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Delaunay Graphs Are Almost as Good as Complete Graphs*

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Abstract. Let S be any set of N points in the plane and let $\mathrm{DT}(S)$ be the graph of the Delaunay triangulation of S. For all points a and b of S, let $d(a, b)$ be the Euclidean distance from a to b and let $\mathrm{DT}(a, b)$ be the length of the shortest path in $\mathrm{DT}(S)$ from a to b. We show that there is a constant c $(\leq((1+\sqrt{5})/2)\pi\approx 5.08)$ independent of S and N such that

$$\frac{\mathrm{DT}(a, b)}{d(a, b)} < c.$$

1. Introduction

Let $\mathrm{DL}_i(S)$ be the Delaunay triangulation of S in the L_i norm $(i=1, 2)$. Chew [Ch] shows that there exists a constant c_1 such that the ratio of shortest distances in $\mathrm{DL}_1(S)$ to straight line (i.e., L_2) distances is bounded above by c_1 where $c_1=\sqrt{10}\approx 3.16\,228$. We extend this result here demonstrating a constant c_2 such that the ratio of shortest distances in $\mathrm{DL}_2(S)$ to straight line distances is bounded above by $c_2=((1+\sqrt{5})/2)\pi\approx 5.08$. The best-known lower bound on c_2 is $\pi/2$ and is also due to Chew.

In his paper, Chew describes applications of his (and our) result to problems of motion planning, polygon visibility, and extensions of Voronoi diagrams/Delaunay triangulations. Our focus is the derivation of c_2 and potential extensions to other problems involving distances in the plane.

In what follows, we provide the definitions and lemmas necessary to prove our main result in Section 2; Section 3 contains the proofs. We conclude with some open problems.

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2. The Main Result

We begin with (informal) definitions of the Voronoi diagram and the Delaunay triangulation. The *Voronoi diagram* for a set S of N points in the plane is a partition of the plane into regions, each containing exactly one point in S, such that, for each point $p \in S$, every point within its corresponding region (denoted $\mathrm{Vor}(p)$) is closer to p than to any other point of S. The boundaries of these regions form a planar graph. The *Delaunay triangulation* of S is the straight-line dual of the Voronoi diagram for S; that is, we connect a pair of points in S if and only if they share a Voronoi boundary. Under the standard assumption that no four points of S are cocircular, the Delaunay triangulation is indeed a triangulation [PS]; we denote its corresponding graph by $\mathrm{DT}(S)$.

For the remainder of this section, fix points $a, b \in S$; we will construct a path in $\mathrm{DT}(S)$ that is not too long in relation to $d(a, b)$. Assume for simplicity that a and b lie on the x-axis, with $x(a) < x(b)$ (we denote the coordinates of a point q in the plane by $x(q)$ and $y(q)$, respectively). We refer to members of S alternatively as points or vertices, and to edges of $\mathrm{DT}(S)$ as edges or line segments, as the context indicates.

Our original idea for the path was simply to use the vertices $a = b_0$, $b_1, \ldots, b_{m-1}, b_m = b$ corresponding to the sequence of Voronoi regions traversed by walking from a to b along the x-axis, as illustrated in Fig. 1, where $m = 4$ (in the case in which a Voronoi edge happens to lie on the x-axis somewhere between a and b, we—arbitrarily—choose that Voronoi region lying above, rather than

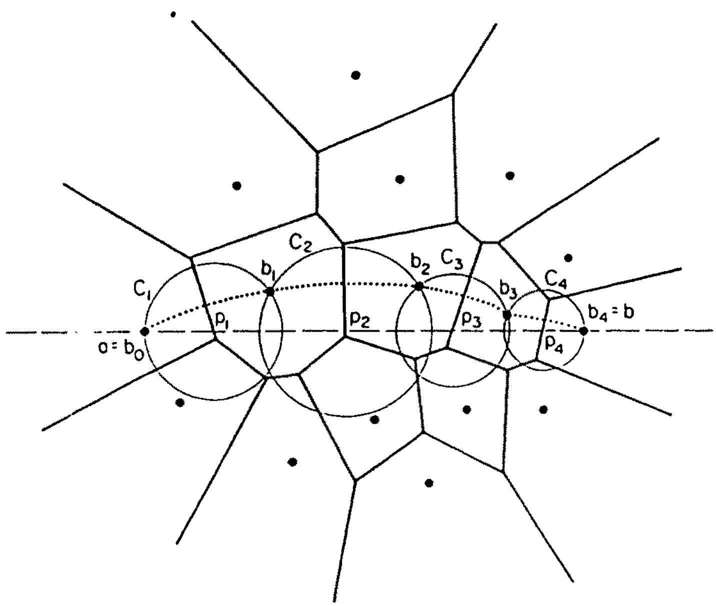

Fig. 1. The Voronoi diagram is shown in solid line, and the direct DT path between a and b in dotted line.

below, the x-axis). In general, we refer to the DT path constructed in this way between some z and z' in S as the *direct DT path* from z to z'. Let p_i denote the point on the x-axis that also lies on the boundary between $\mathrm{Vor}(b_{i-1})$ and $\mathrm{Vor}(b_i)$, for $i = 1, 2, \ldots, m$. The definition of the Voronoi diagram immediately gives that p_i is the center of a circle C_i passing through b_{i-1} and b_i but containing no points of S in its interior.

Two simple properties of direct DT paths are:

Lemma 1. $x(b_0) \leq x(b_1) \leq \cdots \leq x(b_m)$.

Lemma 2. *For all i, $0 \leq i \leq m$, b_i is contained within, or on the boundary of,* circle(a, b) *(by which we denote the circle with a and b diametrically opposed).*

Note in Fig. 1 that all the b_i happen to be in the same half-plane defined by the line connecting a and b (i.e., $y(b_i) \geq 0$ for all $0 \leq i \leq m$). In such cases, we say that the direct path between the two points is *one-sided.* One-sided paths are fortuitous for our purposes, because the ratio of the path length to the Euclidean distance is at most $\pi/2$; this is a simple consequence of Lemma 1 above and the following:

Lemma 3. *Let $D_1, D_2, \ldots, D_k$ be circles all centered on the x-axis such that $D = \bigcup_{1 \leq i \leq k} D_i$ is connected. Then* boundary(D) *has length at most $\pi \cdot (x_r - x_l)$, where x_l and x_r are the least and greatest x-coordinates of D, respectively (see Fig. 2).*

Lemma 3 applies to the one-sided paths because the half of boundary(C) (where C is defined as $\bigcup_{1 \leq k \leq m} C_k$) that lies above the x-axis has length at least as great as the path itself (because the b_i are monotonic in x).

The trouble with this approach is that the path is not necessarily even close to being one-sided; the path may zig-zag across the x-axis (as is illustrated in Fig. 3) $\Theta(N)$ times.

Our modified approach, then, is to try to stay above the x-axis. Should the direct path dip below the x-axis, we determine how costly the dip will be. If dipping below is not too expensive (in a sense defined below) then we follow the direct path below the x-axis and then back up. Otherwise, we construct a shortcut between the two points above the x-axis. Most of the proof consists of showing that the shortcut is not too long. The exact path we take is made more precise in the proof of the following:

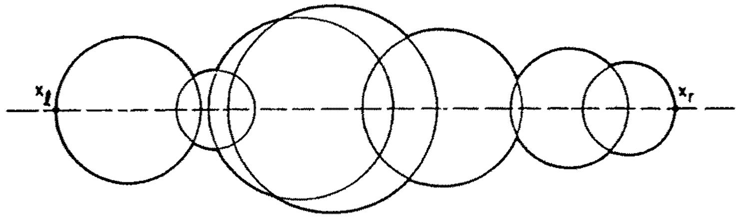

Fig. 2. Illustration for Lemma 3.

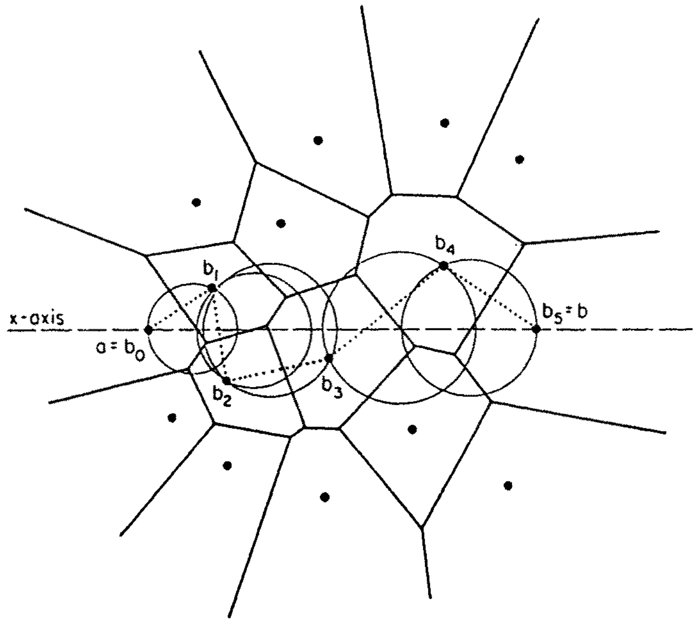

Fig. 3. A direct DT path that is *not* one-sided.

Theorem. *There exists a DT path from a to b of length*

$$\leq((1+\sqrt{5})/2)\pi \cdot d(a, b).$$

Proof. We present an algorithm for constructing a DT path from $a = b_0$ to $b = b_m$, and then analyze the length of the path it produces. Assume that the path so far has brought us to some b_i such that (1) $y(b_i) \geq 0$ (initially, $i = 0$), (2) $i < m$ (meaning we are not finished), and (3) $y(b_{i+1}) < 0$. Thus the direct path would dip below the x-axis for a while after b_i. Let j be the least number greater than i such that $y(b_j) \geq 0$ (e.g., in Fig. 4, if $i = 2$ then $j = 4$). Let T denote the path along the boundary of C clockwise from b_i to b_j. Let w denote the length of the projection of T onto the x-axis (thus $w = x(b_j) - x(b_i)$). Define $h = \min\{y(q): q \text{ lies on } T\}$. Now if $h \leq w/4$ then continue along the direct path to b_j

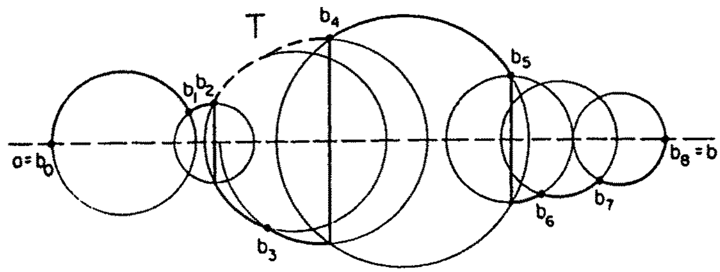

Fig. 4. An upper bound on the length of the direct DT path.

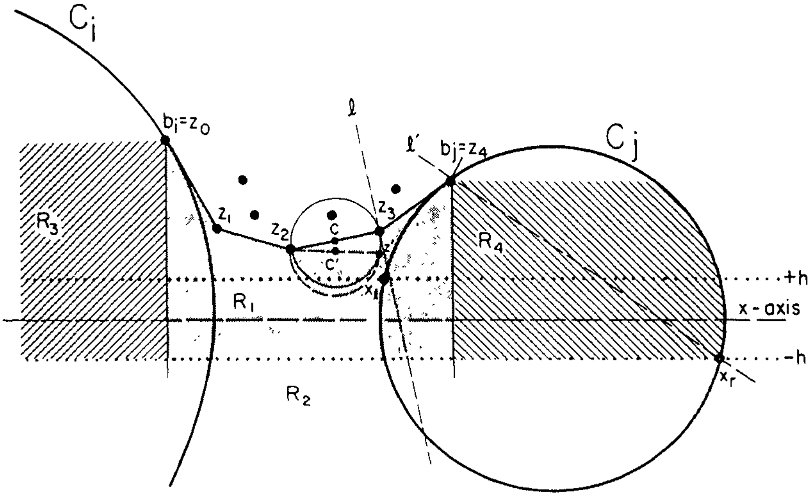

Fig. 5. The shortcut from b_i to b_j. Here $k=2$.

(i.e., use edges $b_ib_{i+1}, b_{i+1}b_{i+2}, \ldots, b_{j-1}b_j$). Otherwise we take a shortcut as follows. Construct the lower convex hull $b_i = z_0, z_1, z_2, \ldots, z_n = b_j$ of the set

$$\{q \in S\colon x(b_i) \leq x(q) \leq x(b_j) \text{ and } y(q) \geq 0 \text{ and } q \text{ lies under } b_ib_j\}$$

(see Fig. 5). Note that these convex hull edges are certainly not on the direct DT path from a to b. Now the shortcut consists of taking the direct DT path from z_k to z_{k+1} for each $0 \leq k \leq n-1$. The key fact (proved in Section 3) is:

Lemma 4. *Let z_kz_{k+1} be an edge of the lower convex hull described above. Then the direct DT path from z_k to z_{k+1} is one-sided.*

Next we analyze the length of the path produced by this algorithm. When proceeding from b_i to b_j, let t denote the length of T. If $h \leq w/4$ then let q_0 be the point of T with least y-value (see Fig. 6), let t_i denote the length of the portion of T from b_i to q_0, and t_j the length of the portion of T from q_0 to b_j (thus $t_i + t_j = t$). Let w_i and w_j denote the lengths of the projections of those two portions of T, respectively (thus $w_i + w_j = w$). Then the path we take (i.e., no shortcuts) has length at most

$$\begin{aligned} t+2(y(b_i)+y(b_j)) &= t+2(2h+(y(b_i)-h)+(y(b_j)-h)) \\ &\leq t+2\left(\frac{w}{2}+(y(b_i)-h)+(y(b_j)-h)\right) \\ &= t+2\left(\frac{w_i}{2}+(y(b_i)-h)+\frac{w_j}{2}+(y(b_j)-h)\right) \\ &\leq t+2\left(\frac{\sqrt{5}}{2}t_i+\frac{\sqrt{5}}{2}t_j\right) = t(1+\sqrt{5}). \end{aligned}$$

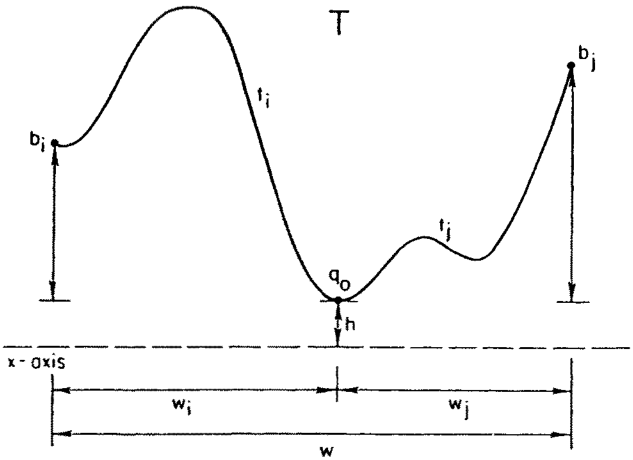

Fig. 6. Analyzing the path length when the shortcut is *not* taken.

The last inequality follows from the (easily proved) fact that

$$\frac{a}{2}+b\leq\frac{\sqrt{5}}{2}c$$

whenever a and b are the legs of a right triangle with hypotenuse c.

On the other hand, if $h>w/4$ then we take the shortcut, which has length at most

$$\sum_{k=0}^{n-1}\text{length of one-sided path from } z_k \text{ to } z_{k+1}$$

(by Lemma 4) which is $\leq\sum_{k=0}^{n-1} d(z_k, z_{k-1})\pi/2\leq t\pi/2$ (by Lemma 3). Hence in either case, the distance we travel in getting from b_i to b_j is at most $(1+\sqrt{5})t$. Therefore summing over all such trips b_i to b_j as well as the trips (for which we travel at most t units) where the direct DT path from a to b stays completely above the x-axis, we get (by Lemma 3) a total path length of at most $d(a, b)((1+\sqrt{5})/2)\pi$. □

3. Proofs of the Lemmas

Proof of Lemma 1. The perpendicular bisector of b_i and b_{i+1} contains p_i. Point b_{i+1} lies to the right of this bisector, and b_i lies to the left; hence $x(b_i)\leq x(b_{i+1})$. □

Proof of Lemma 2. Let c denote the midpoint of segment ab; let k be such that c lies in the Voronoi region of b_k. Then

$$d(b_0, c)\geq d(b_1, c)\geq\cdots\geq d(b_k, c)$$

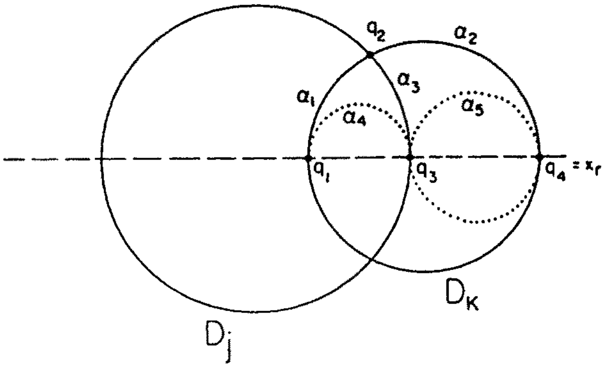

Fig. 7. Illustration for the proof of Lemma 3.

and

$$d(b_k, c) \leq d(b_{k+1}, c) \leq \cdots \leq d(b_m, c). \qquad \square$$

Proof of Lemma 3. By induction on k. The claim is easy if $k=1$; so let $k \geq 2$ and assume it for $k-1$. Let q_1 and q_4 denote the leftmost and rightmost points of D_k, respectively (see Fig. 7), and assume without loss of generality that $q_4 = x_r$. Let q_2 be the rightmost point at which D_k intersects another circle D_j (thus $j < k$); let q_3 be the rightmost point of D_j. We can assume that D_k does not entirely contain any circle D_i ($i \neq k$), since otherwise D_i would not contribute to boundary(D) and hence the induction would be trivial. Denote by α_1 (α_2) the length of the arc on circle D_k clockwise from q_1 to q_2 (resp. q_2 to q_4). Let α_3 be the length of the arc on circle D_j clockwise from q_2 to q_3. Finally, let $\alpha_4 = (\pi/2)(x(q_3) - x(q_1))$ and let $\alpha_5 = (\pi/2)(x(q_4) - x(q_3))$. Then a simple convexity argument shows that

$$\alpha_1 + \alpha_3 \geq \alpha_4.$$

Also, we have

$$\alpha_4 + \alpha_5 = \alpha_1 + \alpha_2.$$

Hence

$$\alpha_1 + \alpha_3 + \alpha_5 \geq \alpha_4 + \alpha_5 = \alpha_1 + \alpha_2,$$

implying $\alpha_3 + \alpha_5 \geq \alpha_2$. Therefore, denoting the length of the boundary of D by bd(D), we have

$$\begin{aligned}
\mathrm{bd}(D) &\leq \mathrm{bd}\left(\mathrm{circle}(q_3, q_4) \cup \bigcup_{1 \leq i \leq k-1} D_i\right) \\
&\leq \mathrm{bd}(\mathrm{circle}(q_3, q_4)) + \mathrm{bd}\left(\bigcup_{1 \leq i \leq k-1} D_i\right) \\
&\leq \pi(x_r - x(q_3)) + \pi(x(q_3) - x_l) \qquad \text{(by the inductive hypothesis)} \\
&\leq \pi(x_r - x_l).
\end{aligned}$$

$\square$

Proof of Lemma 4. By Lemma 2, the direct DT path from z_k to z_{k+1} lies entirely within circle(z_k, z_{k+1}). We now show that there are no points of S within the lower semicircle of circle(z_k, z_{k+1}), so the path must be one-sided.

Let q be an arbitrary point in this lower semicircle; we must show $q \notin S$. If $x(b_i) \leq x(q) \leq x(b_j)$ and $y(q) \geq -h$ (i.e., q lies in region R_1 in Fig. 5) then we claim $q \notin S$. To see this, note that if $y(q) \geq h$ then it lies outside the lower convex hull; whereas if $-h < y(q) < h$ then q lies in the interior of $\bigcup_{i \leq k \leq j} C_k$.

We next show that $y(q) > -h$ (that is, $q \notin R_2$). Assume without loss of generality that $y(z_k) \leq y(z_{k+1})$. Since $z_k \in S$ it must lie directly above some point of T, since the area below T and above the x-axis is contained in C and therefore contains no members of S. Therefore $y(z_k) \geq h > w/4$. Let z' be the point with coordinates $(x(z_{k+1}), y(z_k))$. Let c and c' denote the midpoints of segments $z_k z_{k+1}$ and $z_k z'$, respectively. Then $y(c') > w/4$. That $q \in$ circle(z_k, z') follows from $q \in$ circle(z_k, z_{k+1}) and $y(q) \leq y(z_k) = y(z')$. Furthermore, $x(z_{k+1}) - x(z_k) \leq w$, since by extending $z_k z_{k+1}$ on both sides we encounter points on T and since T is connected (and hence the projection of T onto the x-axis is at least as long as the projection of $z_k z_{k+1}$ onto the x-axis). Therefore radius(circle(z_k, z')) $\leq w/2$. Hence

$$y(q) \geq y(c') - \text{radius}(\text{circle}(z_k, z')) > w/4 - w/2 = -w/4.$$

Note that $x(q) \geq x(b_i)$ (that is $q \notin R_3$), because of our assumption $y(z_k) \leq y(z_{k+1})$.

Finally, we assume $x(q) > x(b_j)$ (hence $q \in R_4$). We show that q lies in the interior of C_j, implying $q \notin S$. Let x_l be the leftmost point of intersection of circle C_j with the line $y = h$. Let x_r be the rightmost point of intersection of C_j with the line $y = -h$. Let l denote the line that passes through z_{k+1} perpendicular to segment $z_k z_{k+1}$, and let l' be the line containing b_j and x_r. Note that both l and l' must have negative slopes. Clearly, the entire circle (z_k, z_{k+1}) lies below l and in particular so does q. We claim that this implies that q lies below l' as well. To see this, first note that our assumption $y(z_k) \leq y(z_{k+1})$ implies $y(z_{k+1}) \leq y(b_j)$, and hence line l intersects the line $x = x(b_j)$ below b_j. Therefore it suffices to show that slope(l) $\leq$ slope(l') (recall that both are negative). The monotonicity of slopes in the lower convex hull gives slope($z_k z_{k+1}$) $\leq$ slope($x_l b_j$). Therefore since l and l' are perpendicular to $z_k z_{k+1}$ and $x_l b_j$, respectively (the latter is because x_l and x_r are diametrically opposed on C_j), we have slope(l) $\leq$ slope(l'). Thus q indeed lies below l'; hence since q is in R_4 it must also be in C_j and therefore not in S. □

4. Related Problems

There are many interesting problems related to that solved here. For example, Raghavan [Ra] suggests that our results extend to a special case problem in 3-space. He conjectures that if S is a set of points on the unit sphere, there is a constant c such that

$$\frac{d_H(a, b)}{d(a, b)} < c,$$

where d_H is the distance along edges of the convex hull and d is the (three-dimensional) Euclidean distance.

The generalization of our result to arbitrary point sets in 3-space and their Delaunay graphs remains open.

In another direction, Feder and others [Fe] have shown that for each $k \geq 7$ there is a constant c such that, for each finite set S of points in the plane, there is a graph G with vertices corresponding to these points, and the following properties:

(1) Each vertex in G has degree at most k.
(2) $d_G(a, b)/d(a, b) < c$, where d_G is the distance along edges of G.

Extensions to the cases $k = 5$ and 6 have been proposed by others. It is not difficult to show that no such constant exists for $k = 2$. What is the minimum k for which such a result is possible?

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