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An Inequality for the Volume of Inscribed Ellipsoids

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Abstract. Let K be a convex body in R^n, and let $x^* \in \operatorname{int} K$ be the center of the ellipsoid of the maximal volume inscribed in the body. An arbitrary hyperplane through x^* cuts K into two convex bodies K^+ and K^-. We show that $w(K^\pm)/w(K) \leq 0.844\ldots$, where $w(\cdot)$ is the volume of the inscribed ellipsoid.

1. Introduction

Let K be a convex body in R^n. It is known [1] that among all the ellipsoids E contained in the body, $E \subseteq K$, there exists a unique ellipsoid $E^* = E^*(K)$ of the maximal volume. We call E^* the ellipsoid inscribed in the body and denote by

$$w(K) = \max\{\operatorname{vol} E \mid E \text{ is an ellipsoid, } E \subseteq K\}$$

the volume of E^* viewed as a function of K. The center $x^* = x^*(K)$ of the inscribed ellipsoid is called the center of the body.

Let x^* be the center of a convex body K. Since $x^* \in \operatorname{int} K$, an arbitrary hyperplane $P = \{x \in R^n \mid p^t(x - x^*) = 0\}$ through x^* splits K into two convex bodies

$$K^+ = \{x \in K \mid p^t(x - x^*) \geq 0\}, \qquad K^- = \{x \in K \mid p^t(x - x^*) \leq 0\}.$$

We show that the size $w(\cdot)$ of the splinters K^+, K^- is always at least $\alpha = 0.844\ldots$ times smaller than the size of the initial body, i.e., for all n, K, and P

$$\max\left\{\frac{w(K^+)}{w(K)}, \frac{w(K^-)}{w(K)}\right\} \leq \alpha = 0.844\ldots. \tag{1}$$

Remark. In [2] inequality (1) was proven with a worse constant of shrinkage, $\alpha = 0.888\ldots$. It may be conjected that the best possible value of α in inequality (1) equals $0.5e^{1/2} = 0.824\ldots$. The latter value is attained as $n \to \infty$ in case K is a spherical cone with P parallel to the base.

Remark. If K is a polyhedra defined by a finite system of linear inequalities, then $w(K)$ can be determined in polynomial time with an arbitrary fixed absolute accuracy [2]. The same computational problem for determining vol(K) is #P-hard.

Remark. Replacing the "lower ellipsoidal volume" $w(K)$ by the "upper ellipsoidal volume" $W(K) = \min\{\text{vol } E \mid E \text{ is an ellipsoid, } K \subseteq E\}$, and taking an n-dimensional Euclidean ball as K, we get

$$\max\left\{\frac{W(K^+)}{W(K)}, \frac{W(K^-)}{W(K)}\right\} \geq 1 - \frac{1}{2n}$$

under an arbitrary choice of x^* and P. Thus, for $W(\cdot)$ inequality (1) can hold for all n only trivially, i.e., with $\alpha = 1$.

2. Proof of Inequality (1)

An arbitrary ellipsoid E in R^n can be given by the pair (a, A), where $a \in R^n$ is the center of the ellipsoid and A is an $n \times n$ symmetric positive definite matrix: $E = \{x \in R^n \mid (x - a)^t A^{-2}(x - a) \leq 1\}$. This ellipsoid is the image of the Euclidean unit ball $\|y\| = (y^t y)^{1/2} \leq 1$ under the transformation A, shifted to the point a, i.e.,

$$E = \{x \in R^n \mid x = a + Ay, \|y\| \leq 1\}.$$

In particular, the support function and the volume of the ellipsoid $E \sim (a, A)$ are given by the expressions

$$g_E(c) = \max\{c^t x \mid x \in E\} = c^t a + \|c^t A\|,$$
$$\text{vol } E = v(n) \det A,$$

where $v(n)$ stands for the volume of the n-dimensional Euclidean unit ball. First we need the following

Lemma 1. *Let $E^* \sim (a^*, A^*)$ be the ellipsoid inscribed in a convex body K and let $E \sim (a, A)$ be an arbitrary ellipsoid contained in the body. Then*

$$\text{vol } E/\text{vol } E^* \leq x \cdot \exp(1 - x)$$

for any x in the interval

$$\min_{c\in R^n} \frac{\|c^tA\|}{\|c^tA^*\|} = \lambda \le x \le \Lambda = \max_{c\in R^n} \frac{\|c^tA\|}{\|c^tA^*\|}.$$

Proof. Since the contents of the lemma is invariant under affine transformations, we may assume that E^* is the unit ball $A^* = I = \operatorname{diag}(1, \ldots, 1)$ and $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_n) > 0$, where $\lambda_1, \ldots, \lambda_n$ are the semiaxes of E. Then $\lambda = \min \lambda_i$ and $\Lambda = \max \lambda_i$, $i \in \{1, \ldots, n\}$. Since the ellipsoids E^* and E are contained in K, it follows that for any $t \in [0, 1]$ the ellipsoid $E(t) \sim (1-t)(a^*, A^*) + t(a, A)$ is contained in K as well. Consider the function

$$f(t) = \ln(\operatorname{vol} E(t)/\operatorname{vol} E(0)) = \sum_{i\in\{1,\ldots,n\}} \ln(1 + t(\lambda_i - 1)).$$

The maximality of the volume of $E^* = E(0)$ implies $f'(0) \le 0$, i.e.,

$$\operatorname{tr} A = \sum_{i\in\{1,\ldots,n\}} \lambda_i \le n.$$

Therefore

$$\operatorname{vol} E/\operatorname{vol} E^* = \prod_{i\in\{1,\ldots,n\}} \lambda_i \le \prod_{i\in\{1,\ldots,n\}} \lambda_i \cdot \exp(1 - \lambda_i).$$

Since $x \exp(1-x) \le 1$ for all x and the minimum of the function $x \cdot \exp(1-x)$ over $[\lambda, \Lambda]$ is attained in the endpoints of the interval, the proof is completed. □

We now want to prove inequality (1). Since $K^\pm \to K^\mp$ as $p \to -p$, it suffices to prove that $w(K^-)/w(K) \le \alpha = 0.844\ldots$. Let E^* and E^- be the ellipsoids inscribed in K and K^-. Inequality (1) is invariant under affine transformations and we may assume without loss of generality that $E^* \sim (a, D)$ and $E \sim (-a, D^{-1})$, where $D = \operatorname{diag}(d_1, \ldots, d_n) > 0$. Since the centers of E^* and E^- are placed in the points a and $-a$, it follows that $0 \in \operatorname{int} K$. In this case the body K can be defined by the system of linear inequalities $K = \{x \in R^n | c^tx \le 1, c \in K^0\}$, where K^0 is the polar set of covectors. The ellipsoids E^* and E^- are contained in K if and only if

$$g_{E^*}(c) = c^ta + \|c^tD\| \le 1, \qquad \forall c \in K^0,$$
$$g_{E^-}(c) = -c^ta + \|c^tD^{-1}\| \le 1, \qquad \forall c \in K^0.$$

Multiplying the last inequalities for a fixed $c \in K^0$ we get

$$c^tc \le \|c^tD\|\,\|c^tD^{-1}\| \le (1 + c^ta)(1 - c^ta) = 1 - (c^ta)^2,$$

i.e.,

$$c^tc + (c^ta)^2 = \|c^t\sqrt{I + aa^t}\| = g_E(c) \le 1, \qquad \forall c \in K^0,$$

where E is the ellipsoid given by the pair $(0, \sqrt{I + aa^t})$. Hence the ellipsoid $E \sim (0, \sqrt{I + aa^t})$ is contained in K. Note that $\operatorname{vol} E = v(n) \cdot \sqrt{1 + a^t a}$. We now wish to apply Lemma 1 to the ellipsoids E^* and E. To this end, let us show that

$$\min_{c \in R^n} \frac{\|c^t \sqrt{I + aa^t}\|}{\|c^t D\|} = \lambda \leq \frac{\sqrt{1 + a^t a}}{d} \leq \Lambda = \max_{c \in R^n} \frac{\|c^t \sqrt{I + aa^t}\|}{\|c^t D\|},$$

where $d = \max(d_1, \ldots, d_n)$. Indeed, let $d = d_1$. Setting $c_1 = (1, 0, \ldots, 0)$ we have

$$\lambda \leq \frac{\|c_1^t \sqrt{I + aa^t}\|}{\|c_1^t D\|} = \frac{\sqrt{1 + a_1^2}}{d} \leq \frac{\sqrt{1 + a^t a}}{d}.$$

On the other hand, the choice $c_2 = a/\|a\|$ yields

$$\frac{\sqrt{1 + a^t a}}{d} \leq \frac{\sqrt{1 + a^t a}}{\|c_2^t D\|} = \frac{\|c_2^t \sqrt{I + aa^t}\|}{\|c_2^t D\|} \leq \Lambda$$

and we can apply Lemma 1:

$$\frac{\operatorname{vol} E}{\operatorname{vol} E^*} = \frac{\sqrt{1 + a^t a}}{d_1 \cdots d_n} \leq \frac{\sqrt{1 + a^t a}}{d} \cdot \exp\left(1 - \frac{\sqrt{1 + a^t a}}{d}\right).$$

To complete the proof the last observation is needed: since the center a of the ellipsoid E^* lies on the boundary of K^-, it follows that

$$a \notin \operatorname{int} E^- = \{x \in R^n | x = -a = D^{-1} y, \|y\| < 1\}.$$

In other words, $2\|aD\| \geq 1$, and consequently $4d^2(a^t a) \geq 1$. Therefore

$$\frac{w(K^-)}{w(K)} = \frac{\operatorname{vol} E^-}{\operatorname{vol} E^*} = \frac{1}{(d_1 \cdots d_n)^2} \leq \frac{1}{d^2} \cdot \exp\left(2 - \frac{2\sqrt{1 + a^t a}}{d}\right)$$
$$\leq \frac{1}{d^2} \cdot \exp\left(2 - \frac{2\sqrt{1 + 1/4d^2}}{d}\right) \leq \max_{d^2} \left\{\frac{1}{d^2} \cdot \exp\left(2 - \frac{\sqrt{4d^2 + 1}}{d^2}\right)\right\}$$
$$= 0.844\ldots,$$

the last maximum being attained in the root $d^2 = 1.517\ldots$ of the cubic equation $4d^6 - 3d^4 - 4d^2 - 1 = 0$.

References

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