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Zero-one Laws for Infinitely Divisible Probability Measures on Groups

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Introduction

Let E be a separable Banach space and $H \subset E$ a measurable subgroup. Then either $\mu(H)=0$ or 1 for each Gaussian probability measure μ on E, for example see C.R. Baker [2] and N.C. Jain [14]. Recently T. Byczkowski [4] proves a zero-one law for Gauss measures (in the sense of Bernstein) on Abelian topological groups. Using a different technique E. Siebert [26] remarks that each Gauss semigroup on a connected Lie group arises from a semigroup of absolutely continuous measures. This result implies a zero-one law.

It is the purpose of this paper to establish zero-one laws and purity laws for continuous convolution semigroups on arbitrary locally compact groups G. In Sect. 2 we start to examine the decomposition of a continuous convolution semigroup $(\mu_t)_{t \geqq 0}$ induced by the Riesz decomposition $M(G)=B \oplus B^{\perp}$ with respect to a prime L-subalgebra B of the Banach space of bounded Radon measures $M(G)$ on G, [9], 2.10. A prime L-subalgebra is a band such that B is a subalgebra of $(M(G), *)$ and $B^{\perp}$ is an ideal. Theorem 1 describes the decomposition in terms of the generating functional of the semigroup. Applying these results on normal semigroups we prove a zero-one law for $\mu_t(xH)$ provided H is a normal measurable subgroup such that the Lévy measure vanishes on H^c. This result includes a new zero-one law for Gauss semigroups on locally compact groups. The zero-one law contained in Corollary 7 seems to be new even for the euclideam space $\mathbb{R}^n$.

By a different technique we prove zero-one laws for generalized Poisson-measures on Abelian topological groups and for infinitely divisible probability measures on locally convex vector spaces E. We show that a measurable subgroup $H \subset E$ has the μ-measure 0 or 1 if the Lévy measure F of μ is unbounded (or F vanishes) on the complement of H. These results can be applied to obtain a direct proof of some known results for Gauss measures and stable measures. The proofs are based on the purity law for generalized convolution products presented in [17].

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1. Preliminaries

Let G always denote a topological Hausdorff group and $\mathscr{U}_G(e)$ the neighbourhood filter of the natural element $e \in G$. We write $C_b(G)$, $C_u(G)$ [$\mathscr{D}(G)$] for the Banach space of bounded real-valued functions (with respect to the sup-norm) on G, the subspace of left uniformly continuous functions [the Bruhat space of infinitely differentiable functions on G provided G is locally compact, [12], 4.4.2]. Let $|\mu| = \mu^+ + \mu^-$ denote the total variation of a bounded Borel measure μ on G. Then the space of tight, signed, bounded Borel measures $M(G) = \{\mu : |\mu|(B) = \sup\{|\mu|(K) : K \subset B$ compact$\}$ for each Borel set $B\}$ forms a real Banach algebra with respect to the norm of total variation $\|\cdot\|$ and the convolution $*$.

Let $M_+(G)$ [$M_1(G)$] be the subset of all positive [probability] measures of $M(G)$. On the other hand $(M(G), \|\cdot\|)$ is an order complete Banach lattice (see [23]). A subset $B \subset M(G)$ is said to be solid if $\nu \in B$ whenever $\mu \in B$ and $|\nu| \leqq |\mu|$ holds for $\nu \in M(G)$. The norm closed solid vector subspaces of $M(G)$ are just the bands lying in $M(G)$, [23], II 10.2. We write $\mu \perp \nu (\mu \ll \nu)$ if μ and ν are mutually singular in $M(G)$ (μ is absolutely continuous with respect to ν). Put $\mu \approx \nu$ iff $\mu \ll \nu$ and $\nu \ll \mu$.

Let $B^\perp = \{\nu \in M(G) : \nu \perp \mu$ for all $\mu \in B\}$ denote the orthogonal band of a band B. Then the relation $M(G) = B \oplus B^\perp$ follows. Following the notation of [9], p. 128 we call a band $B \subset M(G)$ a prime L-subalgebra if (i) B is a subalgebra (closed under $*$) and (ii) $B^\perp$ is an ideal (this means $\mu * \nu \in B^\perp$ whenever μ or $\nu \in B^\perp$)).

For each measure ν on G define the adjoint measure $\tilde{\nu}(A) = \nu(A^{-1})$. Put $f^*(x) = f(x^{-1})$ if f is a function on G. Let $f(\nu)$ be the image measure of ν with respect to a measurable function f on G and let ε_x be the Dirac measure for $x \in G$. Finally $e(\nu) = \exp(-\|\nu\|) \sum_{n=0}^{\infty} \nu^n/n!$ defines the Poisson measure of $\nu \in M_+(G)$, $\nu^0 = \varepsilon_e$, $\nu^1 = \nu$ and $\nu^{n+1} = \nu * \nu^n$ if $\nu(\{e\}) = 0$. A subset $(\mu_t)_{t \in (0,\infty)}$ of probability measures of $M(G)$ is called a continuous convolution semigroup if $\mu_s * \mu_t = \mu_{s+t}$ holds for all $s, t \in (0, \infty)$ and $\lim_{t \to 0} \mu_t = \varepsilon_e$ with respect to the weak topology $\sigma(M(G), C_b(G))$.

Let now G be locally compact. Then every continuous convolution semigroup is uniquely determined by the generating functional $A(f) = \left.\frac{d^+ \mu_t(f)}{dt}\right|_{t=0}$ for $f \in \mathscr{D}(G)$[1]. The functional A has the canonical representation $A(f) = \psi_1(f) + \psi_2(f) + \int_{G^*} [f - f(e) - \Gamma(f)]\, d\eta$ (Lévy-Hinčin formula [12], p. 308), $G^* = G - \{e\}$. ψ_1 is a primitive form, ψ_2 a quadratic form on $\mathscr{D}(G)$, Γ is a fixed Lévy mapping and η, called Lévy measure, is a (possibly unbounded) positive real Radon-measure on G^* determined by

$$\int_{G^*} f\, d\eta = \lim_{t \downarrow 0} t^{-1} \int_G f\, d\mu_t$$

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1 Sometimes we shall consider A on the functions $f \in C_b(G)$ which are locally in $\mathscr{D}(G)$ ([10]) or on a dense subspace of $C_u(G)$

for all continuous real-valued functions f with compact support on G^*, denoted by $\mathscr{K}(G^*)$. Let us always consider the regular extension of η on the Borel σ-field of G^*. There is a one-to-one correspondence between continuous semigroups, the positive cone of generating functionals and the canonical representation given by (ψ_1, ψ_2, η). For more information see [12]. Every bounded Radon-measure ν with $\nu \geqq 0 = \nu(\{e\})$ determines a continuous convolution semigroup $(e(t\nu))_{t \geqq 0}$ (with generator $A = \nu - \|\nu\| \varepsilon_e$), called the Poisson semigroup of ν. For every continuous convolution semigroup the following statements are equivalent ([12], 6.1.5):

(i) $(\mu_t)_{t \geqq 0}$ is a Poisson semigroup,
(ii) A is bounded on $\mathscr{D}(G)$ with respect to the sup-norm,
(iii) η is bounded and $A = \eta - \|\eta\| \varepsilon_e$.

If (iii) is satisfied, A is called a Poisson generator.

A convolution semigroup is said to be normal (symmetric) if $\mu_t * \tilde{\mu}_t = \tilde{\mu}_t * \mu_t$ ($\mu_t = \tilde{\mu}_t$) for each $t > 0$. If $(\mu_t)_{t \geqq 0}$ is normal then $\nu_t = \mu_t * \tilde{\mu}_t$ defines a symmetric continuous convolution semigroup on G with generating functional $f \mapsto A(f) + A(f^*)$) and Lévy-measure $\eta + \tilde{\eta}$ (compare for example with [10], p. 20).

Finally let ω_G denote the left Haar measure on G and B^c the complement of a set B. We define 1_B to be the indicator function of a set B.

2. Zero-one Laws for Continuous Convolution Semigroups on Locally Compact Groups

In this section G denotes a locally compact group. Let $\mathscr{E}_x(tA)$ be the continuous convolution semigroup determined by a generating functional A. The structure of the convolution semigroup generated by the sum of two generating functionals A_1 and A_2 is given by the equation $\mathscr{E}_x(t(A_1 + A_2)) = \mathscr{E}_x(tA_1) * \mathscr{E}_x(tA_2)$ if the measures $\mathscr{E}_x(tA_1)$ and $\mathscr{E}_x(tA_2)$ commute for all $t > 0$. In general $\mathscr{E}_x(t(A_1 + A_2))$ is determined by a Lie-Trotter product formula, see [10].

If $A_2 = c(\nu - \varepsilon_e)$ ($c > 0$, $\|\nu\| = 1$) is a Poisson generator we can compute $\mathscr{E}_x(t(A_1 + A_2))$ with the help of the perturbation series

$$\text{(I)} \quad \mathscr{E}_x(t(A_1 + A_2)) = \exp(-ct) \sum_{k \geqq 0} u_k(t, A_1, A_2), \; u_0(t, A_1, A_2) = \mathscr{E}_x(tA_1),$$

$$u_{k+1}(t, A_1, A_2) = c \int_0^t \mathscr{E}_x(rA_1) * \nu * u_k(t - r, A_1, A_2)\, dr,$$

which yields a norm convergent series in $M(G)$, see [10], p. 61. The formula (I) is the key of the next theorem.

Theorem 1. *Let $B \subset M(G)$ be a prime L-subalgebra and $(\mu_t)_{t > 0}$ a continuous convolution semigroup with a generating functional A. One of the following statements is fulfilled.*

(i) *$\mu_t \in B$ for some $t > 0$ (and hence for all $t > 0$),*
(ii) *$\mu_t \in B^\perp$ for some $t > 0$ (and hence for all $t > 0$),*

(iii) *There exists a generating functional* A_1 *and a Poisson generator* A_2 *so that* $A=A_1+A_2$. *For all* $t\geqq 0$ μ_t *is given by the perturbation series* (I) *and*

a) $\mathscr{E}_x(tA_1)\in B$,

b) $\sum_{k\geqq 1} u_k(t, A_1, A_2)\in B^{\perp}$ *holds.*

Proof. Let $\mu_t=\mu_t^1+\mu_t^2$ be defined by the band projection with $\mu_t^1\in B$ and $\mu_t^2\in B^{\perp}$. The equality $\mu_{t+s}^1+\mu_{t+s}^2=\mu_t^1*\mu_s^1+\mu_t^2*\mu_s^1+\mu_t^1*\mu_s^2+\mu_t^2*\mu_s^2$ shows $\mu_{t+s}^1=\mu_t^1*\mu_s^1$ for all $t,s>0$ since B is a prime L-subalgebra. If $\|\mu_t^1\|=1$ $(=0)$ holds for some $t>0$ the statement (i) ((ii)) follows immediately. Let us now assume $0<\|\mu_t^1\|<1$ for some $t>0$. We conclude $\|\mu_{t+s}^1\|=\|\mu_t^1\|\,\|\mu_s^1\|$ for all $t,s>0$. A standard argument shows $\|\mu_t^1\|=\exp(-\alpha t)$ for all $t>0$ and some $\alpha>0$. This fact yields $\mu_t^2\mapsto 0$ and $\mu_t^1\mapsto\varepsilon_e$ for $t\to 0$ with respect to the $\sigma(M(G), C_b(G))$-topology. Plainly, $(\exp(t\alpha)\,\mu_t^1)_{t>0}$ is a second continuous convolution semigroup. Let A_1 be the generating functional

$$A_1(f)=\frac{d^+}{dt}(\exp(\alpha t)\,\mu_t^1(f))|_{t=0}\qquad\text{for } f\in\mathscr{D}(G).$$

The relation $\mu_t=\mu_t^1+\mu_t^2$ implies:

$$\lim_{t\to 0}\frac{\mu_t^2(f)}{t}=\frac{d^+}{dt}\,\mu_t^2(f)|_{t=0}\qquad\text{exists for } f\in\mathscr{D}(G).$$

Hence

$$\begin{aligned}A(f)&=\frac{d^+}{dt}(\exp(-t\alpha)(\exp(t\alpha)\,\mu_t^1(f)))|_{t=0}+\frac{d^+}{dt}\,\mu_t^2(f)|_{t=0}\\&=A_1(f)-\alpha f(e)+\frac{d^+}{dt}\,\mu_t^2(f)|_{t=0}.\end{aligned}$$

We show: $\mathscr{D}(G)\ni f\mapsto A_2(f)=\frac{d^+}{dt}\,\mu_t^2(f)-\alpha f(e)$ is a Poisson generator. It suffices to prove ([12], 4.4.18):

1) A_2 is almost positive, i.e.: $A_2(f)\geqq 0$ for all $f\in\mathscr{D}(G)$ with $f\geqq f(e)=0$. The assertion holds since $\mu_t^2(f)\geqq 0$.

2) A_2 is normed, i.e.: there is an open neighbourhood U of e satisfying $\sup\{A_2(f): f\in H(U)\}=0$ if we denote $H(U)=\{f\in\mathscr{D}(G): 1_U\leqq f\leqq 1\}$. Let U be any open neighbourhood of e. The inequality $1-\alpha t\leqq\exp(-\alpha t)$ implies $0\leqq\frac{\mu_t^2(f)}{t}\leqq\frac{1-\exp(-\alpha t)}{t}\leqq\alpha$ for all $t>0$ and $f\in H(U)$. Plainly $\frac{d^+}{dt}\,\mu_t^2(f)|_{t=0}\leqq\alpha$ and $A_2(f)\leqq 0$ holds. Let $U(U_1)$ be an open neighbourhood of e such that $\sup\{A(f): f\in H(U)\}=0$ $(\sup\{A_1(f): f\in H(U_1)\}=0)$ is fulfilled. If we put $U_2=U\cap U_1$ it is easy to compute $0\leqq\sup\{A_2(f): f\in H(U_2)\}$ if we observe $A=A_1+A_2$.

3) A_2 is bounded. Let $\|\cdot\|$ be the sup-norm on $\mathscr{D}(G)$ and $\mathscr{K}(G^*)$. We show that A_2 is bounded. Using the inequality of part 2) we conclude

$$\left|\frac{\mu_t^2(f)}{t}-\alpha f(e)\right|\leqq\alpha\|f\|+\frac{\mu_t^2(|f|)}{t}\leqq\alpha\|f\|+\frac{(1-\exp(-\alpha t))\,\|f\|}{t}\leqq 2\alpha\|f\|$$

and $|A_2(f)| \leqq 2\alpha \|f\|$ for all $f \in \mathscr{D}(G)$. The generating functional A_2 has the form $A_2(f) = \int\limits_{G^*} (f - f(e))\, d\eta_2$ and $\|\eta_2\| < \infty$. Let $\eta(\eta_1)$ be the Lévy measure of the canonical representation of $A(A_1)$. We conclude

$$\eta_1(f) = \lim_{t \to 0} \frac{\exp(\alpha t)\, \mu_t^1(f)}{t} = \lim_{t \to 0} \frac{\mu_t^1(f)}{t} \quad \text{for all } f \in \mathscr{K}(G^*),$$

$$\eta = \eta_1 + \eta_2 \quad \text{and} \quad \eta_2(f) = \lim_{t \to 0} \frac{\mu_t^2(f)}{t} \leqq \alpha \|f\|.$$

This relation implies $\|\eta_2\| \leqq \alpha$. We are now able to apply the perturbation theory and formula (I):

$$\mu_t = \exp(-t \|\eta_2\|)(\exp(t\alpha)\, \mu_t^1 + \sum_{k \geqq 1} u_k(t, A_1, A_2)).$$

The definition of μ_t^1 implies $\exp(-t(\|\eta_2\| - \alpha))\, \mu_t^1 \leqq \mu_t^1$ and we conclude: $\|\eta_2\| \geqq \alpha$. Hence $\|\eta_2\| = \alpha$. The band projection shows $\mu_t^2 = \exp(-t\alpha) \cdot \sum_{k \geqq 1} u_k(t, A_1, A_2)$. □

Corollary 2. *If in the situation of Theorem* 1(iii) *the Lévy measure η_2 of A_2 commutes with each measure $\mathscr{E}_x(tA_1)$ for $t > 0$ the relations $\eta_2 \in B^\perp$ and $\mu_t = \mathscr{E}_x(tA_1) * e(t\eta_2)$ are valid.*

Proof. We have only to observe that under these assumptions $\mathscr{E}_x(tA_1)$ and $e(t\eta_2)$ commute and the perturbation series reduces to the ordinary convolution product of the two factors. Furthermore, Theorem 1 implies $\mu_t^1 = \exp(-t\|\eta_2\|) \mathscr{E}_x(tA_1)$ and $\exp(-t\|\eta_2\|)(\mathscr{E}_x(tA_1) * t\eta_2) \leqq \mu_t^2$. Since B is a prime L-subalgebra we conclude $\eta_2 \in B^\perp$. □

A continuous convolution semigroup $(\mu_t)_{t>0}$ in $M(G) - \{\varepsilon_x : x \in G\}$ is called a Gauss semigroup if $\lim_{t \to 0} \frac{1}{t} \mu_t(G - U) = 0$ holds for every open neighbourhood U of e. Our Theorem 1 implies a purity law for Gauss semigroups.

Corollary 3. *Let $B \subset M(G)$ be a prime L-subalgebra and $(\mu_t)_{t>0}$ a Gauss semigroup. One of the following statements is true.*

(i) *$\mu_t \in B$ for some $t > 0$ (and hence for all $t > 0$),*

(ii) *$\mu_t \in B^\perp$ for some $t > 0$ (and hence for all $t > 0$).*

We have only to remark that the Lévy measure of the canonical representation of a Gauss semigroup vanishes. □

Lemma 4. *Let X be a topological group and let $H \subset X$ be a measurable subgroup. For each $\mu \in M_1(X)$ the following assertions are valid:*

a) *$\mu * \tilde{\mu}(H) = 1$ iff there exists a coset Hx satisfying $\mu(Hx) = 1$ and $\mu(Hy) = 0$ for each coset $Hy \neq Hx$.*

b) *$\mu * \tilde{\mu}(H) = 0$ iff $\mu(Hy) = 0$ for all $y \in X$.*

Proof. a) The equation $1 = \mu * \tilde{\mu}(H) = \int \mu(Hy)\, d\mu(y)$ shows that there exists a point $x \in X$ with $\mu(Hx) = 1$.

b) We remark that $\mu * \tilde{\mu}(H) = \int \mu(Hz)\, d\mu(z) \geqq \mu(Hy)^2$ implies the assertion. □

Let τ be a topological group topology on G finer than the original topology. We denote the new topological group by G_τ and it is called a refinement of G. It is well-known that the set of all bounded and tight Borel-measures on G_τ

$$M(G_\tau) = \{\mu\colon |\mu|(B) = \sup\{|\mu|(K), B \supset K\ \tau\text{-compact}\} \text{ for all Borel sets } B \subset G_\tau\}$$

is a prime L-subalgebra of $M(G)$. We remark that a measure $\mu \geqq 0$ belongs to $M(G_\tau)^\perp$ if and only if μ vanishes on all τ-compact sets. For example, J.F. Méla [22] studies G_τ for Abelian groups. Further examples of prime L-subalgebras on Abelian groups appear in [9], 5.1. Let G_τ be a refinement of G and η be the Lévy measure of the semigroup $(\mu_t)_{t>0}$. There exists an uniquely determined decomposition of η in two Lévy measures η_a, η_b satisfying $\eta_a + \eta_b = \eta$ with $\eta_{a|G-U} \in M(G_\tau)$ and $\eta_{b|G-U}\, M(G_\tau)^\perp$ for every open set $U \in \mathscr{U}_G(e)$[2]. Let us now examine the Poisson generator A_2 and the Lévy measure η_2 of A_2 with respect to the decomposition described in Theorem 1 for $B = M(G_\tau)$.

Corollary 5. a) *$\mu_t \in M(G_\tau)^\perp$ for all $t>0$ if $\eta_b(G^*) = \infty$.*

b) *If $\mu_t \notin M(G_\tau)^\perp$ for some $t>0$ then*

(i) $\eta_b \leqq \eta_2$,

(ii) *$\eta_b = \eta_2$ provided $(\mu_t)_{t \geqq 0}$ is a normal semigroup.*

c) *If $(\mu_t)_{t \geqq 0}$ is normal and $\eta = \eta_a$ then either $\mu_t \in M(G_\tau)$ for all $t>0$ or $\mu_t \in M(G_\tau)^\perp$ for all $t>0$.*

Proof. The first assertion follows from [16], Satz 4.

b)(i) The assertion $\eta_b \leqq \eta_2$ is equivalent to $\eta_a \geqq \eta - \eta_2$. If this inequality is false there exists a Lévy measure $0 \neq \eta_3 \leqq \eta - \eta_2$, $\eta_3 \leqq \eta_b$ with $\eta_3 \in M(G_\tau)^\perp$. We put $A_3 = \eta_3 - \|\eta_3\|\, \varepsilon_e$ and write $A_1 = (A_1 - A_3) + A_3$. The perturbation formula (I) yields: $\exp(-t\|\eta_3\|)\, u_1(t, A_1 - A_3, A_3) \leqq \mathscr{E}_x(tA_1)$. But this is impossible since $u_1(t, A_1 - A_3, A_3)$ belongs to $M(G_\tau)^\perp$, compare with [16], Satz 4. (ii) At first we consider a symmetric semigroup $(\mu_t)_{t \geqq 0}$. Let us suppose $\nu = \eta_2 - \eta_b > 0$. There exists an increasing sequence of τ-compact symmetric sets $K_n \subset G$ satisfying $\nu(G - K_n) < 1/n$ and $\mathscr{E}_x(A_1)(G - K_n) < 1/n$. We consider the subgroup $H = \bigcup_{n \in \mathbb{N}} (K_n)^n$ and put $\nu_t = \mathscr{E}_x(tA_1)$. It is easy to see that ν_t is a symmetric semigroup. At first we prove that each ν_t is concentrated on H for $t \in [0, 1]$. $1 = \nu_1(H) = \int \nu_{t/2}(Hx)\, \nu_{1-t/2}(dx)$. Hence there is a point $x \in G$ with $\nu_{t/2}(Hx) = 1$. Lemma 4 implies $\nu_t(H) = 1$. Finally, we consider the perturbation series (I) which shows $u_1(1, A_1, A_2)\ (H) \geqq \|\nu\| > 0$. But this is a contradiction since $u_1(1, A_1, A_2)$ is contained in $M(G_\tau)^\perp$ and H is a σ-compact subgroup of G_τ. Let $(\mu_t)_{t \geqq 0}$ be a normal semigroup and $\tilde{A}$ the generator of the adjoint semigroup $(\tilde{\mu}_t)_{t \geqq 0}$. Then $\nu_t = \mu_t * \tilde{\mu}_t$ admits the generator $A + \tilde{A}$. Following the notation of the proof

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[2] Let $\nu_{|V}$ $(f_{|V})$ denote the restriction of a measure ν (function f) on the set V

of Theorem 1 we conclude $\|\mu_t^1\| = \|\tilde{\mu}_t^1\| = \exp(-t\|\eta_2\|)$ which shows $\|v_t^1\| = \exp(-2t\|\eta_2\|)$. Hence the norm of the Lévy measure belonging to the Poisson generator $(A+\tilde{A})_2$ is equal to $2\|\eta_2\|$. The semigroup v_t admits the Lévy measure $\eta+\tilde{\eta}$. This yields $(\eta+\tilde{\eta})_b = \eta_b + (\eta_b)^{\sim}$. Since v_t is symmetric we conclude

$$\|v_t^1\| = \exp(-t\|(\eta+\tilde{\eta})_b\|) \quad \text{and} \quad \|\eta_b + (\eta_b)^{\sim}\| = 2\|\eta_2\|.$$

But now $\|\eta_b\| = \|\eta_2\|$ and (i) proves $\eta_b = \eta_2$. □

Our Corollary 5 shows that $\eta_b = 0$ if $\mu_t \in M(G_\tau)$ for some $t > 0$. But for non-normal semigroups the converse assertion is false (compare with example 1).

Let $H \subset G$ be a normal subgroup. Then there exists an uniquely determined refinement G_τ of G such that H is a closed and open subgroup of G_τ and the restriction of both topologies on H coincides (apply [11], 4.5). We remark that G_τ is locally compact if $H \subset G$ is closed. If H is measurable then the prime L-subalgebra $M(G_\tau)$ has the form

$$M(G_\tau) = \{\mu \in M(G) \colon \|\mu\| = \sum_{xH} |\mu|(xH)\}$$

and

$$M(G_\tau)^\perp = \{\mu \in M(G) \colon |\mu|(xH) = 0 \text{ for all } x \in G\}.$$

The proof is obvious. We have only to remark that each τ-compact set is contained in the union of a finite number of cosets of H. The topological group G_τ is called the refinement induced by H. Corollary 5b)(ii) completely determines the structure of a normal convolution semigroup if the behaviour of $\mathscr{E}_x(tA_1)$ is known. Therefore we consider the case $\eta = \eta_a$.

Theorem 6. *Let H be a measurable normal subgroup of G and let G_τ be the refinement of G induced by H. Let $(\mu_t)_{t \geq 0}$ be a symmetric continuous convolution semigroup on G with generating functional A and Lévy measure η. If $\eta_{|U^c} \in M(G_\tau)$ for every open set $U \in \mathscr{U}_G(e)$ then the following assertion are valid:*

a) *If $\eta(H^c) = \infty$ then $\mu_t \in M(G_\tau)^\perp$ for all $t > 0$.*
b) *If $\mu_t \notin M(G_\tau)^\perp$ for some $t > 0$ then*
(i) *$(\mu_t)_{t \geq 0}$ is a $\sigma(M(G_\tau), C_b(G_\tau))$-continuous convolution semigroup and*
(ii) $\exp(-t\eta(H^c)) \leq \mu_t(H) \leq 1 - t\eta(H^c)\exp(-t\eta(H^c)).$

Proof. b) If $\mu_{t_0} \notin M(G_\tau)^\perp$ for $t_0 > 0$ then $(\mu_t)_{t \geq 0}$ is a symmetric semigroup on G_τ. Since each μ_t is concentrated on a countable number of cosets of H the canonical projection $\varphi \colon G \to G/H$ into the discrete group G/H is μ_t-measurable. Hence the family of measures $(\varphi(\mu_t))_{t > 0}$ forms a symmetric semigroup of discrete measures on G/H. Applying [12], 6.3.17 we see that there exists a finite subgroup $H_1 \subset G/H$ and a H_1-Poisson semigroup such that

$$\varphi(\mu_t) = \exp(-t\|v\|)[\omega_{H_1} + \sum_{k \geq 1} (\omega_{H_1} * v * \omega_{H_1})^k t^k/k!]$$

where ω_{H_1} denotes the normed Haar measure on H_1 and v is a positive discrete and finite measure on H_1^c.

1. In a first step we show $|H_1| = k = 1$.

Let us choose $H_1=\{x_1,\dots,x_k\}$, $M_i=\varphi^{-1}(\{x_i\})$ with $M_1=H$. Then $\mu_t(M_i)\geqq\frac{1}{k}\exp(-t\|v\|)$ and $\mu_{t|M_i}\to\frac{1}{k}\varepsilon_e$ for $t\to 0$ with respect to the $\sigma(M(G), C_b(G))$ topology (apply the Portemanteau theorem). We remark that

$$\left\|\mu_h-\sum_{i=1}^k \mu_{h|M_i}\right\|\to 0 \quad \text{as } h\to 0.$$

Regarding $\mu_{t+h}=\mu_t*\mu_h$ we receive

$$\mu_{t+h|M_1}=\sum_{i=1}^k \mu_{t|M_i}*\mu_{h|M_i^{-1}}+r_{t,h}$$

where $r_{t,h}$ is a suitable measure with $\|r_{t,h}\|\to 0$ as $h\to 0$. Hence

$$(*)\qquad \mu_{t+h|M_1}\to\frac{1}{k}\sum_{i=1}^k \mu_{t|M_i}=\frac{1}{k}\mu_{t|\varphi^{-1}(H_1)} \quad \text{as } h\to 0.$$

Let us show that this relation implies $k=1$. Therefore, we choose an increasing sequence of symmetric compact sets $K_n\subset H$ with $\mu_1(H-K_n)\to 0$. Then $Z=\bigcup_{n\in\mathbb{N}}(K_n)^n$ defines a subgroup of H. For all $t\in(0,1)$ we compute:

$$\max_{x\in G}\mu_t(xZ)\geqq\int\mu_t(xZ)\,\mu_{1-t}(dx)=\mu_1(Z)\geqq\frac{1}{k}\exp(-\|v\|)$$

and

$$\mu_t(Z)\geqq\sum_{xZ}(\mu_{t/2}(xZ))^2\geqq\left(\frac{1}{k}\exp(-\|v\|)\right)^2$$

if we take the symmetry of μ_t into account. Now $t\mapsto\mu_t(Z)>c$ holds for all $t\in[0,1]$ if we put $c=\frac{1}{2}\left(\frac{1}{k}\exp(-\|v\|)\right)^2$. But $t\mapsto\mu_t((K_n)^n)$ is upper semicontinuous for each $n\in\mathbb{N}$. The category theorem of Baire implies that there exists a natural number n and an open subset U of $[0,1]$ satisfying $U\subset\{t\in[0,1]\colon \mu_t((K_n)^n)\geqq c\}$. Applying (*) we compute for each

$$t\in U\cap[0,1)\colon c\leqq\limsup_{h\to 0}\mu_{t+h}((K_n)^n)\leqq\frac{1}{k}\mu_t((K_n)^n).$$

Repeating these arguments we receive $ck^m\leqq\mu_t((K_n)^n)$ which shows $k=1$.

2. $((\mu_t)_{t\geqq 0}, \mu_0=\varepsilon_e)$ is a continuous convolution semigroup on G_τ. It suffices to prove that μ_t is continuous in $t=0$. Let $V\subset G_\tau$ be an open set containing e. Then there exists an open set $U\subset G$ with $V\cap H=U\cap H$. The inequality

$$\liminf_{t\to 0}\mu_t(V\cap H)\geqq\liminf_{t\to 0}\mu_t(U)-\lim_{t\to 0}\mu_t(H^c)=1$$

proves the statement.

3. In a third step we prove $\mu_t(H)=1$ provided $\mu_t\in M(G_\tau)$ and $\eta(H^c)=0$. At first we assume that $H\subset G$ is closed. Then $(\mu_t)_{t\geqq 0}$ is continuous on the locally compact group G_τ and the whole representation theory can be applied. Let η'

be the Lévy measure of the semigroup on G_τ^*. Then $\eta'(H^c)=0$ because for any compact set $K\subset G_\tau - H$ there is a function $f\in C_b(G)$ satisfying $f_{|H}=0$, $f_{|K}=1$, $0\leqq f\leqq 1$. This fact shows: $\eta'(f)=\eta(f)=0$.

The semigroup admits the generating functional

$$A'(f)=\psi_2'(f)+\int(\tfrac{1}{2}(f+f^*)-f(e))\,d\eta'$$

on $\mathscr{D}(G_\tau)$ where f^* is defined by $f^*(x)=f(x^{-1})$, compare with [10]. Let $U_\alpha\subset G_\tau$ be the family of symmetric, open, relative compact set containing e (directed by $\subset$). Putting $A_\alpha(f)=\psi_2'(f)+\int(f-f(e))\,d\eta'_{|U_\alpha^c}$ we see that $\mathscr{E}_x(tA_\alpha)(H)=1$ holds which follows from formula (I) and [12], 6.2.3. Following the arguments of W. Hazod [10], p. 58 we conclude

$$1\leqq\limsup_\alpha\ \mathscr{E}_x(tA_\alpha)(H)\leqq\mu_t(H).$$

If H is not closed then $\bar{H}$ is a closed normal subgroup of G. We can now assume that $G=\bar{H}$ and H is a dense subset of G[3].

Let us assume that $\mu_t(H)<1$ for some t. Then the semigroup $\varphi(\mu_t)$ admits a Lévy measure $\nu\neq 0$ on G/H. Hence

$$\text{(1)}\qquad \lim_{t\to 0}\frac{\mu_t(H)-1}{t}=-\|\nu\| \quad\text{and}\quad \lim_{t\to 0}\frac{\mu_t(\varphi^{-1}\{x\})}{t}=\nu(\{x\})$$

for each $x\in G/H$, $x\neq H$. For each open symmetric set $U_\alpha\subset G$ with $e\in U_\alpha$ we introduce the semigroup $(\mu_t^\alpha)_{t\geqq 0}$ on G with the generating functional $A_\alpha(f)=A(f)-\int(f-f(e))\,d\eta_{|U_\alpha^c}$. Then for every real-valued bounded measurable function f on G satisfying $f=f1_{H^c}$ formula (I) yields:

$$\mu_t(f)=\exp(-t\eta(U^c))[\mu_t^\alpha(f)+u_1(t,A-A_\alpha,A_\alpha)(f)+o(t)].$$

Regarding $aH^c=H^c a$ for $a\in G$ and $H^c x=H^c$ for $x\in H$ we compute:

$$\begin{aligned}u_1(t,A-A_\alpha,A_\alpha)(H^c)&=\int_H\int_0^t\int\mu_{t-r}^\alpha(x^{-1}y^{-1}H^c)\,d\mu_r^\alpha(y)\,dr\,d\eta_{|U_\alpha^c}(x)\\&=\eta(U_\alpha^c)\,\mu_t^\alpha(H^c)\,t.\end{aligned}$$

This shows:

$$\text{(2)}\qquad \lim_{t\to 0}\frac{\mu_t(f)}{t}=\lim_{t\to 0}\frac{\mu_t^\alpha(f)}{t}$$

if one limit exists (and $f=f1_{H^c}$) since $\mu_t^\alpha(H^c)\to 0$ for $t\to 0$. Let us consider an open set $V\in\mathscr{U}_G(e)$. Then we choose a set U_α with $\bar{U}_\alpha\subset V$ and a continuous real-valued function f satisfying $f_{|\bar{U}_\alpha}=0$, $f_{|V^c}=1$ and $0\leqq f\leqq 1$. Hence

$$0\leqq\lim_{t\to 0}\frac{\mu_t^\alpha(V^c\cap H^c)}{t}\leqq\lim_{t\to 0}\frac{\mu_t^\alpha(f)}{t}=\eta_{|U_\alpha}(f)=0.$$

---

[3] Let $\bar{V}$ denote the closure of a set V

Together with (2) we conclude:

$$\lim_{t\to 0} \frac{\mu_t(V^c \cap H^c)}{t} = 0. \tag{3}$$

We are now ready to study the strongly continuous contraction semigroup of convolution operators $(T_{\mu_t})_{t\geqq 0}$ on $C_u(G_\tau)$ defined by $T_{\mu_t} f(x) = \int f(xy)\, d\mu_t(y)$. We apply the theory of semigroups of operators [13] which shows that $(T_{\mu_t})_{t\geqq 0}$ admits an infinitesimal generator $(N, \mathcal{N})$ with a norm dense domain $\mathcal{N} \subset C_u(G_\tau)$. We consider a function $f = \sum_{M\in G/H} f_M$, $f_M = f 1_M$ where f_M is an uniformly continuous function on $M \subset G$. Since H is dense there exists an uniquely determined function $\bar{f}_M \in C_u(G)$ satisfying $\bar{f}_M 1_M = f_M$.

A set $\{f_\alpha : \alpha \in I\} \subset C_u(G)$ is said to be equi uniformly continuous (e.u.c.) if (i) the set is uniformly bounded ($|f_\alpha| \leqq K$) and (ii) for each $\varepsilon > 0$ there exists a set $V \in \mathcal{U}_G(e)$ satisfying $|f_\alpha(x) - f_\alpha(y)| < \varepsilon$ for all $\alpha \in I$ and all x, y with $x^{-1} y \in V$.

(4) Let $\{g_M : M \in G/H\} \subset C_u(G)$ be e.u.c. Then

$$f = \sum_{M\in G/H} g_M 1_M \in C_u(G_\tau).$$

(5) If $f = \sum_{M\in G/H} f_M$ with $f \in C_u(G_\tau)$ and $f_M = f 1_M$ then

$$\{\bar{f}_M : M \in G/H\} \quad \text{is e.u.c.}$$

(6) If $\{f_\alpha : \alpha \in I\} \subset C_u(G)$ is e.u.c. then

$$\{T_{\mu_t} f_\alpha : \alpha \in I\} \subset C_u(G) \quad \text{is e.u.c.}$$

(7) If $\{f_\alpha : \alpha \in I\} \subset C_u(G)$ is e.u.c. then

$$\|T_{\mu_t} f_\alpha - f_\alpha\| \to 0 \quad \text{for } t \to 0 \text{ uniformly with respect to } \alpha \in I.$$

Let us sketch the proof of (5). For each $\varepsilon > 0$ we choose a symmetric set $V \subset H$, $V \in \mathcal{U}_{G_\tau}(e)$ with $|f(x) - f(y)| < \frac{\varepsilon}{2}$ for all x, y satisfying $x^{-1} y \in V$. Choose a symmetric set $U \in \mathcal{U}_G(e)$ with $U^3 \cap H \subset V$. Let $x, y \in G$ satisfying $x^{-1} y \in U$. For each coset $M \in G/H$ there are h_1, $h_2 \in M$ such that $h_1^{-1} x \in U$, $h_2^{-1} y \in U$ and $|\bar{f}_M(h_1) - \bar{f}_M(x)| \leqq \frac{\varepsilon}{4}$, $|\bar{f}_M(h_2) - \bar{f}_M(y)| \leqq \frac{\varepsilon}{4}$. This fact implies $h_1^{-1} h_2 \in U^3 \cap H$ which implies $h_1^{-1} h_2 \in V$. Consequently

$$\begin{aligned} |\bar{f}_M(x) - \bar{f}_M(y)| &\leqq |\bar{f}_M(x) - \bar{f}_M(h_1)| + |\bar{f}_M(h_1) - \bar{f}_M(h_2)| \\ &\quad + |\bar{f}_M(h_2) - \bar{f}_M(y)| < \varepsilon. \end{aligned}$$

The proofs of the other statements are easy.

In the next step we prove for $f \in C_u(G_\tau)$ the statement

$$\lim_{t\to 0} \frac{\mu_t\big(\sum_{M \neq H} f_M\big)}{t} = \sum_{M \neq H} \nu(\varphi(M))\, \bar{f}_M(e) \tag{8}$$

where M runs over all cosets $M \in G/H - \{H\}$. Assume $|f| \leqq K$. We can choose an open set $U \in \mathscr{U}_G(e)$ satisfying $|\bar{f}_M(x) - \bar{f}_M(y)| < \varepsilon$ for all M and x, y with $x^{-1} y \in U$. Then

$$\begin{aligned}
&\left|\frac{1}{t}\mu_t\Big(\sum_{M \neq H} f_M\Big) - \sum_{M \neq H} \nu(\varphi(M)) \bar{f}_M(e)\right| \\
&\leqq \left|\frac{1}{t}\Big(\mu_t\Big(\sum_{M \neq H} f_M\Big) - \mu_t\Big(1_U \sum_{M \neq H} f_M\Big)\Big)\right| + \left|\frac{1}{t}\mu_t\Big(1_U\Big(\sum_{M \neq H}(f_M - \bar{f}_M(e) 1_M)\Big)\Big)\right| \\
&\quad + \left|\frac{1}{t}\mu_t\Big(\sum_{M \neq H} \bar{f}_M(e)(1_M 1_U - 1_M)\Big)\right| + \left|\sum_{M \neq H} \frac{\mu_t(\bar{f}_M(e) 1_M)}{t} - \nu(\varphi(M)) \bar{f}_M(e)\right| \\
&\leqq K \frac{\mu_t(U^c \cap H^c)}{t} + \varepsilon \frac{\mu_t(U \cap H^c)}{t} + K \frac{\mu_t(U^c \cap H^c)}{t} + K \sum_{M \neq H}\left|\frac{\mu_t(1_M)}{t} - \nu(\varphi(M))\right|.
\end{aligned}$$

We regard the last term.

$$\begin{aligned}
\sum_{M \neq H}\left|\frac{\mu_t(1_M)}{t} - \nu(\varphi(M))\right| &= \sum_{M \neq H}\left|\sum_{n=1}^{\infty} \frac{1}{t} \frac{\nu^n(\varphi(M))}{n!} t^n - \nu(\varphi(M))\right| \\
&= \sum_{M \neq H} \sum_{n=2}^{\infty} \frac{\nu^n(\varphi(M))}{n!} t^{n-1} \leqq \sum_{n=2}^{\infty} \frac{t^{n-1}}{n!} \|\nu\|^n \to 0 \quad \text{for } t \to 0.
\end{aligned}$$

Statement (3) and (1) imply (8) since $\frac{\mu_t(U \cap H^c)}{t}$ is bounded. For every function f contained in the domain $\mathscr{N}$ of N $\lim\limits_{t \to 0} \frac{\mu_t(f 1_H) - f(e)}{t}$ exists since $\lim\limits_{t \to 0} \frac{\mu_t(f 1_{H^c})}{t}$ exists. For each $f \in \mathscr{N}$ with $f = \sum\limits_{M \in G/H} f_M$ we compute by applying (8):

$$\begin{aligned}
\lim_{t \to 0} \frac{\mu_t(f_H) - f(e)}{t} &= \lim_{t \to 0} \frac{\mu_t(\bar{f}_H) - \bar{f}_H(e)}{t} - \lim_{t \to 0} \frac{\mu_t(\bar{f}_H 1_{H^c})}{t} \\
&= A(\bar{f}_H) - \|\nu\| f(e).
\end{aligned}$$

Hence $\bar{f}_H$ lies in the domain of the generating functional A. Together with (8) we get: For all $f \in \mathscr{N}$ the following assertion is valid

$$\lim_{t \to 0} \frac{\mu_t\Big(\sum\limits_{M \in G/H} f_M\Big) - f_H(e)}{t} = A(\bar{f}_H) + \sum_{M \neq H} \nu(\varphi(M)) \bar{f}_M(e) - \|\nu\| f_H(e). \tag{9}$$

Let L_x be the operator on $C_u(G_\tau)$ defined by the left translation with respect to $x \in G$, $L_x f(y) = f(xy)$. Clearly $N(L_x f)(e) = N f(x)$ for $f \in \mathscr{N}$. By definition let $B_{1,2}: \mathscr{N} \to \mathbb{R}$ be the operators $B_1(f) = A(\bar{f}_H)$ and $B_2(f) = \sum\limits_{M \neq H} \nu(\varphi(M)) \bar{f}_M(e)$. Putting $C_i f(x) = B_i(L_x f)$ for $i = 1, 2$ and $f \in \mathscr{N}$ we conclude:

$$N f = C_1 f + C_2 f - \|\nu\| f \quad \text{for each } f \in \mathscr{N}. \tag{10}$$

The equality (10) is a result of statement (9). We remark

(11) $$\overline{L_x f_M} = L_x \bar{f}_M.$$

Let us now examine the operator C_1. We introduce

$$R_t\Big(\sum_{M\in G/H} f_M\Big)(x) = \sum_{M\in G/H} T_{\mu_t}\bar{f}_M(x)\, 1_M(x) \quad \text{for } f = \sum_{M\in G/H} f_M \in C_u(G_\tau),$$

$f_M = f 1_M$. Then

a) $(R_t)_{t\geqq 0}$ is a strongly continuous contraction semigroup of positive operators on $C_u(G_\tau)$ satisfying $R_t f(x) = R_t(L_x f)(e)$.

b) $(R_t)_{t\geqq 0}$ admits the generator C_1 introduced above. Let us give the proof. From (6) we get $R_t(f)\in C_u(G_\tau)$ and (7) implies that R_t is strongly continuous for $t\to 0$. The relation (11) implies $R_t(L_x f)(e) = T_{\mu_t}(\overline{L_x f_M})(e)$ if $x\in M$ which proves

$$R_t f(x) = \sum_{M\in G/H} T_{\mu_t}(\overline{L_x f_M})(e)\, 1_M(x) = R_t(L_x f)(e),$$

$$\begin{aligned} R_t R_s f(x) &= R_t\Big(\sum_{M\in G/H} T_{\mu_s}(\overline{L_x f_M})(e)\, 1_M(x)\Big) \\ &= \sum_{M\in G/H} T_{\mu_t} T_{\mu_s}(\overline{L_x f_M})(e)\, 1_M(x) \\ &= \sum_{M\in G/H} T_{\mu_{t+s}}(\overline{L_x f_M})(e)\, 1_M(x) = R_{t+s} f(x). \end{aligned}$$

b) Let D be the infinitesimal generator of $(R_t)_{t\geqq 0}$. Then $Df(x) = D(L_x f)(e)$ holds for each f contained in the domain of D and

$$\lim_{t\to 0} \frac{R_t f(e) - f(e)}{t} = \lim_{t\to 0} \frac{T_{\mu_t}\bar{f}_H(e) - f(e)}{t} = A(\bar{f}_H)$$

which shows $D = C_1$.

For $f\in C_u(G)$ the definition of R_t yields $R_t f = T_{\mu_t} f$ and $R_t 1_H(x) = 1_H(x)$. Let $t>0$ be fixed. Then $R(f) = R_t f(e)$ defines a positive linear functional on $C_u(G_\tau)$ satisfying $R(1) = 1$ and $R(1_H) = 1$. Since C_2 is a bounded positive operator and R_s is positive we receive

(12) $$\int f\, d\mu_t \geqq \exp(-t\,\|v\|)\, R(f) \quad \text{for each positive } f\in C_u(G_\tau).$$

(Regard (10) and apply the perturbation series for operator semigroups, see for example [10], p. 11).

Let us choose compact sets $K_1\subset H$, $K_2\subset H^c$ with $\mu_t(H-K_1)<\varepsilon$ and $\mu_t(H^c - K_2)<\varepsilon$. There exists a function $f\in C_u(G)$ with compact support and $0\leqq f\leqq 1$, $f_{|K_1} = 0$, $f_{|K_2} = 1$. Thus

$$\int f\, d\mu_t = R(f) = R(f 1_H) + R(f 1_{H^c}) \quad \text{and} \quad 0 \leqq R(f 1_{H^c}) \leqq R(1_{H^c}) = 0$$

imply $R(f 1_H) = \int f\, d\mu_t$. Therefore (12) shows

$$\varepsilon > \int f 1_H\, d\mu_t \geqq \exp(-t\,\|v\|) \int f\, d\mu_t \geqq \exp(-t\,\|v\|)(\mu_t(H^c) - \varepsilon).$$

Hence $\mu_t(H^c)=0$ proves $\nu=0$.

a) Suppose $\mu_t \notin M(G_\tau)^\perp$ for some $t>0$. The first part of the proof shows that $\varphi(\mu_t)_{t\geqq 0}$ is a Poisson semigroup on G/H and $\lim\limits_{t\to 0}\frac{\mu_t(H^c)}{t}=\|\nu\|$. For each open symmetric set $U_\alpha \in \mathscr{U}_G(e)$ we introduce the generating functional $B_\alpha(f)=\int (f-f(e))\,d\eta_{|U_\alpha^c \cap H^c}$ and $A_\alpha=A-B_\alpha$. Formula (I) implies:

$$\begin{aligned}\exp(t\eta(U_\alpha^c\cap H^c))\,\mu_t(H^c)&\geqq u_1(t, A_\alpha, B_\alpha)(H^c)\\&=\int\int_0^t\int \mathscr{E}_x((t-r)\,A_\alpha)(y^{-1}z^{-1}H^c)\,\mathscr{E}_x(rA_\alpha)(dz)\,dr\,\eta_{|U_\alpha^c\cap H^c}(dy)\\&=\int t\mathscr{E}_x(tA_\alpha)(y^{-1}H^c)\,\eta_{|U_\alpha^c\cap H^c}(dy)\geqq \eta(U_\alpha^c\cap H^c)\,t\mathscr{E}_x(tA_\alpha)(H).\end{aligned}$$

Observe that H is normal and note that $\mathscr{E}_x(tA_\alpha)$ is a symmetric semigroup in $M(G_\tau)$ with $\lim\limits_{t\to 0}\mathscr{E}_x(tA_\alpha)(H)=1$ (regard (1)). Thus

$$\lim_{t\to 0}\frac{\mu_t(H^c)}{t}=\|\nu\| \quad \text{implies} \quad \|\nu\|\geqq \eta(H^c\cap U_\alpha^c).$$

For each compact set $K\subset H^c$ we can choose a set U_α with $K\subset U_\alpha^c$. The regularity of η proves $\eta(H^c)<\infty$.

b) (ii) Consider the decomposition $\eta=\eta_{|H}+\eta_{|H^c}$ of the Lévy measure. If $\mu_t\notin M(G_\tau)^\perp$ for some $t>0$ then part 3 yields

$$\mathscr{E}_x(t(A-(\eta_{|H^c}-\eta(H^c)\,\varepsilon_e)))(H)=1.$$

Hence $\mu_t(H)\geqq \exp(-t\eta(H^c))$. The arguments used in the proof of part a) show

$$\mu_t(H^c)\geqq \exp(-t\eta(H^c))\,t\eta(H^c). \quad \square$$

Corollary 7. *Let $(\mu_t)_{t\geqq 0}$ be any normal continuous convolution semigroup in $M_1(G)$. For each normal measurable subgroup H the following assertions are valid:*

a) *If $\eta(H^c)=\infty$ then $\mu_t(xH)=0$ for all $x\in G$ and each $t>0$.*

b) *If $\eta(H^c)=0$ then*

(i) *either $\mu_t(xH)=0$ for all $x\in G$ and each $t>0$ or*

(ii) *for each $t>0$ there exists a point $x(t)\in G$ satisfying*

$$x(t)\,x(s)=x(t+s) \bmod H \quad \textit{and} \quad \mu_t(x(t)\,H)=1.$$

If $(\mu_t)_{t\geqq 0}$ is symmetric we can choose $x(t)=e$.

c) *Suppose that $(\mu_t)_{t>0}$ is symmetric and $0<\eta(H^c)<\infty$. Put $a=\sum\limits_{xH\neq H}\eta(xH)$. Then:*

(i) *either $\mu_t(H)=0$ for each $t>0$ or*

(ii) $\exp(-t\eta(H^c))\leqq \mu_t(H)\leqq \exp(-t(\eta(H^c)-a))-ta\exp(-t\eta(H^c))$.

Proof. Regard the symmetric semigroup $\nu_t=\mu_t*\tilde{\mu}_t$ which has the Lévy measure $\eta+\tilde{\eta}$. Let G_τ be the refinement induced by H.

a) Corollary 5 implies $\nu_t(H)=0$ for $t>0$ if $(\eta+\tilde{\eta})_b(G^*)=\infty$. If $(\eta+\tilde{\eta})_b(G^*)<\infty$ then $(\eta+\tilde{\eta})_a(H^c)=\infty$ is satisfied. Hence $\nu_t(\mathrm{H})=0$ by Corollary 5c) and Theorem 6 for $t>0$. Thus Lemma 4 shows a).

b) The semigroup $(\nu_t)_{t\geq 0}$ fulfils the assumptions of Theorem 6 since $(\eta+\tilde{\eta})(H^c)=0$. Thus either $\nu_t(H)=0$ for all $t>0$ or $\nu_t(H)=1$ for every $t>0$ is valid. Lemma 4 proves the assertion. Part c) follows from Theorem 1, Corollary 5b)(ii) and Theorem 6b). □

Theorem 6 and Corollary 7 include new results for Gauss semigroups since the Lévy measure η vanishes.

For a measure $\mu\in M_1(G)$ the set $\mathrm{ar}(\mu)=\{x\in G: \mu*\varepsilon_x\approx\mu\}$ forms a measurable subgroup of G (see [15], Chap. II). Let us call $ar(\mu)$ the set of equivalent (right) translates of μ. It is well known that $ar(\mu)$ is closed if μ is absolutely continuous. (Note that $x\mapsto\mu(Ax^{-1})$ is continuous for each measurable set A provided μ is absolutely continuous.) The next Theorem deals with symmetric Gauss semigroups having normal sets $ar(\mu_t)$. For example $ar(\mu_t)$ is normal if $(\mu_t)_{t\geq 0}$ is a central semigroup which means $\mu_t=\varepsilon_{y^{-1}}*\mu_t*\varepsilon_y$ for each $y\in G$ (cf. [12], 6.4).

Ch. Berg [3] gives an example of a symmetric Gauss semigroup on the infinite-dimensional torus group with the following interesting property: There exists a point t_0 such that μ_t is singular with respect to Haar measure up to time $t<t_0$ and μ_t is absolutely continuous for $t>t_0$. In [15] and [18] the author shows that absolutely continuity of symmetric convolution semigroups can be described in terms of the set of equivalent translates. Let's remember Berg's example. Then $\mu_t(ar(\mu_t))=1$ $(t>0)$ iff μ_t is absolutely continuous.

Theorem 8. *Let G be a second countable locally compact group and $(\mu_t)_{t\geq 0}$ a symmetric Gauss semigroup on G. Suppose that $ar(\mu_t)$ is normal for each $t>0$. Then either assertion* a) *or* b) are valid:

a) $\mu_t(ar(\mu_t))=0$ *for all* $t>0$,

b) (i) *There exists a normal subgroup H of G and a locally compact refinement H_τ of the relative topology on H such that $(\mu_t)_{t\geq 0}$ is concentrated on H and $(\mu_t)_{t\geq 0}$ forms a continuous Gauss semigroup on H_τ.*

(ii) *There exists a $t_0\in[0,\infty)$ satisfying $\mu_t(ar(\mu_t))=0$ for each $0<t<t_0$ and $ar(\mu_t)=H$ for $t>t_0$. Hence $\mu_t((ar(\mu_t))=1$ for $t>t_0$.*

(iii) $\mu_{t|H}\approx\omega_{H_\tau}$ *for* $t>t_0$.

(iv) $t_0=\inf\{t: \mu_t\ll\omega_{H_\tau}\}$.

Proof. Corollary 7 shows: either $\mu_t(ar(\mu_t))=0$ or $\mu_t(ar(\mu_t))=1$ for fixed t. Suppose $t_0=\inf\{t: \mu_t((ar(\mu_t))=1\}<\infty$ and $\mu_t(ar(\mu_t))=1$. Then a standard argument used several times implies $\mu_s(ar(\mu_t))=1$ for all $s\geq 0$ (see Lemma 4).

Putting $H=\bigcup_{t_0<s} ar(\mu_s)$ we remark that $ar(\mu_s)\subset ar(\mu_t)$ holds for $s\leqq t$. Moreover, $ar(\mu_s)=ar(\mu_t)$ for each $s\geqq t$. Assume that there is a point $x\in ar(\mu_s)-ar(\mu_t)$. Then $\mu_s(ar(\mu_t)x)=0$ yields a contradiction since $\mu_s(ar(\mu_t)xx^{-1})=1$ is true. Hence $H=ar(\mu_t)$. Since t was arbitrary and $ar(\mu_s)$ is increasing we receive $H=ar(\mu_s)$ for all $s>t_0$. For $t>t_0$ $\mu_{t|H}$ is a quasi-invariant probability measure on the analytic measurable group H if we observe that $H\subset G$ is measurable. Theorem 7.1 of Mackey [21] shows that there exists a locally compact topological group topology $\tau(t)$ on H such that the Borel structure is the same as before and the null sets of the Haar $\omega_{H_{\tau(t)}}$ and μ_t coincide. Since $\mu_s\in M_1(H_{\tau(t)})$ $((\mu_s)_{s\geq 0}, \mu_0=\varepsilon_e)$ becomes a continuous convolution semigroup on $H_{\tau(t)}$ (apply

[12], 6.1.24 and note that $\mu_s \in M_1(G)$ has no idempotent right factor, see E. Siebert [25], Lemma 1 and Theorem 5). Finally let us show that $H_{\tau(t)}$ is a refinement on H and that $\tau(t)$ doesn't depend on t.

Let V be an open set of $\mathscr{U}_G(e)$ and $U \in \mathscr{U}_G(e)$ with $U^2 \subset V$. Then $W = \{x \in H: \mu_{t|U \cap H}((U \cap H)x) > 0\} \subset V \cap H$. Hence $W \in \mathscr{U}_{H_{\tau(t)}}(e)$ since $\mu_t(U \cap H) > 0$ because e belongs to the support of μ_t (with respect to G). The measure $\mu_{t|U \cap H}$ is absolutely continuous on $H_{\tau(t)}$ which shows that $W \subset H_{\tau(t)}$ is open (cf. [11], § 19, 20). If $s > t_0$ we start with the semigroup defined on $H_{\tau(t)}$ and apply the arguments used above a second time which proves that $\tau(s)$ is finer than $\tau(t)$. At last we observe that $(\mu_t)_{t \geqq 0}$ is a Gauss semigroup on H_τ. Let $K \subset H_\tau - \{e\}$ be a τ-compact set. Then $K \subset G^*$ is compact. There exists a test function $f \in \mathscr{D}(G)$ with $0 \leqq f \leqq 1$, $f(e) = 0$ and $f_{|K} = 1$. Hence $\lim\limits_{t \to 0} \frac{\int f\, d\mu_t}{t} = 0$. If η' denotes the Lévy measure of $(\mu_{t|H})_{t \geqq 0}$ then $\eta'(K) \leqq \eta'(f\, 1_H) = \lim\limits_{t \to 0} \frac{\int f\, 1_H\, d\mu_t}{t} = 0$ implies $\eta' = 0$.

(iv) Recall that for each symmetric continuous convolution semigroup the support of μ_t forms a closed subgroup which doesn't depend on $t > 0$ (cf. [25]). Suppose now $\mu_t \ll \omega_H$. Choose an arbitrary $r > 0$. Then we shall prove $ar(\mu_{t+r}) = H$. Let A be a Borel subset of H. Consider $0 = \mu_{t+r}(A) = \int \mu_t(Ax^{-1})\, d\mu_r(x)$. Since $x \mapsto \mu_t(Ax^{-1})$ is continuous on H_τ we get $\mu_t(Ax^{-1}) = 0$ for x contained in the support (with respect to H_τ) which is equal to H. Hence $\mu_{t+r}(Ay^{-1}) = 0$ for all $y \in H$. □

Example 1. Let $G = \mathbb{R} \circledS \{-1, 1\}$ be the semidirect product of $(\mathbb{R}, +)$ and the discrete group $(\{-1, 1\}, \cdot)$ defined by $(x, k)\,(y, n) = (nx + y, kn)$ for $x, y \in \mathbb{R}$, $k, n = \pm 1$. Put $\varphi: \mathbb{R} \to G$, $\varphi(t) = (t, 1)$. Then

$$A(f) = \frac{d^+}{dt}\,(f \circ \varphi)|_{t=0} + \alpha(\varepsilon_{(0,-1)} - \varepsilon_{(0,1)})(f)$$

defines a generating functional on $\mathscr{D}(G)$ for $\alpha > 0$. The semigroup $\mu_t = \mathscr{E}_x(tA)$ is important for the probabilistic approach to the solution of the telegraph equation [19]. Let τ be the discrete topology on G and consider $M(G) = M(G_\tau) \oplus M(G_\tau)^\perp$ which describes the decomposition of a measure in a discrete and continuous part. It is easy to see that $(\mu_t)_{t \geqq 0}$ is not normal and $\mu_t = \exp(-\alpha t)[\varepsilon_{(0,t)} + r_t]$ where r_t denotes a continuous measure (apply formula (I) and see [24], Theorem 3). Consequently

$$A_1(f) = \frac{d^+}{dt}\,(f \circ \varphi)_{|t=0} \quad \text{and} \quad \eta_2 = \alpha \varepsilon_{(0,-1)} + \eta_b.$$

We regard the non-normal subgroup $H = \{(0, 1), (0, -1)\}$. Hence $\eta(H^c) = 0$ but $\mu_1(H) = \exp(-\alpha t)$ doesn't satisfy the zero-one law.

Example 2 ([12], 5.5.8). Let $(\mu_t)_{t \geqq 0}$ be symmetric semigroup of normal distributions on $\mathbb{R}$ and let $G = \Pi^2$ be the product of the torus group. Consider the Gauss semigroup $(f(\mu_t))_{t \geqq 0}$ where $f: \mathbb{R} \to G$ is defined by $f(x) = (\exp(ix), \exp(idx))$ for irrational $d \in \mathbb{R}$. Then assertion 8b) is valid for $H = f(\mathbb{R})$ and $t_0 = 0$. But the topology H_τ is strict finer than the relative topology on H.

Remarks. 1. Zero-one laws for Gauss measures (in the sense of Bernstein, see [12]) on Abelian topological groups have been established by T. Byczkowski [4].

2. For each symmetric Gauss semigroup on a connected Lie group there exists a measurable subgroup $H \subset G$ and a locally compact refinement H_τ of the subspace topology such that each $\mu_t (t>0)$ is absolutely continuous with respect to ω_{H_τ}. Note that H is not necessarily normal. This result is due to E. Siebert [26], Theorem 3. He also deduces a zero-one law for Gauss semigroups (and not necessarily normal subgroups H) on connected Lie groups.

3. Zero-one laws for the set of equivalent translates for product measures appear in [18].

3. Zero-one Laws for Measures on Abelian Topological Groups

Let $(G, +)$ be an Abelain topological Hausdorff group and let $(\nu_i)_{i\in I}$ be an upward directed family of measures in $M_+(G)$ with $\nu_i(\{e\})=0$. If $(x_i)_{i\in I}$ is a family of points in G satisfying $\lim\limits_{i\in I} e(\nu_i) * \varepsilon_{x_i} = \mu \in M_1(G)$ (with respect to the weak topology) then μ is called a generalized Poisson measure. The abstract measure $\sup\limits_{i\in I} \nu_i = F$ is said to be a Lévy-measure of μ[4].

Theorem 9. *Let H be a measurable subgroup of G with $F(H^c)=0$ and $x\in G$. Then either $\mu(x+H)=0$ or $\mu(x+H)=1$.*

Let now E be a locally convex vector space and let $\mu\in M_1(E)$ be infinitely divisible (i.e.: for each natural number n there exists a probability measure $\mu_n \in M_1(E)$ with $\mu=(\mu_n)^n$). Then μ has the form $\mu=\gamma * \nu$ where γ denotes a Gauss measure (in the sense of [5]) and ν is a generalized Poisson measure on G, (cf. [5], Satz 1.9). The Lévy-measure F of ν (which is uniquely determined) is said to be the Lévy-measure of μ.

Theorem 10. *Let $H \subset E$ be a measurable subgroup of a locally convex vector space E. Suppose that μ is an infinitely divisible probability measure on E such that the Lévy measure F fulfils $F(H^c)=\infty$. Then $\mu(H+x)=0$ for each $x\in E$.*

Corollary 11. *Let μ be an infinitely divisible probability measure on a separable Banach space E with Lévy-measure F. If $F(H^c)=0$ for a measurable subgroup $H \subset E$ and $x \in E$ then either $\mu(x+H)=0$ or $\mu(x+H)=1$.*

The proof of Theorem 9 is based on the next Lemma which is a result of the technique developed in [17]. The proof of [17], Theorem 6 carries over. By definition we put $\nu * \mathcal{N} = \{\nu * \mu : \mu \in \mathcal{N}\}$ if $\mathcal{N} \subset M(G)$ and $\nu \in M(G)$.

Lemma 12. *Let G be a (not necessarily Abelian) topological group. Suppose that $\mu \in M_1(G)$ is a weak limit point of a net $(\mu_i)_{i\in I}$ in $M_1(G)$ with upward directed index set $(I, \leqq)$ satisfying $\mu_j = \mu_i * \mu_{i,j}$ for some $\mu_{i,j} \in M_1(G)$ for all $i \leqq j$ $(i, j \in I)$. Let $B \subset M(G)$ be a band such that $\mu_i * B \subset B$ and $\mu_i * B^\perp \subset B^\perp$ for each $i \in I$. Then either $\mu \in B$ or $\mu \in B^\perp$.*

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[4] Put $\sup\limits_{i\in I} \mu_i(C) = F(C)$ for each Borel set C

We are now able to give the proof of Theorem 9. Let B be the band $B = \{\nu \in M(G): |\nu|(H^c)=0\}$. Then $B^{\perp} = \{\nu \in M(G): |\nu|(H)=0\}$. Choose $\mu_i = e(\nu_i + \tilde{\nu}_i)$. The continuity of the convolution implies $\lim_{i\in I} \mu_i = \mu * \tilde{\mu}$. The net $(\mu_i)_{i\in I}$ fulfils the assumptions of the Lemma (Put $\mu_{i,j} = e(\nu_j + \tilde{\nu}_j - (\nu_i + \tilde{\nu}_i))$ and observe $\mu_i(H) = 1$.) Hence either $\mu * \tilde{\mu}(H) = 0$ or $\mu * \tilde{\mu}(H) = 1$ and Lemma 4 proves the assertion. □

The proof of Theorem 10:

I. Let $\hat{E}$ denote the completion of E and $i: E \hookrightarrow \hat{E}$ the canonical injection. It suffices to prove the assertion for generalized Poisson measures μ with the Lévy measure F. Note that $i(\mu)$ is a generalized Poisson measure on $\hat{E}$ with Lévy measure $i(F)$. Moreover there is a σ-compact subgroup $H' \subset H$ such that $\mu(H - H') = 0$. If $F(H^c) = \infty$ holds then $i(F)\ (i(H')^c) = \infty$ follows. Hence it is sufficient to prove the assertion for generalized Poisson measures μ on complete spaces E. Suppose that E is complete. Then

1. By [27], Satz 5(iv) there exists an uniquely determined continuous convolution semigroup $(\mu_t)_{t\geqq 0}$ on E with $\mu_1 = \mu$.

2. Following the notation of E. Dettweiler [5], §1 we call a Poisson measure $e(\rho)$ a Poisson factor of μ is there exists an infinitely divisible probability measure ν such that $\mu = \nu * e(\rho)$. If $e(\rho)$ is a Poisson factor of μ then $\mu_t = \nu_t * e(t\rho)$ for all $t \geqq 0$ provided $(\nu_t)_{t\geqq 0}$ denotes the continuous convolution semigroup induced by $\nu (= \nu_1)$. The proof of Satz 1.9 [5] shows:

$$F = \sup\{\rho: e(\rho) \text{ is a Poission factor of } \mu\}.$$

Furthermore we observe that $e(\rho_{|H^c})$ is a Poisson factor of if $e(\rho)$ has this property.

3. Let $B \subset M(E)$ be the prime L-subalgebra

$$B = \{\nu \in M(E): |\nu|(E) = \sum_{x+H} |\nu|(x+H)\}$$

and let $\mu_t = \mu_t^1 + \mu_t^2$ be the decomposition of μ_t induced by B with $\mu_t^1 \in B$ and $\mu_t^2 \in B^{\perp}$. If $\mu_t^1 \neq 0$ for some $t > 0$ then $\|\mu_t^1\| = \exp(-t\alpha)$ and $\exp(t\alpha)\,\mu_t^1$ becomes a continuous convolution semigroup in $M_1(E)$. If $(\mu_t)_{t\geqq 0}$ is a continuous symmetric semigroup and $\varphi: E \to E/H$ denotes the canonical projection then $\varphi(\exp(t\alpha)\,\mu_t^1)$ is a discrete symmetric $\{e\}$-Poisson semigroup on E/H. (Observe that the proof of Theorem 6 part 1 carries over). This fact yields:

$$\lim_{t\to 0} (\exp(t\alpha)\,\mu_t^1(H)) = 1$$

and

$$\lim_{t\to 0} \frac{\mu_t^1(H^c)}{t} = \lim_{t\to 0} \frac{\exp(t\alpha)\,\mu_t^1(H^c)}{t} = K$$

for some $K \geqq 0$. Hence

$$\lim_{t\to 0} \frac{\mu_t(H^c)}{t} = \lim_{t\to 0} \frac{\mu_t^1(H^c)}{t} + \lim_{t\to 0} \frac{1-\exp(-t\alpha)}{t} = K + \alpha$$

since $\mu_t^2(H^c) = \|\mu_t^2\|$.

4. We are now able to show $\mu \in B^{\perp}$ if $F(H^c)=\infty$. Suppose $\mu_t^1 \neq 0$ for some $t>0$. Then we choose a Poisson factor $e(\rho)$ of μ with $\rho(H)=0$. Let us consider $\mu_t = v_t * e(t\rho)$ and $\mu_t * \tilde{\mu}_t = v_t * \tilde{v}_t * e(t(\rho+\tilde{\rho}))$. We remark that $\mu_t^1 \neq 0$ if $(\mu_t * \tilde{\mu}_t)^1 \neq 0$ holds. Applying part 3 we see $(v_t * v_t)^1 \neq 0$ and $\lim_{t\to 0} v_t * \tilde{v}_t(H)=1$. Moreover we note

$$\mu_t * \tilde{\mu}_t(H^c) \geqq \exp(-t\|\rho+\tilde{\rho}\|)t \int_{H^c} v_t * \tilde{v}_t(H^c - x)\, d(\rho+\tilde{\rho})(x)$$
$$\leqq \exp(-t\|\rho+\tilde{\rho}\|)t\, v_t * \tilde{v}_t(H)(\rho+\tilde{\rho})(H^c)$$

which implies

$$K_1 = \lim_{t\to 0} \frac{\mu_t * \tilde{\mu}_t(H^c)}{t} \geqq (\rho+\tilde{\rho})(H^c).$$

But now part 2 shows $F(H^c) \leqq K_1$.

The proof of the Corollary: Let us at first regard the Gaussian part γ. By [1], Theorem 6.8 there exists a system of vectors $(e_i)_{i\in\mathbb{N}}$ in E and a sequence of independent standard normal distributed random variables $(\phi_i)_{i\in\mathbb{N}}$ such that $X = \sum_{i=1}^{\infty} \phi_i e_i$ converges a.s. in E and X has the distribution γ. Hence γ is an infinite convolution product. Let $\gamma_n(\beta_n)$ be the distribution of $\sum_{i=1}^{n} \phi_i e_i$ $\left(\sum_{i=n+1}^{\infty} \phi_i e_i\right)$. Then $\gamma = \gamma_n * \beta_n$ and either $\gamma_n(H)=1$ or $\gamma_n(y+H)=0$ holds for all $y\in E$. (We conclude $\gamma_n * \tilde{\gamma}_n(H)=0$ or 1 since $\gamma_n * \tilde{\gamma}_n$ is a Gaussian measure on a finite dimensional subspace. Then Lemma 4 and the symmetry prove the assertion). Put $B=\{\rho\in M(E): |\rho|(x+H)=0\}$. Suppose $\gamma_n(H)=1$ for each natural number n. Then $\gamma_n * B \subset B$, $\gamma_n * B^{\perp} \subset B^{\perp}$ and Lemma 12 yields $\gamma(x+H)=0$ or 1. If $\gamma_n(H)=0$ for some $n\in\mathbb{N}$ then $\gamma(x+H)=\gamma_n * \beta_n(x+H)=0$ since $\gamma_n(y+H)=0$ for all $y\in E$. Observe that $\mu=\varepsilon_{x_0} * \gamma * v$ holds where v denotes a generalized Poisson measure [1], p. 137. Since γ and v fulfil the assertion the same result is true for μ. □

Example 3. Let μ be a stable measure on a locally convex vector space E (in the sence of [6], D.2.1) such that μ is no Gauss measure. Suppose that $H\subset E$ is a measurable subgroup. If $x\in E$ then either $\mu(x+H)=0$ or 1.

This result is an application of Theorem 9 and 10. Let us assume that μ is no Dirac measure. By regarding the system of finite dimensional marginal distributions it is easy to see that μ is a generalized Poisson measure, see [5], S. 1.9 and [6], L. 2.2, S. 2.3. Suppose that μ has the Lévy measure F. We remark that it suffices to give the proof for symmetric stable laws (Lemma 4). In the following discussion let μ be symmetric.

Let $H_{s,0}$ denote the mapping $x\mapsto sx$ on E if s is a real number. It is well-known that there exists an unique scalar $\beta>1/2$ such that the Lévy measure F fulfils the equality $tF=H_{t^{\beta},0}(F)$. This result can be deduced by regarding Théorèm 3 of A. Tortrat [29]. For separable Banach spaces compare with [1], p. 156. It is also possible to prove this relation by regarding the system of finite dimensional marginal distributions if we note that μ is embeddable into a continuous convolution semigroup $(\mu_t)_{t\geqq 0}$. (Observe that the statements I and

1 of the proof of Theorem 10 imply that $i(\mu)$ is embeddable. Since μ is symmetric it is easy to see that μ is embeddable and $\mu_t = H_{t^\beta, 0}(\mu)$ holds). The value $\alpha = \beta^{-1}$ is called the order of μ. Now choose $t = 2^{-\alpha} < 1$. Then $2H \subset H$ implies $2H^c \supset H^c$ and $2^{-\alpha} F(H^c) = F((2^{-\alpha})^{-\beta} H^c) = F(2H^c) \geqq F(H^c)$ shows $F(H^c) = 0$ or ∞. □

It should be noted that W. Krakowiak [20] proved this zero-one law (under the additional assumption $x=0$) for stable measures on separable Banach spaces by applying a different technique.

Remarks. 1. For Gaussian probability measures Corollary 11 is well known, compare for example with C.R. Baker [2], T. Byczkowski [4], N.C. Jain [14]. Only to arrive completeness and to give another application of Lemma 12 we prove the assertion of Corollary 11 for γ. The author is indebted to A. Tortrat for the hint to consider Theorem 6.8 of [1] and further helpful comments.

2. Zero-one laws for stable measures appear in the papers of R.M. Dudley and M. Kanter [7], X.M. Fernique [8] and W. Krakowiak [20].

3. The concept of generalized Poisson measures on Abelian topological groups appeared in the paper of A. Tortrat [28].

4. If $(\mu_t)_{t \geqq 0}$ is a continuous convolution semigroup without Gaussian part (which means $\psi_2 = 0$) on an Abelian locally compact group then μ_t is a generalized Poisson measure [10]. Hence Lemma 12 gives another approach to some zero-one laws appearing in Sect. 2. But for non-Abelian groups this proof doesn't carry over.

5. If H is a non-measurable subgroup the zero-one laws carry over for Gauss measures and stable measures if we consider the inner measure $\mu_*(H+x)$. Observe that there exists a σ-compact subgroup $H' \subset H$ such that $\mu(H'+x) = \mu_*(H+x)$ holds (see for example the proof of Corollary 5).

6. Further results of A. Tortrat and the author will appear in the Proceedings of the 6th conference on probability measures on groups, Oberwolfach 1981. We shall show how to extend Example 3 for semistable and self-decomposable measures on locally convex spaces. The papers yield further applications of Theorem 9 and 10.

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