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Embeddings of Graphs in Euclidean Spaces

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Abstract. The dimension of a graph $G=(V, E)$ is the minimum number d such that there exists a representation $x \to \bar{x} \in R^d$ $(x \in V)$ and a threshold t such that $xy \in E$ iff $\bar{x}\bar{y} \geq t$. We prove that $\mathrm{d}(G) \leq n-\chi(G)$ and $\mathrm{d}(G) \leq n-\sqrt{n}$ where $n=|V|$ and $\chi(G)$ is the chromatic number of G; we present upper bounds for the dimension of graphs with a large girth and we show that the complement of a forest can be represented by unit vectors in R^6. We prove that $\mathrm{d}(G) \geq \frac{1}{15}n$ for most graphs and that there are 3-regular graphs with $\mathrm{d}(G) \geq c \log n/\log\log n$.

Introduction

We consider the geometric dimension of graphs introduced in [7]. Let $G=(V, E)$ be a finite graph without loops and multiple edges. The dimension $\mathrm{d}(G)$ of G is the minimum number d such that G admits a representation in R^d; a representation consists of an assignment $x \to \bar{x} \in R^d$ $(x \in V)$ and a threshold $t \in R$ such that for every couple $x, y \in V$, $x \neq y$,

$$xy \in E \quad \text{iff} \quad \bar{x}\bar{y} \geq t,$$

where $\bar{x}\bar{y}$ is the scalar product of vectors $\bar{x}, \bar{y}$.

In [7] we also considered a related notion—the spherical dimension $\mathrm{sd}(G)$ of a graph G, which is defined analogously but with an additional condition $\|\bar{x}\|=1$ for all $x \in V$. Clearly, $\mathrm{sd}(G) \geq \mathrm{d}(G)$ for every graph G.

The notion of spherical dimension is closely related to that of sphericity which was introduced in [4]. The sphericity is defined as the minimum number d such that there exists an assignment $x \to \bar{x} \in R^d$ and a threshold $\rho \geq 0$ such that

$$xy \in E \quad \text{iff} \quad \|\bar{x}-\bar{y}\| \leq \rho.$$

In [7] it is noted that

$$\mathrm{sd}(G)-1 \le \mathrm{sph}(G) \le \mathrm{sd}(G)$$

for every graph G.

Let us list some bounds for $\mathrm{d}(G)$, $\mathrm{sd}(G)$, and $\mathrm{sph}(G)$:

(1) $\mathrm{d}(G) \le n-\alpha(G)$ ($n=|V|$, $\alpha(G)$ is the size of a maximal independent set in G).
(2) $\mathrm{sph}(G) \le n-\omega(G)$ (where $\omega(G)$ is the size of a maximal clique in G).
(3) $\mathrm{sd}(G) \le cd \log(d)$ if the complement $\bar{G}$ of G has maximum degree $\le d$.
(4) $\mathrm{d}(T) \le 3$ if T is a tree.
(5) $\mathrm{sph}(T) \le \frac{1}{2}(k+1)\log_k|T|$ if T is a tree with degree $\le k$.
(6) $\mathrm{sph}(T) \le 108 \log|T|$ if T is a tree.
(7) $\mathrm{sph}(F) \le 8\lceil \log|F| \rceil$ if F is a forest.
(8) $c_1 n/\log n \le \mathrm{sph}(Q_n) \le c_2 n/\log n$ where Q_n is the graph of the n-dimensional cube.
(9) $\mathrm{sd}(G) \le cd^2 \log n$ if G has maximum degree $\le d$.
(10) $\mathrm{sph}(G_1+G_2+\cdots+G_m) \le 2(n-1)$ where $n=\max V(G_i)$.
(11) $\mathrm{d}(\bigcup_i G_i) \le \max \mathrm{d}(G_i)+1$ where the G_i's are disjoint.
(12) $\mathrm{d}(G \cup K_A) \le \mathrm{d}(G)+1$ where $A \subset V$ and K_A is a clique on A.
(13) $\mathrm{sph}(G) < 12(2c-1)^2 \log|G|$ for $\lambda_{\min} \ge -c$ ($c \ge 2$) and $|G| > [12(2c-1)^2 \log|G|]^2$, where $\lambda_{\min}$ is the minimal eigenvalue of G.
(14) $\mathrm{sph}(L(G)) < 108 \log m$ where $m=|E|$ and $m > (108 \log m)^2$.
(15) $\mathrm{sph}(G) \ge \log \alpha(G)/(\log(2r(G)+1))$ where $\mathrm{r}(G)$ is the radius of G.

For these and related further results see [1]-[4] and [7].

The aim of this paper is to present some other upper and lower results for $\mathrm{d}(G)$. In Section 1 we prove that $\mathrm{d}(G) \le n-\chi(G)$ where $\chi(G)$ is the chromatic number of G which, together with (1) above, yields $\mathrm{d}(G) \le n-\sqrt{n}$. In Section 2 we consider the dimension of forests and graphs with a large girth. In Section 3 we deal with lower bounds for $\mathrm{d}(G)$. We prove that most of the graphs on n vertices have $\mathrm{d}(G) \ge n/15$ for sufficiently large n. We also prove that, contrary to (3) above, there are graphs G_n with maximum degree ≤ 3 but with $\mathrm{d}(G_n) > \log n/18 \log\log n$.

1. General Upper Bounds

1.1. Theorem. *For every graph on n vertices,*

$$\mathrm{d}(G) \le n-\chi(G).$$

Proof. 1. The set V of vertices of $G=(V,E)$ with $\chi(G)=\chi$ can be written as

$$V = K \cup A_{k+1} \cup A_{k+2} \cup \cdots \cup A_\chi,$$

where K is a clique of size k, each A_i has size ≥ 2, and A_i is a maximal independent set in the induced subgraph G_i on $K \cup A_{k+1} \cup \cdots \cup A_i$, $k+1 \leq i \leq \chi$.

Indeed, let $A_1 \cup \cdots \cup A_\chi$ be the decomposition of V corresponding to a coloring of G such that $|A_1| \leq |A_2| \leq \cdots \leq |A_\chi|$. If A_χ is not maximal independent in G_χ, we move suitable vertices of $A_1 \cup \cdots \cup A_{\chi-1}$ to A_χ to make it such. Then we proceed in the same way with $A_{\chi-1}, A_{\chi-2}, \ldots, A_1$. Thus if $k+1$ is the least integer with $|A_{k+1}| \geq 2$ then $K = A_1 \cup \cdots \cup A_k$ is a clique of size k.

2. Let $|A_{k+1}| = d+1$. Then G_{k+1} can be represented in R^d with a threshold $t<0$. Indeed, consider the vertices $\bar{z}_0, \ldots, \bar{z}_d$ of the regular simplex in R^d centered at the origin with

$$\bar{z}_i\bar{z}_j = -1 \quad (i \neq j), \qquad \bar{z}_i^2 = d.$$

Let $K = \{x_1, \ldots, x_k\}$, $A_{k+1} = \{u_0, \ldots, u_d\}$. Put

$$\bar{x}_i = \sum_{r \in I_i} \bar{z}_r \qquad (i = 1, \ldots, k),$$

where

$$I_i = \{r | x_i u_r \in E\}$$

and

$$\bar{u}_j = 2d^2 \bar{z}_j \qquad (j = 0, \ldots, d).$$

We have:

(a) $\bar{x}_i\bar{x}_j = \sum \{\bar{z}_r\bar{z}_s | r \in I_i, s \in I_j\} \geq -d^2$.

(b) $\bar{u}_i\bar{u}_j = -4d^4 < -d^2$ for $i \neq j$.

(c) If $x_i u_j \in E$ then

$$\begin{aligned}\bar{x}_i\bar{u}_j &= 2d^2 \sum \{\bar{z}_r\bar{z}_j | r \in I_i\} \\ &= 2d^3 + 2d^2 \sum \{\bar{z}_r\bar{z}_j | r \in I_i - \{j\}\} \geq 2d^3 - 2d^3 = 0.\end{aligned}$$

(d) If $x_i u_j \notin E$ then $\bar{x}_i\bar{u}_j = -2d^2|I_i| < -d^2$ for $I_i \neq \emptyset$ because of the maximality of A_{k+1}.

Thus vectors $\bar{x}_i$, $\bar{u}_j$ form a representation of G_{k+1} in R^d with the threshold $t = -d^2$.

3. Suppose G_h can be represented in R^d with a threshold $t<0$. Then G_{h+1} can be represented in R^{d+s}, where $s+1 = |A_{h+1}|$, with a negative threshold. Indeed, let $x_1, \ldots, x_m$ be vertices of G_h and $u_0, \ldots, u_s$ vertices of A_{h+1}. Let $\bar{x}_1, \ldots, \bar{x}_m$ form the representation of G_h in R^d with a threshold $t<0$. Thus there is $t_1 < t$ such that

$$\bar{x}_i\bar{x}_j \geq t \quad \text{if } x_i x_j \in E \quad \text{and} \quad \bar{x}_i\bar{x}_j \leq t_1 \quad \text{if } x_i x_j \notin E.$$

Again, choose vectors $\bar{z}_0, \ldots, \bar{z}_s$ in R^s with

$$\bar{z}_i\bar{z}_j = -1 \qquad \text{for} \quad i \neq j, \qquad \bar{z}_i^2 = s.$$

Put

$$\tilde{x}_i = \left(\bar{x}_i, \varepsilon \sum_{r \in I_i} z_r\right) \in R^{d+s} \quad (i = 1, \ldots, m),$$

$$\tilde{u}_j = (\bar{0}, \alpha\bar{z}_j) \in R^{d+s} \qquad (j = 0, \ldots, s),$$

where I_i is defined as in 2 above and $\alpha > 0$, $\varepsilon > 0$ will be specified later. Then we have:

(a) If $x_i x_j \in E$ then

$$\tilde{x}_i \tilde{x}_j = \bar{x}_i \bar{x}_j + \varepsilon^2 \sum \{\bar{z}_r \bar{z}_u | r \in I_i,\ u \in I_j\} \geq t - \varepsilon^2 s^2.$$

(b) If $x_i x_j \notin E$ then

$$\tilde{x}_i \tilde{x}_j \leq t_1 + \varepsilon^2 \sum_{r=0}^{s} \bar{z}_r^2 \leq \varepsilon^2 s(s+1) + t_1.$$

(c) If $x_i u_j \in E$ then

$$\begin{aligned} \tilde{x}_i \tilde{u}_j &= \alpha\varepsilon \sum \{\bar{z}_r \bar{z}_j | r \in I_i\} \\ &= \alpha\varepsilon \bar{z}_j^2 + \alpha\varepsilon \sum \{\bar{z}_r \bar{z}_j | r \in I_i - \{j\}\} \geq \alpha\varepsilon s - \alpha\varepsilon s = 0. \end{aligned}$$

(d) If $x_i u_j \notin E$ then

$$\tilde{x}_i \tilde{u}_j = -\alpha\varepsilon |I_i| \leq -\alpha\varepsilon$$

because the inclusion maximality of A_{h+1} in G_{h+1} forces $|I_i| > 0$.

(e) $\tilde{u}_i \tilde{u}_j = -\alpha^2$ for $i \neq j$.

We are going to show that α, ε can be chosen such that vectors $\tilde{x}_i$, $\tilde{u}_j$ form a representation of G_{h+1} with any threshold t' where $t_1 < t' < t < 0$. Indeed, first choose $\varepsilon > 0$ sufficiently small such that

$$t - \varepsilon^2 s^2 > t' \qquad \text{(see (a))},$$

$$t_1 + \varepsilon^2 s(s+1) < t' \qquad \text{(see (b))}.$$

Then choose α sufficiently large such that

$$-\alpha\varepsilon < t' \qquad \text{(see (d))},$$

$$-\alpha^2 < t' \qquad \text{(see (e))}.$$

4. It follows from 2 and 3 above that by induction $G = G_\chi$ can be represented in R^d where

$$\begin{aligned} d &= (|A_{k+1}| - 1) + (|A_{k+2}| - 1) + \cdots + (|A_\chi| - 1) \\ &= |V - K| - (\chi - k) = n - \chi. \end{aligned}$$

This conclusion the proof of the theorem. □

1.2. Corollary. *For every graph G on n vertices,*

$$\mathrm{d}(G) \leq n - \sqrt{n}.$$

Proof. In [6] the threshold dimension $\theta(G)$ of a graph G is introduced and shown to satisfy $\theta(G) \leq n - \alpha(G)$ where $\alpha(G)$ is the size of a maximal independent set in G. Following [7], $\mathrm{d}(G) \leq \theta(G)$ for every graph G. Thus $\mathrm{d}(G) \leq n - \alpha(G)$. As $\alpha(G) \cdot \chi(G) \geq n$, we have $\max(\alpha(G), \chi(G)) \geq \sqrt{n}$, hence

$$\mathrm{d}(G) \leq \min(n - \alpha(G), n - \chi(G)) \leq n - \sqrt{n}. \qquad \square$$

2. Upper Bounds for Graphs Without Cycles

2.1. Proposition. *If G is a forest then* $\mathrm{d}(G) \leq 3$; *the representation can be chosen to have a positive threshold.*

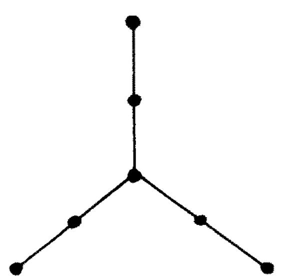
Proof. This is proved in [7] for G a tree; in more detail, for every $\varepsilon > 0$, a tree $G = (V, E)$ can be represented in R^3 with the threshold $t = 1$ such that vectors $\bar{x}$ representing vertices $x \in V$ satisfy $1 < \|\bar{x}\| < 1 + \varepsilon$. Then $\mathrm{diam}\{\bar{x} | x \in V\} \to 0$. Thus, as the tree G is connected, its representation can be found to be placed in an arbitrarily small ball with center on the unit sphere S in R^3. Now, if G is a forest with components $G_1, \ldots, G_m$, then choose distinct points $c_1, \ldots, c_m$ and balls B_i with centers c_i, each with a radius r. Represent G_i in B_i; if r is small enough, this yields a representation of G in R^3 with the threshold $t = 1$.

2.2. Proposition. *If G is a forest that does not contain the tree T_0 below as an induced subgraph then* $\mathrm{d}(G) \leq 2$ *where the representation in R^2 can be chosen to have a positive threshold t.*

Proof. In fact, this is proved in [7] for $G = (V, E)$ with $t' = 1 - \varepsilon$ $(0 < \varepsilon < 1)$ where ε can be arbitrarily small, and vectors $\bar{x} \in R^2$ representing vertices of V satisfy $1 \geq \|\bar{x}\| \geq 1 - \varepsilon$. Thus the representation can be constructed with $\mathrm{diam}\{\bar{x} | x \in V\}$ arbitrarily small. Using the same argument as in the preceding proposition, this can be used to represent any forest without induced T_0 in R^2 with a positive threshold (Fig. 1).

2.3. Proposition. *A graph G on n vertices can be represented in R^d where*:

I. $d \leq \frac{2}{7}n + 1$ if G has a girth at least 7,
II. $d \leq \frac{1}{3}n + \frac{2}{3}$ if G has a girth at least 6,
III. $d \leq \frac{2}{5}n + \frac{3}{5}$ if G has a girth at least 5.



T_0

Fig. 1

Remark. Proposition 2.3 yields linear upper bounds for $\mathrm{d}(G)$ for G without short cycles, which, however, can be further improved (to appear in a subsequent paper). Using a probabilistic approach we can, for example, show an upper bound of the form $c_1\rho(\log n)^{c_2}$ for any C_4-free graph with edge density ρ. As it is well known that $\rho \le \sqrt{n}$ for any C_4-free graph, this clearly improves Proposition 2.3 for $n \ge n(c_1, c_2)$. To prove Proposition 2.3 we need the following.

Lemma. *Let a graph $G=(V, E)$ contain a set C of vertices spanning one of the graphs C_5, C_6, C_7, C_8, P_7 such that each vertex in $V-C$ is adjacent to at most one of the vertices of C. Let the induced subgraph G' of G on $V-C$ be represented in R^d with a positive threshold. Then G can be represented in R^{d+2} with a positive threshold.*

Proof. Let $C=\{v_0, \ldots, v_{r-1}\}$ span C_r, $5 \le r \le 8$. Put

$$\bar{z}_i = \left(\cos\frac{2\pi i}{r}, \sin\frac{2\pi i}{r}\right) \qquad (i=0, \ldots, r-1).$$

Then

$$\bar{z}_i^2 = 1, \qquad \bar{z}_i\bar{z}_{i+1} = \cos\frac{2\pi}{r} \ge \cos\frac{2\pi}{5} > 0.$$

$$\bar{z}_i\bar{z}_j \le 0 \qquad \text{for} \quad i \ne j \pm 1 \bmod r, \quad i \ne j.$$

Let $x \to \bar{x} \in R^d$ be a representation of G' in R^d with a positive threshold t, without loss of generality, $t=1$. Then, if $\alpha > C$ is sufficiently small, vectors

$$\tilde{x} = (\bar{x}, \alpha\bar{z}_i) \in R^{d+2} \qquad (x \in V-C,\ v_i x \in E),$$
$$\tilde{x} = (\bar{x}, \bar{0}) \in R^{d+2} \qquad (x \in V-C,\ v_i x \notin E \text{ for no } i)$$

form a representation of G' in R^{d+2} with a threshold t', where

$$\cos\frac{2\pi}{r} < t' < 1.$$

Put

$$\tilde{v}_i = (\bar{0}, \bar{z}_i/\alpha).$$

Then, for sufficiently small $\alpha > 0$,

$$\tilde{v}_i\tilde{v}_{i+1} = \frac{1}{\alpha^2}\cos\frac{2\pi}{r} > 1,$$
$$\tilde{v}_i\tilde{v}_j \le 0 \qquad \text{if} \quad i \ne j \pm 1 \bmod r, \quad i \ne j,$$
$$\tilde{x}\tilde{v}_i = 1 > t' \qquad \text{if} \quad x \in V-C \quad \text{is adjacent to } v_i,$$
$$\tilde{x}\tilde{v}_j \le \cos\frac{2\pi}{r} < t' \qquad \text{if} \quad x \in V-C \quad \text{is not adjacent to } v_j.$$

This proves that vectors $\tilde{x}$ $(x \in V-C)$ and $\tilde{v}_i$ $(i=0,\dots,r-1)$ form a representation of G in R^{d+2} with the threshold t'.

If C spans P_7, add a new vertex x_0 to G to be adjacent just to an endpoint of P_7; the resulting graph G' contains a copy of C_8 on $C \cup \{x_0\}$ and hence G' can be represented in R^{d+2} with a positive threshold by the preceding part of the proof.

Proof of Proposition 2.3. We shall prove by induction on n that G has a representation in R^d with a positive threshold where d is bounded by $\frac{2}{7}n+1$ or $\frac{1}{3}n+\frac{2}{3}$ or $\frac{2}{5}n+\frac{3}{5}$ in cases I, II, and III, respectively. This is certainly true if $n \leq 3$. Let $n > 3$.

(a) Let G be a forest. Then $\mathrm{d}(G) \leq 3 \leq \frac{2}{7}n+1 \leq \frac{1}{3}n+\frac{2}{3} \leq \frac{2}{5}n+\frac{3}{5}$ if $n \geq 7$, see 2.1 above. If $4 \leq n < 7$ then $\mathrm{d}(G) \leq 2 \leq \frac{1}{3}n+\frac{2}{3} \leq \frac{2}{7}n+1 \leq \frac{2}{5}n+\frac{3}{5}$ by 2.2.

(b) Let $G=(V,E)$ not be a forest. Let D be the shortest cycle in G of length k. If $k>8$, choose $C \subset D$ spanning P_7; if $k \leq 8$, let $C=D$. Due to the minimality of D, every vertex of the induced subgraph G' on $V-C$ is adjacent to at most one vertex of C. If $k \geq 7$ or $k=6$ or $k=5$ then G' has $\leq n-7$ or $n-6$ or $n-5$ vertices, respectively. By the induction assumption, G can be represented in $R^{d'}$ where

$$d' \leq 2(n-7)/7+1$$

or

$$d' \leq (n-6)/3+\tfrac{2}{3}$$

or

$$d' \leq 2(n-5)/5+\tfrac{3}{5},$$

respectively, with a positive threshold. By the lemma, G can be represented in R^d with a positive threshold where $d=d'+2$, hence $d \leq \frac{2}{7}n+1$, or $d \leq \frac{1}{3}n+\frac{2}{3}$ or $d \leq \frac{2}{5}n+\frac{3}{5}$, respectively.

2.4. The following is an essential improvement of a result of Frankl and Maehara [2] who proved

$$\mathrm{sd}(G) \leq 8 \log n$$

if $\bar{G}$ is a forest. Let us remark that $\mathrm{sd}(G)$ is unbounded for trees. It was proved in [7] that

$$\mathrm{sd}(B_n) \geq c \log n/\log\log n,$$

where B_n is the "complete binary" tree on $n=2^s-1$ vertices with s levels.

Proposition. *If the complement $\bar{G}$ of G is a forest then*

$$\mathrm{sd}(G) \leq 6.$$

Proof. Let $G=(V,E)$ and $\bar{G}$ be a forest. In each component K_i of $\bar{G}$, choose a vertex r_i to be a root of the tree K_i. This defines decomposition of each K_i into levels. Also, neighbors of each vertex x that is not a root consist of a father and of a family of sons. Let V_1 be the set of all vertices on odd levels (thus all roots r_i are in V_1) and $V_2=V-V_1$. Then V_2 can be written as $V_2=\{x_1,\dots,x_n\}$ in such a way that sons of each $y\in V_1$ form a segment of the form

$$x_j, x_{j+1}, \dots, x_k.$$

Choose reals α_i,

$$0<\alpha_1<\frac{\pi}{n}<\alpha_2<\frac{2\pi}{n}<\cdots<\frac{(n-1)\pi}{n}<\alpha_n<\pi.$$

Fix a $y\in V_1$ with sons as above and with the father x_r, define a complex polynomial g_y,

$$g_y(z)=\left(z-\exp\left(i\frac{\pi(j-1)}{n}\right)\right)\left(z-\exp\left(i\frac{k\pi}{n}\right)\right)\exp\left(-i\frac{\pi(j+k-1)}{2n}\right)$$
$$\times\left(z-\exp\left(i\frac{\pi(r-1)}{n}\right)\right)\left(z-\exp\left(i\frac{\pi r}{n}\right)\right)\exp\left(-i\frac{\pi(2r-1)}{2n}\right).$$

We can verify that the coefficients of g_y are of the form

$$g_y(z)=c_2+c_1z+c_0z^2+\bar{c}_1z^3+\bar{c}_2z^4, \qquad c_0\in R.$$

Put $b_i^*=2\operatorname{Re} c_i$, $a_i^*=2\operatorname{Im} c_i$ $(i=1,2)$, and $d_0^*=c_0$. For a real α, define $f_y(\alpha)=g_y(\exp i\alpha)\cdot\exp(-2i\alpha)$. Then

$$f_y(\alpha)=c_2\,e^{-2i\alpha}+c_1\,e^{-i\alpha}+c_0+\bar{c}_1\,e^{i\alpha}+\bar{c}_2\,e^{2i\alpha}$$
$$=d_0^*+a_1^*\sin\alpha+b_1^*\cos\alpha+a_2^*\sin 2\alpha+b_2^*\cos 2\alpha.$$

Thus f_y is real valued, $f_y(\alpha)=(d_0^*,a_1^*,b_1^*,a_2^*,b_2^*)\cdot(1,\sin\alpha,\cos\alpha,\ \sin 2\alpha,\cos 2\alpha)$. As the only roots of g_y are $\exp(i(\pi(j-1)/n))$, $\exp(i(\pi k/n))$, $\exp(i(\pi(r-1)/n))$, $\exp(i(\pi r/n))$ we see that the only roots of f_y are $\pi(j-1)/n$, $\pi k/n$, $\pi(r-1)/n$, and $\pi r/n$. Hence f_y has the same sign at $\alpha_j,\alpha_{j+1},\dots,\alpha_{k-1},\alpha_k$, and α_r, and the opposite sign at the other α_i's. Put

$$\varepsilon=-\operatorname{sgn} f_y(\alpha_r)\sqrt{3}/\sqrt{d_0^{*2}+a_1^{*2}+b_1^{*2}+a_2^{*2}+b_2^{*2}},$$
$$h_y=\varepsilon f_y,\quad d_0^y=\varepsilon d_0^*,\quad a_i^y=\varepsilon a_i^*,\quad b_i^y=\varepsilon b_i^* \qquad (i=1,2).$$

Then $h_y(\alpha_i)<0$ for $j\le i\le k$ and for $i=r$ and $h_y(\alpha_i)>0$ otherwise.

Put

$$\bar{y} = (d_0^y, a_1^y, b_1^y, a_2^y, b_2^y, 2),$$
$$\bar{x}_i = (1, \sin\alpha_i, \cos\alpha_i, \sin 2\alpha_i, \cos 2\alpha_i, -2).$$

Then

$$\bar{x}_i\bar{y} = h_y(\alpha_i) - 4 < -4 \quad \text{for} \quad j \le i \le k, \quad i = r, \quad \text{i.e., for} \quad x_i y \in E,$$
$$\bar{x}_i\bar{y} = h_y(\alpha_i) - 4 > -4 \quad \text{otherwise},$$
$$\bar{x}_i\bar{x}_j > -1 + 4 > -4 \quad \text{for all } i, j,$$
$$\|\bar{x}_i\| = \|\bar{y}\| = \sqrt{7}.$$

Using the same construction for all $y \in V_1$, we also see that, for $y_1, y_2 \in V_1$,

$$\bar{y}_1\bar{y}_2 > -3 + 4 > -4.$$

Thus we have a representation of G in R^6 by vectors with the same norm $\sqrt{7}$ and with the threshold $t = -4$ which concludes the proof. □

3. Lower Bounds

3.1. We shall prove that most graphs on n vertices have dimension $\ge n/15 - 1$. However, the only graphs of that dimension we explicitly know are those containing K_{rr} for $r > n/15$ [7].

Theorem. *Let t_n be the total number of graphs G on $n \ge 38$ vertices. Then at least $(1 - 1/n)t_n$ of them have*

$$d(G) \ge n/15 - 1.$$

Proof. 1. Let $G_1, G_2, \ldots, G_{t_n}$ be a list of all graphs on the set $V = \{x_1, \ldots, x_n\}$ of vertices where $d(G_1) \le d(G_2) \le \cdots \le d(G_{t_n})$. Let $r = \lfloor t_n/n \rfloor$, $d = d(G_{r+1}) + 1$. Then $G_1, \ldots, G_r$ can be represented in R^d with the threshold 1; in fact, if $x_i \to \bar{x}_i \in R^{d-1}$ is a representation in R^{d-1} with a threshold $t < 1$ (or $t \ge 1$) then $x_i \to (\bar{x}_i, \sqrt{1-t})$ or $(x_i \to (\bar{x}_i/t, 0)$, resp.) is a representation in R^d with the threshold 1. Also, using small changes of coordinates we can modify the representations in such a way that $\bar{x}_i\bar{x}_j > 1$ rather than $\bar{x}_i\bar{x}_j \ge 1$ if $x_i x_j$ is an edge, and that vectors $\bar{x}_1, \ldots, \bar{x}_d$ representing vertices $x_1, \ldots, x_d$ are linearly independent. Finally, if $\tilde{x}_i = (\tilde{x}_{i1}, \ldots, \tilde{x}_{id}) \in R^d$ is the vector of coordinates of x_i with respect to the orthonormal basis obtained from the base $\bar{x}_1, \ldots, \bar{x}_d$ by the Gramm-Schmidt orthonormalization process, then $\bar{x}_i\bar{x}_j = \tilde{x}_i\tilde{x}_j$; moreover, obviously $\tilde{x}_{i,i+1} = \tilde{x}_{i,i+2} = \cdots = \tilde{x}_{id} = 0$ for $i = 1, \ldots, d-1$. To summarize our consideration, we have representations $x_i \to \tilde{x}_i^G$ of graphs $G = G_1, \ldots, G_r$ in R^d such that

$$\tilde{x}_i^G\tilde{x}_j^G > 1 \quad \text{if } x_i x_j \text{ is an edge in } G,$$
$$\tilde{x}_i^G\tilde{x}_j^G < 1 \quad \text{if } x_i x_j \text{ is a nonedge in } G,$$
$$\tilde{x}_{ij}^G = 0 \quad \text{if} \quad 1 \le i < j \le d.$$

2. Define polynomials P_{ij} $(1 \le i < j \le n)$ in $(nd - (d(d-1)/2))$ variables x_{11}, $x_{21}, x_{22}, x_{31}, x_{32}, x_{33}, \ldots, x_{d1}, \ldots, x_{dd}, x_{d+1,1}, \ldots, x_{d+1,d}, \ldots, x_{n1}, \ldots, x_{nd}$ by

$$P_{ij}(x_{uv})_{u,v} = \sum_{k=1}^{d} x_{ik}x_{jk} - 1$$

(where $x_{ik} = 0$ if $1 \le i < k \le d$ and analogously for x_{jk}). Then for $\tilde{x}^G = (\tilde{x}^G_{uv})_{u,v} \in R^N$ (where $N = nd - (d(d-1)/2)$) we have

$$P_{ij}(\tilde{x}^G) > 0 \quad \text{if } x_i x_j \text{ is an edge in } G,$$

$$P_{ij}(\tilde{x}^G) < 0 \quad \text{if } x_i x_j \text{ is a nonedge in } G.$$

3. Following a result of Warren [5], given arbitrary polynomials $P_1, \ldots, P_M$ on N variables of degree $\le D$ where $M \ge N$, the total number of sign sequences $(\operatorname{sgn} P_1(x), \ldots, \operatorname{sgn} P_M(x))$ that consist of terms $+1$, -1 does not exceed $(4eDM/N)^N$.

4. Let us apply this estimate to our polynomials P_{ij}; as distinct graphs $G \ne G'$ induce distinct sign sequences of P_{ij} at $\tilde{x}^G$ and $\tilde{x}^{G'}$, we have

$$r \le (4eDM/N)^N.$$

Here $r = 2^{\binom{n}{2}}/n$, $D = 2$, $M = \binom{n}{2}$, and $N = nd - (d(d-1)/2)$. Hence

$$r \le \left(8e\binom{n}{2}\Big/N\right)^N \le \left(\frac{4e}{N/n^2}\right)^N,$$

$$r^{1/n^2} \le \left(\frac{4e}{y}\right)^y \qquad \text{where} \quad y = \frac{N}{n^2} = \frac{nd - (d(d-1)/2)}{n^2} \in (0, 1),$$

$$\frac{\ln r}{n^2} \le y^{\ln(4e/y)}. \tag{$*$}$$

We have

$$\frac{\ln r}{n^2} = \frac{(\ln 2)\binom{n}{2} - \ln n}{n^2} = \frac{\ln 2}{2} - \frac{\ln 2}{2n} - \frac{\ln n}{n^2} \ge \frac{\ln 2}{2} - \frac{1}{85.9}$$

for $n \ge 38$. The numerical solution of the equation $y \ln(4e/y) - (\ln 2))2 + 1/85.9 = 0$ on $(0, 1)$ is $y \doteq 1/15.26175$, and the function $y \ln(4e/y)$ is increasing on $(0, 1)$. Hence $(*)$ implies $y \ge 1/15.262$ for $n \ge 38$. Putting $x = d/n$ we have

$$y = \frac{d}{n} - \frac{1}{2}\left(\frac{d}{n}\right)^2 + \frac{1}{2n}\left(\frac{d}{n}\right) = x - \tfrac{1}{2}x^2 + \frac{1}{2n}\,x \le x - \tfrac{1}{2}x^2 + \tfrac{1}{76}x$$

for $n \geq 38$. Thus

$$\tfrac{77}{76}x - \tfrac{1}{2}x^2 \geq \frac{1}{15.262}.$$

A numerical solution then gives $x \geq 1/14.95 > \frac{1}{15}$, hence $d > n/15$. Then $\mathrm{d}(G) > n/15 - 1$ for all $G = G_{r+1}, G_{r+2}, \ldots, G_{t_n}$ which concludes the proof. □

Remark. The constant $\frac{1}{15}$ can be improved for large n but not to $\frac{1}{14}$ using our method.

3.2. In [7] we proved that if the complement $\bar{G}$ of a graph G has degree bounded by d then $\mathrm{d}(G) \leq 2d \log(8d)$, independently of the number of vertices of G. We show that this is not the case if G has bounded degree:

Theorem. *If n is sufficiently large then there exists a bipartite 3-regular graph G on $2n$ vertices with*

$$\mathrm{d}(G) \geq \frac{\log n}{18 \log \log n}.$$

To prove the theorem we state two auxiliary lemmas.

Lemma 1. *For sufficiently large n there exists a bipartite graph $G = (V, E)$ on $2n$ vertices such that:*

(i) *G is 3-regular.*
(ii) *For every partition $V = A \cup B$, $|A| = |B| = n$ there exists a matching M, $|M| = cn \geq n/34$ with e having one endpoint in A and another in B for every $e \in M$.*
(iii) *G does not contain induced cycles of length $\leq 2t_o$ where $t_0 = \frac{1}{8}\log n$.*

Proof. We will proceed by random construction. Let $V = X \cup Y$ and $|X| = |Y| = n$. Let π_1, π_2, π_3: $X \to Y$ be three random bijections, each taken with probability $1/n!$ independently of the choice of the others. Let $\mathbf{G} = (V, E)$ be the "random graph" with edge set $E = \{\{x, \pi_i(x)\};\ x \in X, i = 1, 2, 3\}$. Thus the probability of a particular graph G equals $\mathrm{pg}(G)/(n!)^3$ where $\mathrm{pg}(G)$ denotes the number of ways that G can be written as a union (not necessarily disjoint) of three perfect matchings.

Claim 1. *Denote by p_n the probability that $\mathbf{G}$ is 3-regular. Then*

$$\varliminf_{n\to\infty} p_n \geq 1/e^3.$$

Proof. The permutation π_1 can be chosen in $n!$ ways. If π_1 is fixed, then $n!$ $(1-1/1!+1/2!-\cdots+(-1)^n/n!)$ of permutations π_2 has the property that $\pi_2(x)\neq\pi_1(x)$ for all $x\in X$, i.e., that edges $x\pi_i(x)$ $(x\in X, i=1,2)$ form a 2-regular graph H. If π_2 is such a permutation, consider the $n\times n$ matrix $(a_{ij})_{i\in X, j\in Y}$ where

$$a_{ij}=0 \quad \text{if } ij \text{ is an edge of } H,$$

$$a_{ij}=1/(n-2) \quad \text{otherwise.}$$

Then $\sum_{i\in X} a_{ij}=1$ for every $j\in Y$ and $\sum_{j\in Y} a_{ij}=1$ for every $i\in X$.

Hence (a_{ij}) is a doubly stochastic matrix. By [8] and [9], for the permanent

$$P=\sum_{\pi\in S_n} a_{1\pi(1)}a_{2\pi(2)}\cdots a_{n\pi(n)}$$

of (a_{ij}) we have $P\geq n!/n^n$. On the other hand, $P=q(1/(n-2))^n$ where q is the number of nonzero summands in P. As the nonzero summands clearly correspond to these π such that π_1,π_2,π induce a 3-regular graph, it follows $q\geq n!/(1-2/n)^n$.

Hence the number of triples π_1,π_2,π_3 inducing a 3-regular graph is $\geq n!\,n!$ $(1-1/1!+1/2!-\cdots+(-1)^n/n!)n!\,(1-2/n)^n$. Then

$$p_n\geq\left(1-\frac{1}{1!}+\frac{1}{2!}-\cdots+\frac{(-1)^n}{n!}\right)\left(1-\frac{2}{n}\right)^n\to 1/e^3$$

which concludes the proof of Claim 1. □

Claim 2. *Denote by q_n the probability that* **G** *satisfies* (ii) *with $c=\frac{1}{33}$. Then $q_n\to 1$ for $n\to\infty$.*

Proof. Let $V=A\cup B$ be a fixed partition, $|A|=|B|=n$, $a=|X\cap A|$, $b=|X-A|$, and $a\geq b$. Then $b=|Y\cap A|$, $a=|Y-A|$, and $a+b=n$. Set $c=\frac{1}{33}$,

$$S_i^A=\{(x,y)\,|\,x\in X\cap Y,\ y\in Y-A,\ \pi_i(x)=y\}.$$

We have

$$\operatorname{Prob}\{|S_i^A|<cn\}\leq\frac{\sum_{0\leq j<cn}\binom{a}{j}\binom{b}{a-j}}{\binom{a+b}{a}}. \tag{1}$$

The right-hand side is nonzero only if $b\geq a-j$ for some j, i.e., only if $a-cn\leq b$ which yields

$$a\leq\lfloor (n/2)(1+c)\rfloor. \tag{2}$$

The right-hand side of (1) may be further bounded from above by

$$\frac{\sum_{0\le j<cn}\binom{a}{j}^2}{\binom{a+b}{a}} \le \frac{\binom{a}{\lceil cn\rceil}^2}{\binom{n}{a}} \le \frac{\binom{\lfloor (n/2)(1+c)\rfloor}{\lceil cn\rceil}^2}{\binom{n}{\lfloor (n/2)(1+c)\rfloor}} \le 2^{-n\varphi(c)+K\log n},$$

where $\varphi(c)=2c\log c+\frac{1}{2}(1-c)\log((1-c)/2)-\frac{3}{2}(1+c)\log((1+c)/2)$ and K is independent of n. As $\varphi(\frac{1}{33})>\frac{2}{3}$,

$$\text{Prob}\{|S_i^A|<cn\}\le 2^{-(2/3)n}$$

for sufficiently large n. Then $\text{Prob}\{|S_i^A|<cn \mid i=1,2,3\}\le 2^{-2n}$. The number of all partitions in question is $\binom{2n}{n}$. Thus the probability that $|S_i^A|<cn$ $(i=1,2,3)$ for some of them is $\le\binom{2n}{n}2^{-2n}\to 0$. □

Claim 3. *Denote by* r_n *the probability that* **G** *has the following property: after deleting* $\le\sqrt{n}$ *edges from* **G**, *the graph "***G** *minus deleted edges" does not contain any cycle of length* $\le 2t_0$ *where* $t_0=\frac{1}{8}\log n$. *Then* $r_n\ge 0.99$ *holds for sufficiently large* n.

Proof. Fix a set C of $2t$ pairs xy $(x\in X, y\in Y)$ forming a cycle, where $t\le t_0$. We will estimate the probability $r_n(C)$ that **G** contains C as a cycle. Let π_1, π_2, π_3 be bijections inducing $\mathbf{G}=(V,E)$ and let E_i be the set of edges induced by π_i $(i=1,2,3)$. Set $M_i=C\cap E_i$; clearly, M_i is a matching. By [10, p. 129] we have $2^{2t}+2$ ways to split C into three matchings M_1, M_2, M_3. For each of them we have $a=(n-t_1)!\,(n-t_2)!\,(n-t_3)!$ triples π_1, π_2, π_3 with $E_i\cap C=M_i$ $(i=1,2,3)$, where $|M_i|=t_i$ $(i=1,2,3)$.

Hence

$$r_n(C)\le(2^{2t}+2)\cdot\text{Max}\frac{(n-t_1)!\,(n-t_2)!\,(n-t_3)!}{(n!)^3}\le\frac{2^{2t+1}}{n^{2t}}$$

for sufficiently large n (as $t_i\ll\sqrt{n}$). The number of cycles of length $2t$ formed by pairs xy, $x\in X$, $y\in Y$, is

$$\binom{n}{t}^2 t!\,(t-1)!\le\frac{n^{2t}}{t}.$$

Thus the expected number of cycles of length $\le 2t_0$ contained in **G** is at most

$$\sum_{t\le t_0}\frac{2^{2t+1}}{t}<\frac{\sqrt{n}}{100}.$$

Hence we infer that the probability ≥ 0.99, **G** contains at most $\sqrt{n}$ cycles of length $\leq 2t_0$. Choosing an edge from each of these cycles we get the required set of edges.

Now we are ready to finish the proof of the lemma. Due to Claims 1-3, for sufficiently large n there exists a bipartite graph $G=(V, E)$ on $2n$ vertices that is 3-regular, has the partition property (ii), and only $\leq \sqrt{n} \cdot 2t_0$ of its edges is contained in induced cycles of length $\leq 2t_0$.

We modify G to get a new graph satisfying (i)-(iii) of the lemma. Consider an edge xy of G that is contained in a cycle C of length $\leq 2t_0$. Choose another edge $x'y'$ such that the distance from x to x' in G is $\geq 2t_0+1$ and that $x'y'$ is contained in no cycle of length $\leq 2t_0$; such $x'y'$ does exist, for at most $\frac{1}{8}n \log n$ vertices are contained in cycles of length $\leq 2t_0$ and at most $3(2^{2t_0+1}-1)\cdot\sqrt{n}$ vertices $x' \neq x$ are joined with x by a path of length $\leq 2t_0+1$. Delete edges xy, $x'y'$ from G and add xy', yx' to G. This does destroy the cycle C but does not create any new cycle of length $\leq 2t_0$; moreover, the resulting graph remains 3-regular. Repeating this procedure at most $\sqrt{n}$ times (once for each short cycle) we get a 3-regular graph without cycles of length $\leq 2t_0$. Also, the new graph satisfies property (ii) with $|M| \geq \frac{1}{33}n - \sqrt{n}$, i.e., $|M| \geq n/34$ for sufficiently large n.

Lemma 2. *The graph $G=(V, E)$ from Lemma 1 has the following property: for every function $f: V \to R$ there is an induced subtree T of G such that:*

(i) diam $T \leq 2t_0$.
(ii) $|L| \geq n^{1/17}$, *where L is the set of leaves of T.*
(iii) $f(x) \geq f(y)$ *whenever x is a leaf and y is a nonleaf in T.*

Proof. For every $x \in V$, let L_x be the set of all vertices of V where the distance from x is $t_1 = \lfloor t_0/2 \rfloor - 1$. As G does not contain cycles of length $\leq 2t_0$ and G is 3-regular, L_x has cardinality

$$S = 3.2^{t_1-1} < \sqrt{n}.$$

Write V as $V=\{v_1, \ldots, v_{2n}\}$ where $f(v_1) \geq f(v_2) \geq \cdots \geq f(v_{2n})$. Choose j such that for $P=\{v_1, \ldots, v_j\}$ and for $B=\bigcup_{x \in P} L_x$ we have

$$n - \sqrt{n} \leq |B| \leq n$$

and put $A = V - B$. Choose $A' \subset A$, $B' \supset B$ with $|A'| = |B'| = n$. By Lemma 1, condition (ii), there exists a matching M joining A' with B' with $|M| \geq \frac{1}{34}n$. Let A_1 be the set of vertices x in A' such that $xy \in M$ for some $y \in B$. As $|B' - B| < \sqrt{n}$ we have $A_1 \geq \frac{1}{34}n - \sqrt{n}$ and for $A_2 = A - A_1$ we have $|A_2| = |A| - |A_1| \leq \frac{33}{34}n + 2\sqrt{n} \leq \frac{34}{35}n$ for sufficiently large n.

(a) We claim that there exists $x_0 \in A$ such that

$$|L_{x_0} - A_2| \geq \frac{S}{35}.$$

Indeed, otherwise $|L_x \cap A_2| > \frac{34}{35}S$ for every $x \in A$. Denote by N the number of pairs (x, y) with $x \in A$, $y \in A_2$, and $y \in L_x$ (equivalently, $x \in L_y$). Then, on one hand,

$$N = \sum_{x \in A} |L_x \cap A_2| > |A|\tfrac{34}{35}S \geq \tfrac{34}{35}nS$$

and, on the other hand,

$$N = \sum_{y \in A_2} |L_y \cap A \leq |A_2|S \leq \tfrac{34}{35}nS,$$

a contradiction.

(b) Let $L_{x_0} - A_2 = \{x_1, \ldots, x_m\}$, $m \geq S/35$. For every $i \in \{1, \ldots, m\}$, we have a path P_i of length t_1 from x_0 to x_i and a path Q_i of length $\leq t_1 + 1$ from x_i to some $z_i \in P$ (the existence of Q_i follows by the definition of B if $x_i \in B$; if $x_i \in A_1$ we use an M-edge $x_i y$ with $y \in B$ and a path of length $= t_1$ from y to a vertex in P). As the x_i's are pairwise distinct and G does not contain cycles of length $2t_0 \geq 4t_1 + 2$, the z_i's are pairwise distinct, too, and the union of P_i's and Q_i's is an induced subtree T of G. Assuming, in addition, that z_i is the only P-vertex on Q_i, the tree T satisfies (iii) due to the definition of P. We have diam $T \leq 4t_1 + 2 \leq 2t_0$ and $|L| = m \geq S/35 = \frac{3}{35}2^{\lfloor t_0/2 \rfloor - 2} \geq 2^{(1/17)\log n} = n^{1/17}$ for sufficiently large n. □

Proof of the Theorem. Let $G = (V, E)$ be the 3-regular bipartite graph on $2n$ vertices from Lemma 2; let $x \to \bar{x}$ $(x \in V)$ be its representation in R^d with $t = 1$. Set $f(x) = \|\bar{x}\|$ for $x \in V$. Let T be the induced subtree of Lemma 2. Thus we have $\gamma > 0$ such that $\|\bar{x}\| \geq \gamma$ or $\|\bar{x}\| \leq \gamma$ if x is a leaf or a nonleaf in T, respectively. Let x_0 be a nonleaf such that the distance from x_0 to each of the leaves is at most t_0. For every leaf x, let f_x and g_x be vertices on the path from x_0 to x where the distance from x is 1 and 2, respectively.

Let L_1 be a maximal set of leaves with the property

$$x, y \in L_1, \qquad x \neq y \quad \text{implies} \quad g_x \neq g_y \quad \text{and} \quad g_x g_y \notin E.$$

If follows easily from the 3-regularity of G and the maximality of L_1 that $|L_1| \geq |L|/8 \geq n^{1/17}/8$. For every $x \in L_1$ set $\tilde{x} = \alpha\bar{x} + (1-\alpha)\bar{g}_x$ where $\alpha \in \langle 0, 1 \rangle$ is chosen so that $\|\tilde{x}\| = \gamma$; such α does exist because $\|\tilde{x}\| = \|\bar{x}\| \geq \gamma$ for $\alpha = 0$ and $\|\tilde{x}\| = \|\bar{g}_x\| \leq \gamma$ for $\alpha = 1$; for $x \in L_1$ set $\tilde{x} = \bar{x}$. Then for $x, y \in L_1$, $x \neq y$,

$$\tilde{x}\tilde{y} = [\alpha\bar{x} + (1-\alpha)\bar{g}_x][\beta\bar{y} + (1-\beta)\bar{g}_y] < 1,$$
$$\tilde{x}\tilde{f}_x = [\alpha\bar{x} + (1-\alpha)\bar{g}_x]\bar{f}_x \geq 1.$$

For every $x \in L_1$, let K_x be the ball with center $\tilde{x}$ and radius $r = \sqrt{2\gamma^2 - 2}/2$. We have for $x, y \in L_1$

$$\|\tilde{x} - \tilde{y}\| = (\tilde{x}^2 - 2\tilde{x}\tilde{y} + \tilde{y}^2)^{1/2} > (2\gamma^2 - 2)^{1/2} = 2r.$$

Hence the balls K_x are pairwise disjoint. On the other hand, if $x_0, x_1, \ldots, x_t = x$ is a path from x_0 to $x \in L$, then

$$\|\tilde{x}_i - \tilde{x}_{i+1}\| = (\tilde{x}_i^2 - 2\tilde{x}_i\tilde{x}_{i+1} + \tilde{x}_{i+1}^2)^{1/2} \leq (2\gamma^2 - 2)^{1/2} = 2r,$$

hence balls K_x are contained in the ball with center x_0 and radius $\leq (2t_0 + 1)r$. Then

$$|L_1| V_d r^d \leq V_d[(2t_0 + 1)r]^d,$$

where V_d is the volume of the unit ball in R^d. It follows

$$n^{1/17}/8 \leq (2t_0 + 1)^d,$$

and thus

$$d \geq \frac{\log(n^{1/17}/8)}{\log(2t_0 + 1)} \geq \frac{\frac{1}{17}\log n - 3}{\log(\log n) - 2} \geq \frac{\log n}{18 \log(\log n)}$$

for sufficiently large n. □

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