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Approximation of Convex Discs by Polygons*

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Abstract. We consider the class of all convex discs with areas and perimeters bounded by given constants. Which disc of this class has the least possible area deviation from a k-gon? This and related questions are the subject of the present paper.

1. Introduction

By a *convex disc* we mean a convex compact subset of the Euclidean plane with interior points. In this paper we shall deal with the approximation of convex discs by convex polygons. There are several methods of measuring the deviation between two convex discs. The following are two of the most usual methods. We write $a(M)$ and $p(M)$ for the area, i.e., the Lebesgue measure, and the perimeter of the set M. If X and Y are convex discs, the *area deviation* between X and Y is defined by

$$\delta^A(X, Y) = a(X \cup Y) - a(X \cap Y), \tag{1}$$

and the *perimeter deviation* by

$$\delta^P(X, Y) = p(X \cup Y) - p(X \cap Y). \tag{2}$$

Equation (1) may also be written in the form

$$\delta^A(X, Y) = a(X \Delta Y),$$

* Dedicated to Professor E. Hlawka on the occasion of his birthday.

where

$$X \Delta Y = (X \setminus Y) \cup (Y \setminus X)$$

is the *symmetric difference* of the two sets. Note that δ^A makes the class of all convex discs into a metric space, but δ^P does not. Various concepts of deviation of two convex bodies are discussed in [17]. Gruber [14] gives an up-to-date review of the results concerning the approximation of a convex body by polytopes. The approximation of convex discs by convex polygons is of interest by itself. Moreover, it is important in its application to problems of packing and covering (see [3, 5, 6, 8–11, 13]).

Throughout this paper let a and p be positive numbers satisfying the isoperimetric inequality

$$\frac{p^2}{a} \geq 4\pi.$$

Let $\mathscr{C}(a, p)$ be the class of all convex discs with area not less than a and perimeter not greater than p. A convex polygon with at most k sides is simply called a k-gon. Let $\mathscr{P}_k$ denote the class of all k-gons. Two measures for the closeness of the approximation of k-gons to discs from $\mathscr{C}(a, p)$ are given by

$$\Delta^A(a, p, k) = \inf \delta^A(C, P) \tag{3}$$

and

$$\Delta^P(a, p, k) = \inf \delta^P(C, P), \tag{4}$$

where the infimum is taken over all $C \in \mathscr{C}(a, p)$ and all $P \in \mathscr{P}_k$. Both functions are interesting only in the case when

$$\frac{p^2}{a} < 4k \tan \frac{\pi}{k}, \tag{5}$$

which means that p is less than the perimeter of a regular k-gon of area a. Otherwise we have $\mathscr{C}(a, p) \cap \mathscr{P}_k \neq \emptyset$, so that $\Delta^A(a, p, k) = \Delta^P(a, p, k) = 0$.

By combining some results from [2], [5] we shall obtain $\Delta^P(a, p, k)$ in Section 2. Using ideas of Besicovitch [1], Eggleston [2], and Fejes Tóth and Florian [5] we will find the supremum of $a(C \cap P)$ taken over all $C \in \mathscr{C}(a, p)$ and all k-gons P of a given area; this is the subject of Section 3 and the main part of this paper. In Section 4 we will apply the results of Section 3 to determine those members of $\mathscr{C}(a, p)$ and $\mathscr{P}_k$ for which $\delta^A(C, P)$ is minimal.

2. The Perimeter Deviation

According to a remarkable result by Eggleston [2], the perimeter deviation of a given convex disc from an arbitrary convex k-gon is minimal for a k-gon inscribed in the disc. Thus we have

$$\Delta^P(a, p, k) = \inf \delta^P(C, P) \qquad (C \in \mathscr{C}(a, p), P \in \mathscr{P}_k, P \subset C). \tag{6}$$

Before we state the result, we need to describe a certain geometrical configuration. Let P^* be a regular k-gon. We join each two consecutive vertices of P^* by congruent circular arcs of radius not less than the circum-radius of P^*. These arcs form the boundary of a convex disc C^* which we call a *regular arc-sided k-gon* with *kernel* P^* (Fig. 1). If (5) is satisfied it can be shown (see [5]) that

(i) there is exactly one regular arc-sided k-gon C^* with area a and perimeter p, and
(ii) the infimum on the right-hand side of (6) is attained only for C^* and its kernel P^*.

Hence

$$\Delta^P(a, p, k) = \delta^P(C^*, P^*). \tag{7}$$

A simple expression for $\delta^P(C^*, P^*)$ can be obtained in the following way (see [5]). Let 2α be the central angle of the circular arcs bounding C^*, where $0 < \alpha \le \pi/k$. We define a function $\Phi(q)$ by the parametric equations

$$\Phi(q) = \frac{\alpha^2}{\sin^2 \alpha}, \qquad q = \frac{\alpha - \sin\alpha\cos\alpha}{\sin^2\alpha} \qquad \left(0 < \alpha \le \frac{\pi}{k}\right) \tag{8}$$

for $0 < q \le \bar{q}$, where $\bar{q}$ corresponds to $\alpha = \pi/k$, and put $\Phi(0) = 1$. Elementary calculation yields the equation

$$\frac{p^2}{a} = 4k \sin\frac{\pi}{k} \frac{\Phi(q)}{\cos(\pi/k) + q\sin(\pi/k)}, \tag{9}$$

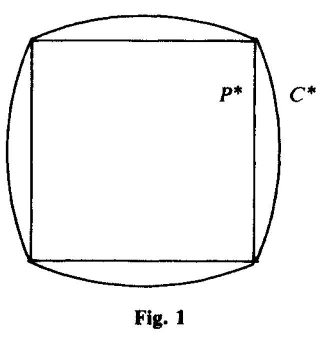

Fig. 1

which has a single root $q \in (0, \bar{q}]$. Since $p(P^*) = (p \sin \alpha)/\alpha$, we finally conclude that

$$\delta^P(C^*, P^*) = p(1-(\Phi(q))^{-1/2}). \tag{10}$$

3. A Maximum Problem

Let $\mathcal{P}_k(a_0)$ denote the class of all k-gons of given area a_0. In this section we shall deal with the following

Problem. Find a member of $\mathscr{C}(a, p)$ and a member of $\mathcal{P}_k(a_0)$ such that their intersection has the greatest possible area.

Accordingly we introduce the function

$$M(a, p; k, a_0) = \max a(C \cap P), \tag{11}$$

where the maximum is to be taken over all discs C from $\mathscr{C}(a, p)$ and all k-gons P from $\mathcal{P}_k(a_0)$. The existence of the maximum follows from the Blaschke selection theorem.

In the particular case when $p^2/a = 4\pi$, the class $\mathscr{C}(a, p)$ consists only of the circle of area a. Fejes Tóth [7] showed that the intersection of a given circle C and a k-gon P of given area has maximal area if P is regular and concentric with C; for alternative proofs of this "momentum lemma" see [4], [12], [15].

In certain cases the solution to our problem can be deduced from two previous results which we now recollect.

Let the function $f_1(a, p, k)$ be defined by

$$f_1(a, p, k) = \begin{cases} \dfrac{p^2}{4\pi t}\dfrac{1}{\Phi(q)} & \text{if } \dfrac{p^2}{a} < 4\pi t, \\ \dfrac{p^2}{4\pi t} & \text{if } \dfrac{p^2}{a} \geq 4\pi t, \end{cases} \tag{12}$$

where $t = t(k) = (k/\pi)\tan(\pi/k)$, and $\Phi(q)$ is given by (8). Let C be a disc from $\mathscr{C}(a, p)$ and P a k-gon with $P \subset C$. It was proved in [5] that

$$a(P) \leq f_1(a, p, k), \tag{13}$$

with equality if and only if either C is a regular arc-sided k-gon of area a and perimeter p, and P is the kernel of C ($p^2/a < 4\pi t$), or $C = P$ is a regular k-gon of perimeter p ($p^2/a \geqq 4\pi t$). By (13) we have

$$\begin{aligned} &M(a, p; k, a_0) = a_0 \quad \text{if} \quad a_0 \leqq f_1(a, p, k), \quad \text{and} \\ &M(a, p; k, a_0) < a_0 \quad \text{if} \quad f_1(a, p, k) < a_0. \end{aligned} \tag{14}$$

Hence in the following we can assume that $a_0 > f_1(a, p, k)$.

Let x be a convex disc with in radius r. $X_{-\rho}$ $(0<\rho<r)$ denotes the inner parallel domain and X_ρ $(0<\rho)$ the outer parallel domain of X at distance ρ. If $\bar{P}$ is a regular k-gon, we call a set of the form $(\bar{P}_{-\rho})_\rho$ a *smooth regular k-gon* with *case* $\bar{P}$ (Fig. 2). It will be convenient to consider both $\bar{P}$ and its in-circle as degenerate smooth regular k-gons with case $\bar{P}$. The corresponding values of ρ are 0 and r.

We define the function $F(p, k, a_0)$ by

$$F(p, k, a_0) = \begin{cases} a_0 & \text{if } a_0 < \dfrac{p^2}{4\pi t}, \\[2ex] \dfrac{p\sqrt{a_0 \pi t} - p^2/4 - a_0\pi}{(t-1)\pi} & \text{if } \dfrac{p^2}{4\pi t} \le a_0 \le \dfrac{p^2 t}{4\pi}, \\[2ex] \dfrac{p^2}{4\pi} & \text{if } \dfrac{p^2 t}{4\pi} < a_0. \end{cases} \tag{15}$$

Let C be a convex disc of perimeter not greater than p and P a k-gon from $\mathcal{P}_k(a_0)$ with $C \subset P$. Fejes Tóth ([6] or [8, p. 175]) proved that

$$a(C) \le F(p, k, a_0). \tag{16}$$

Let $p^2/4\pi t \le a_0 \le p^2 t/4\pi$. Then equality holds in (16) if and only if C is a (possibly degenerate) smooth regular k-gon with perimeter p, and P is the case of C.

Let C be a disc from $\mathcal{C}(a, p)$ and P a k-gon with $C \subset P$. Since F is a strictly increasing function of a_0 for $0 < a_0 \le p^2 t/4\pi$, it follows from (15) and (16) that

$$a(P) \ge f_2(a, p, k), \tag{17}$$

where

$$f_2(a, p, k) = \begin{cases} \dfrac{t}{4\pi}\left(p - \sqrt{(p^2 - 4a\pi)(1-t^{-1})}\right)^2 & \text{if } \dfrac{p^2}{a} < 4\pi t, \\[2ex] a & \text{if } \dfrac{p^2}{a} \ge 4\pi t. \end{cases} \tag{18}$$

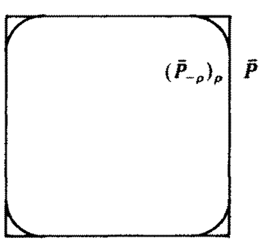

Fig. 2

If $p^2/a \le 4\pi t$ and $a(P) = f_2(a, p, k)$, then C is a (possibly degenerate) smooth regular k-gon of area a and perimeter p, and P is the case of C. Thus, by (18) we have

$$f_2(a, p, k) \ge a \tag{19}$$

and

$$M(a, p; k, a_0) \ge a \qquad \text{if} \quad a_0 \ge f_2(a, p, k). \tag{20}$$

Returning to our problem, we distinguish the cases $p^2/a < 4\pi t$ and $p^2/a \ge 4\pi t$, and begin with the simpler

Case (a). $p^2/a \ge 4\pi t$

From (12) and (18) we see that

$$f_2(a, p, k) \le f_1(a, p, k).$$

Let $a_0 > f_1(a, p, k)$, and let C and P be members of $\mathscr{C}(a, p)$ and $\mathscr{P}_k(a_0)$ such that

$$a(C \cap P) = M(a, p; k, a_0). \tag{21}$$

Inequalities (13) and (20) imply that $P \not\subset C$ and

$$a(C \cap P) \ge a. \tag{22}$$

We shall see in the proof of Theorem 1 (Lemma 4) that, whenever (21) together with the assumptions $P \not\subset C$ and $C \not\subset P$ are satisfied, then $a(C) = a$, whence $a(C \cap P) < a$. By (22) this last inequality is impossible. Thus we conclude that $C \subset P$, and from (16) and (21) it follows that

$$a(C \cap P) = F(p, k, a_0). \tag{23}$$

Equations (14) and (23) can be summarized in

Remark 1. If $p^2/a \ge 4\pi t$ we have

$$M(a, p; k, a_0) = F(p, k, a_0).$$

From the suppositions (21) and $p^2/4\pi t \le a_0 \le p^2 t/4\pi$ it follows that C is a (possibly degenerate) smooth regular k-gon of perimeter p, and P is the case of C.

Case (b). $p^2/a < 4\pi t$

Because of (12) and (19) we obtain the inequality

$$f_1(a, p, k) < f_2(a, p, k), \tag{24}$$

which is contrary to case (a).

Let $a_0 \geq f_2(a, p, k)$, and let $C \in \mathscr{C}(a, p)$ and $P \in \mathscr{P}_k(a_0)$ satisfy (21). By repeating the argument used in case (a) we again come to the conclusion that C is contained in P and that (23) holds in case (b) as well.

Remark 2. If $p^2/a < 4\pi t$ we have

$$M(a, p; k, a_0) = \begin{cases} a_0 & \text{if } a_0 \leq f_1(a, p, k), \\ F(p, k, a_0) & \text{if } f_2(a, p, k) \leq a_0. \end{cases}$$

From (21) and $a_0 = f_1(a, p, k)$ it follows that C is a regular arc-sided k-gon of area a and perimeter p, and P is the kernel of C. From (21) and $f_2(a, p, k) \leq a_0 \leq p^2 t/4\pi$ it follows that C is a (possibility degenerate) smooth regular k-gon of perimeter p, and P is the case of C.

We now proceed to find the maximum of a $(C \cap P)$ with $C \in \mathscr{C}(a, p)$ and $P \in \mathscr{P}_k(a_0)$ in the more difficult case when

$$f_1(a, p, k) < a_0 < f_2(a, p, k).$$

To describe the extremal configuration we consider the outer parallel domain C of a regular arc-sided k-gon at some distance ρ. We first assume that C is not a circle. Then C is bounded by k equal circular arcs of radius $\hat{r}$ and k equal circular arcs of radius r, where $\hat{r} < r$. The lines joning the endpoints of every arc of radius r enclose a regular k-gon P which we call the *central k-gon of* C (Fig. 3). By a central k-gon of a circle C we mean any regular k-gon concentric with C.

Theorem 1. *Let*

(i) $$f_1(a, p, k) < a_0 < f_2(a, p, k)$$

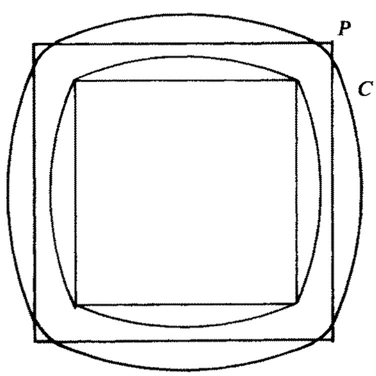

Fig. 3

and let C and P be such members of $\mathscr{C}(a, p)$ and $\mathscr{P}_k(a_0)$ that

(ii) $$a(C \cap P) = M(a, p; k, a_0).$$

Then C is an outer parallel domain of a regular arc-sided k-gon, and P is the central k-gon of C. Furthermore, C has area a and perimeter p.

We shall, in fact, prove Theorem 1 by making the weaker assumptions (ii) and

(iii) $$C \not\subset P, \qquad P \not\subset C$$

instead of (i) and (ii). Theorem 1 together with Remarks 1 and 2 solve the problem set at the beginning of this section. Observe that a regular arc-sided k-gon and its kernel as well as a smooth regular k-gon and its case may be regarded as a degenerate parallel domain of a regular arc-sided k-gon and its central k-gon.

Proof of Theorem 1. Let C and P satisfy suppositions (ii) and (iii). We will develop the properties of C and P in the following 12 lemmas. The last lemma shows that C and P correspond with the statement of our theorem.

First, we remark that by (ii) and (iii) there is a vertex of P outside C, and there is a side of P intersecting the interior of C.

Lemma 1. *P has exactly k vertices.*

Proof. Suppose that P has fewer than k vertices. Let A_1 be a vertex of P outside C, and let A_iA_{i+1} be a side of P containing interior points of C. We cut off from P a sufficiently small triangle with vertex A_1 and displace A_iA_{i+1} toward the exterior of P such that the new k-gon P' obtained from P by this process has area a_0. Then we have

$$a(C \cap P') > a(C \cap P) \tag{25}$$

in contradiction to assumption (ii). □

We denote the vertices of P in the anticlockwise sense by $A_1, A_2, \ldots, A_k$ and set $A_{k+1} = A_1$, $A_0 = A_k$.

Lemma 2. *No vertex of P lies in the interior of C.*

Proof. Suppose that A_1 is outside C and A_2 is an interior point of C. Let the side A_1A_2 rotate in the clockwise sense about a point between A_1 and A_2 such that the new k-gon $P' = A_1'A_2'A_3 \cdots A_k$ has area a_0. If the angle of rotation is small, A_1' is exterior to C and A_2' is an interior point of C. This again leads to inequality (25).

Suppose that A_1 lies on the boundary and A_2 in the interior of C. Let the side A_1A_2 rotate about A_1 in the clockwise sense through a small angle into the new position A_1A_2' with A_2' in the interior of C. If A_i is one of the vertices outside C we displace A_iA_{i+1} toward the interior of P such that the k-gon P' finally obtained has area a_0. Since P' satisfies (25), the supposition was wrong, and Lemma 2 is proved. □

We shall use the symbol AB to denote both the segment AB and its length.

Lemma 3. (a) *A side A_1A_2 of P that contains interior points of C meets the boundary of C at points U_{11}, U_{12}, where U_{12} is between U_{11} and A_2, such that*

$$A_1U_{11} = U_{12}A_2. \tag{26}$$

(b) *Any side A_iA_{i+1} of P intersects C at the points of a segment $U_{i1}U_{i2}$ such that*

$$\frac{U_{i1}U_{i2}}{A_iA_{i+1}} \geq \frac{U_{11}U_{12}}{A_1A_2}. \tag{27}$$

Proof. (a) Suppose that, contrary to (26),

$$A_1U_{11} > U_{12}A_2. \tag{28}$$

This implies that A_1 is outside C and

$$MU_{11} < MU_{12}, \tag{29}$$

where M is the midpoint of A_1A_2. Let A_1A_2 rotate in the clockwise sense through a small angle φ into the new position $A_1'A_2'$, such that $P' = A_1'A_2'A_3 \cdots A_k$ has area a_0. The segments A_1A_2 and $A_1'A_2'$ intersect at, say, M'. Denoting the triangle with vertices X, Y, Z by XYZ, we have

$$a(C \cap P') - a(C \cap P) = a(C \cap M'A_2A_2') - a(C \cap M'A_1A_1'). \tag{30}$$

If M' is outside or on the boundary of C, then $a(C \cap M'A_2A_2') > 0$ and $a(C \cap M'A_1A_1') = 0$, so that P' satisfies (25). Thus we can assume that M' is an inner point of C. Since M' approaches M as φ tends to zero, M belongs to C and we obtain

$$\lim_{\varphi \to 0} \frac{a(C \cap M'A_1A_1')}{a(C \cap M'A_2A_2')} = \frac{MU_{11}^2}{MU_{12}^2}. \tag{31}$$

From (29), (30) and (31) we see that P' satisfies (25) if φ is sufficiently small. Because (25) is impossible, part (a) of Lemma 3 is proved.

(b) It suffices to show (27) for $2 \leq i \leq k-1$. We displace the side A_1A_2 of P outward parallel to itself through a small distance $\eta_1 > 0$. Let $P' = A_1'A_2' \cdots A_k'$ be the new k-gon, where $A_j' = A_j$ for $j = 3, \ldots, k$. From P' we obtain the further k-gon P'' by displacing the side $A_i'A_{i+1}'$ inward parallel to itself through a small distance $\eta_2 < 0$, such that $a(P'') = a_0$. Write $P'' = A_1'' \cdots A_k''$ with $A_j'' = A_j'$ for $j \neq i, i+1$. We shall use the notations

$$C \cap P = S, \qquad C \cap P' = S', \qquad C \cap P'' = S'' \tag{32}$$

and note that $S \subset S'$ and $S'' \subset S'$. From the definitions of P' and S' we obtain the relations

$$\lim_{\eta_1 \to 0} \frac{a(P') - a(P)}{\eta_1} = A_1A_2, \qquad \lim_{\eta_1 \to 0} \frac{a(S') - a(S)}{\eta_1} = U_{11}U_{12}. \tag{33}$$

Because $A_i'A_{i+1}' \geq A_iA_{i+1}$, we have

$$\begin{aligned} a(P) - a(P') = a(P'') - a(P') &= \tfrac{1}{2}(A_i'A_{i+1}' + A_i''A_{i+1}'')\eta_2 \\ &< \tfrac{1}{2}A_iA_{i+1}\eta_2 < 0. \end{aligned} \tag{34}$$

Together with $\lim_{\eta_1 \to 0} a(P') = a(P)$ this implies that

$$\lim_{\eta_1 \to 0} \eta_2 = 0. \tag{35}$$

By the definitions of P' and P'' there are constants γ_1 and γ_2 such that

$$A_i'A_{i+1}' = A_iA_{i+1} + \gamma_1\eta_1, \qquad A_i''A_{i+1}'' = A_i'A_{i+1}' + \gamma_2\eta_2. \tag{36}$$

From (33) to (36) we deduce that

$$\lim_{\eta_1 \to 0} \frac{\eta_2}{\eta_1} = -\frac{A_1A_2}{A_iA_{i+1}}. \tag{37}$$

Observe that the intersection of C with A_iA_{i+1} is nonempty. Otherwise we would have

$$a(S'') = a(S') > a(S)$$

if η_1 is sufficiently small, and this contradicts the maximum property of P. Thus $C \cap A_iA_{i+1}$ is a segment, say $U_{i1}U_{i2}$. We now proceed to prove that

$$\lim_{\eta_1 \to 0} \frac{a(S') - a(S'')}{\eta_2} = -U_{i1}U_{i2}. \tag{38}$$

By (32) we have

$$a(S') - a(S'') = a(S' \backslash S'') = a(C \cap (P' \backslash P'')). \tag{39}$$

Decomposing $P' \backslash P''$ into two disjoint sets according to

$$P' \backslash P'' = [(P' \backslash P'') \cap P] \cup [(P' \backslash P'') \cap (P' \backslash P)], \tag{40}$$

we distinguish two cases. In both we assume, as we clearly may, that η_1 is sufficiently small.

(α) If $i=2$ then $(P'\backslash P'')\cap(P'\backslash P)$ is a parallelogram with sides A_2A_2' and $A_2'A_2''$. Hence

$$a[(P'\backslash P'')\cap(P'\backslash P)]=O(\eta_1\eta_2). \tag{41}$$

(β) If $i>2$ then

$$(P'\backslash P'')\cap(P'\backslash P)=\varnothing. \tag{42}$$

Let l be the line joining A_i and A_{i+1}, and let $L(\eta_2)$ be the parallel strip bounded by l and the line through A_i'' and A_{i+1}''. Let $s(x)$ denote the length of the intersection of $C\cap P$ with the line parallel to l at distance $x>0$ from l. Then we have

$$a[C\cap P\cap(P'\backslash P'')]=a[C\cap P\cap L(\eta_2)]=-\eta_2 s(x) \tag{43}$$

for some x between 0 and $-\eta_2$. Because $\lim s(x)=U_{i1}U_{i2}$ as x tends to zero, the required equation (38) follows from (35) and (39) to (43).

Supposing the contrary to (27), we obtain by (33), (37) and (38) that

$$\lim_{\eta_1\to 0}\frac{a(S')-a(S'')}{a(S')-a(S)}=\frac{U_{i1}U_{i2}/A_iA_{i+1}}{U_{11}U_{12}/A_1A_2}<1,$$

which implies by (32) that

$$a(C\cap P'')>a(C\cap P). \tag{44}$$

The contradiction of (44) to assumption (ii) of Theorem 1 completes the proof of part (b). □

Corollary 1. *Any side of P intersects C at the points of a segment of positive length. A side of P which intersects the interior of C has its endpoints outside C.*

Following the notation used in Lemma 3 we shall constantly denote the points at which the side A_iA_{i+1} meets the boundary of C by U_{i1} and U_{i2}. The points $U_{11}, U_{12}, \ldots, U_{k1}, U_{k2}$ are round the boundary of C in the anticlockwise sense.

In the proofs of the above lemmas C is a given convex disc and P runs through the class $\mathcal{P}_k(a_0)$. In the proofs of the following seven lemmas P is a given k-gon while C varies on $\mathcal{C}(a,p)$.

Lemma 4. *$p(C)=p$ and $a(C)=a$.*

Proof. If $p(C)<p$ we choose $\rho>0$ such that $p(C_\rho)<p$. Thus C_ρ belongs to $\mathscr{C}(a,p)$. Since the boundary of C intersects the interior of P we have

$$a(C_\rho \cap P)>a(C\cap P)$$

in contradiction to the maximum property of C.

Suppose that $a(C)>a$. Since there are interior points of C outside P, we can find a proper convex subset of C, say C', such that C' is a member of $\mathscr{C}(a,p)$ and

$$a(C'\cap P)=a(C\cap P).$$

Because $p(C')<p$ this is impossible. □

Lemma 5. *Suppose that the side A_1A_2 contains interior points of C. Let s_1 be the arc on the boundary of C between U_{11} and U_{12}, which is outside P. s_1 is a circular arc.*

Proof. Let c be the circular arc on the same side of the line A_1A_2 as s_1 that has endpoints U_{11} and U_{12} and the same length as s_1. Let us assume that $s_1\neq c$. Denoting the convex hull of the set M by $\operatorname{conv} M$ and the (not necessarily convex) disc $(C\backslash \operatorname{conv} s_1)\cup \operatorname{conv} c$ by D we have

$$p(D)=p(C), \qquad a(D\cap P)=a(C\cap P). \tag{45}$$

We now refer to the well-known fact that the area of a convex disc which is bounded by a given straight segment and an arc of given length attains its maximum if and only if the disc is a circular segment. Thus

$$a(\operatorname{conv} s_1)<a(\operatorname{conv} c), \tag{46}$$

whence

$$a(D)>a(C). \tag{47}$$

From (45) and (47) it follows that $C'=\operatorname{conv} D$ belongs to $\mathscr{C}(a,p)$ and

$$a(C'\cap P)\geq a(C\cap P).$$

By (47), however, we have $a(C')>a$, which contradicts Lemma 4. Thus the assumption $s_1\neq c$ was wrong and Lemma 5 is proved. □

Lemma 6. *C is strictly convex.*

Proof. Suppose that the straight segment V_1V_2 is part of the boundary of C. Let S be a circular segment of $C\backslash P$, the chord of which has length less than V_1V_2. We cut S off from C and join it to V_1V_2, obtaining a nonconvex disc D with $a(D)=a(C)$, $p(D)=p(C)$ and a $a(D\cap P)\geq a(C\cap P)$. Then $C'=\operatorname{conv} D$ has the properties

$$p(C')<p, \qquad a(C')>a$$

and

$$a(C'\cap P)\geq a(C\cap P)$$

in contradiction to Lemma 4. Thus Lemma 6 is proved. □

From Lemma 3, Corollary 1 and Lemma 6 we infer

Corollary 2. (i) *Every vertex of P is exterior to C.*

(ii) *Every side of P intersects the interior of C.*

(iii)
$$A_iU_{i1}=U_{i2}A_{i+1} \qquad \textit{for} \quad i=1,\ldots,k. \tag{48}$$

(iv)
$$\frac{U_{11}U_{12}}{A_1A_2}=\cdots=\frac{U_{k1}U_{k2}}{A_kA_1}. \tag{49}$$

Lemma 7. *Let s_i be the circular arc on the boundary of C between U_{i1} and U_{i2}, which is outside P. s_i is less than a semicircle, for $i=1,\ldots,k$.*

Proof. Suppose that s_1 is greater than or equal to a semicircle. Let t_1 and t_2 be the tangent lines to s_1 at U_{11} and U_{12}, respectively. By (48) and (49) the segments $U_{12}U_{21}$ and $U_{11}U_{22}$ are parallel. Since C is contained in the set bounded by s_1, t_1 and t_2, s_1 is at most a semicircle. In this case U_{21} lies on t_2, and U_{22} on t_1. Thus A_1 and A_3 are separated from C by t_1, showing that the sides $A_3A_4,\ldots,A_kA_1$ do not intersect C. But by Corollary 2(ii) this is impossible. □

Lemma 8. *The circular arcs $s_1,\ldots,s_k$ considered in Lemma 7 have the same radius, say r.*

Proof. Suppose that s_1 and s_2 have different radii. Let c_1 and c_2 be two chords of the arcs s_1 and s_2, other than $U_{11}U_{12}$ and $U_{21}U_{22}$, and having equal lengths. Let s_1' and s_2' be the respective subarcs of s_1 and s_2. From C we obtain a new disc, say D, by exchanging the positions of the circular segments $\operatorname{conv} s_1'$ and $\operatorname{conv} s_2'$. This means that we cut them off from C and join them to c_2 and c_1, respectively. Obviously,

$$p(D)=p(C), \qquad a(D)=a(C), \qquad a(D\cap P)=a(C\cap P). \tag{50}$$

Since s_1 and s_2 have different radii and $c_1\neq U_{11}U_{12}$, $c_2\neq U_{21}U_{22}$, D is not convex. Thus by (50) we have for $C'=\operatorname{conv} D$

$$C'\in\mathscr{C}(a,p), \qquad a(C')>a, \qquad a(C'\cap P)\geq a(C\cap P). \tag{51}$$

The contradiction to Lemma 4 proves the lemma. □

Lemma 9. *Let $\hat{s}_i$ be the arc on the boundary of C between $U_{i-1,2}$ and U_{i1}, which is contained in P, for $i=1,\ldots,k$.*

(i) *$\hat{s}_i$ is a circular arc;*
(ii) *the arcs $\hat{s}_1,\ldots,\hat{s}_k$ have the same radius, say $\hat{r}$;*
(iii) *if r is the radius of the arcs considered in Lemma 8, then*

$$\hat{r}\leq r. \tag{52}$$

Proof. (i) Let V_1, V_2 be two distinct points of $\hat{s}_2 = \widehat{U_{12}U_{21}}$ other than U_{12} and U_{21}. In order to show that $\hat{s}_2$ is a circular arc it suffices to prove this for the arc $\widehat{V_1V_2}$. Observe that $\widehat{V_1V_2}$ has positive distance, say d, from the boundary of P. We cover $\widehat{V_1V_2}$ by a finite number of its subarcs such that each of them overlaps the following and has length less than d. By Lemma 6, not one is a straight segment. If any of these subarcs is not a circular arc, we replace it by a circular arc c of the same length in exactly the same way as in the proof of Lemma 5. By construction, the convex hull of c is contained in P. We obtain a (not necessarily convex) disc D with

$$p(D) = p(C), \qquad a(D) > a(C), \qquad a(D \cap P) > a(C \cap P).$$

Thus $C' = \operatorname{conv} D$ satisfies (51) which is impossible. Hence each of the subarcs is circular, and so is $\widehat{V_1V_2}$.

(ii) Suppose that $\hat{s}_1$ and $\hat{s}_2$ have different radii. We obtain a nonconvex disc D satisfying (50) by exchanging the positions of two small circular segments, the arcs of which are part of $\hat{s}_1$ and $\hat{s}_2$, respectively. Since the proof is quite similar to that of Lemma 5 we omit the details.

(iii) Suppose that $\hat{r} > r$. Similarly, as in the proof of Lemma 5, we exchange the positions of two small circular segments, the arcs of which are part of $\hat{s}_1$ and s_1, respectively. We obtain a nonconvex disc D with $p(D) = p(C)$, $a(D) = a(C)$, and $a(D \cap P) > a(C \cap P)$. Hence $C' = \operatorname{conv} D$ satisfies (51), and the proof of Lemma 9 is complete. □

We next prove that C has a smooth boundary.

Lemma 10. *Through every boundary point of C there passes exactly one support line.*

Proof. Let t and $\hat{t}$ be the tangent lines to the arcs s_1 and $\hat{s}_2$ at U_{12}. To prove Lemma 10 we have to show that $t = \hat{t}$. Suppose that $t \neq \hat{t}$. Let XY be a chord of C parallel to $\hat{t}$, where X lies on $\hat{s}_2$ and Y on s_1 (Fig. 4). We denote the convex

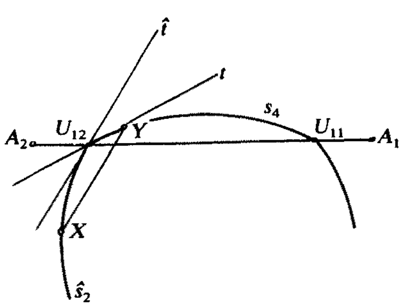

Fig. 4

hull of $\widehat{XU_{12}Y}$ by S_1, the angle between XT and the arc $\widehat{XU_{12}}$ by $\sphericalangle X$, and the angle between YX and the arc $\widehat{YU_{12}}$ by $\sphericalangle Y$. Let $\widehat{VW}$ be a subarc of $\hat{s}_2$ contained in the interior of P and such that $VW = XY$. Write S_2 for the convex hull of $\widehat{VW}$, and $\sphericalangle V$ for the angle of S_2 at the vertex V. By exchanging the positions of S_1 and S_2 we obtain the sets T_1 and T_2, which are congruent to S_1 and S_2, respectively.

If the segment XY has a sufficiently small distance from $\hat{t}$, the following conditions are satisfied:

(i) $\sphericalangle X < \sphericalangle Y$;
(ii) $S_1 \cap S_2 = T_1 \cap T_2 = \varnothing$;
(iii) $T_1 \subset P$.

Since $t \neq \hat{t}$, we have $\sphericalangle V < \sphericalangle X$, and by (i) $\sphericalangle V < \sphericalangle Y$. Hence

$$T_2 \subset S_1. \tag{53}$$

By cutting S_1 and S_2 off from C and replacing them by T_2 and T_1 we obtain a nonconvex disc D with

$$p(D) = p(C), \qquad a(D) = a(C).$$

By using (ii), (iii), and (53) we find

$$\begin{aligned} a(D \cap P) &= a(C \cap P) - a(S_1 \cap P) - a(S_2 \cap P) + a(T_1 \cap P) + a(T_2 \cap P) \\ &= a(C \cap P) + a(S_1 \setminus T_2) - a((S_1 \setminus T_2) \cap P) \\ &\geq a(C \cap P), \end{aligned}$$

which shows that $C' = \operatorname{conv} D$ satisfies (51). Thus the supposition $t \neq \hat{t}$ was wrong, and Lemma 10 is proved. □

In the case when $\hat{r} = r$ all the arcs s_i, $\hat{s}_1$ $(i = 1, \ldots, k)$ have the same radius. Lemma 10 shows that C is a circle. By (48) P is inscribed in a circle concentric with C, and from (49) it follows that P is regular, as stated in Theorem 1. By (52) we can from now on suppose that

$$\hat{r} < r. \tag{54}$$

We shall denote the center of the circle to which the arc $\hat{s}_i$ belongs by M_i, for $i = 1, \ldots, k$.

Lemma 11. *$P_0 = M_1 \cdots M_k$ is a convex k-gon inscribed in a circle that has its center O in the interior of P_0. Let C_0 be the convex disc obtained from P_0 by joining each two consecutive vertices by circular arcs of radius $r - \hat{r}$. Then C is the outer parallel domain of C_0 at distance $\hat{r}$. P and P_0 are homothetic with respect to O.*

Proof. Let O_1 be the center of the circle to which $s_1 = \widehat{U_{11}U_{12}}$ belongs. By (48), O_1 lies on the perpendicular bisector b of the segment A_1A_2. In view of Lemma 7, O_1 and the k-gon P are on the same side of the line A_1A_2. From Lemma 10 and (54) it follows that M_1 is between O_1 and U_{11}, and M_2 between O_1 and U_{12}. Because $M_1U_{11} = M_2U_{12} = \hat{r}$, we see that M_1M_2 is parallel to A_1A_2 and

$$\frac{M_1M_2}{U_{11}U_{12}} = 1 - \frac{\hat{r}}{r}. \tag{55}$$

Writing $U_{11}U_{12}/A_1A_2 = q$, we have by (49)

$$\frac{U_{i1}U_{i2}}{A_iA_{i+1}} = q \qquad \text{for} \quad i = 1, \ldots, k \tag{56}$$

and by (55)

$$\frac{M_1M_2}{A_1A_2} = \left(1 - \frac{\hat{r}}{r}\right) q. \tag{57}$$

Since M_1 and M_2 are symmetric with respect to b, the lines A_1M_1 and A_2M_2 intersect at a point, say O, on b. Hence $OA_1 = OA_2$. Because $M_1M_2 < A_1A_2$, O and P are on the same side of the line A_1A_2, and the line M_1M_2 separates O and A_1A_2. Equation (57) implies that

$$OM_1 = OM_2 = \left(1 - \frac{\hat{r}}{r}\right) q \cdot OA_2. \tag{58}$$

In the same way it can be shown that the lines A_2M_2 and A_3M_3 intersect at a point, say O', on the same side of the line A_2A_3 as P such that $O'A_3 = O'A_2$ and

$$O'M_3 = O'M_2 = \left(1 - \frac{\hat{r}}{r}\right) q \cdot O'A_2. \tag{59}$$

From (58) and (59) we infer that $O' = O$, $OM_1 = OM_2 = OM_3$, and $OA_1 = OA_2 = OA_3$. By applying this argument to $A_3A_4, \ldots$ we conclude that

$$OM_1 = \cdots = OM_k \tag{60}$$

and

$$OA_1 = \cdots = OA_k. \tag{61}$$

By (58) P_0 is obtained from P by homothety of center O and ratio $(1 - \hat{r}/r)q$. In view of the construction, O is an inner point of P_0. Since $OM_1 < O_1M_1$, the circular arcs of radius $r - \hat{r}$ joining each two consecutive vertices of P_0 form the boundary of a convex disc C_0. C is the outer parallel domain of C_0 at distance $\hat{r}$ as required. □

The following lemma completes the proof of Theorem 1. We shall use the same notation as in Lemma 11.

Lemma 12. *P is regular.*

Proof. It suffices to show that

$$A_1A_2 = A_2A_3. \tag{62}$$

The segments OA_1 and OA_3 meet the boundary of C at two points, say A_1' and A_3', belonging to $\hat{s}_1$ and $\hat{s}_3$, respectively. Observe that by (60) $OA_1' = OA_3'$, and by (61) $OA_1 = OA_3$. Thus A_1A_3 and $A_1'A_3'$ have the same perpendicular bisector t passing through O. Write X^t for the convex set obtained from the convex set X by Steiner symmetrization about the line t. If T denotes the triangle $A_1A_3A_2$, then $P' = (P \setminus T) \cup T^t$ is a k-gon which is, by (61), convex. Since $a(T^t) = a(T)$, P' is a member of $\mathcal{P}_k(a_0)$.

The chord $A_1'A_3'$ dissects C into two convex subsets. Let C_1 be that subset which contains the arc $\hat{s}_2$. Since $a(C_1^t) = a(C_1)$ and $p(C_1^t) \le p(C_1)$, the (possibly nonconvex) set $D = (C \setminus C_1) \cup C_1^t$ has the properties

$$a(D) = a(C), \qquad p(D) \le p(C). \tag{63}$$

We proceed to show that

$$a(D \cap P') \ge a(C \cap P). \tag{64}$$

Let l be any line perpendicular to t. We have to consider three possible cases:

(i) l meets the interiors of C_1 and T. Denoting the length of the segment s by $|s|$ we have

$$\begin{aligned} |l \cap C \cap P| &= |l \cap C_1 \cap T| \le \min\{|l \cap C_1|, |l \cap T|\} \\ &= |l \cap C_1^t \cap T^t| = |l \cap D \cap P'|. \end{aligned} \tag{65}$$

(ii) l meets neither the interior of C_1 nor that of T. Then

$$|l \cap C \cap P| = |l \cap D \cap P'|. \tag{66}$$

(iii) l meets either the interior of C_1 or that of T. Let t_1 and t_3 be the tangents to $\hat{s}_1$ and $\hat{s}_3$ at A_1' and A_3' respectively. t_1, t_3, A_1A_3 and $A_1'A_3'$ enclose a (possibly degenerate) trapezium S that is symmetric with respect to t

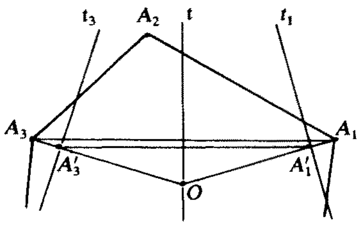

Fig. 5

(see Figs. 5 and 6). Because OA_1A_3 is contained in P, so is S. Since t_1 and t_3 are support lines of C, S contains the intersection of C with the parallel strip bounded by the lines A_1A_3 and $A_1'A_3'$. Thus, if l meets the interior of C_1 we have

$$|l \cap C \cap P| = |l \cap C| = |l \cap C_1| = |l \cap C_1^t| = |l \cap D \cap P'|, \tag{67}$$

and if l meets the interior of T

$$|l \cap C \cap P| = |l \cap C| = |l \cap C \cap T^t| = |l \cap D \cap P'|. \tag{68}$$

Now (64) follows from (65) to (68).

By (63) and (64), $C' = \operatorname{conv} D$ is a member of $\mathscr{C}(a, p)$ satisfying

$$a(C' \cap P') \geq a(C \cap P). \tag{69}$$

In view of supposition (ii) of Theorem 1 equality holds in (69). Using Lemma 4 we obtain from (63) that

$$p = p(C') \leq p(D) \leq p(C) = p,$$

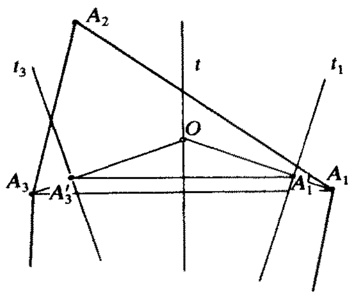

Fig. 6

and $p(D)=p(C)$ implies that

$$p(C_1^t)=p(C_1). \tag{70}$$

Since C_1 is not contained in a line perpendicular to t we conclude from (70) (see [16, p. 208]) that C_1 is symmetric with respect to a line t' parallel to t. Since $A_1'A_3'$ is symmetric with respect to t we see that $t'=t$. By (49) the lines $U_{12}U_{21}$ and $U_{11}U_{22}$ are perpendicular to t. Hence U_{12} and U_{21} as well as U_{11} and U_{22} are pairs of symmetric points, so that

$$U_{11}U_{12}=U_{21}U_{22}. \tag{71}$$

Equation (62) follows from (71) and (49). This completes the proof of Lemma 12 and that of Theorem 1. □

Let C be a parallel domain of a regular arc-sided k-gon, and let P be the central k-gon of C. We conclude this section by showing that C is uniquely determined by the parameters $a(C)=a$, $p(C)=p$, $a(P)=a_0$.

Let $C=(C_1)_\rho$, where C_1 is a regular arc-sided k-gon, and let $a(C)=a$ and $p(C)=p$ be given. If $a(C_1)=a_1$, $p(C_1)=p_1$, and 2α is the central angle of the arcs bounding C_1, we have by (9)

$$\frac{p_1^2}{a_1}=4k\frac{\Phi(q)}{u+q}. \tag{72}$$

q and $\Phi(q)$ are given by (8), and $u=\cot(\pi/k)$. The discs C form an array joining the smooth regular k-gon, corresponding to $\alpha=0$, with the regular arc-sided k-gon, corresponding to $\alpha=\alpha^*$, where $q=q(\alpha^*)$ is determined by (8) and (9). By applying Steiner's formulas to $(C_1)_\rho$ we obtain from (72) that

$$\rho(\alpha)=\frac{1}{2\pi}\left(p-\sqrt{\frac{(p^2-4a\pi)\Phi}{\Phi-\pi(u+q)/k}}\right). \tag{73}$$

Differentiation yields

$$\rho'(\alpha)=-\frac{\sqrt{p^2-4a\pi}(v-u)}{2k[\Phi-\pi(u+q)/k]^{3/2}}\,\frac{\sin\alpha-\alpha\cos\alpha}{\sin^2\alpha}, \tag{74}$$

where $v=\cot\alpha$. Since $\alpha<\pi/k$, we can remark for later use that

$$\rho'(\alpha)<0. \tag{75}$$

By using Steiner's formula and (73) we find the in-radius of the kernel of C_1

$$r_1(\alpha)=\frac{u}{2k}\sqrt{\frac{p^2-4a\pi}{\Phi-\pi(u+q)/k}}, \tag{76}$$

and observe that

$$r_1'(\alpha) = -\frac{u\sqrt{p^2-4a\pi}(\alpha-\pi/k)}{2k[\Phi-\pi(u+q)/k]^{3/2}}\,\frac{\sin\alpha-\alpha\cos\alpha}{\sin^3\alpha}>0. \tag{77}$$

For the in-radius of P

$$r_P(\alpha)=r_1+\rho\cos\alpha \tag{78}$$

we obtain by (74) and (77)

$$\begin{aligned}
&2k\sin\frac{\pi}{k}\left[\Phi-\frac{\pi(u+q)}{k}\right]^{3/2}\sin^3\alpha(r_P'+\rho\sin\alpha)\\
&\quad=-\sqrt{p^2-4a\pi}(\sin\alpha-\alpha\cos\alpha)\cos\alpha\cos\left(\frac{\pi}{k}-\alpha\right)\\
&\qquad\times\left[\tan\left(\frac{\pi}{k}-\alpha\right)-\left(\frac{\pi}{k}-\alpha\right)+\left(\frac{\pi}{k}-\alpha\right)\tan\left(\frac{\pi}{k}-\alpha\right)\tan\alpha\right].
\end{aligned}$$

Hence

$$r_P'(\alpha)<0. \tag{79}$$

Thus $a(P)$ is a strictly decreasing function of α, and α is uniquely determined by $a(P)=a_0$. This proves the above statement.

4. The Area Deviation

We now turn to the problem of finding such members of $\mathscr{C}(a,p)$ and $\mathscr{P}_k$ for which $\delta^A(C,P)$ is minimal. In view of a remark made in Section 1, we have to consider only such values of a, p, and k that

$$\frac{p^2}{a}<4k\tan\frac{\pi}{k}.$$

For a disc C from $\mathscr{C}(a,p)$ and a k-gon P from $\mathscr{P}_k(a_0)$ we have by (1)

$$\delta^A(C,P)=a(C)+a_0-2a(C\cap P). \tag{80}$$

Because $a(C)\geq a$, it follows from (80) and (11) that

$$\delta^A(C,P)\geq a+a_0-2M(a,p;k,a_0). \tag{81}$$

If $a_0\leq f_1(a,p,k)$, Remark 2 implies that

$$\delta^A(C,P)\geq a-f_1(a,p,k),$$

with equality if and only if C is a regular arc-sided k-gon of area a and perimeter p, and P is the kernel of C. Let P' be the k-gon obtained from P by displacing a side of P outward parallel to itself through a sufficiently small distance. By using (33) it follows easily that

$$\delta^A(C, P') < \delta^A(C, P),$$

which shows that $\delta^A(C, P)$ is not minimal. Thus we can assume in the following that $a_0 > f_1(a, p, k)$.

Since $a(C) \geq a(C \cap P)$, we conclude from (80) and (11) that

$$\delta^A(C, P) \geq a_0 - M(a, p; k, a_0). \tag{82}$$

If $a_0 \geq f_2(a, p, k)$ we have by (82) and Remark 2

$$\delta^A(C, P) \geq a_0 - F(p, k, a_0), \tag{83}$$

where F is given by (15). As can be shown by differentiation, the function of a_0 on the right-hand side of (83) is strictly increasing for $a_0 \geq f_2(a, p, k)$. This function thus attains its minimum for $a_0 = f_2(a, p, k)$. Therefore, we need to consider only such values of a_0 for which

$$f_1(a, p, k) < a_0 \leq f_2(a, p, k). \tag{84}$$

We shall again make use of (81) and observe that, by Theorem 1 and Remark 2, equality occurs in (81) if and only if C is an outer parallel domain of a regular arc-sided k-gon of area a and perimeter p, and P is the central k-gon of C. If $a_0 = f_2(a, p, k)$ C is degenerate, which means that C is a smooth regular k-gon with case P.

Let us first assume that $\delta^A(C, P)$ is minimal for some a_0 from the interior of the interval (84). Resuming the notation used in Section 3, we can state that (see also [2, p. 363])

$$\frac{U_{11}U_{12}}{A_1A_2} = \frac{1}{2} \qquad \text{if} \quad f_1(a, p, k) < a_0 < f_2(a, p, k). \tag{85}$$

Otherwise we could reduce $\delta^A(C, P)$ by displacing A_1A_2 parallel to itself through a small distance. This follows from (33). Second, if we assume that $\delta^A(C, P)$ is minimal for $a_0 = f_2(a, p, k)$, the same argument as above shows that

$$\frac{U_{11}U_{12}}{A_1A_2} \geq \frac{1}{2} \qquad \text{if} \quad a_0 = f_2(a, p, k). \tag{86}$$

Using the notation introduced at the end of Section 3 we have

$$A_1A_2 = 2r_P \tan\frac{\pi}{k}$$

and

$$U_{11}U_{12} = 2\left(r_1 \tan\frac{\pi}{k} + \rho \sin\alpha\right).$$

Hence by (78)

$$\frac{U_{11}U_{12}}{A_1A_2} = \frac{r_1 \tan(\pi/k) + \rho \sin\alpha}{r_1 + \rho\cos\alpha} \cot\frac{\pi}{k} = g(\alpha), \tag{87}$$

where $\rho(\alpha)$ and $r_1(\alpha)$ are given by (73) and (76). From

$$g'(\alpha)(r_1+\rho\cos\alpha)^2 \sin\frac{\pi}{k} = r_1\rho' \sin\left(\alpha - \frac{\pi}{k}\right) + r_1\rho\cos\left(\alpha - \frac{\pi}{k}\right) + r_1'\rho\sin\left(\frac{\pi}{k} - \alpha\right) + \rho^2\cos\frac{\pi}{k},$$

and (75) and (77) we see that

$$g'(\alpha) > 0. \tag{88}$$

Thus we have to consider two cases.

(i) If $g(0) < \frac{1}{2}$, (86) is impossible, and the minimum of $\delta^A(C, P)$ is attained in the case indicated by (85). C is a parallel domain of a (proper) regular arc-sided k-gon, and P is the central k-gon of C.

(ii) If $g(0) \geq \frac{1}{2}$, (85) is impossible and the minimum of $\delta^A(C, P)$ is attained in the case indicated by (86). C is a smooth regular k-gon, and P is the case of C.

$g(0)$ can easily be evaluated by (87), (73), and (76). Writing $(k/\pi)\tan(\pi/k) = t$ and referring to (3) we can summarize the result of this section in

Theorem 2. *Suppose that $p^2/4a\pi < t$. There is exactly one disc C from $\mathscr{C}(a, p)$ and one k-gon P such that*

$$\delta^A(C, P) = \Delta^A(a, p, k).$$

C and P are characterized by the following properties:

(i) $a(C) = a$, $p(C) = p$.

(ii) *if $p^2/4a\pi < (1+t)^2/(1+3t)$, C is a parallel domain of a regular arc-sided k-gon, and P is the central k-gon of C. Any side of P, say A_1A_2, meets the boundary of C at points U_{11}, U_{12} such that*

$$A_1U_{11} = U_{12}A_2 = \tfrac{1}{4}A_1A_2.$$

(iii) *if* $p^2/4a\pi \geq (1+t)^2/(1+3t)$, *C is a smooth regular k-gon, and P is the case of C.*

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