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The Markov Property at Co-Optional Times

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1. Introduction

Let X be a "nice" Markov process; say a right process for definiteness. Let R be a random time; that is R is a positive random variable. We say that R has the *Markov property* if the pre-R field $\mathscr{F}_R$ and the post-R field $\mathscr{G}_R$ are conditionally independent given X_R. See Sect. 2 for precise definitions of all unfamiliar terms used in this introduction. By the very definition of a Markov process, constant times have the Markov property, and the strong Markov property implies that stopping times have the Markov property. It has been known for some time (see [11] or [12]) that coterminal times have the Markov property. If R is either a stopping time or a coterminal time much more is known: namely the post-R process $(X_{R+t}, t>0)$ defined on $\{R<\infty\}$ is itself a strong Markov process.

In this paper our main concern is to investigate the situation for co-optional times. Section 2 collects the necessary definitions and preliminaries, and the main results follow in Sect. 3. First of all it is easy to see that not all co-optional times have the Markov property and we give a simple example at the end of Sect. 3. We begin Sect. 3 by reformulating the Markov property in terms of the dual optional projection of certain increasing processes associated with R. We then give two sufficient conditions that a co-optional time L have the Markov property. The first of these (Proposition 3.7) states that if L is disjoint from all stopping times in the sense that for all μ, $P^\mu[0<L=T<\infty]=0$ when T is a stopping time, then L has the Markov property. Following Dynkin we say that L is *reconstructable* if there exists a decreasing sequence (L_n) of co-optional times with $L_n \downarrow L$ and $L_n>L$ on $\{0<L<\infty\}$. Reconstructable co-optional times are closely related to the co-predictable return times discussed by Azéma in [1]. Proposition 3.8 states that every reconstructable co-optional time has the Markov property. This result is very reminiscent of the fact that a process reversed from a co-optional time has the moderate Markov property. Finally in Sect. 3

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we give a new proof in the spirit of the present paper that coterminal times have the Markov property.

In Sect. 4 we apply the result on reconstructable co-optional times to the space time process in order to prove that if L is co-optional, then the process $\tilde{Y}_t = (L, X_{tL})$, $0<t<1$ defined on $\{L<\infty\}$ is an inhomogeneous Markov process. This generalizes one of the main results in [8].

In [6] a theory of "splitting times" was developed. This was based on the fact that the end of a homogeneous optional set (i.e. a coterminal time) has the Markov property. Clearly the results of that paper can be extended to co-optional times having the Markov property.

It would be very interesting to characterize precisely which co-optional times have the Markov property, or to investigate general properties of the class of all times having the Markov property.

2. Preliminaries

Our basic datum is a Markov process $X=(\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, P^x)$ with state space $(E, \mathscr{E})$ which satisfies the right hypotheses as stated in [5]. After a change, if necessary, to the Ray topology these hypotheses amount to saying that X is a right continuous strong Markov process with a Borel transition function on $(E, \mathscr{E})$ where E is a universally measurable subset of a compact metric space $\bar{E}$ and $\mathscr{E}$ is the Borel σ-algebra of the metric space E. In addition $t \to X_t$ has left limits in $\bar{E}$ on $(0, \infty)$ almost surely, but we shall have no need for these left limits in this paper and so this condition could be omitted from our hypotheses. We do not single out a particular state to act as a cemetery, and so no questions of a lifetime ζ arise.

In applying the general theory of processes to the system $(\Omega, \mathscr{F}, \mathscr{F}_t, P^\mu)$ for all μ simultaneously, one needs to be precise about the meanings of familiar sounding objects. A process $Z=(Z_t)$ is *optional* if for each initial measure μ it is P^μ indistinguishable from a process Z^μ that is optional over the filtration $(\Omega, \mathscr{F}^\mu, \mathscr{F}_t^\mu, P^\mu)$ in the usual sense of the general theory [3]. *Measurable* and *predictable* processes are defined similarly. Two processes are indistinguishable if they are P^μ indistinguishable for all μ. A process Z is *homogeneous* if it is measurable and $Z_{t+s}(\omega) = Z_t(\theta_s \omega)$ for all $t>0$, $s \geqq 0$, and $\omega \in \Omega$. The restriction that t be *strictly* positive in this definition is crucial. An *optional function* f on E is a universally measurable function such that the process $f(X_t)$ is optional. Every Borel function and every α-excessive function is an optional function.

A *raw additive functional* (RAF) is an increasing right continuous measurable process $A=(A_t)_{t \geqq 0}$ with $A_0=0$ and satisfying $A_{t+s} = A_t + A_s \circ \theta_t$ identically. An *additive functional* (AF) is a RAF that is adapted to $(\mathscr{F}_t)$. Note that this definition corresponds to what is often called a perfect AF or RAF since we permit no exceptional set in the shift identity. In view of the perfection results of Walsh [14] and Meyer [9] this causes no essential loss in generality and we have decided to suppress the adjective "perfect". The same remark applies to the definition of co-optional times in the next paragraph.

A *random time* R is an $\mathscr{F}$ measurable random variable with values in $\mathbb{R}^+ = [0, \infty]$. A *co-optional time* L is the *end* of a homogeneous set; that is, L

$=\sup\{t: (t,\omega)\in H\}$ where $H\subset\mathbb{R}^+\times\Omega$ is homogeneous. (A set is homogeneous provided its indicator function is a homogeneous process.) It is easy to see that a random time L is a co-optional time if and only if $L\circ\theta_t=(L-t)^+$ identically where $r^+=\sup(r,0)$ for $r\in\mathbb{R}^+$. A *coterminal time* is the end of an optional homogeneous set. This differs from the definition of coterminal time given in [11] which required killing operators. However, an exact coterminal time L as defined in [11] is the end of an optional homogeneous set. Although this is a very simple fact we shall prove it for completeness at the end of this section in Proposition 2.3. The definition of a coterminal time as the end of an optional homogeneous set seems to be more convenient and has the advantage of emphasizing the crucial difference between co-optional and coterminal times.

With any random time R one associates a σ-field $\mathscr{F}_R$ by saying that an $\mathscr{F}$ measurable random variable F is $\mathscr{F}_R$ measurable provided that for each μ there exists an optional process $Z^\mu=(Z_t^\mu)$ relative to the filtration $(\Omega,\mathscr{F}^\mu,\mathscr{F}_t^\mu,P^\mu)$ such that $F=Z_R^\mu$ on $\{R<\infty\}$. Clearly R is $\mathscr{F}_R$ measurable and it is easy to see that so is $f(X_R)\,1_{\{R<\infty\}}$ for any universally measurable f on E. We shall call $\mathscr{F}_R$ the σ-field of events before R. Recall that $\mathscr{F}^*$ is the σ-field of universally measurable sets over $(\Omega,\mathscr{F}^0)$. Observe that if $F\in\mathscr{F}^*$, then $F\circ\theta_R\,1_{\{R<\infty\}}\in\mathscr{F}$. We define the σ-field of events after R, $\mathscr{G}_R$, as follows: an $\mathscr{F}$ measurable F is $\mathscr{G}_R$ measurable provided there exists $G\in\mathscr{F}^*$ with $F=G\circ\theta_R$ on $\{R<\infty\}$. If $F\in\mathscr{F}^*$, then $F\circ\theta_R\,1_{\{R<\infty\}}\in\mathscr{G}_R$. In particular, if $f\in\mathscr{E}^*$, then $f(X_R)\,1_{\{R<\infty\}}\in\mathscr{G}_R$.

We turn now to showing that an exact coterminal time as defined in [11] is the end of an optional homogeneous set. Let L be a coterminal time as defined in [11]. We do not assume that L is exact. Then according to Definition 4.1 and Proposition 4.1 of [11], L is a random time with the following properties:

(2.1) (i) $L\circ\theta_s=(L-s)^+$; i.e. L is co-optional
(ii) $L\circ k_s=L$ on $\{L<s\}$
(iii) $L\circ k_s\leqq s$
(iv) $L\circ k_s\leqq L$
(v) $t\to L\circ k_t$ is increasing on $[0,\infty)$.

In (2.1) the k_t are killing operators and the statements involving s hold for every $s\in\mathbb{R}^+=[0,\infty)$. We refer the reader to [11] for the properties of the killing operators. Define

(2.2) $L'=\sup_{t>0} L\circ k_t=\lim_{t\uparrow\infty} L\circ k_t.$

Then $L'\leqq L$ and L' is called the *exact regularization* of L. One says that L is *exact* if $L=L'$. It is easy to see that the process $(L\circ k_t)$ is predictable and so if we define

$$L_t=\inf_{s>t} L\circ k_s=\lim_{s\downarrow\downarrow t} L\circ k_s,$$

then L_t is a right continuous $(\mathscr{F}_t)$ adapted process and hence optional.

(2.3) **Proposition.** *Let $H=\{(t,\omega): L_t=t\}$. Then H is a homogeneous optional set and $M=\sup\{t: t\in H\}=L'$, the exact regularization of L.*

Proof. Clearly H is optional. If $t \geqq 0$, then

$$(2.4) \quad L_t \circ \theta_s = \lim_{u \downarrow \downarrow t} L \circ k_u \circ \theta_s = \lim_{u \downarrow \downarrow t} L \circ \theta_s \circ k_{u+s}$$
$$= \lim_{u \downarrow \downarrow t} (L \circ k_{u+s} - s)^+ = (L_{t+s} - s)^+.$$

Thus if $t > 0$, $t = L_t \circ \theta_s$ if and only if $t + s = L_{t+s}$, and so H is homogeneous. If $L(\omega) = a < \infty$ and $s > a$ then $L(k_s \omega) = a$ by (2.1-ii) and so $L_t(\omega) = a$ for every $t \geqq a$. Hence $M(\omega) = a$. Thus $M = L$ on $\{L < \infty\}$. In view of (2.1-iii) and (2.2) in order to complete the proof it suffices to show that $M \circ k_t \to M$ as $t \uparrow \infty$. Let $H(k_t) = \{s: L_s \circ k_t = s\}$ so that $M \circ k_t = \sup H(k_t)$. Since

$$L_s \circ k_t = \lim_{u \downarrow \downarrow s} L \circ k_u \circ k_t = \lim_{u \downarrow \downarrow s} L \circ k_{u \wedge t},$$

we see that $L_s \circ k_t \leqq t$ and if $s < t$, then $L_s \circ k_t = L_s$ while if $s \geqq t$, $L_s \circ k_t = L \circ k_t$. Combining these observations with $L \circ k_t \leqq L_t \leqq t$ for all t we see that

$$(2.5) \quad H(k_t) = \{s: L_s \circ k_t = s\} \subset \{s \leqq t: L_s = s\}$$

and

$$(2.6) \quad [0, t) \cap H(k_t) = \{s < t: L_s \circ k_t = s\} = \{s < t: L_s = s\}.$$

From (2.5) we obtain $M \circ k_t \leqq M$, and from (2.6) we observe that

$$M \circ k_t \geqq \sup\{[0, t) \cap H(k_t)\} = \sup\{s < t: L_s = s\}$$

and the last term increases to M as $t \uparrow \infty$. This proves that $M \circ k_t \to M$ as $t \uparrow \infty$.

3. The Markov Property and Co-Optional Times

In this section we shall develop two sufficient conditions for a co-optional time to have the Markov property. We begin with the following general definition.

(3.1) *Definition. A random time R has the Markov property if for every $F \in b\mathscr{F}^*$ (i.e. F is bounded and $\mathscr{F}^*$ measurable) there exists $f \in b\mathscr{E}^*$ such that for every bounded optional process Z and every initial measure μ one has*

$$(3.2) \quad E^\mu[Z_R F \circ \theta_R; 0 < R < \infty] = E^\mu[Z_R f(X_R); 0 < R < \infty].$$

The set $\{0 < R < \infty\}$ is in $\mathscr{F}_R$ but not in $\mathscr{G}_R$ in general. Since $f(X_R) 1_{\{R < \infty\}}$ is in both $\mathscr{G}_R$ and $\mathscr{F}_R$, (3.2) implies that under every P^μ the traces of $\mathscr{F}_R$ and $\mathscr{G}_R$ on $\{0 < R < \infty\}$ are conditionally independent given X_R.

If R is a stopping time, then the strong Markov property states that R has the Markov property in a slightly stronger form, with $\{0 < R < \infty\}$ replaced by $\{R < \infty\}$ in (3.2). Although less familiar it is also known that a coterminal time L also has this stronger form of the Markov property. See [11] or [12]. This last fact will be given a new proof later in this section. See Proposition 3.13.

The following result is an immediate consequence of (3.1) and standard results on the construction of kernels. See [4], for example.

(3.3) **Proposition.** *Let R be a random time having the Markov property. Then there exists a sub Markov kernel $K(x, d\omega)$ from $(E, \mathscr{E}^*)$ to $(\Omega, \mathscr{F}^*)$ such that for each initial measure μ, $F \in b\mathscr{F}^*$, and bounded optional process Z one has*

$$E^\mu[Z_R F \circ \theta_R;\ 0<R<\infty] = E^\mu[Z_R K(X_R, F);\ 0<R<\infty].$$

We begin by giving a very simple reformulation of the Definition (3.1). To this end fix a random time R. If $F \geqq 0$ is in $b\mathscr{F}^*$ define

(3.4) $B_t^F = F \circ \theta_R \, 1_{\{0<R\leqq t\}}$.

Thus $B^F = (B_t^F)$ is an increasing right continuous process that is constant except for a single jump of magnitude $F \circ \theta_R$ at $t=R$ when $0<R<\infty$. If $R=0$ or $R=\infty$, then $B_t^F = 0$ for all t. When $F=1$ we write simply B for B^1. It is known [13] (see also [10]) that there exists a right continuous adapted increasing process A^F such that for each μ, A^F is a version of the dual optional projection of B^F relative to P^μ. Again we write A for A^1 the dual optional projection of $B = B^1$. Finally if $C = (C_t)$ is an increasing process and $f \geqq 0$ is in $b\mathscr{E}^*$ we write $f * C$ for the increasing process $t \to \int_{(0,t]} f(X_s)\, dC_s$. With these concepts we may reformulate (3.1) as follows.

(3.5) **Lemma.** *Let R be a random time. Then R has the Markov property if and only if for each $F \in \mathscr{F}^*$ with $0 \leqq F \leqq 1$ there exists an $f \in \mathscr{E}^*$ with $0 \leqq f \leqq 1$ such that $A^F = f * A$ where A^F and A are defined above. Of course, this equality means that A^F and $f * A$ are indistinguishable.*

Proof. Fix $F \in \mathscr{F}^*$ with $0 \leqq F \leqq 1$ and an optional process Z with $0 \leqq Z \leqq 1$. Also fix an initial measure μ. Then from the very definitions

(3.6) $E^\mu[Z_R F \circ \theta_R;\ 0<R<\infty] = E^\mu \int Z_t \, dB_t^F = E^\mu \int Z_t \, dA_t^F$.

Suppose firstly that $A^F = f * A$. Using the measures on E defined by

$$\nu_1(h) = E^\mu[Z_R h(X_R);\ 0<R<\infty]$$

$$\nu_2(h) = E^\mu \int Z_t h(X_t)\, dA_t$$

and the universal measurability of f, there exists a Borel function g on E such that $\nu_1(g) = \nu_1(f)$ and $\nu_2(g) = \nu_2(f)$. Therefore

$$\begin{aligned} E^\mu \int Z_t \, dA_t^F &= E^\mu \int Z_t f(X_t)\, dA_t = E^\mu \int Z_t g(X_t)\, dA_t \\ &= E^\mu[Z_R g(X_R);\ 0<R<\infty] = E^\mu[Z_R f(X_R);\ 0<R<\infty]. \end{aligned}$$

Combining this with (3.6) shows that R has the Markov property. Conversely if R has the Markov property, then just reversing the above argument shows that $A^F = f * A$ where f corresponds to F in (3.1).

We turn now to co-optional times L. If L is co-optional, then using the facts that on $\{L \leqq t\}$, $L \circ \theta_t = 0$ while on $\{L > t\}$, $L \circ \theta_t = L - t$ and $\theta_L \circ \theta_t = \theta_{t + L \circ \theta_t} = \theta_L$, it is easily checked that the increasing process $B_t^F = F \circ \theta_L \, 1_{\{0<L\leqq t\}}$ defined in (3.4) is a RAF. It is known [2] or [13] that in this case one may choose A^F, the dual optional projection of B^F, to be an AF.

If $0 \leqq F \leqq 1$, then $A^F + A^{1-F} = A$ because of the additivity of dual optional projections. Therefore if A is continuous, then according to the absolute continuity theorem for continuous additive functionals proved in [2] there exists an optional f with $0 \leqq f \leqq 1$ and $A^F = f * A$. If T is a stopping time, then $1_{[T]}$ is optional where $[T]$ is the graph of T. Therefore

$$\begin{aligned} E^\mu[\Delta A_T;\ T<\infty] &= E^\mu \int 1_{[T]}(t)\, dA_t \\ &= E^\mu \int 1_{[T]}(t)\, dB_t = P^\mu[0<L=T<\infty]. \end{aligned}$$

Hence A is continuous if and only if $P^\mu[0<L=T<\infty]=0$ for every stopping time T and every initial measure μ. Combining these remarks with (3.4) we obtain the following result.

(3.7) **Proposition.** *Let L be a co-optional time. If the dual optional projection A of $1_{\{0<L\leqq t\}}$ is continuous, then L has the Markov property. Moreover, A is continuous if and only if*

$$P^\mu[0<L=T<\infty]=0$$

for every stopping time T and every initial measure μ.

Recall that a co-optional time is *reconstructable* provided there exists a decreasing sequence (L_n) of co-optional times such that almost surely $L_n \downarrow L$ and $L_n > L$ on $\{0<L<\infty\}$. We say that the sequence (L_n) reconstructs L. Here is our second result.

(3.8) **Proposition.** *A reconstructable co-optional time has the Markov property.*

Proof. Fix $F \in \mathscr{F}^*$ with $0 \leqq F \leqq 1$ and let B, B^F, A, and A^F be as above. If T is a stopping time and $\Lambda \in \mathscr{F}_T$, then $t \to 1_\Lambda 1_{[T]}(t)$ is an optional process, and so one obtains

$$(3.9) \quad \Delta A_T^F\, 1_{\{0<T<\infty\}} = E^\mu\{\Delta B_T^F\, 1_{\{0<T<\infty\}} | \mathscr{F}_T\}.$$

Let (L_n) be a sequence reconstructing L. Then

$$\Delta B_T^F\, 1_{\{0<T<\infty\}} = F \circ \theta_L\, 1_{\{0<L=T<\infty\}} = F \circ \theta_T\, 1_{\{0<L=T<\infty\}},$$

and

$$\begin{aligned} \{0<L=T<\infty\} &= \{T \leqq L, 0<T<\infty\} - \{T<L, 0<T<\infty\} \\ &= \{T<L_n \,\forall\, n, 0<T<\infty\} - \{L \circ \theta_T > 0, 0<T<\infty\} \\ &= \{L_n \circ \theta_T > 0 \;\forall\, n,\, L \circ \theta_T = 0, 0<T<\infty\}. \end{aligned}$$

Defining $h^F(x) = E^x(F; L_n > 0 \,\forall\, n, L=0)$, one may then write (3.9) in the form

$$(3.10) \quad \Delta A_T^F = h^F(X_T) \text{ a.s. on } \{0<T<\infty\}.$$

We would like to conclude from (3.10) that the processes ΔA^F and $t \to h^F(X_t)\, 1_{\{t>0\}}$ are indistinguishable. This would follow from the section theorem if we knew that h^F was optional. However, all that is clear is that h^F is universally measurable.

To get around this let $h = h^1$. Then

$$h(x) = P^x(L_n > 0 \; \forall n) - P^x(L > 0)$$
$$= \lim_n P^x(L_n > 0) - P^x(L > 0),$$

the limit existing since (L_n) is a decreasing sequence. But if M is co-optional, the function $c(x) = P^x(M > 0)$ is excessive since

$$P_t c(x) = P^x(M \circ \theta_t > 0) = P^x(M > t) \uparrow c(x).$$

Consequently h is an optional function, and so it follows that ΔA_t and $h(X_t)\, 1_{\{t>0\}}$ are indistinguishable. Therefore

$$C_t = A_t - \sum_{0<s\leq t} h(X_s)$$

is a continuous additive functional. In particular $h(X_s) > 0$ for at most countably many values of s. Hence $\int h(X_s)\, dC_s = 0$, and so C is carried by the optional set $D = \{h = 0\}$. Since $\{t\colon X_t \in D^c\}$ is countable and optional it follows from VI-T33 of [3], that given μ there exists a sequence (T_n) of stopping times with disjoint graphs such that $\{t\colon X_t \in D^c\}$ and $\bigcup [T_n]$ are P^μ indistinguishable. But $h^F \leq h$ and so a.s. P^μ

$$h^F(X_t) = \sum_n h^F(X_{T_n})\, 1_{[T_n]}(t) \tag{3.11}$$

for all t. Now $h^F(X_{T_n})$ is $\mathscr{F}_{T_n}$ measurable and so it follows from (3.11) that h^F is, in fact, optional, and we may conclude from (3.10) that ΔA^F and $t \to h^F(X_t)\, 1_{\{t>0\}}$ are indistinguishable.

Finally let $C_t^F = A_t^F - \sum_{0\leq s\leq t} h^F(X_s)$ be the continuous part of A^F. Since $0 \leq F \leq 1$, A^F is strongly dominated by A, and hence C^F is strongly dominated by C. Hence C^F is carried by D, and by the absolute continuity theorem [2], there exists an optional function φ such that $C^F = \varphi * C$. If we now define

$$f(x) = \frac{h^F(x)}{h(x)}\, 1_{D^c}(x) + \varphi(x)\, 1_D(x),$$

then $A^F = f * A$. Therefore L has the Markov property by (3.5).

(3.12) *Remark.* Note that we actually proved that if L is reconstructable or satisfies the condition in (3.7), then one may choose f to be an optional function in (3.2).

As mentioned before the following fact is known but we shall sketch a proof based on Lemma 3.5.

(3.13) **Proposition.** *Let L be a coterminal time. Then for every $F \in b\mathscr{F}^*$ there exists a bounded universally measurable function f such that for each initial measure μ and bounded optional process Z one has*

$$E^\mu[Z_L F \circ \theta_L;\, L < \infty] = E^\mu[Z_L f(X_L);\, L < \infty].$$

Proof. Let $L = \sup\{t: M_t = 1\}$ where M is the indicator of a homogeneous optional set that may be assumed closed without loss of generality.

Let $R = \inf\{t > 0: M_t = 1\}$. Since M is homogeneous and optional, R is a terminal time. Therefore $\varphi(x) = E^x(e^{-R})$ is 1-excessive and hence optional. For $1 \leqq n < \infty$, let $\Lambda_n = \left\{x: \frac{n-1}{n} \leqq \varphi(x) < \frac{n}{n+1}\right\}$ and let $\Lambda_\infty = \{x: \varphi(x) = 1\}$. Then $E = \bigcup\{\Lambda_n;\ 1 \leqq n \leqq \infty\}$. Fix $F \in \mathscr{F}^*$ with $0 \leqq F \leqq 1$. Since $M_L = 1$ if $L < \infty$ it follows that $M_t\, dA_t^F = dA_t^F$. Also

$$A^F = 1_{\Lambda_\infty} * A^F + \sum_{1 \leqq n < \infty} 1_{\Lambda_n} * A^F,$$

and since each Λ_n is optional, $1_{\Lambda_n} * A^F$ is the dual optional projection of $1_{\Lambda_n} * B^F$, $1 \leqq n \leqq \infty$ where the notation is that introduced above.

If T is a stopping time, then for any μ

$$(3.14)\quad E^\mu[1_{\Lambda_\infty}(X_T)\, \Delta A_T^F;\ 0 < T < \infty] = E^\mu[1_{\Lambda_\infty}(X_T)\, F \circ \theta_T;\ L = T, 0 < T < \infty].$$

But almost surely on $\{X_T \in \Lambda_\infty,\ 0 < T < \infty\}$ one has $R \circ \theta_T = 0$ and hence $L \circ \theta_T > 0$; that is $L > T$. Hence the right side of (3.14) is zero and so $1_{\Lambda_\infty} * A^F$ is continuous.

For $1 \leqq n < \infty$, the process $M_t^n = 1_{\Lambda_n}(X_t)\, M_t$ is the indicator of a homogeneous optional set. Let $R_n = \inf\{t > 0: M_t^n = 1\}$ be its debut. Clearly $R \leqq R_n$ and one has

$$E^x(e^{-R_n}) \leqq E^x(e^{-R}) = \varphi(x).$$

But on Λ_n, $\varphi \leqq n/n+1 < 1$. A familiar argument now shows that $M_t^n\, 1_{\{t>0\}}$ is indistinguishable from the indicator of the set of iterates R_n^k, $k \geqq 1$ of R_n ($R_n^1 = R_n$ and $R_n^{k+1} = R_n^k + R_n \circ \theta_{R_n^k}$ for $k \geqq 1$). See, for example, the proof of (3.5) in [7]. Since M is the indicator of a closed set, $M_L = 1$ if $L < \infty$. Therefore

$$M_t^n\, dB_t^F = M_t\, 1_{\Lambda_n}(X_t)\, dB_t^F = 1_{\Lambda_n}(X_t)\, dB_t^F,$$

and so $1_{\Lambda_n}(X_t)\, dA_t^F = M_t^n\, dA_t^F$. Hence $1_{\Lambda_n} * A^F$ is carried by the discrete set (R_n^k). Now for every μ

$$\begin{aligned} &1_{\Lambda_n}(X_{R_n})\, \Delta A_{R_n}^F\, 1_{\{0 < R_n < \infty\}} \\ &\quad = E^\mu[F \circ \theta_{R_n};\ X_{R_n} \in \Lambda_n, L = R_n, 0 < R_n < \infty \mid \mathscr{F}_{R_n}] \\ &\quad = E^{X(R_n)}(F;\ L = 0)\, 1_{\Lambda_n}(X_{R_n})\, 1_{\{0 < R_n < \infty\}} \end{aligned}$$

because on $\{0 < R_n < \infty\}$ one has $L \geqq R_n$ and so on this set $L = R_n$ if and only if $L \circ \theta_{R_n} = 0$. Let $h^F(x) = E^x(F; L = 0)$. It now follows by iteration that the processes $1_{\Lambda_n}(X_t)\, \Delta A_t^F$ and $1_{\Lambda_n}(X_t)\, h^F(X_t)\, M_t$ are indistinguishable. Note that if $x \in \Lambda_\infty$, $h^F(x) = 0$ since $P^x(L > 0) = 1$. Therefore summing on n, $1 \leqq n < \infty$, ΔA_t^F and $h^F(X_t)\, M_t$ are indistinguishable because $1_{\Lambda_\infty} * A^F$ is continuous. Let $h = h^1$ and set

$$(3.15)\quad f = \frac{h^F}{h}\, 1_{\{h > 0\}} + \varphi\, 1_{\{h = 0\}}$$

where φ is an optional function such that $C^F=\varphi * C$, C^F and C being the continuous parts of A^F and A. Then $A^F=f*A$, and so from (3.5) one obtains for any bounded optional process Z and initial measure μ

$$E^\mu[Z_L F\circ\theta_L;\ 0<L<\infty]=E^\mu[Z_L f(X_L);\ 0<L<\infty].$$

But using the Markov property at zero and (3.15)

$$\begin{aligned} E^\mu[Z_L f(X_L); L=0] &= E^\mu[Z_0 f(X_0); L=0] \\ &= E^\mu[Z_0 f(X_0)\, h(X_0)] = E^\mu[Z_0\, h^F(X_0)] \\ &= E^\mu[Z_0 E^{X(0)}(F; L=0)] = E^\mu[Z_0 F; L=0] \\ &= E^\mu[Z_L F\circ\theta_L;\ L=0], \end{aligned}$$

completing the proof of (3.13).

(3.16) *Example.* This is an example of a co-optional time which does not have the Markov property. Note firstly that the minimum of two co-optional times is co-optional, but the minimum of two coterminal times is not, in general, coterminal. We shall exhibit two coterminal times whose minimum does not have the Markov property. Let $E=\{a,b,c,\Delta\}$. Let X be a pure jump process with Δ a trap and all other transitions possible. Let $J_{ac}=\{t>0\colon X_{t-}=a,\ X_t=c\}$. Let $L_{ac}=\sup J_{ac}$. Since J_{ac} is an optional homogeneous set, L_{ac} is a coterminal time. Define J_{bc} and L_{bc} similarly and let $L=L_{ac}\wedge L_{bc}$. Then L is co-optional. Note that $L<\infty$ and that $P^x(L>0)>0$ if $x\neq\Delta$. Let $T=\inf J_{ac}$ and $F=1_{\{T<\infty\}}$. Since $J_{ac}\in\mathscr{B}(\mathbb{R}^+)\times\mathscr{F}^0$, $F\in\mathscr{F}^*$. Suppose that L has the Markov property and that f corresponds to F in (3.2). Let Z^a be the indicator of J_{ac} and note that $Z^a_L=1$ if and only if $L=L_{ac}>0$ and that $F\circ\theta_L=0$ on $\{L=L_{ac}>0\}$. Since $X_L=c$ on $\{0<L<\infty\}$ one has

$$\begin{aligned} 0 &= E^a[Z^a_L F\circ\theta_L;\ 0<L] = E^a[Z^a_L f(X_L);\ 0<L] \\ &= f(c)\, P^a[0<L=L_{ac}], \end{aligned}$$

and consequently $f(c)=0$. Now use the Markov property with Z^b the indicator of J_{bc}. Then

$$\begin{aligned} 0 &< P^a[0<L=L_{bc};\ T\circ\theta_{L_{bc}}<\infty] \\ &= E^a[Z^b_L F\circ\theta_L; 0<L] = f(c)\, P^a[0<L=L_{bc}], \end{aligned}$$

and hence L can not have the Markov property. In this example the AF, A, associated with L is carried by the discrete set $J_{ac}\cup J_{bc}$ and, consequently, A is purely discontinuous.

Remark. It is easy to see that the set $\{0<L<\infty\}$ can not be replaced by $\{L<\infty\}$ in the statement of the Markov property for co-optional times. For example, let $E=\{b,\Delta\}$ where b is a holding point with transition to Δ possible and Δ is a trap. Let $L'=\sup\{t\colon X_t=b\}$ and $L=L'\circ\theta_s=(L'-s)^+$ for a fixed $s>0$. Then L is reconstructable, but one easily checks that the Markov property does not hold on $\{L=0\}$.

Remark. If L is a co-optional time having the Markov property, one might wonder if A has the following property – compare with (3.5): given an AF, B strongly dominated by A, then B has the form $B=f*A$. However, it is very easy to construct a coterminal time L for which this fails.

4. Random Time Dilations

In this section we are going to generalize a result that we proved in [8] under much stronger hypotheses. However, under the stronger hypotheses of [8] we obtained more explicit results.

(4.1) **Proposition.** *Let L be a co-optional time. Then under P^μ for each initial measure μ the process $\tilde{Y}_t=(L, X_{tL})$, $0<t<1$ defined on $\{L<\infty\}$ is an inhomogeneous Markov process relative to the filtration $(\mathscr{F}_{tL})$.*

Proof. For the proof we introduce the space time process associated with X. To be explicit for $r\in\mathbb{R}^+$ let $\tau_t(r)=r+t$. Let $\tilde{\Omega}=\mathbb{R}^+\times\Omega$ and define $\tilde{X}_t(\tilde{\omega})=\tilde{X}_t(r,\omega)=(\tau_t(\omega), X_t(\omega))$ and $\tilde{\theta}_t\tilde{\omega}=\tilde{\theta}_t(r,\omega)=(r+t,\theta_t\omega)$. If for each $(r,x)\in\mathbb{R}^+\times E$ we define $\tilde{P}^{(r,x)}=\varepsilon_r\otimes P^x$, then $(\tilde{\Omega},\tilde{\mathscr{F}},\tilde{\mathscr{F}}_t,\tilde{X}_t,\tilde{\theta}_t,\tilde{P}^{(r,x)})$ is a right process with state space $\mathbb{R}^+\times E$. Here $\tilde{\mathscr{F}}$ and $\tilde{\mathscr{F}}_t$ have their usual meanings relative to process $\tilde{X}$. This is discussed in a more general setting in [6]. For each $t\in(0,1)$ define $\tilde{L}_t$ on $\tilde{\Omega}$ by

$$\tilde{L}_t(r,\omega)=t\,L(\theta_{\frac{1-t}{t}r}\omega)=(t\,L(\omega)-(1-t)\,r)^+=(t(L(\omega)+r)-r)^+.$$

One may then check that for each fixed $t\in(0,1)$, $\tilde{L}_t$ is co-optional for $\tilde{X}$. That is, $\tilde{L}_t\circ\tilde{\theta}_s=(\tilde{L}_t-s)^+$. Moreover, if $0<\tilde{L}_t(r,\omega)<\infty$, then for $t<s<1$,

$$\tilde{L}_s(r,\omega)=(s(L(\omega)+r)-r)^+>\tilde{L}_t(r,\omega).$$

In addition, $\tilde{L}_s(r,\omega)$ decreases to $\tilde{L}_t(r,\omega)$ as $s\downarrow t$. Therefore, each $\tilde{L}_t$ is a reconstructable co-optional time for $\tilde{X}$. Because of Proposition (3.8) (see also (3.3)) there exists a kernel $K_t(r,x;d\tilde{\omega})$ such that for every initial law ν on $\mathbb{R}^+\times E$, every bounded process Z that is optional for $\tilde{X}$ and every $F\in b\tilde{\mathscr{F}}^*$

$$(4.2)\quad \tilde{E}^\nu\{Z(\tilde{L}_t)\,F\circ\theta_{\tilde{L}_t};\,0<\tilde{L}_t<\infty\}=\tilde{E}^\nu\{Z(\tilde{L}_t)\,K_t(\tilde{X}(\tilde{L}_t);\,F);\,0<\tilde{L}_t<\infty\}.$$

Specializing (4.2) to the case $\nu=\varepsilon_0\otimes\mu$ where μ is an initial law on E, Z is optional for $(\mathscr{F}_t)$ and $F\in b\mathscr{F}^*$ gives

$$(4.3)\quad E^\mu\{Z(t\,L)\,F\circ\theta_{tL};0<L<\infty\}=E^\mu\{Z(t\,L)\,K_t(t\,L,X(t\,L);F);0<L<\infty\}.$$

From (4.3) it follows that the σ-fields $\mathscr{F}(t\,L)$ and $\mathscr{G}(t\,L)$ are conditionally independent on $\{0<L<\infty\}$, given $(L, X(t\,L))$. From this, the inhomogeneous Markov property of $\tilde{Y}_t=(L, X_{tL})$, $0<t<1$, relative to the filtration $(\mathscr{F}_{tL})$ is clear, because if $0<s<t<1$, $X_{tL}=X_{uL}\circ\theta_{sL}$ where $u=(t-s)/(1-s)$.

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