} d_1(\mathrm{st}_B x_i, x_i)$ vanishes by virtue of observation (ii) and Lemma 4. So $\rho(K) = Y + Z = [\rho(K_0) - \frac{1}{2}\rho(K_2)] + [\rho(K_1) - \frac{1}{2}\rho(K_2)] = \rho(K_0) + \rho(K_1) - \rho(K_2)$. The proof of Lemma 1, and hence of Theorems A and A′, is thus complete.

The kindred result Theorem A″ is easily established by a slightly modified version of this reasoning. The key point is the following lemma, analogous to Lemma 1.

**Lemma 5.** *Let $\rho$ be a locally determined numerical PL-homeomorphism invariant for compact PL n-manifolds. Let $K$ be a compact PL n-manifold of the form $K = K_0 \cup K_1$ where $K_0, K_1$ are themselves compact n-manifolds and where $K_2 = K_0 \cap K_1$ is a codimension 0 submanifold of both $\partial K_0$ and $\partial K_1$. Then $\rho(K) = \rho(K_0) + \rho(K_1) - \rho(K_2 \times I)$.*

First, we see quite easily that Lemma 5 implies Theorem A″. For a compact PL $n$-manifold $M$, let $h(M)$ denote the dimension of the highest dimensional handle in a handlebody-decomposition of $M$ where this highest dimension is minimal (with respect to all possible handlebody structures). Our proof runs by induction on $h(M)$.

If $h(M) = 0$, then $M$ is the disjoint union of some finite number $m$ of $n$-disks, whence $\rho(M) = m = \chi(M)$.

Now suppose A″ holds for all compact manifolds $M$ with $h(M) \leq j < n$. Consider $M_1$ with $h(M_1) = j + 1$. Then $M_1 = M \cup ((j+1)\text{-handles})$ where $h(M) \leq j$, whence $\rho(M) = \chi(M)$ by inductive assumption. Let $N$ denote the union of all the $(j + 1)$-handles of $M_1$, i.e., if there are $m$ such handles, $N$ is the disjoint union of $m$ $n$-disks, and so $\rho(N) = m = \chi(N)$. Let $L = M \cap N \subseteq \partial M, \partial N$. $L$ is the disjoint union of $m$ copies of $S^j \times D^{n-j-1}$, hence $h(L \times I) = j$ and $\rho(L \times I) = \chi(L \times I) = m\,(1 + (-1)^j)$. By lemma 5,

$$\begin{aligned}\rho(M_1) &= \rho(M) + \rho(N) - \rho(L \times I) = \chi(M) + \chi(N) - \chi(L \times I)\\ &= \chi(M) + \chi(N) - \chi(L) = \chi(M_1).\end{aligned}$$

The induction is thus complete.

As for Lemma 5 itself, we note that the argument for Lemma 1 goes through almost word for word. Note that lemma 2 holds in the context of regular cell-complex decompositions of PL manifolds. The analogue of Lemma 3 holds where $M$ is now a compact triangulated $(n - 1)$-manifold. The modified result reads

$$\sum_{i=1}^{k} d_1(\mathrm{st}_{M\times I}\, u_i, u_i) = \sum_{i=1}^{k} d_1(\mathrm{st}_{M\times I}\, w_i, w_i) = \tfrac{1}{2}\rho(M \times I).$$

Lemma 4 holds as well in this context.

The computations leading to Lemma 1 now serve equally well for Lemma 5. Here it need only be observed that if $K$ is a compact PL $n$-manifold decomposed as $K_0 \cup K_1$ (with $K_2 = K_0 \cap K_1$ a codimension-0 submanifold of $\partial K_0$ and $\partial K_1$), then $B$ (as defined in the proof of Lemma 1) is now a compact PL $n$-manifold PL homeomorphic to $K$ while $B_0$, $B_1$ are PL manifolds homeomorphic, respectively, to $K_0$ and $K_1$. The proof then goes through substituting $\rho(K_2 \times I)$ for $\rho(K_2)$ as appropriate.

## 3. Proof of Theorem B

As noted in the introduction, the Euler characteristic $\chi$ is locally determined by

$$e(L) = 1 + \sum_{i=0}^{\dim L} \frac{(-1)^{i+1}}{i+2} \text{ (number of } i\text{-simplices of } L).$$

We now show that no other function on simplicial isomorphism classes of finite complexes can locally determine $\chi$.

To this end, let $d$ be some other function for which $\chi(K) = \sum_v d(\mathrm{lk}\ V)$ for all finite simplicial complexes $K$. Given any simplicial complex $J$ with $\sigma$ a simplex, we call $\sigma$ a *maximal* simplex of $J$ if and only if $\sigma$ is not a face of any larger simplex. Note that any finite complex is the union of its maximal simplices.

Our proof that $d(L) \equiv e(L)$ proceeds via induction on the number of maximal simplices of $L$. In the case where $L$ has but one maximal simplex, it is clear that $L$ must be isomorphic to a standard simplex, say $\Delta^k$ for some $k \geq 0$. Consider $K = v * L$ for some disjoint vertex $v$. $v * L$ is of course isomorphic to $\Delta^{k+1}$ so $1 = \chi(v * L) = d(L) + \sum_{i=0}^{k} d(\mathrm{lk}_K\, v_i)$ (where $v_0, \ldots, v_k$ are the vertices of $L$) since $L = \mathrm{lk}_K\, v$. But $\mathrm{lk}_K\, v_i$ is obviously a $k$-simplex for $i = 0, \ldots, k$. So we have $d(\mathrm{lk}_K\, v_i) = d(L)$, whence $1 = (k+2)d(L)$, $d(L) = 1/(k+2) = e(L)$.

Now suppose $d(L) = e(L)$ for all $L$ having $\leq j$ maximal simplices. Let $L$ have $j+1$ maximal simplices and let $v_1, \ldots, v_r$ denote the vertices of $L$. Consider $K = v * L \cong cL$ where $v$ is a vertex distinct from $v_1, \ldots, v_r$. $L = \mathrm{lk}_K\, v$. Consider $\mathrm{lk}_K\, v_i$. This is clearly $v * \mathrm{lk}_L\, v_i \cong c\ \mathrm{lk}_L\, v_i$. Thus $\mathrm{lk}_K\, v_i$ is isomorphic to $\mathrm{st}_L\, v_i$. However, $\mathrm{st}_L\, v_i$ is clearly the union of maximal simplices of $L$. Hence $\mathrm{lk}_K\, v_i$ has $\leq j+1$ maximal simplices.

Now because $K$ is contractible, $1 = \chi(K) = d(L) + \sum_{i=1}^{r} d(\mathrm{lk}_K\, v_i)$. Segregate the $v_i$ into two classes, $u_i$, $i = 1, \ldots, s$, and $w_i$, $i = 1, \ldots, t$, with $s + t = r$, by the criterion that $\mathrm{lk}_K\, u_i$ has $\leq j$ maximal simplices whereas $\mathrm{lk}_K\, w_i$ has $j+1$ maximal simplices. Thus $d(\mathrm{lk}_K u_i) = e(\mathrm{lk}_K u_i)$ for $i = 1, \ldots, s$. Thus since

$$\begin{aligned}\chi(K) &= d(L) + \sum_{i=1}^{s} d(\mathrm{lk}_K u_i) + \sum_{i=1}^{t} d(\mathrm{lk}_K w_i)\\ &= e(L) + \sum_{i=1}^{s} e(\mathrm{lk}_K u_i) + \sum_{i=1}^{t} e(\mathrm{lk}_K w_i),\end{aligned}$$

we have

$$d(L) + \sum_{i=1}^{t} d(\mathrm{lk}_K w_i) = e(L) + \sum_{i=1}^{t} e(\mathrm{lk}_K w_i).$$

Note that $v$ and $w_i$, $i = 1, \ldots, t$, may be characterized as those vertices of $K$ which are common to all the maximal simplices of $K$. It follows that $v, w_1, \ldots, w_t$ are the vertices of $\tau = \bigcap_\alpha \sigma_\alpha$, where $\{\sigma_\alpha\}$ is the set of maximal simplices of $K$. For any pair of these vertices, it is clear that there is thus a simplicial automorphism of $K$ carrying the first into the second. In other words, $L = \mathrm{lk}_K v \cong \mathrm{lk}_K w_i$, $i = 1, \ldots, t$. Hence we see that $(t + 1)d(L) = (t + 1)e(L)$ whence $d(L) = e(L)$. This completes the proof.

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