

Reconstructing Plane Sets from Projections

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Abstract. We give some uniqueness results for the problem of determining a finite set in the plane knowing its projections along m directions. We apply the results to the problem of the reconstruction of a homogeneous convex body with a finite set of spherical disjoint holes. If m X-ray pictures with directions in some plane are given, then the problem is well posed provided the number of the holes is less than or equal to m and the set of the directions satisfies a suitable condition.

1. Introduction

The problem of determining the structure of an object knowing its projections along straight lines arises in a variety of optical contexts (see [1], [6], and [8]). Here we consider the reconstruction of a homogeneous plane body K . We assume that we know the projections of K along the complete set of straight lines parallel to m given coplanar directions θ_i , $i = 1, \dots, m$. In mathematical terms the problem is to determine the characteristic function of K from the values of its integral along each straight line in the directions θ_i . Such integrals are the projections of K along the corresponding straight lines. We assume that we know such integrals without error. Under such assumptions the authors in [2], [3], [5], and [9] are able to prove some uniqueness and stability results for reconstructing a homogeneous plane convex body H . In particular, Gardner and McMullen [2] proved that H is uniquely determined by its projections in m directions θ_i if the following condition holds:

- (i) *The set $\{\theta_i\}$ is not linearly equivalent to a subset of directions of diagonals of a regular polygon.*

Let us observe that the set of directions of diagonals of any regular polygon is “equally spaced” and that any equally spaced set of directions arises this way. More precisely a subset of directions of diagonals of a regular polygon is a set

of directions given by angles at rational multiples of π . Sets which are affinely equivalent to such sets will be called *affinely rational*. Hence the set $\{\theta_i\}$ satisfies the Gardner–McMullen condition (i) if and only if it is *not* affinely rational.

Let us now consider the reconstruction of a homogeneous convex body K with holes. For reconstructing K we first have to determine the shape and the position of each hole and then we reconstruct the boundary of K . The determination of the centers of gravity of each hole suggests the following problem (Problem A):

Reconstruct a finite set C in the plane knowing its projections along the complete set of straight lines parallel to m given directions θ_i , $i = 1, \dots, m$.

Here the projection of C along a line l in the direction θ_i is the number of points of C lying on l . If the number $|C|$ of the points in C is less than the number m of the directions θ_i , the set C is uniquely determined (Proposition 1). Further, when $|C| = m$ we are able to prove that C is uniquely determined if the set $\{\theta_i\}$ is not affinely rational or C is not the set of the vertices of an affinely regular polygon (Proposition 2). Should $|C| = m + h$ while h is positive and m is sufficiently large with respect to h , a similar result holds (Proposition 3). When no conditions are placed on $|C|$ we can construct an example in which the uniqueness property does not hold for Problem A (Proposition 4).

In Section 3 we apply these results to the following continuous reconstruction problem (Problem B):

Reconstruct a homogeneous plane body K obtained from a convex body by the deletion of a finite number of disjoint circular disks from its interior, knowing its projections in m directions θ_i , $i = 1, \dots, m$.

In Theorems 1 and 2 we prove that Problem B is well posed if the set $\{\theta_i\}$ is not affinely rational and the following *a priori* assumption holds: the number of the holes in K is at most m .

2. Reconstruction of Finite Sets

Proposition 1. *Let $\theta_1, \theta_2, \dots, \theta_m$ be m given directions in the plane, and let C be a finite plane set consisting of n points. If $n < m$ then the projections in the directions θ_i , $i = 1, \dots, m$, uniquely determine C .*

We give two different proofs:

Proof 1. By contradiction, let A and B be two distinct sets with fewer than m points and with the same projections in the directions θ_i . Let x belong to $A \setminus B$; then for each direction θ_i there exists a point y_i in the set B such that $y_i - x$ is parallel to θ_i for each i , $i = 1, \dots, m$.

Since the points y_i must be distinct, the set B contains at least m points, which contradicts the assumption $n < m$. \square

The following proof (see [7]) is constructive.

Proof 2. For any direction θ_i let us denote by $\theta_i(C)$ the set of lines with direction θ_i through the points of C . Let r_i and s_i be the two lines in $\theta_i(C)$ that are "extremal" in the sense that they bound a closed strip S_i containing C . Each side of the convex polygon

$$P = \bigcap_{i=1}^m S_i \tag{2.1}$$

contains at least one point of C and $P \supset C$. Since C contains n points and $n < m$, it follows that P has fewer than $2m$ sides. As the extremal lines $r_i, s_i, i = 1, \dots, m$, are $2m$ in number, it follows that three extremal lines intersect in a vertex z of P . Moreover, one of these three lines intersects P only in z and, since $P \supset C$, it follows that z belongs to C ; thus z is explicitly determined by the projections of C . By eliminating z and each line of the set $\theta_i(C)$ that contains it and then repeating the above argument, we may explicitly reconstruct C . \square

When $n = m$ let us consider the following example: let V and W be two congruent and concentric regular n -gons. The set of vertices of V and the set of vertices of W have the same projections in the n directions determined by the $2n$ sides of the convex hull of $V \cup W$ (see Fig. 1(a) for $n = 4$). Let us observe that two convex polygons affinely equivalent to V and W also have the same property (see Fig. 1(b)).

In fact, we now show that this is the only way that two configurations can determine the same projections in the case $n = m$, that is:

Proposition 2. Let $\theta_1, \theta_2, \dots, \theta_m$ be m given directions in a cyclic order and let

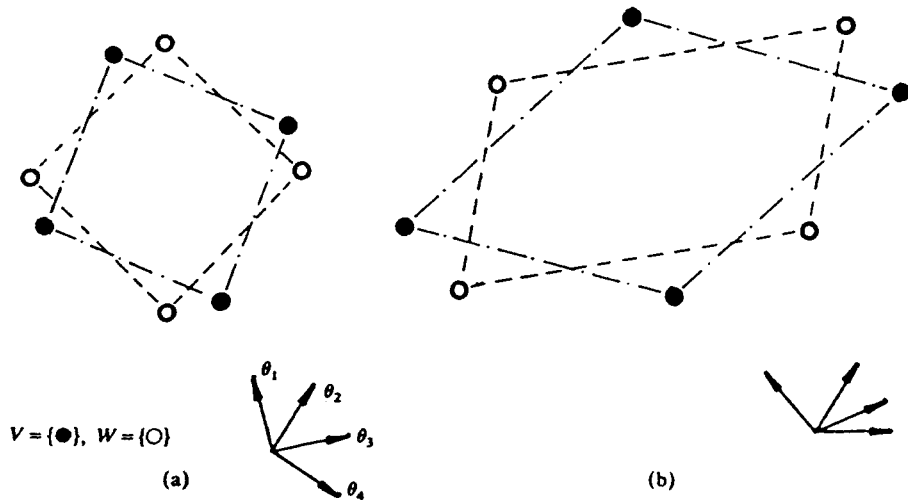


Fig. 1

C be a set consisting of m points. The projections in the directions $\theta_i, i = 1, \dots, m$, fail to determine uniquely C if and only if the following conditions hold:

- (ii)(a) there exists an affine map T in the plane such that $T(C)$ is a regular polygon;
- (ii)(b) the directions $T(\theta_i)$ are different from the directions of the sides of $T(C)$;
- (ii)(c) the directions $T(\theta_i)$ are equally spaced.

Let us observe that from condition (ii)(c) it follows that the set $\{\theta_i\}$ is affinely rational.

In the sequel \parallel will denote parallelism. To prove Proposition 2 we need the following lemma:

Lemma 1. *Let P be a convex polygon of $2m$ vertices $z_j, j = 1, \dots, 2m$, in a cyclic order. Let W and V be the convex polygons of vertices z_{2i} and $z_{2i-1}, i = 1, \dots, m$, respectively. If for all $j, j = 1, \dots, 2m$, the following condition holds:*

$$z_j z_{j+1} \parallel z_{j-1} z_{j+2} \parallel z_{j-2} z_{j+3}, \tag{2.2}$$

then there exists an affine map T such that $T(W)$ and $T(V)$ are two congruent, concentric regular m -gons.

Proof of Lemma 1. Since by the assumptions

$$z_{i+2} z_{i+3} \parallel z_{i+1} z_{i+4}, \quad z_{i+2} z_{i+5} \parallel z_{i+1} z_{i+6}, \quad z_{i+4} z_{i+5} \parallel z_{i+3} z_{i+6},$$

we have that the hexagon $z_{i+5} z_{i+4} z_{i+1} z_{i+6} z_{i+3} z_{i+2}$ is a Pascal hexagon for each i . Hence z_1, \dots, z_6 belong to a nondegenerate conic δ . Similarly z_2, \dots, z_7 belong to a conic, which must coincide with δ since z_2, \dots, z_6 belong to δ . It follows that z_i belongs to δ for each i . If δ is a parabola there exists j such that $z_j, z_{j+1}, \dots, z_{2m}, z_1, \dots, z_{j-1}$ are ordered on δ . It is easy to see now that $z_j z_{j-3}$ is not parallel to $z_{j-1} z_{j-2}$, which contradicts the assumptions. Similarly, if δ is a hyperbola it follows from the convexity of P that the vertices of P belong to the same branch of the hyperbola and the argument above can be repeated. So we have that δ is an ellipse. Thus there exists an affine map T such that $T(\delta)$ is a circle D . By the assumptions we get

$$T(z_j) T(z_{j+1}) \parallel T(z_{j-1}) T(z_{j+2})$$

and since the points $T(z_i), i = 1, \dots, 2m$, belong to the same circle D we derive that

$$d(T(z_{i-1}), T(z_i)) = d(T(z_{i+1}), T(z_{i+2})),$$

where d is the Euclidean distance. It follows that $T(W)$ and $T(V)$ are two congruent concentric regular m -gons. □

Proof of Proposition 2. Let P be the polygon defined by (2.1). The boundary of P consists of at most $2m$ sides. If the number of sides of P is less than $2m$, by repeating the argument in proof 2 of Proposition 1, we derive that C is uniquely determined. Therefore if C is not uniquely determined P has exactly $2m$ sides. Let $z_j, j = 1, \dots, 2m$, be the vertices of P in a cyclic order and let W and V be the polygons of vertices z_{2i} and $z_{2i-1}, i = 1, \dots, m$, respectively. Since $|C| = m$ and each side of P contains at least one point of C it follows that either $C = W$ or $C = V$. Since C is not uniquely determined W and V have the same projections in the directions θ_i . So for each direction θ_i there exists a side $z_j z_{j+1}$ such that $z_j z_{j+1} \parallel \theta_i$ and (2.2) holds.

Lemma 1 holds that there exists an affine map T in the plane such that $T(W)$ and $T(V)$ are congruent and concentric regular m -gons. Since either $C = W$ or $C = V$ we derive (ii)(a). Moreover, the directions of the sides of $T(P)$, that is to say the directions $T(\theta_i)$, are equally spaced and different from the directions of the sides of $T(C)$. This proves (ii)(b) and (ii)(c). Conversely, it is easily seen that if the conditions (ii)(a), (ii)(b), and (ii)(c) hold, C is not uniquely determined. This completes the proof. \square

Proposition 3. *Let C be a set consisting of $m + h$ points, with m and h positive integers. If*

$$m > 4h^2 + 11h, \tag{2.3}$$

then the projections of C in the directions $\theta_i, i = 1, \dots, m$, fail to determine C uniquely if and only if (ii)(a) and (ii)(b) hold and

(ii)(d) the set of directions $\{T(\theta_i)\}$ is a subset of a set of $m + h$ equally spaced directions.

First we give a definition and three lemmas.

Definition. Let J be a set of consecutive integers, with $|J| \geq 6$. Let $Q = \{q_j\}_{j \in J}$ be an ordered set of points. Q is *regular* if the following conditions hold:

any five consecutive points of Q are the vertices ordered counterclockwise of a convex pentagon (in the strict Euclidean sense: all vertex angles must be less than π);

$$\begin{aligned} q_i q_{i+1} \parallel q_{i-1} q_{i+2} & \text{ for } i, \quad J \supset \{i-1, i, i+1, i+2\}; \\ q_i q_{i+1} \parallel q_{i-2} q_{i+3} & \text{ for } i, \quad J \supset \{i-2, i, i+1, i+3\}. \end{aligned} \tag{2.4}$$

Lemma 2. *Let $Q = \{q_j\}_{j \in J}$ be a regular set with g the first index in J . Then*

$$q_i q_{i+1} \parallel q_j q_k \text{ for } i, j, k, \quad j < k, \quad 2i+1 = j+k, \quad J \supset \{i, j, k\}. \tag{2.5}$$

Moreover, there exists a unique point q_{g-1} such that $\{q_{g-1}\} \cup Q$ is a regular set.

Remark. Let us observe that the set $\{q_1, \dots, q_n\}$ of the vertices of a regular n -gon is a regular set for $n \geq 6$. In this case the point $q_{g-1} = q_0$ in Lemma 2 coincides with q_n .

Proof of Lemma 2. As in the proof of Lemma 1 we have that the set Q is inscribed in a conic δ . Moreover, let us observe that (2.4) implies (2.5) when $k-j < 6$. We argue by induction. Assume that (2.5) holds for $k-j < l$, with $l \geq 6$, and consider the hexagon $q_j q_{j+1} q_{k-3} q_{k-2} q_{k-1} q_k$, with $k-j = l$. We have $q_{k-1} q_{k-2} \parallel q_{k-3} q_k$. Since $(k-3) - (j+1) < l$, $(k-2) - j < l$, by induction $q_{i-1} q_i \parallel q_{j+1} q_{k-3}$ and $q_{i-1} q_i \parallel q_j q_{k-2}$. Therefore $q_j q_{k-2} \parallel q_{j+1} q_{k-3}$. Since the hexagon $q_j q_{j+1} q_{k-3} q_{k-2} q_{k-1} q_k$ is inscribed in δ by applying the Pascal theorem we derive that $q_j q_k \parallel q_{j+1} q_{k-1}$; since by induction $q_i q_{i+1} \parallel q_{j+1} q_{k-1}$ we have (2.5) for $k-j = l$.

We now prove that q_{g-1} is uniquely determined. In fact, let r be a line through q_{g+2} parallel to $q_g q_{g+1}$ and let s be a line through q_{g+4} parallel to $q_{g+1} q_{g+2}$. In order to satisfy (2.4) for $i = g$ and $i = g+1$ we have $q_{g-1} \in \{r \cap s\}$. So q_{g-1} is uniquely defined. Moreover, it is easily seen that $q_{g-1} q_g q_{g+1} q_{g+2} q_{g+3}$ is a convex pentagon and therefore $\{q_{g-1}\} \cup Q$ is a regular set. This concludes the proof. \square

Notation. Let A and B be two disjoint sets, each with $m+h$ points and with the same projections in the directions θ_i , $i = 1, \dots, m$. Let P be the convex hull of $A \cup B$, P^0 the interior of P , and ∂P the boundary of P . Let

$$\begin{aligned} a &= |A \cap \partial P|, & b &= |B \cap \partial P|, & \alpha &= |A \cap P^0|, & \beta &= |B \cap P^0|, \\ c &= a + b, & \gamma &= \alpha + \beta. \end{aligned} \tag{2.6}$$

By the assumptions it follows that

$$a + \alpha = b + \beta = m + h.$$

We denote by z_1, \dots, z_c the points of $(A \cup B) \cap \partial P$ in a cyclic order and by $\theta_i(z_j)$ the line through z_j in the direction θ_i .

Lemma 3. *Let θ_i be a fixed direction. Let z_{r-1}, z_t, z_u, z_w be vertices of P with $t+1 < u < w$, $z_{r-1} \in B$, $z_w \in A$. If $z_t z_u \parallel \theta_i$ and*

$$z_t, z_{t+1}, \dots, z_u \text{ are not collinear,} \tag{2.7}$$

$$\{z_{u+1}, z_{u+2}\} \cap A \neq \emptyset \text{ and } z_{r-1} \notin \theta_i(z_v) \text{ for each } z_v \in A \text{ with } u < v < w, \tag{2.8}$$

then one of the following conditions holds:

$$z_w \in \theta_i(z_{r-1}); \tag{2.9}$$

$$\theta_i(z_{r-1}) \cap A \cap P^0 \neq \emptyset; \tag{2.10}$$

$$\theta_i(z_w) \cap B \cap P^0 \neq \emptyset. \tag{2.11}$$

Proof of Lemma 3. Let r be the line (parallel to θ_i) containing the segment $z_t z_u$. First we prove that if (2.9) does not hold then $z_{r-1} \notin r$ and $z_w \notin r$. By contradiction let us assume that (2.9) does not hold and $z_{r-1} \in r$. We distinguish two cases:

- (a) $z_{r-1} \in z_t z_u$;
- (b) $z_{r-1} \notin z_t z_u$.

In each case $\partial P \supset z_i z_u$. In the first case, since P is convex, $z_{u+1}, z_{u+2} \in z_i z_u$ and, since (2.9) does not hold, $z_{u+1}, z_{u+2} \neq z_w$. This contradicts (2.8). In the second case, since P is convex, the vertices z_i, z_{i+1}, \dots, z_u belong to $z_i z_u$, and this contradicts (2.7). The proof that $z_w \notin r$ follows similarly.

Now it is easily seen that exactly one of the following cases will occur:

- (c) $z_w \in \theta_i(z_{i-1})$.
- (d) The endpoint of $\theta_i(z_w) \cap P$ different from z_w belongs to the open side $z_{i-1} z_i$.
- (e) The endpoint of $\theta_i(z_{i-1}) \cap P$ different from z_{i-1} belongs to the relative interior of the polygonal path z_u, \dots, z_w .

In the first case (2.9) holds. Let us consider the third case. As A and B have the same projections in the direction θ_i then $\theta_i(z_{i-1}) \cap A \neq \emptyset$. But, by (2.8), $\theta_i(z_{i-1}) \cap A \cap \partial P = \emptyset$ and then (2.10) follows. Similarly, in the second case (2.11) follows. This concludes the proof. \square

Lemma 4. *Let z_1, \dots, z_c be the points of $(A \cup B) \cap \partial P$. Let $d = \gamma + 3$ and let e be a positive integer with $2(d + e) \leq c + 2$. Let*

$$J = \{-d - e, \dots, 0, \dots, d + e + 1\}, \quad J' = \{-d - e - 1\} \cup J,$$

and

$$z_{-d+i-1} z_{-d+i} \parallel \theta_i \quad \text{for } i, \quad 1 \leq i \leq 2d + 1.$$

If $Z = \{z_j\}_{j \in J}$ is regular and

$$z_{2i+1} \in A, \quad z_{2j} \in B \quad \text{for } i, j, \quad 2i + 1 \in J, \quad 2j \in J, \quad (2.12)$$

then $Z' = \{z_j\}_{j \in J'}$ is regular and (2.12) holds for $i, j, 2i + 1 \in J', 2j \in J'$.

We recall that $|(A \cup B) \cap \partial P| = c$. Therefore, if $2(d + e) + 2 > c$ the points of Z are not all distinct; for instance, if $c = 2(d + e)$ then $z_{-d-e} = z_{d+e}$.

Proof of Lemma 4. By repeatedly applying Lemma 2 there exist $2\gamma + 1$ points $q_{-d-e-(2\gamma+1)}, \dots, q_{-d-e-1}$ such that the set

$$\{q_{-d-e-(2\gamma+1)}, \dots, q_{-d-e-1}, z_{-d-e}, z_{-d-e+1}, \dots, z_{d+e+1}\} \quad (2.13)$$

is regular.

Let $f = -d + e - 1$. Since by assumption $z_{-d+i-1} z_{-d+i} \parallel \theta_i$ for $i, 1 \leq i \leq 2d + 1$, then (2.13) implies

$$z_{-d-e} \in \theta_i(z_{f+2i}) \quad \text{for } i, \quad i \geq 1, \quad f + 2i \leq d + e + 1; \quad (2.14)$$

$$q_{-d-e-1} \in \theta_i(z_{f+2i+1}) \quad \text{for } i, \quad i \geq 1, \quad f + 2i + 1 \leq d + e + 1; \quad (2.15)$$

$$q_{-d-e-2} \in \theta_i(z_{f+2i+2}) \quad \text{for } i, \quad i \geq 1, \quad f + 2i + 2 \leq d + e + 1; \quad (2.16)$$

$$q_{-d-e-3} \in \theta_i(z_{f+2i+3}) \quad \text{for } i, \quad i \geq 1, \quad f + 2i + 3 \leq d + e + 1. \quad (2.17)$$

We assume that $z_{-d-e} \in A$; that is, $d+e$ is odd. In the other case the proof follows similarly.

To prove Lemma 4 we prove four statements.

(I) *Let us assume that $z_{-d-e-1} \in B$ and Z' is not regular. Then $q_{-d-e-1} \in B \cap P^0$.*

If Z' is not regular then

$$z_{-d-e-1} \neq q_{-d-e-1}. \quad (2.18)$$

Let i be such that $i \geq 2$, $f+2i \leq d+e$. Since $d = \gamma + 3 = (\alpha + \beta) + 3$ and $f = -d + e - 1$, then i can take $\alpha + \beta + 2$ different values. Let us consider the four points z_{-d-e-1} , z_{-d-e} , z_{f+2i} , and z_{f+2i+1} . Lemma 3 and (2.14) imply that one of the following cases will occur:

- (a_i) $z_{-d-e-1} \in \theta_i(z_{f+2i+1})$;
- (b_i) $\theta_i(z_{f+2i+1}) \cap B \cap P^0 \neq \emptyset$;
- (c_i) $\theta_i(z_{-d-e-1}) \cap A \cap P^0 \neq \emptyset$.

If there exist i, j , $i \neq j$, such that (a_i) and (a_j) hold, then

$$z_{-d-e-1} \in \theta_i(z_{f+2i+1}) \cap \theta_j(z_{f+2j+1})$$

and, by (2.15), we derive $z_{-d-e-1} = q_{-d-e-1}$, contradicting (2.18).

Therefore,

$$(a_i) \text{ holds at most for one index.} \quad (2.19)$$

Furthermore, as $\alpha = |A \cap P^0|$, then

$$(c_i) \text{ holds at most for } \alpha \text{ indices.} \quad (2.20)$$

Form (2.19) and (2.20) it follows that (b_i) must hold at least for $\beta + 1$ indices.

We now prove that there exist i, j , $i \neq j$, such that (b_i) and (b_j) hold and

$$\theta_i(z_{f+2i+1}) \cap \theta_j(z_{f+2j+1}) \cap B \cap P^0 \neq \emptyset. \quad (2.21)$$

Otherwise, for each i, j , $i \neq j$, such that (b_i) and (b_j) hold $\theta_i(z_{f+2i+1}) \cap B \cap P^0$ and $\theta_j(z_{f+2j+1}) \cap B \cap P^0$ are disjoint sets. Therefore $\beta + 1 \leq |B \cap P^0|$ contradicting the definition of β .

By (2.15), $\theta_i(z_{f+2i+1}) \cap \theta_j(z_{f+2j+1}) = \{q_{-d-e-1}\}$, therefore (2.21) implies that $q_{-d-e-1} \in B \cap P^0$. This concludes the proof of statement (I).

(II) *Let us assume that $z_{-d-e-1} \in B$ and Z' is not regular. Then $q_{-d-e-3} \in B \cap P^0$.*

Let i be such that $i \geq 2$, $f+2i+2 \leq d+e$. Since $d = \gamma + 3 = (\alpha + \beta) + 3$ and $f = -d + e - 1$, then i can take $\alpha + \beta + 1$ different values. First we prove that one of the following cases will occur:

- (d_i) $z_{-d-e-1} \in \theta_i(z_{f+2i+3})$;
- (e_i) $\theta_i(z_{f+2i+3}) \cap B \cap P^0 \neq \emptyset$;
- (f_i) $\theta_i(z_{-d-e-1}) \cap A \cap P^0 \neq \emptyset$.

We fix an index i and distinguish two cases: (a_i) does not hold and (a_i) holds.

If (a_i) does not hold, let us consider the four points z_{-d-e-1} , z_{-d-e} , z_{2i+f} , and z_{2i+f+3} . In this case $z_{-d-e-1} \notin \theta_i(z_v)$ for each $z_v \in A$ with $2i+f < v < 2i+f+3$. Therefore by Lemma 3 and (2.14) it follows that either (d_i) , (e_i) or (f_i) holds.

If (a_i) holds, by (2.15), z_{2i+f+1} , z_{-d-e-1} , and q_{-d-e-1} are on the same straight line parallel to θ_i ; therefore $|\theta_i(z_{-d-e-1}) \cap B| \geq 2$ and, since A and B have the same projections in the direction θ_i , (f_i) follows. In any case either (d_i) , (e_i) or (f_i) holds.

We observe that

$$q_{-d-e-3} \neq z_{-d-e-1}. \tag{2.22}$$

Otherwise (2.13) implies that the polygon $z_{-d-e-1}q_{-d-e-1}z_{-d-e}z_{-d-e+1}z_{-d-e+2}$ is convex. Since z_{-d-e-1} , z_{-d-e} , z_{-d-e+1} , and z_{-d-e+2} are consecutive points in the boundary of the convex polygon P , by the definition of P it follows that $q_{-d-e-1} \notin P^0$, contradicting statement (I).

If there exist i, j , $i \neq j$, such that (d_i) and (d_j) hold then

$$z_{-d-e-1} \in \theta_i(z_{f+2i+3}) \cap \theta_j(z_{f+2j+3})$$

and, by (2.17), we derive $z_{-d-e-1} = q_{-d-e-3}$, contradicting (2.22). Therefore

$$(d_i) \text{ holds at most for one index.} \tag{2.23}$$

Furthermore, as $\alpha = |A \cap P^0|$, then

$$(f_i) \text{ holds at most for } \alpha \text{ indices.} \tag{2.24}$$

From (2.23) and (2.24) it follows that (e_i) has to hold at least for β indices.

We now prove that there exist i, j , $i \neq j$, such that (e_i) and (e_j) hold and

$$\theta_i(z_{f+2i+3}) \cap \theta_j(z_{f+2j+3}) \cap B \cap P^0 \neq \emptyset. \tag{2.25}$$

Otherwise, for each i, j , $i \neq j$, such that (e_i) and (e_j) hold $\theta_i(z_{f+2i+3}) \cap B \cap P^0$ and $\theta_j(z_{f+2j+3}) \cap B \cap P^0$ are disjoint sets.

Furthermore, if (e_i) holds and $q_{-d-e-1} \in \theta_i(z_{f+2i+3}) \cap B \cap P^0$ then, by (2.15), z_{2i+f+1} , z_{2i+f+3} , and q_{-d-e-1} are on the same straight line parallel to θ_i . Since $z_{2i+f+1}, z_{2i+f+3} \in A \cap \partial P$, $q_{-d-e-1} \in B \cap P^0$, and A and B have the same projections in the direction θ_i , then

$$|\theta_i(z_{f+2i+3}) \cap B \cap P^0| \geq 2.$$

In conclusion, if (2.25) does not hold then the set

$$\bigcup_{i: (e_i) \text{ holds}} \{\theta_i(z_{f+2i+3}) \cap B \cap P^0\} \cup \{q_{-d-e-1}\}$$

contains at least $\beta + 1$ points. Therefore $\beta + 1 \leq |B \cap P^0|$, which contradicts the definition of β .

By (2.25) and (2.17) it follows that $q_{-d-e-3} \in B \cap P^0$.

(III) Let $z_{-d-e-1} \in B$. Then $z_{-d-e-1} = q_{-d-e-1}$; that is, Z' is regular.

Let us assume that Z' is not regular. By induction, the same argument as in the proof of statement (I) and (II) shows that

$$q_{-d-e-1}, q_{-d-e-3}, \dots, q_{-d-e-(2\beta+1)} \in B \cap P^0.$$

This contradicts the assumption that $\beta = |B \cap P^0|$.

(IV) $z_{-d-e-1} \in B$.

We argue by contradiction. Let us assume that $z_{-d-e-1} \in A$. The same argument as in the proof of statement (III) shows that

$$z_{-d-e-1} = q_{-d-e-2}. \tag{2.26}$$

We observe that (2.26), (2.14), and (2.16) imply that

$$z_{-d-e} \in \theta_i(z_{f+2i}), \quad z_{-d-e-1} \in \theta_i(z_{f+2i+2}),$$

for $i, i \geq 1, f+2i+2 \leq d+e+1$. Since the line $\theta_i(z_{f+2i+1})$ lies between the lines $\theta_i(z_{f+2i})$ and $\theta_i(z_{f+2i+2})$ then $\theta_i(z_{f+2i+1}) \cap \partial P = \{z_{2i+f+1}, z\}$ with z in the open segment $z_{-d-e-1}z_{-d-e}$. Since the open segment does not contain points of $A \cup B$ and A and B have the same projections in the direction θ_i , then

$$\theta_i(z_{f+2i+1}) \cap B \cap P^0 \neq \emptyset. \tag{2.27}$$

We now prove that there exist $i, j, i \neq j, 1 \leq i \leq d, 1 \leq j \leq d$, such that

$$\theta_i(z_{f+2i+1}) \cap \theta_j(z_{f+2j+1}) \cap B \cap P^0 \neq \emptyset. \tag{2.28}$$

Otherwise $\theta_i(z_{f+2i+1}) \cap B \cap P^0$ and $\theta_j(z_{f+2j+1}) \cap B \cap P^0$ are disjoint sets for each $i, j, i \neq j$, and this implies that $|B \cap P^0| \geq d = \gamma + 3$ contradicting the definition of γ .

Inequalities (2.28) and (2.15) imply that

$$q_{-d-e-1} \in B \cap P^0. \tag{2.29}$$

Equations (2.26) and (2.13) imply that the polygon

$$z_{-d-e-1}q_{-d-e-1}z_{-d-e}z_{-d-e+1}z_{-d-e+2}$$

is convex. Therefore, since $z_{-d-e-1}, z_{-d-e}, z_{-d-e+1}$, and z_{-d-e+2} are consecutive points in the boundary of the convex polygon P , by the definition of P it follows that $q_{-d-e-1} \notin P^0$. This contradicts (2.29). This contradiction concludes the proof of statement (IV) and of Lemma 4. □

Proof of Proposition 3. First let us observe that each line l in the direction θ_i that supports P must contain at least one point of A and one point of B . Therefore

$$m \leq a, \quad m \leq b, \quad 2m \leq a + b, \tag{2.30}$$

and since $a + \alpha = m + h$, $b + \beta = m + h$ we have

$$\alpha \leq h, \quad \beta \leq h, \quad \gamma \leq 2h. \tag{2.31}$$

In particular, P has at least $2m$ sides and, since $h > 0$, (2.3) implies that P has at least 30 sides.

We now prove that there exists an affine map T such that

$$T(A \cap \partial P) \text{ and } T(B \cap \partial P) \text{ are two congruent and concentric regular polygons.} \tag{2.32}$$

In the sequel we denote by R_i the interior of the convex hull of the six points $z_{i-2}, z_{i-1}, z_i, z_{i+1}, z_{i+2}, z_{i+3}$. Since P has more than ten sides, any five consecutive such hexagons intersect and any five nonconsecutive hexagons have empty intersection.

For each $x \in (A \cup B) \cap P^0$ let us define

$$F_x = \bigcup_{i: x \in R_i} \{z_i z_{i+1}\}.$$

Let us observe that F_x consists of consecutive segments $z_i z_{i+1}$ and it consists of at most five such segments. Let

$$F = \bigcup_{x \in (A \cup B) \cap P^0} F_x;$$

we have that F consists of at most γ connected components and of 5γ segments $z_i z_{i+1}$.

Let s_j and r_j be the lines parallel to θ_j that support P . On $s_j \cap \partial P$ we choose a segment $z_j z_{j+1}$ with one end in A and the other in B . Similarly, we choose on $r_j \cap \partial P$ another such segment. Let E be the union of the segments above for $j = 1, \dots, m$, let $G = \partial P \setminus E$, and let

$$D = \partial P \setminus (G \cup F). \tag{2.33}$$

Since G contains $a + b - 2m$ segments $z_i z_{i+1}$ then $G \cup F$ consists of at most $a + b - 2m + \gamma$ connected components and of at most $a + b - 2m + 5\gamma$ segments $z_i z_{i+1}$. Since $a + \alpha = b + \beta = m + h$, $\alpha + \beta = \gamma$, then $G \cup F$ consists of at most $2h$ connected components. Therefore, by (2.33), D too consists of at most $2h$ connected components.

Since $D = E \setminus F$ and by definition E contains at least $2m$ segments $z_i z_{i+1}$, we derive that D contains at least $2m - 5\gamma$ segments $z_i z_{i+1}$. Therefore there exists a connected component Z of D which contains a number of consecutive segments $z_i z_{i+1}$ greater than or equal to $(2m - 5\gamma)/2h$. From (2.3) and (2.31) Z contains at least $2\gamma + 7$ consecutive segments $z_i z_{i+1}$ of P .

We can assume that the set of vertices of Z have the order $z_{-d}, z_{-d+1}, \dots, z_{d+1}$, with $d = \gamma + 3$. Furthermore, since the segments in Z belong to E , we can assume that

$$\begin{aligned} z_{-d+i-1} z_{-d+i} &\parallel \theta_i && \text{for } i = 1, \dots, 2\gamma + 7, \\ z_{2i+1} &\in A && \text{for } i, \quad -d \leq 2i + 1 \leq d + 1, \\ z_{2i} &\in B && \text{for } i, \quad -d \leq 2i \leq d + 1. \end{aligned}$$

Let us consider the four points z_{j-1}, z_j, z_{j+1} , and z_{j+2} for $j, -d + 1 \leq j \leq d - 1$. Since $z_j z_{j+1} \parallel \theta_{-d+j+1}$, since by (2.33) $R_j \cap (A \cup B) \cap P^0 = \emptyset$ and since A and B have the same projections in the direction θ_{-d+j+1} , it follows that $z_{j-1} z_{j+2} \parallel \theta_{-d+j+1}$. Similarly,

$$z_{j-2} z_{j+3} \parallel \theta_{-d+j+1} \quad \text{for } j, -d + 2 \leq j \leq d - 2.$$

Therefore $\{z_{-d}, z_{-d+1}, \dots, z_{d+1}\}$ is a regular set.

We now apply Lemma 4 to show that $\{z_{-d-1}, z_{-d}, \dots, z_{d+1}\}$ is a regular set and $z_{-d-1} \in A$. Similarly, $\{z_{-d-1}, z_{-d}, \dots, z_{d+1}, z_{d+2}\}$ is a regular set and $z_{d+2} \in B$.

By repeating the argument above we get that $z_{2i+1} \in A, z_{2i} \in B$ for each i , that $c = a + b$ is even, and that $\{z_{-(c/2)-2}, z_{-(c/2)-1}, \dots, z_{(c/2)+3}\}$ is regular. This implies that P satisfies the assumptions of Lemma 1. Therefore there exists an affine map T satisfying (2.32). We conclude that $A \cap \partial P$ and $B \cap \partial P$ have the same projections in the directions $\theta_i, i = 1, \dots, m$. Then $A \cap P^0$ and $B \cap P^0$ also have the same projections in the directions $\theta_i, i = 1, \dots, m$. Since

$$|A \cap P^0| = |B \cap P^0| < m,$$

from Proposition 1 it follows that

$$A \cap P^0 = B \cap P^0 = \emptyset.$$

This proves (ii)(a), (ii)(b), and (ii)(d). Conversely, it is easily seen that if conditions (ii)(a), (ii)(b), and (ii)(d) hold, C is not uniquely determined. This completes the proof. □

We conjecture that Proposition 3 holds even if in (2.3) the lower bound for m is decreased. However, this bound cannot be too low. For instance, for $h = 1$ and $m = 4$ Proposition 3 does not hold. In fact, for any set of four direction $\theta_1, \theta_2, \theta_3, \theta_4$ there exist two sets A and B consisting of five points, with the same projections in the directions θ_i . In Fig. 2 A consists of black points, B consists of white points, $c = \sin(\alpha + \beta)/\cos \alpha \cdot \sin \beta, d = -(c \cot \alpha + \tan \beta), \alpha, \beta \in$

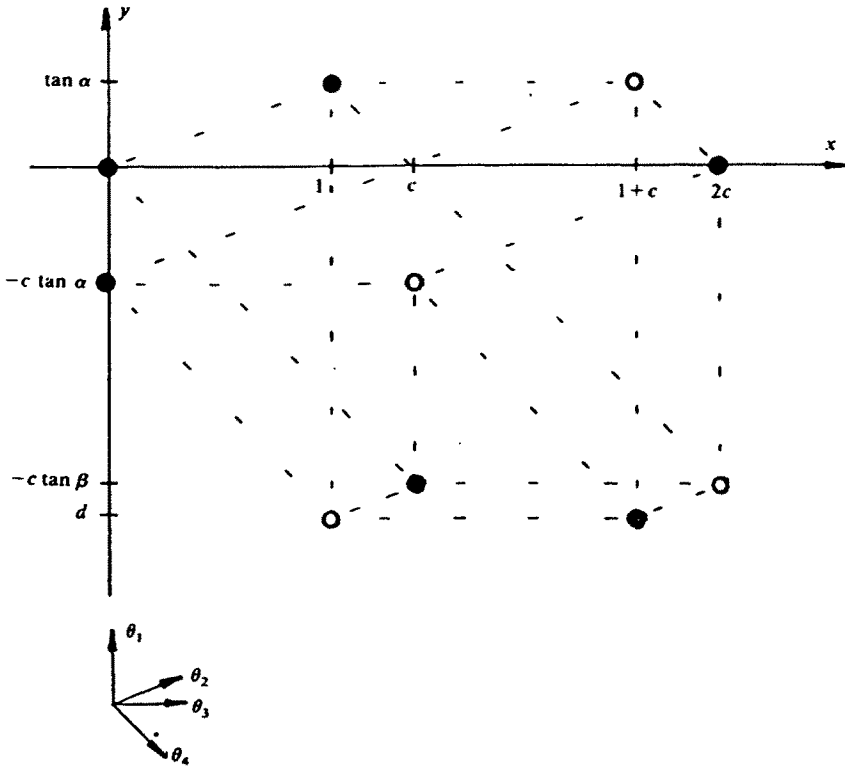


Fig. 2

$(0, \pi/2)$, and the directions $\theta_1, \theta_2, \theta_3$ and θ_4 are, respectively, given by the vectors $(0, 1), (\cos \alpha, \sin \alpha), (1, 0)$, and $(\cos \beta, -\sin \beta)$.

The following proposition shows that when there is no *a priori* bound on $|C|$, then for any finite set of directions $\{\theta_i\}$ the uniqueness property for Problem B does not hold.

Proposition 4. *Let $\{\theta_1, \theta_2, \dots, \theta_m\}$ be an arbitrary finite set of directions in the plane. Then there exist two distinct finite sets A and B with the same projections in the directions θ_i .*

Proof. First let us observe that if $m = 1$ Proposition 4 is trivial. For $m > 1$ we argue by induction. Let A and B be two finite sets with the same projections in the directions $\theta_i, i = 1, \dots, m - 1$, and let r be a fixed vector with direction θ_m . It is easy to see that the sets

$$\bar{A} = A \cup \{B + r\}, \quad \bar{B} = B \cup \{A + r\}$$

have the same projections in the directions $\theta_i, i = 1, \dots, m$ (see Fig. 3(a) and (b)). □

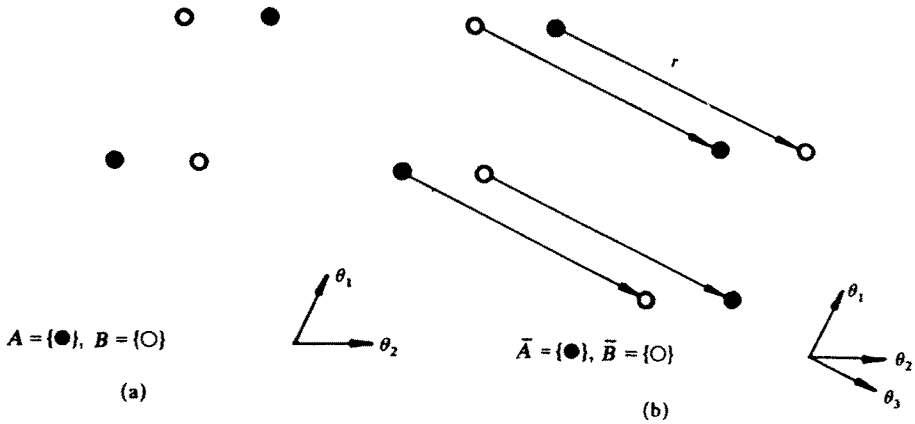


Fig. 3

Let us observe that the cardinality of the sets A and B in the proof of Proposition 4 is equal to 2^{m-1} .

The following result is related to the classic nonuniqueness theorem for Radon transforms; it was observed by Lorentz [4].

Corollary. *Let $\{\theta_1, \theta_2, \dots, \theta_m\}$ be an arbitrary finite set of directions in the plane. Then there exist two distinct sets with nonempty interior and with the same projections in the directions θ_i .*

Proof. Let A and B be as in Proposition 4. Let us consider two families Γ_1 and Γ_2 of disjoint homogeneous and congruent disks C_i with center the points of A and B , respectively. Let

$$F = \bigcup_{C_i \in \Gamma_1} C_i, \quad G = \bigcup_{C_i \in \Gamma_2} C_i;$$

then F and G have the same projections. □

The results in this section are also connected with projections of a finite number of mass points, that is, points in which positive masses are concentrated [7].

3. Reconstruction of Convex Bodies with Holes

In this section we prove two theorems which provide conditions for the reconstruction of a homogeneous convex body K with a finite number of disjoint holes (Problem B).

Definition. Let n be a positive integer and let K_n be the class of plane convex

bodies with at most n disjoint circular holes. More precisely, let

$$K_n = \left\{ M \setminus \bigcup_{h=1}^t Q_h : M \text{ is a plane convex body, } t \leq n, Q_h \text{ is a disk and } M \supset Q_h \text{ for } h = 1, \dots, t, Q_h \cap Q_k = \emptyset \text{ for } h \neq k \right\}.$$

Similarly we define

$$\tilde{K}_n = \left\{ H \setminus \bigcup_{i=1}^s C_i : H \text{ is a plane convex body, } s \leq n, C_i \text{ is a disk and } H^0 \supset C_i \text{ for } i = 1, \dots, s, C_i \cap C_j = \emptyset \text{ for } i \neq j \right\}.$$

Recall that knowing the projection of K in the direction θ_i is equivalent to knowing the values of the integral of the characteristic function of K along each straight line in the direction θ_i .

First we prove the following lemma.

Lemma 5. *Let K belong to \tilde{K}_n and let W belong to K_n . Let us assume that K and W have the same projections in the directions $\theta_i, i = 1, \dots, m$. Then the holes of W coincide with those of K if one of the following conditions holds:*

$$m > n; \tag{3.1}$$

$$m = n \text{ and the set } \{\theta_1, \theta_2, \dots, \theta_m\} \text{ is not affinely rational.} \tag{3.2}$$

Proof. We have

$$K = H \setminus \bigcup_{i=1}^s C_i \tag{3.3}$$

and

$$W = M \setminus \bigcup_{h=1}^t Q_h, \tag{3.4}$$

with M and H plane convex bodies, C_i and Q_h disks, $t \leq n$, and $s \leq n$. Let θ_j be a fixed direction. We assume that θ_j is orthogonal to the x -axis. The projection of H in the direction θ_j is a concave function $h_j(x)$, defined on a compact interval $[m_j, d_j]$. Moreover, the projection of a disk C_i in the direction θ_j is given by the function $g_{i,j}(x)$ where

$$g_{i,j}(x) = \begin{cases} 2\sqrt{r_i^2 - (x - a_i)^2} & \text{if } a_i - r_i \leq x \leq a_i + r_i, \\ 0 & \text{otherwise,} \end{cases}$$

where a_i and r_i denote, respectively, the abscissa of the center and the radius of C_i . Therefore the projection $f_j(x)$ of K in the direction θ_j is

$$f_j(x) = h_j(x) - \sum_{i=1}^s g_{i,j}(x), \quad x \in [m_j, d_j]. \tag{3.5}$$

Similarly, if we consider the projection of W , from (3.4) we have

$$f_j(x) = v_j(x) - \sum_{h=1}^t u_{h,j}(x), \quad x \in [m_j, d_j], \quad (3.6)$$

where $v_j(x)$ is the projection of M in the direction θ_j and

$$u_{h,j}(x) = \begin{cases} 2\sqrt{p_h^2 - (x - b_h)^2} & \text{if } b_h - p_h \leq x \leq b_h + p_h, \\ 0 & \text{otherwise,} \end{cases}$$

where b_h denotes the abscissa of the centre of Q_h and p_h its radius.

Since $h_j(x)$ and $v_j(x)$ are concave functions in (m_j, d_j) , from (3.5) and (3.6) we infer that f_j has an unbounded one-sided derivative in (m_j, d_j) at the points $a_i \pm r_i$, $i = 1, \dots, s$, and at the points $b_h \pm p_h$, $h = 1, \dots, t$. In other words, from the projection of K in the direction θ_j we determine the sets L_j and R_j of the lines parallel to θ_j which are tangent from the left and from the right (resp.) to some disk C_i and to some disk Q_h (see Fig. 4).

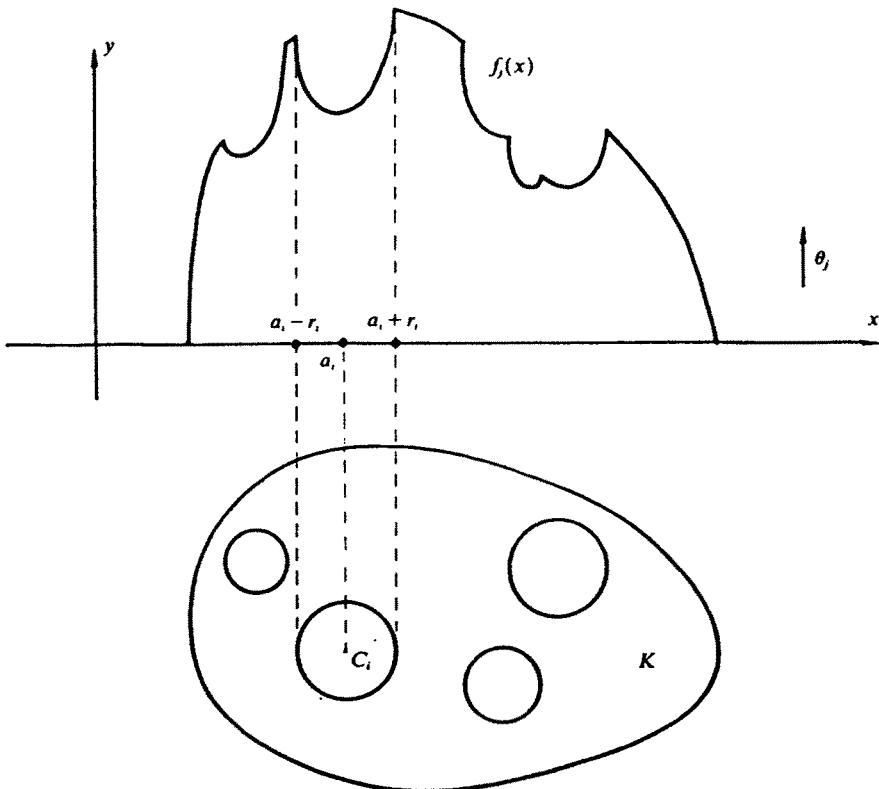


Fig. 4

But we have more. Let l belong to L_j , let $E(l)$ be the family of disks C_i tangent from the right to l , and let $F(l)$ be the corresponding family of disks Q_h . Since $K \in \tilde{K}_n$, l intersects the x -axis in a point with abscissa x in (m_j, d_j) . By differentiating (3.5) we get

$$D^+ f_j(x + \varepsilon) = D^+ h_j(x + \varepsilon) - \frac{1}{\sqrt{2\varepsilon}} \sum_{i: a_i - r_i = x} \sqrt{r_i} + O(\sqrt{\varepsilon}), \quad \varepsilon > 0, \quad (3.7)$$

where $O(\sqrt{\varepsilon})$ goes to zero when ε tends to zero, and D^+ denotes right differentiation. Similarly, by differentiating (3.6) we get

$$D^+ f_j(x + \varepsilon) = D^+ v_j(x + \varepsilon) - \frac{1}{\sqrt{2\varepsilon}} \sum_{h: b_h - p_h = x} \sqrt{p_h} + O(\sqrt{\varepsilon}), \quad \varepsilon > 0. \quad (3.8)$$

From (3.7) and (3.8) we obtain

$$\sum_{i: C_i \in E(l)} \sqrt{r_i} = \sum_{h: Q_h \in F(l)} \sqrt{p_h}. \quad (3.9)$$

Let us observe that if there exists a circle D such that

$$D \in \{C_i\} \cap \{Q_h\},$$

then $K \cup D$ and $W \cup D$ satisfy the assumptions of Lemma 5 and (3.1).

We argue by contradiction and we assume that $\{C_i\} \neq \{Q_h\}$; from the remark above it follows that we may assume

$$\{C_i\} \cap \{Q_h\} = \emptyset. \quad (3.10)$$

Let C be a fixed circle of $\{C_i\}$, with radius r . Let

$$L(C) = \left\{ l \in \bigcup_{j=1}^m L_j, l \text{ tangent to } C \right\},$$

$$R(C) = \left\{ l \in \bigcup_{j=1}^m R_j, l \text{ tangent to } C \right\}.$$

We have

$$|L(C)| = |R(C)| = m. \quad (3.11)$$

Let us assume now that (3.1) holds. From (3.11), by the pigeonhole principle, we derive

There exists at least one circle Q in $\{Q_h\}$ such that Q has three tangent lines in common with C , and each of the tangent lines supports Q and C from the same side. (3.12)

From (3.12) we get that $Q = C$, which contradicts (3.10). This concludes the proof when (3.1) holds.

Let us assume now that (3.2) holds. Statement (3.12) cannot occur, otherwise $Q = C$ which contradicts (3.10). By the pigeonhole principle and (3.11) we derive that $s = t = m$ and the following situation occurs:

Each circle Q_h has exactly two tangent lines in common with C , for each circle C in $\{C_i\}$; conversely, each circle C_i has exactly two tangent lines in common with Q , for each circle Q in $\{Q_h\}$. (3.13)

From (3.13) it follows that each line l in L_j is tangent from the left just to one circle Q of $\{Q_h\}$ and just to one circle C of $\{C_i\}$. From (3.9) we deduce that the radius of Q is equal to the radius of C and therefore we get that all the circles C_i and Q_h have the same radius.

Let A be the set of the centers of $\{C_i\}$ and B of $\{Q_h\}$. We know that for each line l in L_j there is exactly one point in A and one point in B on the same side of l and at the same distance from l . Therefore we conclude that A and B have the same projections in the directions $\theta_j, j = 1, \dots, m$. So the problem has been reduced to Problem A. Since the set $\{\theta_1, \theta_2, \dots, \theta_m\}$ is not affinely rational we conclude that $A = B$. This concludes the proof. □

From Lemma 5 and the Gardner-McMullen result [2] quoted in the Introduction, Theorem 1 follows.

Theorem 1. *Any set $K \in \tilde{K}_n$ is uniquely determined by its projections in m directions $\theta_1, \theta_2, \dots, \theta_m$ if $m \geq n$ and if the set $\{\theta_1, \theta_2, \dots, \theta_m\}$ is not affinely rational.*

Notation. Given $K \in \tilde{K}_n$ and a direction θ_i we denote by $S_i(K)$ the Steiner symmetral of K in the direction θ_i .

$S_i(K)$ is a symmetric set with respect to the line through the origin perpendicular to the direction θ_i ; furthermore,

$S_i(K) \cap r$ is a connected set, possibly empty, for any line r in the direction θ_i , $S_i(K)$ has the same projection as K in the direction θ_i .

Let us observe that $S_i(K)$ provides the same data for problem B in a different form. We denote by Λ the class of the sets which are the Steiner symmetrals of some K in \tilde{K}_n with respect to some direction in the plane. Given m directions $\theta_1, \theta_2, \dots, \theta_m$ we introduce the mapping S :

$$S(K) = (S_1(K), S_2(K), \dots, S_m(K)), \quad K \in \tilde{K}_n,$$

from \tilde{K}_n to $(\Lambda)^m$. We put both on \tilde{K}_n and on Λ the topology induced by the Hausdorff distance

$$d(H, K) = \max \left(\sup_{x \in H} \inf_{y \in K} \|x - y\|, \sup_{x \in K} \inf_{y \in H} \|x - y\| \right).$$

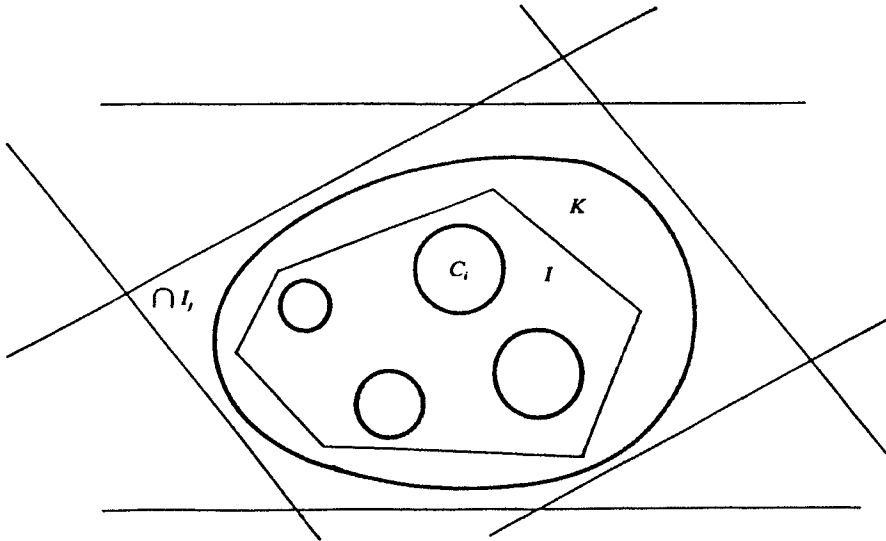


Fig. 5

Now we prove the well posedness for Problem B.

Theorem 2. *If $m \geq n$ and the set $\{\theta_1, \theta_2, \dots, \theta_m\}$ is not affinely rational then the mapping S is continuous and continuously invertible on $S(\tilde{K}_n)$.*

Proof. Volčič [9] has proved Theorem 2 when $n = 0$, that is for homogeneous convex bodies without holes. The general proof follows by a similar argument. Therefore we outline only the principal differences.

Let $K = H \setminus \bigcup_{i=1}^s C_i$, with H a convex body and C_i disjoint disks. Let I_j be a strip parallel to θ_j , containing K in its interior (see Fig. 5). Let us assume that there exists a sequence $\{A_t\}$, $A_t \in \tilde{K}_n$, such that $\{S(A_t)\}$ converges to $S(K)$ and $\{A_t\}$ does not converge to K .

For t large enough $I_j \supset S_j(A_t)$, $j = 1, \dots, m$; therefore $\bigcap_{j=1}^m I_j \supset A_t$. Since $\bigcap_{j=1}^m I_j$ is a compact set there exists a subsequence $\{A_{\lambda_t}\}$ of $\{A_t\}$ converging to a set $W \in K_n$, $W \neq K$. Since S is a continuous mapping, W has the same projections of K in the directions θ_i , and Lemma 3 implies that the holes of W coincide with those of K . Therefore $W \in \tilde{K}_n$ and by Theorem 1, $W = K$, contrary to the assumptions. This concludes the proof. \square

In addition we point out a nonuniqueness result for Problem B when we have no *a priori* bound on the number of holes.

Proposition 5. *Let $\{\theta_1, \theta_2, \dots, \theta_m\}$ be an arbitrary finite set of directions in the plane. Then there exist two distinct sets K_1 and K_2 in \tilde{K}_n , n large enough, with the same projections in the directions θ_i .*

Proof. Let F and G be as in the proof of Proposition 4. For this proof it is sufficient to consider the set K_1 and K_2 defined by

$$K_1 = H \setminus F, \quad K_2 = H \setminus G,$$

where H is a convex body containing in its interior F and G . □

Finally, we mention the problem of reconstructing the spatial trajectories of elementary particles. This arises in bubble chamber experiments, where trajectories are photographed from several viewpoints (see [6]). The reconstruction for any planar section, when the optic axes are coplanar, suggests studying the following analogue of Problem A:

Given a set of m directions $\{\theta_1, \theta_2, \dots, \theta_m\}$, let l_i be a straight line orthogonal to θ_i , passing through the origin of the axis. The problem consists in reconstructing a finite set C in the plane knowing the orthogonal projection $\pi_i(C)$ of C on each line l_i . $\pi_i(C)$ is the union of the orthogonal projections of each point in C on the line l_i . Unlike Problem A it happens that for each point $x \in \pi_i(C)$ we do not know the number of the points in C with the same orthogonal projection x .

We are able to prove that Propositions 1 and 2 of Section 2 hold for this problem too.

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