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Toughness and Delaunay Triangulations*

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Abstract. We show that nondegenerate Delaunay triangulations satisfy a combinatorial property called 1-toughness. A graph G is *1-tough* if for any set P of vertices, $c(G - P) \leq |G|$, where $c(G - P)$ is the number of components of the graph obtained by removing P and all attached edges from G, and $|G|$ is the number of vertices in G. This property arises in the study of Hamiltonian graphs: all Hamiltonian graphs are 1-tough, but not conversely. We also show that all Delaunay triangulations T satisfy the following closely related property: for any set P of vertices the number of interior components of $T - P$ is at most $|P| - 2$, where an interior component of $T - P$ is a component that contains no boundary vertex of T. These appear to be the first nontrivial properties of a purely combinatorial nature to be established for Delaunay triangulations. We give examples to show that these bounds are best possible and are independent of one another. We also characterize the conditions under which a degenerate Delaunay triangulation can fail to be 1-tough. This characterization leads to a proof that all graphs that can be realized as polytopes inscribed in a sphere are 1-tough. One consequence of the toughness results is that all Delaunay triangulations and all inscribable graphs have perfect matchings.

1. Introduction

The connection between Delaunay triangulations and Hamiltonian graphs has been a question of some interest. In his thesis, Shamos posed a variant of the question by asking whether every Delaunay triangulation contained a traveling salesman cycle for its sites [45]. The answer to this question was shown to be negative in [19]. The question of when Delaunay triangulations are Hamiltonian

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has arisen in the contexts of pattern recognition and shape representation [38], [43]. Consider the problem of constructing a "reasonable" simple curve through a given planar set of points. One approach that has been suggested is to construct the Delaunay triangulation of the points, and then to construct a cycle through this triangulation either by "growing" a single triange [43] or by "sculpting" the convex hull [8]. These algorithms will be successful only if the Delaunay triangulation has a Hamiltonian cycle.

It has been shown [20], [22] that not all nondegenerate Delaunay triangulations are Hamiltonian.[1] A degenerate example was presented in [31]. Nevertheless, O'Rourke and Boissonnat both report that their algorithms appear to work in practice. In fact, Boissonnat has run a number of simulations with randomly generated point sets containing up to 2000 points, and all his examples have yielded Hamiltonian Delaunay triangulations [9]. Thus there is evidence that Delaunay triangulations are Hamiltonian with high probability. In this paper we establish two results that may partially explain this phenomenon.

The simplest way to construct a non-Hamiltonian triangulation is to draw a graph such as the one in Fig. 1.1(a), and then to argue as follows. Suppose there were a Hamiltonian cycle. Every time the cycle passed from one of the dark vertices to another, it would have to pass through one of the light vertices (A, B, and C), visiting a different one each time. Since there are four dark vertices, and only three light vertices, this is impossible, so the graph must be non-hamiltonian. An argument essentially similar to this has been used to prove the non-Hamiltonicity of triangulations arising in several different contexts [5], [31], [36], [47]. The key property used in the preceding argument, that of *1-toughness*, was first identified by Chvátal [13]. A graph is *1-tough* if, for any k, removing k vertices splits the graph into at most k components. It is easy to show that any Hamiltonian graph is 1-tough, essentially by the above argument. The converse is not true, even for triangulations. Examples of non-Hamiltonian, 1-tough maximal planar graphs appear in [22], [23], and [40]. The connection between Hamiltonicity and toughness is discussed in [7], [13] and [14].

In Section 3 of this paper we show that all nondegenerate Delaunay triangulations are 1-tough (Theorem 3.2). This result partially explains why constructing non-Hamiltonian Delaunay triangulations has been somewhat difficult. It is, to our knowledge, the first property of a purely combinatorial (graph-theoretical) nature to be established for Delaunay triangulations.

For any value of k, it is easy to construct a triangulation with the property that removing k vertices splits it into $2k - 2$ components. The construction is as follows. Start with any triangulation T having k vertices, and insert new vertices, one inside each triangle and one just outside each segment of the convex hull. Add segments to T to create a new triangulation, U. It follows from Euler's formula that $2k - 2$ new vertices have been added. If the k vertices of the original triangulation are removed from U, each of the new vertices will become a separate component. This construction, which is illustrated for $k = 5$ in Fig. 1.1(b), shows that general triangulations can fail rather badly to be 1-tough.

[1] A Delaunay triangulation is nondegenerate if there is no ambiguity in its construction, degenerate otherwise. A more precise definition is given in the next section.

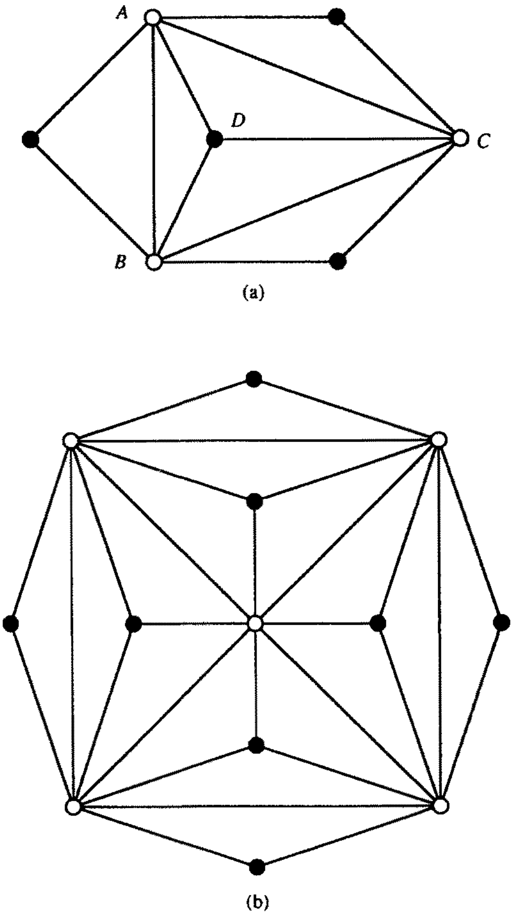

Fig. 1.1. Examples of triangulations that are not 1-tough. (a) Removing A, B, and C splits this graph into four components. (b) An example of a triangulation that can be split into $2k - 2$ components by removing k vertices, for $k = 5$.

Related to the 1-toughness result is Theorem 3.1, which says that removing k vertices can split a Delaunay triangulation into at most $k - 2$ components that do not contain a boundary vertex of the triangulation. In Section 4 we give examples to show that neither this result nor the 1-toughness result implies the other, and that neither result can be sharpened.

In Section 5 we show that when the nondegeneracy assumption is removed, 1-toughness "almost" still holds, and we can characterize the circumstances under which it fails (Therorem 5.3). This leads to a proof, in Section 6, that graphs inscribable in a sphere are 1-tough. In Section 7 we show that all Delaunay triangulations and inscribable graphs have perfect matchings. In the final section we discuss some open problems.

Much has been written in recent years about the efficient construction and applications of Delaunay triangulations and Voronoi diagrams. References to this literature can be found in the books [26] and [44] and the survey paper [34]. More closely related to the spirit of this paper is research directed at proving geometric properties of Delaunay triangulations. Shamos' thesis [45] discussed the applicability of Delaunay triangulations and Voronoi diagrams to many problems in computational geometry and raised several interesting questions. The relationships among the greedy triangulation, the minimum-weight triangulation, and the Delaunay triangulation are examined in [12], [33] and [35]–[37]. It is shown in [1] that a Delaunay triangulation need not contain a minimum-weight perfect matching. Geometric characterizations of Voronoi diagrams and some generalizations are given in [2], [3] and [28]. Dobkin *et al.* have shown that the shortest path connecting two vertices in a Delaunay triangulation is at most a constant multiple of the Euclidean distance between them [25]; recently, Keil and Gutwin have improved the constant [32]. De Floriani *et al.* have shown [17] that given any point in a Delaunay triangulation, it is possible to order the triangles so that any ray from the point intersects the triangles in increasing order. Edelsbrunner [27] has generalized this property to d dimensions.

While the concept of toughness was initially introduced primarily because of its connection with Hamiltonicity, it has recently been of interest as a measure of vulnerability of a network [4]. Thus Theorem 3.2 can be viewed as a statement about the fault-tolerance properties of networks configured as Delaunay triangulations, complementing the results bounding path lengths in such networks in [25] and [32]. Bauer *et al.* have shown that recognizing 1-tough graphs is NP-hard [6]. Dawes and Rodrigues have shown [16] that all k-connected, k-regular graphs are 1-tough.

2. Mathematical Preliminaries

The number of elements of a set S is denoted by $|S|$. Except as noted, we use the same graph-theoretical terminology as [10]. For a graph G, $V(G)$ represents the set of vertices of G, $|G| = |V(G)|$, and $c(G)$ represents the number of components of G. If $P \subseteq V(G)$, the subgraph of G *induced* by P, denoted $G[P]$, is the graph with vertex-set P in which two vertices are joined by an edge if they are joined by an edge in G. The graph $G - P$ is obtained from G by removing the set P (and all edges with at least one endpoint in P).

A *walk* in G is a sequence of two or more (not necessarily distinct) vertices such that each pair of consecutive vertices is connected by an edge. A *path* is a walk in which all vertices are distinct. A *cycle* is a walk $p_0 \cdots p_n$ in which $p_0 = p_n$ but the vertices are otherwise distinct. A *Hamiltonian cycle* in a graph G is a cycle that visits each vertex of G. A graph is *Hamiltonian* if it contains a Hamiltonian cycle. A graph G is *t-tough* [13] if, for any $P \subseteq V(G)$, $t \cdot c(G - P) \leq |P|$. In particular, G is *1-tough* if, for any $P \subseteq V(G)$, $c(G - P) \leq |P|$.

A *planar graph* is a graph that can be drawn in the plane so that its vertices are points of the plane and such that if two edges intersect, they do so only at a common endpoint. A *plane graph* is a graph that is already drawn in such a fashion. By Fary's theorem (see, for example, [41]), there is no loss of generality in

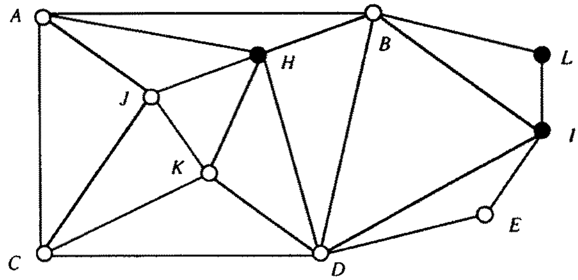

Fig. 2.1. Classification of edges and components in plane graphs.

assuming that all edges in a plane graph are line segments. A plane graph separates its complement (i.e., the rest of the plane) into regions, which are called *faces*. The unique unbounded face is called the *exterior face*, and all other faces are called *interior faces*. Two plane graphs are *combinatorially equivalent* if there is a bijection between the vertices, edges, and faces which preserves all incidence relations and the identity of the exterior face. If v is the number of vertices in a plane graph, e the number of edges, and f the number of interior faces, then *Euler's formula* asserts that in a connected plane graph, $e = v + f - 1$.

The set of all edges that are incident with a face is called the *boundary* of the face. (Notice, since a face is a subset of the complement of the graph, it does not contain its boundary.) The boundary of the exterior face is called the *boundary* of the graph. An edge is called an *exterior edge* if it is only incident to the exterior face; a *boundary edge* if it is incident to the exterior face and an interior face; and an *interior edge* if it is incident only to interior faces (which may or may not be distinct). If $P \subseteq V(G)$, a component of $G - P$ is an *interior component* if it does not contain any vertex on the boundary of G and a *boundary component* if it contains at least one vertex on the boundary of G. For example, if G is the graph of Fig. 2.1, BD is an interior edge of G, AC is a boundary edge, and G has no exterior edges. If $P = \{A, B, C, D, E, J, K\}$, then $\{AJ, CJ, CK, DK, JK\}$ is the set of interior edges of $G[P]$, $\{AB, AC, BD, CD\}$ is the set of boundary edges of $G[P]$, and DE is the only exterior edge of $G[P]$. Here $\{H\}$ is an interior component of $G - P$ and $\{I, L\}$ is a boundary component of $G - P$.

A 2-connected plane graph T is called a *triangulation* if every interior face is bounded by a triangle and the boundary of T is a convex polygon. A triangulation T is called a *maximal planar graph* if, in addition, the boundary of T is a triangle. An *elementary triangle* is a triangle that is the boundary of a face. For example, in Fig. 1.1(a), ABD is an elementary triangle but ABC is not. If XYZ is an elementary triangle, we say that vertex Y, or angle XYZ, is *opposite* the edge XZ.

Let S be a set of distinguished points in the plane, which we call sites. We assume throughout this paper than not all sites are collinear.[2] For each $s \in S$, the *Voronoi*

[2] This assumption is merely a convenience. If we consider the collinearity of all points of S to be a form of degeneracy, then Theorem 5.3 still holds. Moreover, if such a set S arises in the proof of Theorem 6.2, the argument given there still works.

region $V(s)$ generated by s is the set of points closer to s than to any other site. The collection of all Voronoi regions generated by sites of S is called the *Voronoi diagram* generated by S. The *Voronoi dual* of S is defined to be the straight-line geometric dual of the Voronoi diagram. If no more than three Voronoi regions meet at any point in the Voronoi diagram generated by S, then the Voronoi dual of S is a triangulation, called the *Delaunay triangulation* of S; in this case, the Delaunay triangulation and the Voronoi dual are said to be *nondegenerate*. An example nondegenerate Voronoi diagram and the corresponding Delaunay triangulation are shown in Fig. 2.2. If the Voronoi dual is *degenerate* (i.e. if more than

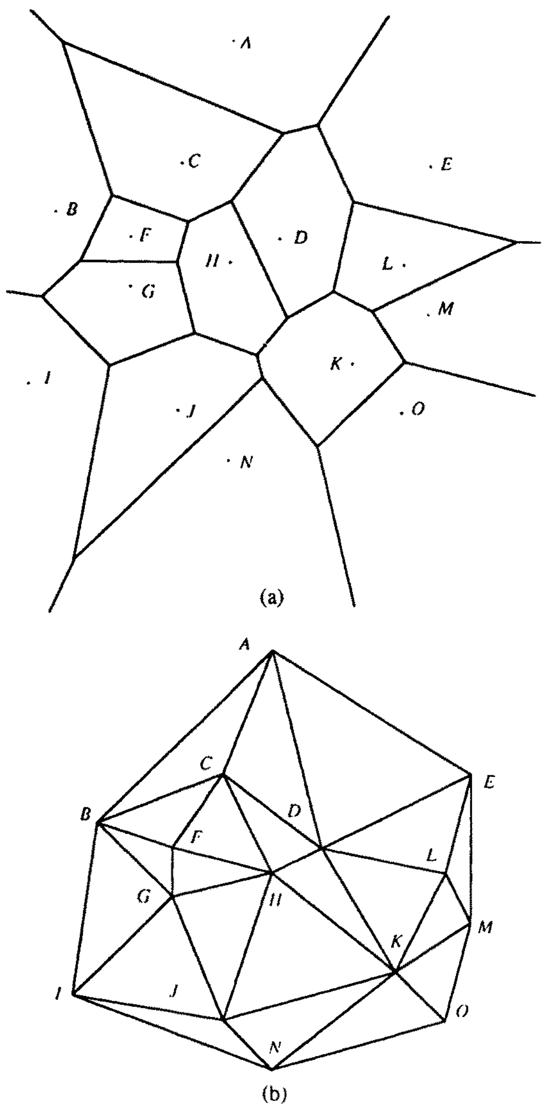

Fig. 2.2. (a) A Voronoi diagram. (b) The corresponding Delaunay triangulation.

three Voronoi regions intersect), then a Delaunay triangulation is any triangulation obtained by adding edges to the Voronoi dual. The following characterization of Delaunay triangulations follows easily from the "General Lemma" of [18]. (Here, and throughout the paper, $\mu(\cdot)$ represents the measure of an angle in degrees.)

Fact 2.1. A triangulation is a Delaunay triangulation if and only if, whenever ABC and ABD are two elementary triangles, the angles ACB and ABD satisfy the inequality

$$\mu(ACB) + \mu(ADB) \leq 180. \tag{2.1}$$

A Delaunay triangulation is nondegenerate iff all inequalities (2.1) are strict.

3. Toughness Conditions for Delaunay Triangulations

In this section we prove the following two theorems concerning the structure of Delaunay triangulations:

Theorem 3.1. *Let T be a Delaunay triangulation and let $P \subseteq V(T)$. Then $T - P$ has at most $|P| - 2$ interior components.*

Theorem 3.2. *Let T be a nondegenerate Delaunay triangulation and let $P \subseteq V(T)$. Then $T - P$ has at most $|P|$ components. In other words, T is 1-tough.*

Theorem 3.1 does not require nondegeneracy, while Theorem 3.2. does. In Section 5 we examine the circumstances under which a degenerate Delaunay triangulation can fail to be 1-tough. The conclusion of Theorem 3.1 is a property that depends only on the combinatorial structure of T, and 1-toughness of a graph is invariant under isomorphism (and hence under combinatorial equivalence). So both these theorems describe necessary properties that a triangulation must satisfy if it has a combinatorially equivalent realization as a Delaunay triangulation.

3.1. Proof of Theorem 3.1

Let T be a triangulation and let $P \subseteq V(T)$. We classify the interior of $T[P]$ into two types. Interior faces with no vertices of $T - P$ in their interior are called *good faces*, and interior faces with one or more vertices of $T - P$ in their interior are called *bad faces*. Good and bad faces are illustrated in Fig. 3.1(a) and (b). Each interior component of $T - P$ is contained in some bad face of $T[P]$. The proof of Theorem 3.1 proceeds in three stages. First we establish (Lemma 3.5) that if $T[P]$ is

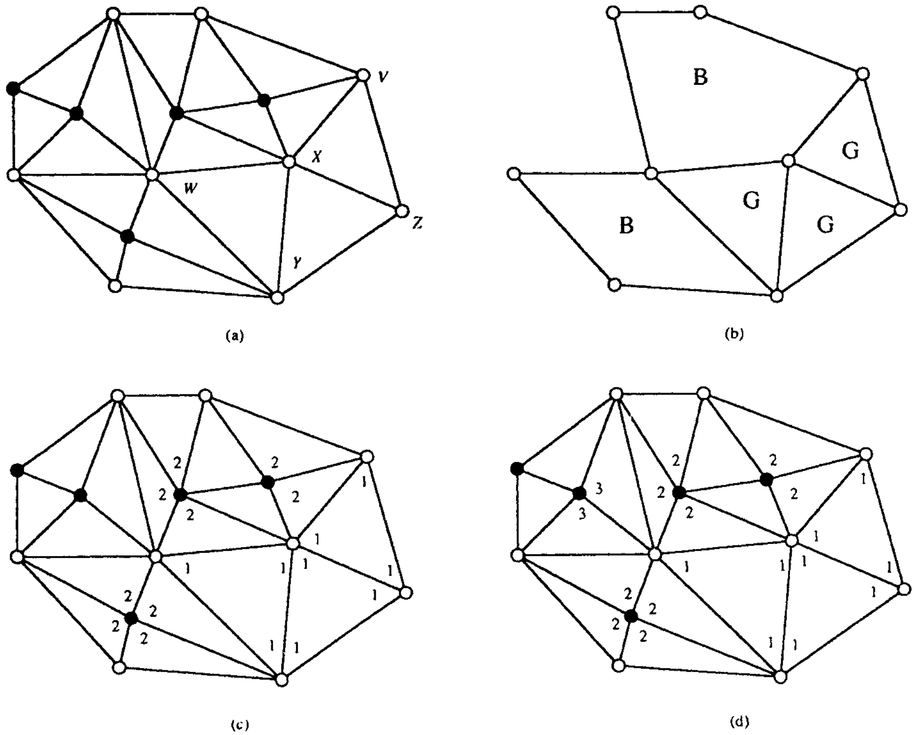

Fig. 3.1. Good and bad faces. (a) A triangulation T, with $P \subseteq V(T)$ indicated by light circles. (b) The good and bad faces of $T[P]$, labeled with G and B, respectively. (c) Type 1 and type 2 angles. (d) Type 1, type 2, and type 3 angles.

connected, then the number of interior components of $T - P$ is equal to the number of bad faces. We then show that in a Delaunay triangulation the number of bad faces of $T - P$ cannot exceed $|P| - 2$, provided $T[P]$ is connected (Lemma 3.8). Finally, we show that the requirement that $T[P]$ be connected is unnecessary.

Before beginning the proof we introduce some basic topological facts and definitions. Our discussion is informal; for a rigorous treatment of the topology of the plane see [39]. In general, the boundary of a face is not necessarily a cycle, nor need it be connected (Fig. 3.2). A *counterclockwise traversal* (resp. *clockwise traversal*) of a component of a boundary of a face F is a closed walk such that the edge $p_1 p_2$ appears in the walk if and only if (1) p_1 and p_2 belong to that component of the boundary of F, and (2) F is the face to the left (resp. right) of the directed edge $p_1 p_2$. For example, the three components of the boundary of face 2 in Fig. 3.2, expressed as counterclockwise traversals, are $RHIR$, JKJ, and $BQALAB$. A *simple face* of a plane graph is a face with the property that, for any closed curve lying entirely within the face, either all vertices of the graph are inside the curve or all vertices of the graph are outside the curve. In the graph of Fig. 3.2, faces 1, 3, and 4 and the exterior face are simple, but face 2 is not simple.

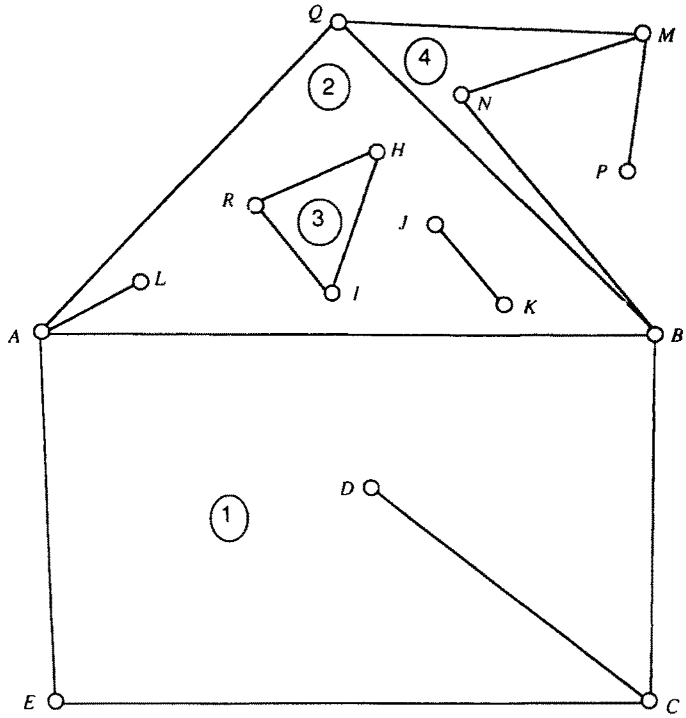

Fig. 3.2. Faces in plane graphs

Let G be a plane graph, and let G_1 and G_2 be components of G. Suppose some face F of G has the following property: it is possible to draw a closed curve in F such that G_1 lies inside the curve, but it is not possible to draw a closed curve in F such that G_2 lies inside the curve. Then we say that G_2 *surrounds* G_1. For example, in Fig. 3.2, the component cntaining the vertices $\{A, B, C, D, E, L, M, N, P, Q\}$ surrounds the other two components. The topological properties of the plane needed in this section can be summarized as follows:

Fact 3.3. Let G be a plane graph. (a) If G is connected, all faces of G are simple. (b) The boundary of any simple face of G is connected. (c) If G_1 and G_2 are components of G and G_1 surrounds G_2, then G_2 does not surround G_1.

Let F be any interior face of a planar graph G. It follows immediately from Fact 3.3(c) that there is a unique component of the boundary of F that is not surrounded by any other component of the boundary of F. We call this component the *outer boundary* of F. Thus, in Fig. 3.2, $AQBALA$ is a clockwise traversal of the outer boundary of face 2. If $T[P]$ is connected, then it follows from Fact 3.3(a) and (b) that the boundary of any face is connected, so the outer boundary of a face is the boundary of the face in this case.

As we indicated above, our first goal is to establish that, in a triangulation, the number of interior components of $T - P$ is equal to the number of bad faces of $T[P]$. The key observation is the following lemma.

Lemma 3.4. *If T is a triangulation, $P \subseteq V(T)$, and $T[P]$ is connected, then any two vertices that lie inside (i.e., in the topological interior of) a common interior face of $T[P]$ can be connected by a path (in T, and hence in $T - P$) lying entirely inside that face.*

Proof. Let F be an interior face of T, and let w and x be two vertices lying inside F. Since T is connected, and the boundary of F is connected (by Fact 3.3(a) and (b)), it follows that w and x may be joined by a path Π that remains on or inside F. We must show that Π can be chosen to be strictly inside F. It suffices to establish the following claim: if y and z are two vertices inside F, each of which is adjacent to a vertex of T lying on the boundary of F, then there is a walk from y to z lying entirely inside F. To establish the claim, let v be a neighbor of y that is on the boundary of F, and consider the following algorithm:

```
p := y;
q := v;
repeat
   r := counterclockwise(p, q);
   if r is inside F then
      output edge (pr);
      p := r;
   else
      q := r;
   endif;
until p = z;
```

Here, the function *counterclockwise*(p, q) (resp. *clockwise*(p, q)) returns the unique vertex r such that qr is the first edge encountered after pq when moving counterclockwise (resp. clockwise) about q. The crucial observation about the algorithm is that since T is a triangulation and pq is not a boundary edge, *counterclockwise*(p, q) = *clockwise*(q, p), so the loop preserves these invariants:

(1) q is on the boundary of F,
(2) p is inside F,
(3) pq is an edge of T, and
(4) after the call to *counterclockwise*, pr and qr are edges of T.

Also, q runs through the vertices of a clockwise traversal of the boundary of F. It follows that p runs through the set of vertices of $T - P$ that are inside F and adjacent to vertices of $T[P]$ on the boundary of F. In particular, p ultimately takes on the value z, so the algorithm terminates and the output is a path from y to z consisting entirely of vertices inside F. This establishes the claim, and hence the lemma. □

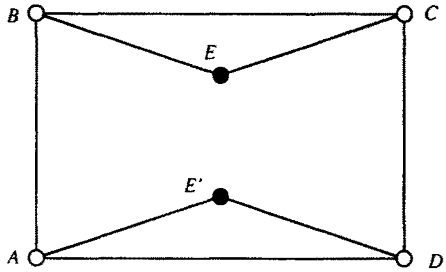

Fig. 3.3. In a graph that is not a triangulation, a single face of $T[P]$ can contain more than one component of $T - P$.

Lemma 3.5. *Suppose T is a triangulation, $P \subseteq V(T)$, and $T[P]$ is connected. Each bad interior face of $T[P]$ contains exactly one component of $T - P$. Furthermore, any good interior face of $T[P]$ is bounded by a triangle.*

Proof. Let F be an interior face of $T[P]$, and let p_1 and p_2 be two adjacent vertices of P such that the edge $p_1 p_2$ forms a portion of the boundary of F. Assume, without loss of generality, that F is to the left of $p_1 p_2$. Let $x = clockwise(p_1, p_2)$. If F does not contain a vertex of $T - P$ (i.e., if F is a good face), then $x \in P$, which implies that F is bounded by the triangle $p_1 p_2 x$ and thus proves the second assertion of the lemma. By Lemma 3.4, the vertices of $T - P$ inside F (if there are any) all belong to the same component of $T - P$, which proves the first assertion. □

Notice that Lemmas 3.4 and 3.5 can fail if T is not a triangulation. For example, let T be the graph of Fig. 3.3. If $P = \{A, B, C, D\}$, the face of $T[P]$ bounded by the cycle $ABCD$ contains two components of $T - P$, namely $\{E\}$ and $\{E'\}$, and there is no path from E to E' lying entirely inside this face. Lemmas 3.4 and 3.5 remain true if the requirement that $T[P]$ be connected is dropped, but we do not use this fact.

Lemma 3.5 implies that Theorem 3.1 will follow (for $T[P]$ connected) if we can bound the number of bad faces in a Delaunay triangulation. In order to do this, we associate with each good and bad face of $T[P]$ certain *distinguished angles*. The distinguished angles are those angles that are opposite an edge of $T[P]$. More precisely:

- For each good face (which must be bounded by a triangle, by Lemma 3.5), we distinguish the three internal angles of the triangle. We call these *type 1 angles*.
- For each bad face, we distinguish all angles of the form AXB, where AB is an edge of the face boundary, and X is a vertex of $T - P$ inside the face such that triangle AXB is an elementary triangle of T. We call these *type 2 angles*.

The type 1 and type 2 angles are illustrated in Fig. 3.1(c). Some important properties of the type 1 and type 2 angles are contained in the following two lemmas.

Lemma 3.6. *Let T be a triangulation with $P \subseteq V(T)$. Each interior edge of $T[P]$ is opposite two distinguished angles, and each boundary edge of $T[P]$ is opposite one distinguished angle.*

Proof. Let $p_1 p_2$ be any interior edge of $T[P]$. Then, by definition, the faces on either side of this edge are interior faces. Hence there are vertices on either side of the edge, say a and b, such that $p_1 a p_2$ and $p_1 b p_2$ are elementary triangles of T. If $a \in P$, then $p_1 a p_2$ is a type 1 angle, otherwise it is type 2, and similarly for angle $p_1 b p_2$. This proves the first half of the lemma. The proof of the second half is similar. □

Lemma 3.7. *If T is a triangulation, $P \subseteq V(T)$, and $T[P]$ is connected, the sum of the type 2 angles associated with a given bad face of $T[P]$ is at least 360°. Equality holds if and only if the face contains exactly one vertex of $T - P$ in its interior.*

Proof. (See Fig. 3.4.) Let F be any bad face of $T[P]$. Let $p_1 p_2 \cdots p_m p_1$ be a counterclockwise traversal of the boundary of F, and let r_i be the vertex to the left of $p_i p_{i+1}$ such that $p_i r_i p_{i+1}$ is an elementary triangle of T (where subscripts are taken modulo m). Since F is a bad face, each $r_i \in T - P$. Notice that the vertices $\{r_i\}$ and $\{p_i\}$ need not all be distinct, as illustrated in Fig. 3.4 where $r_3 = r_4$ and $p_4 = p_6$. A simple continuity argument combined with the well-known formula for the sum of the interior angles of polygons yields

$$\sum_{i=1}^{m} \mu(p_i p_{i+1} p_{i+2}) = 180(m-2). \tag{3.1}$$

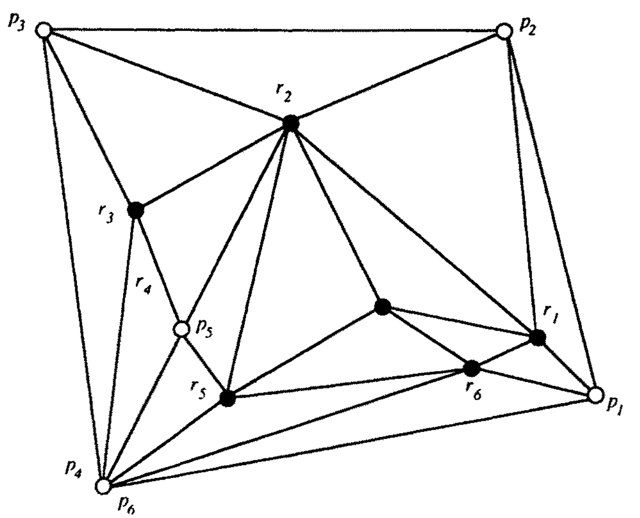

Fig. 3.4. Proof of Lemma 3.7.

For all i, if we move clockwise about p_{i+1} from the edge $p_{i+1}p_i$ to the edge $p_{i+1}p_{i+2}$, we encounter edge $p_{i+1}r_i$ either before or simultaneously with edge $p_{i+1}r_{i+1}$. Hence,

$$\mu(r_{i+1}p_{i+1}p_{i+2}) + \mu(r_i p_{i+1} p_i) \leq \mu(p_i p_{i+1} p_{i+2}). \tag{3.2}$$

By (3.1) and (3.2),

$$\begin{aligned}
\sum_{i=1}^{m} \mu(p_i r_i p_{i+1}) &= \sum_{i=1}^{m} \{180 - (\mu(r_i p_i p_{i+1}) + \mu(r_i p_{i+1} p_i))\} \\
&= \sum_{i=1}^{m} \{180 - (\mu(r_{i+1} p_{i+1} p_{i+2}) + \mu(r_i p_{i+1} p_i))\} \\
&\geq 180m - \sum_{i=1}^{m} \mu(p_i p_{i+1} p_{i+2}) \\
&= 180m - 180(m-2) \\
&= 360.
\end{aligned}$$

This proves the first statement. Equality holds in (3.2) if and only if $r_i = r_{i+1}$. Hence the sum of the type 2 angles is exactly 360° if and only if $r_i = r_{i+1}$ for all i, which proves the second statement. □

Lemma 3.8. *Let T be a Delaunay triangulation, let $P \subseteq V(T)$, and assume $T[P]$ is connected. Then $T - P$ can have at most $|P| - 2$ interior components.*

Proof. By Lemma 3.5, it is enough to show that $T[P]$ can have at most $|P| - 2$ bad faces. The proof is essentially a counting argument. We establish a lower bound on the total value of the distinguished angles in terms of the number of bad faces of $T[P]$. We then establish an upper bound on the same quantity, using the Delaunay triangulation characterization of Fact 2.1. By comparing these two bounds, we are able to derive the required bound on the number of bad faces.

Let g be the number of good faces, and let b be the number of bad faces. Let d denote the total measure of all distinguished angles. Each good face contributes three distinguished angles of total measure 180° (by Lemma 3.5), and each bad face contributes several distinguished angles of total measure at least 360° (by Lemma 3.7). Hence

$$d \geq 180 \cdot g + 360 \cdot b. \tag{3.3}$$

Let f be the number of interior faces of $T[P]$ (so $f = b + g$). Let e be the number of edges of $T[P]$. By Euler's formula, we have

$$e = |P| + f - 1 = |P| + b + g - 1. \tag{3.4}$$

Define two distinguished angles to be *paired* if they are opposite a common edge. A distinguished angle is *unpaired* if it is not paired with another distinguished angle. For example, in Fig. 3.1(a), angles XZY and XWY are paired with each other, while angle VXZ is unpaired. Since T is a Delaunay triangulation, the sum of two paired distinguished angles is at most 180°. Each unpaired angle is less than 180°, as it is an interior angle in a triangulation. Since only interior components of $T[P]$ are being considered, not all distinguished angles are paired, as any distinguished angle facing a boundary edge of $T[P]$ is unpaired. Since the distinguished angles are exactly those angles that face an edge of $T[P]$ (by Lemma 3.6), it follows that

$$d < 180 \cdot e. \tag{3.5}$$

Combining (3.3), (3.4), and (3.5) we have

$$180 \cdot g + 360 \cdot b < 180(|P| + b + g - 1),$$

which simplifies to

$$b < |P| - 1. \tag{3.6}$$

Since b and $|P|$ are integers, $b \leq |P| - 2$. □

To prove Theorem 3.1 we need only prove that Lemma 3.8 remains true if we drop the assumption that $T[P]$ is connected. The proof is by induction on the number of components of $T[P]$. Lemma 3.8 establishes the result when this number is 1.

Assume $T[P]$ has j components, where $j > 1$. It follows from Lemma 3.3(c) that some component of $T[P]$ does not surround any other component of P. Let P_0 be the set of vertices of this component of $T[P]$, and let P' be the set obtained by deleting P_0 from P. Then, by Lemma 3.8, $T - P_0$ has at most $|P_0| - 2$ interior components, so by removing P_0 from P we are decreasing the number of interior components of $T - P$ by at most $|P_0| - 2$. Hence

$$c_i(T - P) \leq c_i(T - P') + |P_0| - 2, \tag{3.7}$$

where $c_i(T - P)$ represents the number of interior components of $T - P$. Since $T[P']$ has $j - 1$ components, it follows from the inductive hypothesis that

$$c_i(T - P') \leq |P'| - 2. \tag{3.8}$$

By (3.7) and (3.8),

$$c_i(T - P) \leq |P'| - 2 + |P_0| - 2 = |P| - 4 < |P| - 2,$$

which completes the proof of Theorem 3.1.

3.2. Proof of Theorem 3.2.

Theorem 3.1 establishes a bound on the number of interior components of $T - P$. To prove Theorem 3.2, it is necessary to obtain a bound on the number of boundary components of $T - P$ as well. This is achieved by introducing a new type of distinguished angle, to supplement the type 1 and type 2 angles defined in Section 3.1. We define a *type* 3 *angle* to be an angle of the form AXB, where AXB is an elementary triangle of T, A and B are vertices of P, and X is a vertex of a boundary component of $T - P$. Type 3 angles are illustrated in Fig. 3.1(d). The following lemma, which is analogous to Lemma 3.7, establishes a lower bound on the total measure of all type 3 angles in an arbitrary triangulation.

Lemma 3.9. *Let T be a triangulation, and let $P \subseteq V(T)$. Then the sum of the measures of all type 3 angles of $T - P$ is at least* $180(c_b - 2)$, *where c_b is the number of boundary components of $T - P$. Moreover, if equality holds, then*

(a) *each boundary component of $T - P$ consists of exactly one vertex, and*
(b) *if $p \in P$ is on the convex hull of T, then the two consecutive segments of the convex hull of T that meet at p form an angle of exactly* $180°$.

Proof. For each boundary component Q of $T - P$, define the *p-boundary* of Q to be the unique walk $p_0 p_1 \cdots p_{r+1}$ through $T[P]$, such that each edge of the walk is the base of an elementary triangle whose apex is a vertex of Q to the left of the edge. Define the *q-boundary* of Q to be the unique walk $q_0 q_1 \cdots q_{s+1}$ from the first vertex of the p-boundary to the last vertex of the p-boundary, such that every vertex (except the first and last vertices) is in Q and such that each edge of the walk is the base of a triangle whose apex is a vertex of P to the right of the edge. (Notice that $p_0 = q_0$ and $p_{r+1} = q_{s+1}$, but it is not necessarily true that $r = s$.) The p-boundary and q-boundary are illustrated in Fig. 3.5, where $r = 5$ and $s = 6$.

Let α be the sum of the measures of the type 3 angles of Q. The first step in the proof is to establish a bound for α in terms of the total measure of the angles at vertices of Q along the q-boundary, namely:

$$\alpha \geq \sum_{j=1}^{s} \mu(q_{j-1} q_j q_{j+1}) - 180(s - 1). \tag{3.9}$$

Define Π to be the walk from p_0 to p_{r+1} that alternates between the p-boundary and the q-boundary in such a way that the angles at the vertices belonging to Q are precisely the distinguished angles. In Fig. 3.5, for example, Π is the walk $p_0 q_1 p_1 q_2 p_2 q_3 p_3 q_4 p_4 q_4 p_5 q_6 p_6$. The path Π can be obtained from the q-boundary by performing the following step for $j = 1, \ldots, s - 1$:

(*) Replace edges $q_j q_{j+1}$ by the edges $q_j p'$ and $p' q_{j+1}$, where p' is the (unique) vertex on the p-boundary to the right of the edge $q_j q_{j+1}$ such that $q_j p' q_{j+1}$ is an elementary triangle of T.

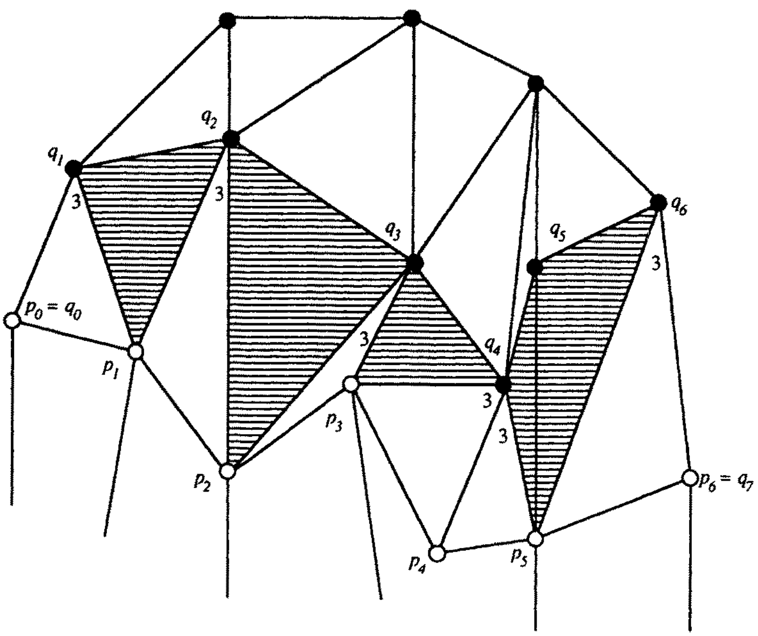

Fig. 3.5. Illustration of p-boundaries and q-boundaries. The shaded triangles indicate the area below the q-boundary and above the walk Π used in the derivation of (3.9). The type 3 angles are as illustrated.

Each time (*) is performed, the total measure of all angles on the walk at vertices belonging to Q is reduced by an amount less than 180° (namely, by the sum of the measure of angles $p'q_jq_{j+1}$ and $p'q_{j+1}q_j$). Since the q-boundary is transformed into Π by performing (*) $s - 1$ times, (3.9) follows. If equality holds in (3.9), then $s = 1$, which means that the component Q consists of a single vertex.

The next step in the proof consists of constructing a polygon, which we call R, obtained by taking the convex hull of $V(T)$ (with the vertices enumerated in clockwise order) and "cutting across it" with q-boundaries. That is, if $v_1, \ldots, v_n, v_1$ is an enumeration of the vertices of the convex hull of $V(T)$, then, for each pair v_i and v_j of vertices that form the opposite ends of a q-boundary, replace the path $v_i \cdots v_j$ with the corresponding q-boundary.

The polygon R has two kinds of vertices—vertices of P that are on the convex hull of $V(T)$, and vertices of $T - P$ that lie on q-boundaries. Assume that there are n_k vertices of $T - P$ along the q-boundary of component number k, and let $n = \sum_{k=1}^{c_b} n_k$. Let β_k be the sum of the internal angles of R at vertices of the q-boundary of component number k, and let $\beta = \sum_{k=1}^{c_b} \beta_k$ (i.e., β is the sum of the measures of all internal angles of R at vertices of $T - P$). Since all vertices of R that are not vertices of $T - P$ are convex vertices (i.e., their interior angles with respect to the polygon R do not exceed 180°), it follows that

$$\beta \geq 180(n - 2). \tag{3.10}$$

Let α_k be the sum of the measures of all type 3 angles at vertices in boundary component number k. Applying (3.9) to each boundary component of $T - P$ and then summing over all boundary components, we obtain

$$\sum_{k=1}^{c_b} \alpha_k \geq \sum_{k=1}^{c_b} \{\beta_k - 180(n_k - 1)\}$$
$$= \beta - 180 \sum_{k=1}^{c_b} (n_k - 1)$$
$$= \beta - 180(n - c_b).$$

Hence, by (3.10), we have

$$\sum_{k=1}^{c_b} \alpha_k \geq 180(n - 2) - 180(n - c_b) = 180(c_b - 2), \tag{3.11}$$

which proves the first conclusion of Lemma 3.9.

To prove the second conclusion, observe that if quality holds in (3.11), it must hold in (3.10) and in all applications of (3.9). It was remarked earlier in the proof that if equality holds in (3.9), the boundary component Q contains exactly one vertex. If equality holds in (3.10), then each angle of R occurring at a vertex of P must be a 180° angle. The result follows from these observations. □

As before, we first establish Theorem 3.2 under the additional assumption that $T[P]$ is connected.

Lemma 3.10. *Let T be a nondegenerate Delaunay triangulation, and suppose $P \subseteq V(T)$ with $T[P]$ connected. Then $T - P$ has at most $|P|$ components.*

Proof. The proof is quite similar to the proof of Lemma 3.8. Let b be the number of bad faces of $T[P]$, let g be the number of good faces of $T[P]$, and let c_b be the number of boundary components of $T - P$. By Lemma 3.5, the number of interior components of $T - P$ is given by b, so it suffices to show that $c_b + b \leq |P|$.

As before, let d be the total measure of all distinguished angles (i.e., all angles of types, 1, 2, and 3). Each good face contributes three type 1 angles of total measure 180. Each bad face contributes several type 2 angles of total measure at least 360. By Lemma 3.9, the total measure of all the type 3 angles is at least 180 $(c_b - 2)$. Hence,

$$d \geq 180(g + 2b + c_b - 2). \tag{3.12}$$

Let e be the number of edges of $T[P]$. Using the same argument as in the proof of Lemma 3.8, we have

$$e = |P| + b + g - 1 \tag{3.13}$$

and

$$d < 180 \cdot e. \tag{3.14}$$

Combining (3.12), (3.13), and (3.14), we have

$$180(g + 2b + c_b - 2) < 180(|P| + b + g - 1),$$

which simplifies to

$$b + c_b < |P| + 1. \tag{3.15}$$

Since b, c_b, and $|P|$ are integers, $b + c_b \leq |P|$, which was to be proved. □

Notice that while nondegeneracy was not necessary to get strict inequality in (3.6), nondegeneracy must be assumed to obtain strict inequality in (3.15). This is because, when type 3 angles are included, it is possible for all distinguished angles to be paired. Hence, if T is degenerate, the strict inequality in (3.14), and hence in (3.15), may become an equality. The toughness properties of degenerate Delaunay triangulations are investigated in Section 5.

To complete the proof of Theorem 3.2, we need only show that the assumption that $T[P]$ is connected in Lemma 3.10 is not essential. The proof is similar to the final step in the proof of Theorem 3.1, with the following difference. When we remove the set P_0 from P, we merge at most $|P_0|$ components into at least one, so the analog of (3.7) is

$$c(T - P) \leq c(T - P') + |P_0| - 1.$$

When we combine this with the inductive hypothesis that $c(T - P') \leq |P'|$, we obtain

$$c(T - P) \leq |P'| + |P_0| - 1 = |P| - 1,$$

which completes the proof.

4. Examples

In this section we present some examples to show that Theorems 3.1 and 3.2 are independent of one another and that they are the best possible results.

Figure 4.1 shows a triangulation that fails to satisfy the conclusion of Theorem 3.1, because removing A, B, C, and D splits it into three internal components. Since the triangulation is 1-tough (in fact, it is Hamiltonian), this shows Theorem 3.1 is not implied by Theorem 3.2. Other examples of 1-tough, maximal planar graphs that fail to satisfy the conclusion of Theorem 3.1 can be found in [23] and [40]. Conversely, the example in Fig. 1.1(a), which is not 1-tough, satisfies the conclusion

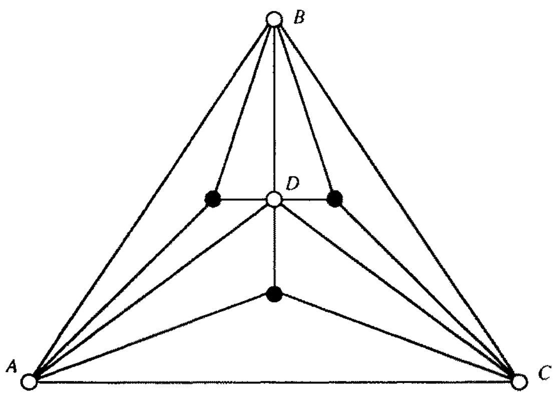

Fig. 4.1. A 1-tough triangulation that fails to satisfy the conclusion of Theorem 3.1.

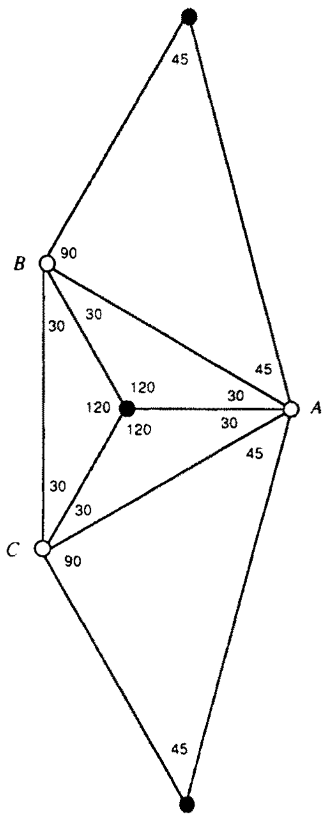

Fig. 4.2. Theorems 3.1 and 3.2 are both sharp.

of Theorem 3.1. These examples show that Theorems 3.1 and 3.2 are indeed independent of one another.

The triangulation in Fig. 4.2 shows that neither Theorem 3.1 nor Theorem 3.2 can be improved. It is easy to verify that the figure satisfies the hypotheses of Fact 2.1, which shows that it is a nondegenerate Delaunay triangulation. Removing the three vertices A, B, and C separates the triangulation into three components, one of which is interior. Hence, bounds on the number of components proved in Section 3 can be attained.

5. Toughness Conditions for Degenerate Delaunay Triangulations and Voronoi Duals

We now examine the conditions under which degenerate Delaunay triangulations and Voronoi duals can fail to be 1-tough. We use the results of this section in our proof that all inscribable graphs are 1-tough. It is important to keep in mind the

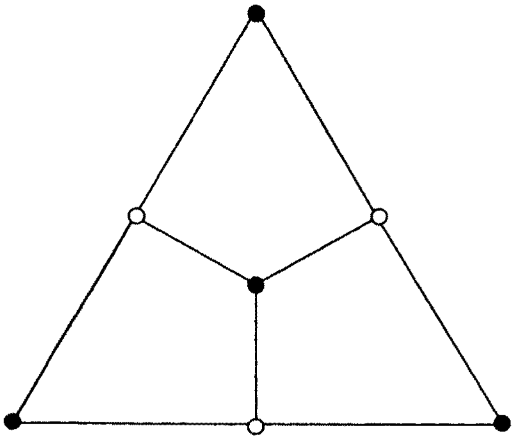

Fig. 5.1. A Voronoi dual that is not 1-tough.

difference between a Voronoi dual and a Delaunay triangulation in the degenerate case: a Voronoi dual is the dual of a Voronoi diagram, while a Delaunay triangulation is a triangulation obtained by adding edges (in some arbitrary fashion) to a Voronoi dual.

A degenerate Voronoi dual need not be 1-tough. An example is provided by the Voronoi dual D shown in Fig. 5.1. This example originally appeared in [31]. The set S of generating sites consist of the vertices of an equilateral triangle, the midpoints of the sides, and the centroid. It fails to be 1-tough because removing the three midpoints of the sides separates it into four components. This graph has several features worth noting. Let P be the set consisting of the three midpoints of the sides. The graph is "almost" 1-tough, because $c(D - P) = |P| + 1$. Each component of $D - P$ consists of a single vertex. Each point of P is on a line segment connecting two points of $S - P$ (i.e., no point of P is an extreme point of S). We show that these properties are characteristic of Voronoi duals that fail to be 1-tough (Theorem 5.3). First we show that the corresponding properties hold for Delaunay triangulations.

Lemma 5.1. *Let T be a Delaunay triangulation, and suppose there exists a set $P \subseteq V(T)$ such that $c(T - P) > |P|$. Then T is degenerate. Moreover, the following properties hold*:

(a) $c(T - P) = |P| + 1$.
(b) *All components of $T - P$ consist of a single vertex.*
(c) *No point of P is an extreme point of $V(T)$.*

Proof. It follows immediately from Theorem 3.2 that T is degenerate. The remaining properties of T and P follow from an analysis of the conditions that cause the various inequalities in Section 3 to become equalities, which we sketch.

Nondegeneracy was used only to get strict inequality in (3.14) and (3.15). In the degenerate case, the inequality (3.15) becomes an equality, which proves (a). If equality holds in (3.15), it also holds in (3.12), which in turn means it holds in all applications of Lemmas 3.7 and 3.9. The last sentence of Lemma 3.9 then implies (c) and also implies that (b) holds for boundary components for $T - P$. The final assertion of Lemma 3.7 implies that (b) holds for interior components of $T - P$ as well. □

In order to show that the conclusion of Lemma 5.1 also holds for a non-1-tough Voronoi dual, we show that any non-1-tough Voronoi dual can be completed to a non-1-tough Delaunay triangulation by adding edges. This is a consequence of the following, somewhat more general lemma.

Lemma 5.2. *Let G be any plane graph with the property that every interior face is a convex region and no three vertices on the boundary of a common interior face are collinear. Let $P \subseteq V(G)$. Then it is possible to add edges to G to obtain a triangulation T in such a way that $c(T - P) = c(G - P)$.*

Proof. The proof proceeds by induction on the length of the longest cycle bounding an interior face of G. If this number is 3, then G is a triangulation, and there is nothing to prove. If this length is $n > 3$, let F be an interior face of G bounded by a cycle of length n. Choose a new edge as follows. If the boundary of F contains two nonconsecutive vertices of P, join them by an edge. Otherwise, the boundary of F must contain vertices belonging to only one component of $G - P$, in which case any edge connecting two nonconsecutive vertices on the boundary of F may be chosen. Because of the convexity condition on the faces, the edge may be taken to be a line segment. Thus the chosen edge splits F into two smaller faces, each of which is convex and has no three collinear vertices on its boundary, and preserves the number of components of $G - P$. Apply this process to each interior face bounded by a cycle of length n to obtain a graph H such that $c(H - P) = c(G - P)$. The result then follows by induction. □

Theorem 5.3. *Let D be a Voronoi dual, and suppose there exists a set $P \subseteq V(D)$ such that $c(D - P) > |P|$. Then*

(a) $c(D - P) = |P| + 1$.
(b) *All components of $D - P$ consist of a single vertex.*
(c) *No point of P is an extreme point of $V(D)$.*

Proof. Since the boundary of each interior face of D is a polygon inscribed in a circle, each interior face is convex and no three vertices on the boundary of a single face are collinear. So by Lemma 5.2, edges can be added to D to create a degenerate Delaunay triangulation T in such a way that the number of components of $D - P$ is unchanged. Properties (a), (b), and (c) then follow from the corresponding properties for T, established in Lemma 5.1. □

There is one case in which the conclusions of Theorem 5.3 (and Theorem 3.2) can be strengthened.

Theorem 5.4. *Let S be a set of sites with exactly three sites on the convex hull. If D is the Voronoi dual generated by S, and $P \subseteq S$, then $c(D - P) \leq |P| - 1$.*

Proof. Let $P \subseteq S$ be given, and let T be a Delaunay triangulation derived from the Voronoi dual D as described in Lemma 5.2. Since there are only three sites on the boundary of T, there can be only one boundary component of $T - P$. By Theorem 3.1, there can be at most $|P| - 2$ interior components of $T - P$. Hence $c(D - P) \leq c(T - P) \leq |P| - 1$. □

6. Toughness Conditions for Inscribable Graphs

A graph is said to be *inscribable* if it can be represented as the edges and vertices of a three-dimensional convex polytope inscribed in a sphere. A graph so represented is said to be *inscribed*. In this section we show that any inscribable graph is 1-tough.

There is a strong connection between inscribable graphs and Voronoi duals that was apparently first discovered by Brown [11]. To describe this connection, we need to introduce the notion of a farthest-point Voronoi diagram. The definition of this structure is analogous to the definition of the Voronoi diagram as presented in Section 2, except that the region associated with each site is the set of points that are *farther* from that site than from any other. These diagrams are described more fully in [26] and [44]. The farthest-point Voronoi dual is a graph in which sites are connected if and only if their corresponding regions in the farthest-point Voronoi diagram share a common boundary.

The connection between inscribable graphs and Voronoi duals is based on a geometrical transformation known as *spherical inversion*. Let B be a sphere with center C and radius ρ. If P is any point (other than C), then the *inversion of P in B*, denoted by $I_B(P)$, is the point on the ray CP whose distance from C is $\rho^2/d(C, P)$. Point $I_B(C)$ is defined to be the "point at infinity." Some basic properties of spherical inversion are proved in [15]. Brown's discovery was as follows. Given a set of sites in the plane, embed the plane in Euclidean 3-space, invert the sites in a sphere whose center does not lie in the plane, and compute the convex hull of the images of the sites (which can be easily shown to lie on a common sphere). Then two images of sites (on the sphere) are joined by an edge of the convex hull if and only if the corresponding sites are joined by an edge of either the Voronoi dual or the farthest-point Voronoi dual (of the given set of sites). An equivalent statement is the following:

Lemma 6.1. *Any inscribable graph is isomorphic to a graph consisting of a set of planar sites, the edges of its Voronoi dual, and the edges of its farthest-point Voronoi dual.*

Theorem 6.2. *Any inscribable graph is 1-tough.*

Proof. By Lemma 6.1, it is sufficient to show that if S is any set of sites in a plane, and if G is the graph obtained by connecting two sites if they are joined by an edge of either the Voronoi dual or the farthest-point Voronoi dual, then G is 1-tough.

Let D be the Voronoi dual of S and let $P \subseteq S$. We must show $c(G - P) \leq |P|$. Since G is obtained from D by adding edges to it, $c(G - P) \leq c(D - P)$, so we only have to worry about the case $c(D - P) > |P|$. Let O be any minimal enclosing circle of S. Since the disk bounded by O is strictly convex [29] and has O as its set of extreme points, any point of $O \cap S$ must be an extreme point of S. Hence, by Theorem 5.3(c), the only sites that can be on O are those belonging to $S - P$. By Theorem 5.3(b), each such site represents a separate component of $D - P$. Since any two consecutive sites along O will be joined by an edge in the farthest-point Voronoi dual, it follows from Theorem 5.3(a) that

$$c(G - P) < c(D - P) = |P| + 1. \tag{6.1}$$

Since P was an arbitrary subset of S and both sides of (6.1) are integers, the 1-toughness of G follows. □

Theorem 6.2 cannot be strengthened without adding additional hypotheses, as there exist inscribable graphs G for which $c(G - P) = |P|$. A simple example is provided by a cube, which is clearly inscribable and has eight vertices. It is easy to find four vertices (namely any vertex v, and the three vertices opposite v on the three faces of the cube to which v belongs) whose removal separates the cube into four components.

Theorem 6.2 strengthens a result proved in [29, p. 285] and attributed there to Steinitz. This result, stated in its dual form, is the following:

Theorem 6.3. *If G is an inscribable graph, then an independent set of vertices of G can have at most $|G|/2$ elements.*

($I \subseteq V(G)$ is *independent* if no two vertices in I are joined by an edge.) Suppose that G did have an independent set of vertices I with $|I| > |G|/2$. Let $P = V(G) - I$. Then each vertex in V would be a separate component of $G - P$, so $|P| < |I| = c(G - P)$, which would imply that G was not 1-tough. It follows by contraposition that Theorem 6.2 implies Theorem 6.3.

Figure 6.1 shows that Theorem 6.2 is stronger than Theorem 6.3. Removing the six light vertices splits the graph into seven components, so it is not 1-tough, and hence not inscribable by Theorem 6.2. However, the graph has 16 vertices, and no independent set can contain more than seven vertices. The proof follows from three easily verified facts. No independent set can contain more than two of the light vertices. If an independent set contains even one light vertex, it can contain at most four dark vertices. No independent set can contain more than seven dark vertices.

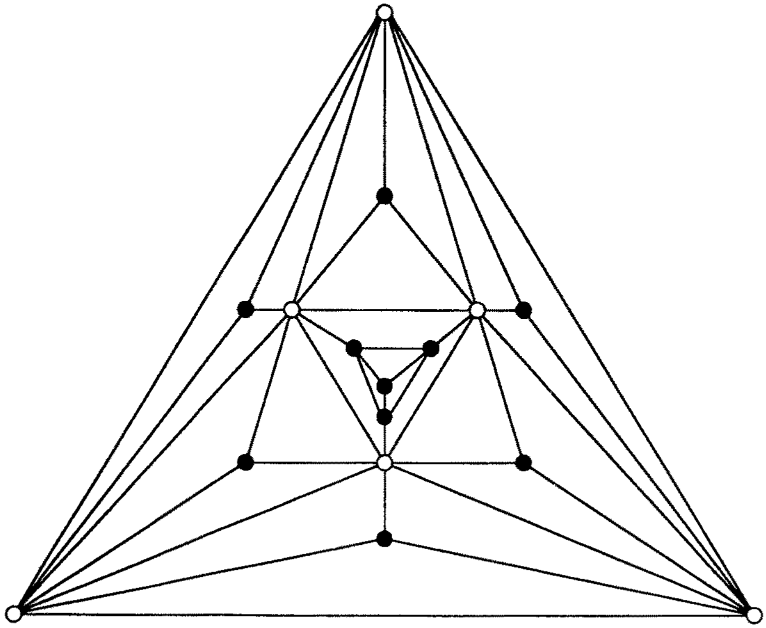

Fig. 6.1. Theorem 6.2 can be used to prove that this graph is not inscribable in a sphere, but Theorem 6.3 cannot.

7. Perfect Matchings in Delaunay Triangulations

In [42], the question was raised whether every Delaunay triangulation has a perfect matching. This question is answered in the affirmative in Theorem 7.2 below. A *perfect matching* in a graph G with $|G| = n$ is a set of $\lfloor n/2 \rfloor$ disjoint edges ($\lfloor \cdot \rfloor$ denotes the floor function). If $|G|$ is even, a perfect matching in G is called a *1-factor*. The following lemma is a restatement of Tutte's classical theorem characterizing graphs with 1-factors [46].

Lemma 7.1. *Let G be a graph, and suppose that, for each $P \subseteq V(G)$,*

$$c_0(G - P) \leq |P| + 1, \tag{7.1}$$

where $c_0(G - P)$ is the number of components of $G - P$ that have odd cardinality. Then G has a perfect matching.

Proof. The usual statement of Tutte's theorem is that if $c_0(G - P) \leq |P|$ for all $P \subseteq V(G)$, then G has a 1-factor. If $|G|$ is even, then $c_0(G - P) - |P|$ must be even, which implies that strict inequality holds in (7.1) and hence there is a 1-factor by Tutte's theorem. If $|G|$ is odd, create a new graph G' by adding a new vertex x to G and connecting x to all vertices of G, and observe that G' satisfies the hypothesis of Tutte's theorem. It follows that G' has a 1-factor, so G has a perfect matching. □

Theorem 7.2. *Every Delaunay triangulation has a perfect matching.*

Proof. Let T be a Delaunay triangulation and let $P \subseteq V(T)$. By Theorem 3.2 and Lemma 5.1, $c_o(T-P) \leq c(T-P) \leq |P| + 1$, so T has a perfect matching by Lemma 7.1. □

The same argument shows that all degenerate Voronoi duals (and hence all inscribable graphs) have perfect matchings.

8. Final Remarks

In this paper we have proved several results about the combinatorial structure of Delaunay triangulations and inscribable graphs. In particular, we have shown that nondegenerate Delaunay triangulations are 1-tough and that degenerate Voronoi duals can fail to be 1-tough only under very special circumstances. We have also shown that all inscribable graphs are 1-tough. One consequence of our results is that all Delaunay triangulations, whether degenerate or not, and all inscribable graphs have perfect matchings.

There are several open questions related to the results of this paper, such as whether it is indeed true that "most" Delaunay triangulations are Hamiltonian (probabilistically, asymptotically as the number of sites tends to infinity), and whether there is a polynomial-time algorithm for recognizing Hamiltonian Delaunay triangulations. Some empirical data related to the first question is presented in [21].

It is unknown whether the conclusions of Theorems 3.1 and 3.2 are (jointly) sufficient to guarantee that a triangulation has a combinatorially equivalent realization as Delaunay triangulation. It is shown in [24] that any inner triangulation of a polygon has such a realization, but this sufficient condition is much more restrictive than the conclusions of Theorems 3.1 and 3.2. The closely related problem of characterizing all inscribable graphs is a long-standing open problem. A brief history of this problem can be found in [10].

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