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Some Results in Probabilistic Geometry

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1. This paper owes its existence to a lecture by Prof. A. RÉNYI in Cambridge in April 1967.

It is possible in a natural way to define a measure on the space of all lines of Euclidean E^2 whose distance d from the origin satisfies $0 < d \leqq 1$. Given such lines $L_1, \ldots, L_n$, let Π be the convex polygon consisting of all points which lie on the same side of each L_i as the origin. Further let E_n be the expectation value of the number of sides of Π when $L_1, \ldots, L_n$ are "chosen at random" with respect to the measure mentioned above. Then RÉNYI and SULANKE [1] have shown that the limit

$$\lim_{n \to \infty} E_n$$

exists, and that it is equal to $\pi^2/2$.

It is the purpose of this note to prove the existence of rather more general limits of this type. We shall not evaluate these limits.

2. We shall be concerned with hyperplanes H of Euclidean space E^k, $k > 1$, which do not contain the origin. The equation of such a hyperplane can uniquely be written

$$\boldsymbol{\omega}\boldsymbol{x} = \omega_1 x_1 + \cdots + \omega_k \omega_k = r \tag{1}$$

where $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_k)$ is a point of the unit sphere S^{k-1} and where $r > 0$. Put $\boldsymbol{\omega} = \boldsymbol{\omega}(H)$, $r = r(H)$. Conversely, for given $\boldsymbol{\omega} \in S^{k-1}$ and given $r > 0$, the equation (1) defines a hyperplane $H = H(\boldsymbol{\omega}, r)$ which does not contain the origin.

Given a hyperplane H as above, write H' for the halfspace of all points $\boldsymbol{x}$ satisfying

$$\boldsymbol{\omega}\boldsymbol{x} \leqq r. \tag{2}$$

In this paper, a *polyhedron* Π will always be the intersection of finitely many such halfspaces. Thus polyhedrons are closed and contain the origin in their interior, but they are not necessarily bounded. The minimum number of inequalities (2) needed to define a polyhedron Π will be called the number of *faces* of Π and denoted by $f(\Pi)$.

We shall study real-valued functions F of polyhedrons which are homogeneous in the sense that

$$F(\lambda\Pi) = F(\Pi) \tag{3}$$

for every $\lambda > 0$ where, as usual, $\lambda\Pi$ consists of all points $\lambda\boldsymbol{x}$ with $\boldsymbol{x} \in \Pi$. We shall make the further assumption that

$$|F(\Pi)| \ll D^{f(\Pi)} \tag{4}$$

for every fixed number $D > 1$.

Now let $\boldsymbol{H}$ be the set of hyperplanes H having $0 < r(H) \leqq 1$. The product of the canonical measure on S^{k-1} normalized such that $\int\limits_{S^{k-1}} d\boldsymbol{\omega} = 1$, and the Lebesgue measure on $0 < r \leqq 1$ induces a measure on $\boldsymbol{H}$. Given a function G defined on $\boldsymbol{H}$ one has

$$\int\limits_{\boldsymbol{H}} G(H)\, dH = \int\limits_{S^{k-1}} d\boldsymbol{\omega} \int\limits_0^1 dr\, G(H(\boldsymbol{\omega}, r))\,, \tag{5}$$

provided the right hand side exists; otherwise the left hand side is not defined. One has

$$\int\limits_{\boldsymbol{H}} dH = 1\,. \tag{6}$$

Now let F be a function of polyhedrons Π. Put

$$E_n(F) = \int\limits_{\boldsymbol{H}} \cdots \int\limits_{\boldsymbol{H}} F\left(\bigcap_{i=1}^{n} H_i'\right) dH_1 \ldots dH_n \quad , \tag{7}$$

provided the right hand side exists. $E_n(F)$ is the "expectation value" of $F\left(\bigcap\limits_{i=1}^{n} H_i'\right)$ if $H_1, \ldots, H_n$ are "chosen at random".

Theorem 1. *Suppose the function $F = F(\Pi)$ is homogeneous and satisfies* (4), *and $E_n(F)$ exists for $n = 1, 2, \ldots$. Then*

$$E(F) = \lim_{n\to\infty} E_n(F)$$

exists. In fact, one has

$$E_n(F) = E(F) + O(c^n) \tag{8}$$

with any constant c greater than

$$c_k = 1 - \frac{V(k-1)}{k\,V(k)} \tag{9}$$

where $V(k)$ denotes the volume of the unit ball in E^k.

For example, the theorem holds if we take F to be the number of faces or the number of vertices of Π. The existence of $E(F)$ in this case when $k = 2$ is the result of Rényi and Sulanke mentioned above. Or one may set $F = F_p$ where

$$F_p(\Pi) = \begin{cases} 1 & \text{if} \quad f(\Pi) = p\,, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2. *Write*

$$N_p^k = E(F_p)$$

where k is the dimension of the space E^k in which we are working. Then one has $N_p^k = 0$ if $p \leqq k$ and $N_p^k > 0$ if $p > k$.

3. Put

$$K(\Pi) = \begin{cases} 1 & \text{if} \quad \Pi \text{ is contained in } |\boldsymbol{x}| \leqq 1\,, \\ 0 & \text{otherwise.} \end{cases}$$

The function K is not homogeneous.

Lemma.

$$E_n(K) = 1 + O\,(c^n)$$

where c is any constant greater than the constant c_k of (9).

Proof. Our first aim will be to evaluate the measure $\mu(\boldsymbol{H}_0)$ of the set $\boldsymbol{H}_0$ of all hyperplanes H of $\boldsymbol{H}$ having

$$(1, 0, \ldots, 0) \notin H' .$$

This property of hyperplanes H remains invariant under rotations about the x_1-axis. It therefore will suffice to study hyperplanes H of the type

$$\omega_1 x_1 + \omega_2 x_2 = r .$$

Such a hyperplane H lies in $\boldsymbol{H}_0$ precisely if $0 < r < \omega_1$. We obtain

$$\mu(\boldsymbol{H}_0) = \int\limits_{\omega_1 \geqq 0} \omega_1 \, d\boldsymbol{\omega}$$

and therefore

$$\mu\,(\boldsymbol{H}_0) = \int\limits_0^1 \omega_1 \frac{(k-1)\,V(k-1)}{k\,V(k)} \, (1-\omega_1^2)^{(k-3)/2} d\omega_1$$
$$= \frac{(k-1)\,V(k-1)}{k\,V(k)} \int\limits_0^{\pi/2} \sin\varphi\,(\cos\varphi)^{k-2}\,d\varphi = \frac{V(k-1)}{k\,V(k)} .$$

Next, let S_ε be the "spherical cap of radius ε and center $(1, 0, \ldots, 0)$" consisting of all points $\boldsymbol{\omega}$ of S^{k-1} having spherical distance at most ε from $(1, 0, \ldots, 0)$. Let $\boldsymbol{H}_\varepsilon$ be the set of all hyperplanes H of $\boldsymbol{H}$ having H' disjoint from S_ε. As can easily be shown, $\mu(\boldsymbol{H}_\varepsilon)$ exists and

$$\lim_{\varepsilon\to 0} \mu\,(\boldsymbol{H}_\varepsilon) = \mu\,(\boldsymbol{H}_0) = \frac{V(k-1)}{k\,V(k)} = 1 - c_k .$$

Now let $c > c_k$ and choose $\varepsilon > 0$ so small that

$$\mu(\boldsymbol{H}_\varepsilon) > 1 - c .$$

S^{k-1} may be covered by a finite number of spherical caps of radius ε, say by $S_\varepsilon^{(1)}, \ldots, S_\varepsilon^{(N)}$. Now for fixed j between 1 and N the set of all n-tuples $H_1, \ldots, H_n$ such that no H_i completely separates $S_\varepsilon^{(j)}$ from the origin has measure $(1 - \mu(\boldsymbol{H}_\varepsilon))^n$. The measure of the set of n-tuples $H_1, \ldots, H_n$ which have this property for some j between 1 and N is

$$\leqq N(1 - \mu(\boldsymbol{H}_\varepsilon))^n \ll c^n .$$

Since $S_\varepsilon^{(1)}, \ldots, S_\varepsilon^{(N)}$ give a covering of $S^{(k-1)}$, the lemma is proved.

Let $\boldsymbol{B}^n$ be the subset of $\boldsymbol{H} \times \cdots \times \boldsymbol{H}$ consisting of n-tuples $H_1, \ldots, H_n$ such that $\bigcap\limits_{i=1}^{n} H_i'$ is *not* contained in the ball $|\boldsymbol{x}| \leqq 1$. Our lemma says precisely that

$$\mu(\boldsymbol{B}^n) \ll c^n . \tag{10}$$

4. The notation

$$H_1 \leqq H_2$$

will mean that $r(H_1) \leqq r(H_2)$. Since each ordering of $H_1, \ldots, H_n$ according to this notation occurs with the same likelihood, we have for any m with $1 < m \leqq n$ the relation

$$E_n(F) = \left(n \binom{n-1}{m-1}\right) \underset{\substack{H_i \leqq H_m \text{ if } i \leqq m \\ H_j \geqq H_m \text{ if } j \geqq m}}{\int \cdots \int} F\left(\bigcap_{i=1}^{n} H_i'\right) dH_1 \ldots dH_n .$$

The hyperplanes $H_1, \ldots, H_{m-1}$ are now of the type $H_i = r(H_m)\, G_i$ where G_i lies in $\boldsymbol{H}$ $(1 \leqq i \leqq m-1)$. Furthermore,

$$dH_1 \ldots dH_{m-1} = r(H_m)^{m-1} dG_1 \ldots dG_{m-1},$$

whence

$$E_n(F) = \left(n \binom{n-1}{m-1}\right) \underset{\substack{H_j \geqq H_m \\ (j=m,\ldots,n)}}{\int \cdots \int} r(H_m)^{m-1} dH_m \ldots dH_n \int \cdots \int dG_1 \ldots dG_{m-1} \tag{11}$$
$$F\left(\left(\bigcap_{i=1}^{m-1} (r(H_m)\, G_i')\right) \cap \left(\bigcap_{j=m}^{n} H_j'\right)\right).$$

Substituting $F = 1$ we obtain

$$1 = \left(n \binom{n-1}{m-1}\right) \underset{\substack{H_j \geqq H_m \\ (j=m,\ldots,n)}}{\int \cdots \int} r(H_m)^{m-1} dH_m \ldots dH_n. \tag{12}$$

By (4) and (10) one has

$$\left| \underset{G_1 \times \cdots \times G_{m-1} \in \boldsymbol{B}^{m-1}}{\int \cdots \int} F(\ldots)\, dG_1 \ldots dG_{m-1} \right| \ll c^m D^n ,$$

uniformly in n and m, for every $c > c_k$ and every $D > 1$. On the other hand, if $G_1 \times \cdots \times G_{m-1}$ is not in $\boldsymbol{B}^{m-1}$, then $\bigcap_{i=1}^{m-1} G_i'$ is contained in the ball $|\boldsymbol{x}| \leqq 1$ and

$$F\left(\bigcap_{i=1}^{m-1} (r(H_m)\, G_i') \cap \bigcap_{j=m}^{n} H_j'\right) = F\left(\bigcap_{i=1}^{m-1} (r(H_m)\, G_i')\right) = F\left(\bigcap_{i=1}^{m-1} G_i'\right).$$

This gives

$$\begin{aligned}
\underset{\boldsymbol{H}}{\int} \cdots \underset{\boldsymbol{H}}{\int} dG_1 \ldots dG_{m-1} F(\ldots) &= \underset{\notin\, \boldsymbol{B}^{m-1}}{\int \ldots \int} F\left(\bigcap_{i=1}^{m-1} G_i'\right) dG_1 \ldots dG_{m-1} + O(c^m D^n) \\
&= \int \cdots \int F\left(\bigcap_{i=1}^{m-1} G_i'\right) dG_1 \ldots dG_{m-1} + O(c^m D^m) + O(c^m D^n) \\
&= E_{m-1}(F) + O\,(c^m D^n),
\end{aligned}$$

by virtue of (4) and (10). Combining this relation with (11) and (12) we get

$$E_n(F) = E_{m-1}(F) + O(c^m D^n)$$

or, replacing m by $m+1$,

$$|E_n(F) - E_m(F)| \ll c^m D^n . \tag{13}$$

This inequality holds for any $c > c_k$ and any $D > 1$, uniformly for all n, m having $1 \leqq m \leqq n$.

Let $c_k < c < 1$. There are $c_0 > c_k$ and $D > 1$ having $c_0 D^2 = c$. One then has for $m \leqq n$,

$$\begin{aligned} |E_n(F) - E_m(F)| &\ll c_0^m D^{2m} + c_0^{2m} D^{4m} + c_0^{4m} D^{8m} + \cdots \\ &\ll c^m + c^{2m} + c^{3m} + \cdots \\ &\ll c^m . \end{aligned} \tag{14}$$

Hence $E_1(F), E_2(F), \ldots$ is a Cauchy sequence and has a limit $E(F)$. The equation (8) follows immediately from (14), and hence Theorem 1 is proved.

5. It remains to show Theorem 2. If $p \leqq k$, there is no bounded polyhedron in E^k with p faces, hence there is no polyhedron with p faces contained in the ball $|\boldsymbol{x}| \leqq 1$. The equation $N_p^k = 0$ therefore follows immediately from the lemma of § 3.

If $p > k$, put $\bar{F}_p(\Pi) = F_p(\Pi) K(\Pi)$. There exists a polyhedron Π of E^k with precisely p faces which is contained in the interior of the ball $|\boldsymbol{x}| \leqq 1$. This implies in particular that $\bar{F}_p(\Pi) = 1$. Continuity arguments show that

$$E_p(\bar{F}_p) = \int_H \ldots \int_H \bar{F}_p\left(\bigcap_{i=1}^{p} H_i'\right) dH_1 \ldots dHp > 0 .$$

Apply (11) with $m = p + 1$, $n > p$, $F = F_p$. The inner integral on the right hand side of (11) becomes

$$\begin{aligned} &\int_H \cdots \int_H dG_1 \ldots dG_p F_p\left(\left(\bigcap_{i=1}^{p} r(H_{p+1}) G_i'\right) \cap \left(\bigcap_{j=p+1}^{n} H_j'\right)\right) \\ &\quad \geqq \int_{\substack{G_1 \times \cdots \times G_p \\ \notin B^p}} \cdots \int dG_1 \ldots dG_p F_p(\ldots) \\ &\quad = \int_{\substack{G_1 \times \cdots \times G_p \\ \notin B^p}} \cdots \int F_p\left(\bigcap_{i=1}^{p} G_i'\right) dG_1 \ldots dG_p = E_p(\bar{F}_p) . \end{aligned}$$

In view of (12) this gives

$$E_n(F_p) \geqq E_p(\bar{F}_p) .$$

The theorem now follows immediately.

Reference

1. RÉNYI, A., and R. SULANKE: Zufällige konvexe Polygone in einem Ringgebiet. Z. Wahrscheinlichkeitstheorie verw. Geb. **9**, 146–157 (1968).

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