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The Correct Measure Function for the Graph of a Transient Stable Process

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1. Introduction

Let $X(t)$, $t \geqq 0$, be a stable process of index α, $0 < \alpha \leqq 2$, in N dimensions. The process is transient when $N > \alpha$ and this is the case considered here. The range of the process up to time t is the set

$$R(t) = R(t, \omega) = \{X(s, \omega) : 0 \leqq s \leqq t\}$$

in R^N, and the graph of the process up to time t is the set

$$G(t) = G(t, \omega) = \{(s, X(s, \omega)) : 0 \leqq s \leqq t\}$$

in R^{N+1}. The purpose of the present paper is to find the correct Hausdorff measure function for the graph, i.e. a measure function φ such that the φ-measure of $G(t, \omega)$ is almost surely positive and finite. The statement and proof of this result is in section 4. The nonsymmetric Cauchy processes ($\alpha = 1$) are not included.

BLUMENTHAL and GETOOR [4] obtained the Hausdorff dimension of the graph when $N = 1$ in the symmetric case. Their methods are easily extended to N dimensions in the symmetric case or even the nonsymmetric case when $\alpha < 1$. This is discussed in section 3. TAYLOR [8] has obtained the correct measure function for the range of the process. Both the techniques and results of these authors will be quite evident in the present paper.

2. Preliminaries

The N-dimensional characteristic function of $X(t)$ has the form $\exp[t\psi(y)]$, where

$$\psi(y) = i(a, y) - \lambda |y|^\alpha \int_{S_N} w_\alpha(y, \theta)\, \mu(d\theta),$$

with $a \in R^N$, $\lambda > 0$,

$$w_\alpha(y, \theta) = [1 - i \operatorname{sgn}(y, \theta) \tan \pi\alpha/2]\, |(y/|y|, \theta)|^\alpha$$

if $\alpha \neq 1$,

$$w_1(y, \theta) = |(y/|y|, \theta)| + (2i/\pi)(y/|y|, \theta) \log |(y, \theta)|,$$

and μ a probability measure on the surface of the unit sphere S_N in R^N [6]. We shall assume that X is a genuine N-dimensional process, i.e. that μ is not supported by a proper subspace of R^N. The element $a \in R^N$ is taken to be zero throughout;

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otherwise the behavior of the linear component of X would be dominant when $\alpha < 1$. The process is called symmetric when μ is uniform. When $\alpha = 1$, we shall consider only the symmetric case. Finally, it is assumed that the process has been defined in such a way that the sample functions $X(t)$ are right continuous and have left limits everywhere.

The density $f(t, x)$ satisfies the scaling property (except in the nonsymmetric case when $\alpha = 1$)

$$f(t, x) = f(rt, r^{1/\alpha} x) r^{N/\alpha}$$

for all $r > 0$, or in terms of the process itself $X(rt)$ and $r^{1/\alpha} X(t)$ have the same distribution. Joint distributions of the process also enjoy this scaling property so that it can be extended to such things as first passage times. (The first passage time out of the sphere of radius ra and r^α times the first passage time out of the sphere of radius a have the same distribution.)

Lemma 2.1. *Let $A(a)$ be a collection of cubes of side a, $a \leqq 1$, in R^N with the property that the number of these cubes which intersect an arbitrary sphere of radius a in R^N is bounded by a constant K which is independent of a and the sphere. (This is the case when the cubes do not overlap too much.) Let $M(a)$ be the number of these cubes hit by the path $X(s)$ at some time $s \in [0, t]$. Then there is a constant c, which is independent of a but not of t, such that*

$$E M(a) \leqq c a^{-\alpha}.$$

Proof. Let $\sigma_0 = 0$ and for $k \geqq 1$, let

$$\tau_k = \tau_k(a) = \inf\{s \geqq \sigma_{k-1}: |X(s) - X(\sigma_{k-1})| > a\}$$
$$\sigma_k = \sigma_k(a) = \min\{\tau_k, \sigma_{k-1} + a^\alpha\}.$$

Then $Y_k = Y_k(a) = \sigma_k - \sigma_{k-1}$ is a sequence of independent, identically distributed random variables. Note that the right continuity of $X(s)$ implies the positivity of Y_k. If $\eta = \eta(a) = \min\{k: \sigma_k \geqq t\}$, then (see, e.g. [5], p. 566)

$$E\eta \cdot E Y_1 = E \sigma_\eta \leqq t + a^\alpha.$$

Now $Y_1(a)$ and $a^\alpha Y_1(1)$ have the same distribution by the scaling property so that $E Y_1(a) = a^\alpha E Y_1(1)$. Thus

$$E\eta(a) \leqq (t + 1)[E Y_1(1)]^{-1} a^{-\alpha}.$$

Finally, we note that $R(t)$ is covered by the spheres of center $X(\sigma_k)$, radius a, for $k = 0, 1, 2, \ldots, \eta - 1$ and that each of these can intersect at most K cubes of A so that $M(a) \leqq K\eta(a)$.

The next lemma is due to Takeuchi [7] in the symmetric case. The proof here must be different since the hitting probabilities can be larger for nonsymmetric processes.

Lemma 2.2. *If S is a sphere of radius a in R^N and X is any stable process of index α with $N > \alpha$ (excluding the nonsymmetric processes when $\alpha = 1$), then there is a constant c such that*

$$P^x[X(t) \in S \quad \text{for some} \quad t \geqq T] \leqq c\left(\frac{a}{T^{1/\alpha}}\right)^{N-\alpha}.$$

Proof. There is no loss in centering S at the origin. We let μ denote the capacitory measure on S. (The definitions and relevant properties of the potential theory used here are given in [8], p. 1234.) Then the hitting probability of S is given by

$$\Phi(y,S) = P^y[X(t) \in S \text{ for some } t \geqq 0] = \int_S \int_0^\infty f(t, z-y)\, dt\, \mu(dz).$$

Therefore

$$\begin{aligned} P^x[X(t) \in S \text{ for some } t \geqq T] &= \int_{R^N} f(T, y-x)\, \Phi(y,S)\, dy \\ &= \int_S \int_0^\infty \int_{R^N} f(T, y-x)\, f(t, z-y)\, dy\, dt\, \mu(dz) \\ &= \int_S \int_0^\infty f(T+t, z-x)\, dt\, \mu(dz) \\ &= \int_S \int_T^\infty f(1, s^{-1/\alpha}(z-x))\, s^{N-\alpha}\, ds\, \mu(dz) \\ &\leq c\,(T^{-1/\alpha})^{N-\alpha}\, \mu(S), \end{aligned}$$

the final inequality being a consequence of the boundedness of $f(1,x)$. Finally $\mu(S) = ca^{N-\alpha}$ by Lemma 3 of [8]

We state as a lemma a remark of B. Fristedt which must be well known and yet its utility has apparently been overlooked.

Lemma 2.3. *Let E be any Borel set in R^N and φ be any measure function. Then*

$$\varphi - m\{\textit{Projection of } E \textit{ on any subspace}\} \leqq \varphi - m(E).$$

Proof. This is just the observation that if one covers E with a collection of spheres, the projection will be covered by the projections of the spheres and the corresponding diameters cannot increase.

3. The Dimension of the Graph

Although the dimension result given in this section is a corollary of the correct measure function result of the next section, it seems worthwhile to consider it separately since the proof is somewhat easier. As noted above the proof is essentially that used by Blumenthal and Getoor [4] in the symmetric, one-dimensional case.

Theorem 3.1. *Let $G(t, \omega)$ be the graph of a stable process of index α in R^N (the nonsymmetric processes with $\alpha = 1$ are excluded). The Hausdorff dimension of $G(t)$ is almost surely* $\max(1, \alpha)$.

Proof. We give the proof here only in the symmetric case, leaving the general result as a corollary of Theorem 4.1. As a consequence of Lemma 2.3, the dimension of the projection of any set E is no larger than the dimension of the set E itself. By projecting the graph on the time axis, we see that its dimension must be at least one. A projection on the range space shows that the dimension must be at least the dimension of the range which is α almost surely [1]. Therefore

$$P[\dim G(t, \omega) \geqq \max(1, \alpha)] = 1.$$

For the other inequality, it suffices to show that if $\beta > \max(1, \alpha)$, then the $x^\beta - m$ of the graph is finite. As in the one-dimensional proof of Blumenthal and Getoor [4], an application of their Theorem 8.4 of [3] to the process $Y(s) = (s, X(s))$ reduces the problem to one of showing that the β-variation of X is finite. This is known to be the case by Theorem 4.1 of [1].

Remarks. The use of the projection argument gives a considerable simplification of the proof of the lower bound for the dimension as given by Blumenthal and Getoor. The present proof of the lower bound is valid in general since the dimension of the range is α even if the process is not symmetric [2]. The given proof of the upper bound is also legitimate when $\alpha < 1$, because when $\alpha < \beta \leqq 1$, the β-variation of X is finite [3], so that one may prove that the x-measure of the graph is finite. However, the question of whether the β-variation is finite in the non-symmetric case with $\max(1, \alpha) < \beta$ is still open so that this proof cannot yet be extended to the non-symmetric processes of index $\alpha \geqq 1$.

4. The Correct Measure Function

The main result is stated in the following

Theorem 4.1. *Let $X(t)$ be a stable process of index α in R^N (the nonsymmetric processes with $\alpha = 1$ are excluded) and*

$$G(t, \omega) = \{(s, X(s, \omega)) \colon 0 \leqq s \leqq t\},$$

the graph of the process up to time t. Let

$$\varphi(h) = \begin{cases} h & \text{if} \quad \alpha \leqq 1 \\ h^\alpha \log\log 1/h & \text{if} \quad \alpha > 1 \end{cases}$$

Then there is a constant $c > 0$ (which is independent of t and ω but may depend on α) such that

$$\varphi - m(G(t)) = ct$$

for all $t \geqq 0$ almost surely. When $\alpha < 1$, the constant $c = 1$.

Proof. We first obtain the lower bound by use of Lemma 2.3. If $\alpha \leqq 1$,

$$\varphi - m(G(t)) \geqq \varphi - m((0, t)) = t$$

by projecting on the time axis. If $\alpha > 1$,

$$\varphi - m(G(t)) \geqq \varphi - m(R(t)) = ct$$

by Theorem 6 of [8], where R denotes the range of the process. For the upper bound, we must consider three cases.

Upper bound, $\alpha < 1$. In this case, an argument involving the variation similar to that used for the upper bound in the last section would suffice if we were not interested in the value of the constant c. But if we are to evaluate the constant, a more economical covering must be used. Let $\varepsilon > 0$ be given and choose γ so that $0 < \gamma < \varepsilon/4$. As in the proof of Lemma 2.1, let $\sigma_0 = 0$ and for $k \geqq 1$,

$$\begin{aligned} \tau_k &= \inf\{s \geqq \sigma_{k-1} \colon |X(s) - X(\sigma_{k-1})| > \gamma\} \\ \sigma_k &= \min\{\tau_k, \sigma_{k-1} + \varepsilon/2\}. \end{aligned}$$

Note that $|X(\tau_k) - X(\sigma_{k-1})| \geqq \gamma$ by the right continuity of the paths. Now if

$$M = \min\{k\colon \sigma_k \geqq t\}$$

and

$$A_k = [\sigma_{k-1}, \sigma_k) \times X[\sigma_{k-1}, \sigma_k),$$

then $\{A_k\}_{k=1}^{M}$ is a cover of $G(t)$. Let

$$\Delta_k = \sup_{u,v \in [\sigma_{k-1}, \sigma_k)} |X(u) - X(v)|.$$

Since

$$|X(u) - X(v)| \leqq |X(u) - X(\sigma_{k-1})| + |X(v) - X(\sigma_{k-1})| \leqq 2\gamma,$$

it follows that $\Delta_k \leqq 2\gamma$. Now

$$\operatorname{diam} A_k \leqq \sigma_k - \sigma_{k-1} + \Delta_k \leqq \varepsilon/2 + 2\gamma < \varepsilon$$

so that our cover is with sets having diameters less than ε. Choose β so that $\alpha < \beta < 1$. When $\sigma_k = \tau_k$, we have

$$\Delta_k \leqq 2\gamma \leqq 2\gamma^{1-\beta} |X(\sigma_k) - X(\sigma_{k-1})|^{\beta},$$

while if $\sigma_k = \sigma_{k-1} + \varepsilon/2$,

$$\Delta_k \leqq 2\gamma = 4\gamma\,\varepsilon^{-1}(\sigma_k - \sigma_{k-1}).$$

In any case, therefore,

$$\Delta_k \leqq 2\gamma^{1-\beta} |X(\sigma_k) - X(\sigma_{k-1})|^{\beta} + 4\gamma\,\varepsilon^{-1}(\sigma_k - \sigma_{k-1}).$$

Using this estimate in the approximating sum,

$$\sum_{k=1}^{M} \operatorname{diam} A_k \leqq \sum_{k=1}^{M} (\sigma_k - \sigma_{k-1}) + \sum_{k=1}^{M} \Delta_k$$

$$\leqq (1 + 4\gamma\,\varepsilon^{-1})(t + \varepsilon/2) + 2\gamma^{1-\beta} \sum_{k=1}^{M} |X(\sigma_k) - X(\sigma_{k-1})|^{\beta}.$$

Now the sum is bounded by the β-variation of X on $[0, t + \varepsilon/2]$ which is finite a.s. by Theorem 4.2 of [3]. Since γ is arbitrary,

$$\varphi - m[G(t)] \leqq t + \varepsilon/2 \quad \text{a.s.}$$

Finally, letting $\varepsilon \to 0$ we have that

$$\varphi - m[G(t)] = t \quad \text{a.s.}$$

when $\alpha < 1$.

Upper bound, $\alpha = 1$ (Symmetric case). Let σ_k, τ_k, Y_k, η be defined as in the proof of Lemma 2.1, and let

$$A_k = [\sigma_{k-1}, \sigma_k) \times X[\sigma_{k-1}, \sigma_k)$$

as above. Then $\{A_k\}_{k=1}^{\eta}$ is a covering of $G(t)$ by sets of diameter at most $3a$. It follows that

$$\varphi - m(G(t)) \leqq \varliminf_{a \to 0} \eta(a) \cdot 3a.$$

By the weak law of large numbers, $\sigma_k(1)/k$ tends to $E\,Y_1(1)$ in probability as k tends to infinity. Now let $\varepsilon > 0$ be given and choose $\delta = E\,Y_1(1)/2$. By a change of scale

$$P\left[\left|\frac{\sigma_k(a)}{k} - a E Y_1(1)\right| \geqq a\,\delta\right] = P\left[\left|\frac{\sigma_k(1)}{k} - E Y_1(1)\right| \geqq \delta\right] \leqq \varepsilon$$

for all $k \geqq K$. On the complement of the first event above,

$$\frac{\sigma_k(\mathrm{a})}{k} \geqq a\,\delta$$

so that for all $a \leqq t(K\,\delta)^{-1}$, we have

$$P[\eta(a) \leqq t(a\,\delta)^{-1} + 1] \geqq 1 - \varepsilon.$$

By letting $\varepsilon_n = 2^{-n}$, there is a sequence $a_n \to 0$ such that

$$P[\eta(a_n) \geqq t(a_n\,\delta)^{-1} + 1] \leqq 2^{-n}.$$

Applying the Borel-Cantelli lemma, we obtain the result that

$$\overline{\lim_{n\to\infty}}\, a_n\,\eta(a_n) \leqq t\,\delta^{-1} \text{ a.s.}$$

This proves that $\varphi - m(G(t))$ is finite a.s. for every $t > 0$. Since it was shown earlier to be positive, the argument of TAYLOR and WENDEL [9], section 7, allows us to conclude that it must have the form $c\,t$ for some positive constant c.

Upper bound, $\alpha > 1$. In this case, one might conjecture that the φ-Hausdorff measures of $G(t)$ and $R(t)$ are equal. But since the actual value of the constant c is unknown even for $R(t)$, we shall make no effort to obtain an "optimal" covering.

As in TAYLOR [8] we consider the collection Λ_n of cubes in R^N of the form

$$\left\{x \in R^N : \frac{j_i - 1}{2^n} \leqq x_i < \frac{j_i + 1}{2^n}, \quad i = 1, 2, \ldots, N\right\},$$

where the j_i are integers. TAYLOR has shown that for all n sufficiently large, there is a covering of $R(t)$ by cubes C_i with the C_i coming from $\bigcup_{j=n}^{6n} \Lambda_j$ such that

$$\sum_i \varphi(\operatorname{diam} C_i) \leqq M,$$

where M is a fixed constant. Now choose an integer k to satisfy

$$k\,\frac{\alpha - 1}{\alpha}(N - \alpha) > 2.$$

We will call a cube C from Λ_m bad if it is hit by X at some time $\tau \leqq t$ and if the time set which X spends in C cannot be covered by k intervals of length $2^{-m+1}\sqrt{N}$.

We can now describe the covering of $G(t)$. First, consider a good cube $C_i \in \Lambda_m$ from Taylor's covering of $R(t)$. Since it is good, either it is not hit by X prior to time t and thus might as well be discarded, or the set of times spent in C_i can be covered by k intervals of length $2^{-m+1}\sqrt{N}$. Form $D_{i1}, D_{i2}, \ldots, D_{ik}$ by taking the Cartesian product of C_i with these k intervals on the time axis. The part of the graph such that X is in a good cube is then covered by the D_{ij} formed in this way, and since $\operatorname{diam} D_{ij} = \sqrt{2}\operatorname{diam} C_i$, we have for all n sufficiently large

$$\sum_{i,j} \varphi(\operatorname{diam} D_{ij}) \leqq 2^\alpha k \sum_i \varphi(\operatorname{diam} C_i) \leqq 2^\alpha k\,M.$$

The rest of the covering of $G(t)$ will consist of all Cartesian products of bad cubes $C_i \in \Lambda_m$ from Taylor's covering of $R(t)$ with intervals of the form

$$\left[\frac{j-1}{2^m}, \frac{j}{2^m}\right], \quad j = 1, 2, \ldots, [t\,2^m] + 1,$$

on the time axis. We shall now prove that the contribution of this part of the sum is negligible.

For a given cube $C \in \Lambda_m$, define a sequence of stopping times as follows: $T_0 = \tau$, the hitting time of C, and for $j \geqq 1$, T_j is the hitting time of the sphere of center $X(\tau)$, radius $2^{-m+1}\sqrt{N}$ starting from $X(T_{j-1} + 2^{-m+1}\sqrt{N})$. Note that if $T_{j-1} = \infty$, then we define $T_j = \infty$ also. Now if $T_k = \infty$, there are k intervals of length $2^{-m+1}\sqrt{N}$, $[T_j, T_j + 2^{-m+1}\sqrt{N}]$, for $j = 0, 1, 2, \ldots, k-1$, which cover the set of times spent in the cube C since the cube is contained in the sphere. Therefore, for all m sufficiently large.

$$\begin{aligned} P[C \text{ is bad}] &= P[C \text{ is hit by time } t] \cdot P[C \text{ is bad} \mid C \text{ is hit by time } t] \\ &\leqq P[C \text{ is hit by time } t] \cdot c \cdot (2^{-m})^{(1-1/\alpha)(N-\alpha)k}, \end{aligned}$$

where Lemma 2.2 has been applied k times to estimate $P[T_k < \infty]$. The constant c here is to be a positive constant which is independent of m but is not the same as in Lemma 2.2. Letting N_m denote the number of bad cubes in Λ_m and M_m the total number of cubes of Λ_m hit by time t, and recalling the manner in which k was chosen, we obtain

$$EN_m \leqq c \cdot 2^{-2m}\, EM_m \leqq c \cdot 2^{-2m} \cdot 2^{m\alpha}$$

by an application of Lemma 2.1. Again the constant c has been changed at the last step to incorporate the constant from Lemma 2.1. Now

$$P[N_m \geqq m^2\, 2^{m\alpha}\, 2^{-2m}] \leqq \frac{c}{m^2}$$

so by the Borel-Cantelli lemma, for almost all ω and all m sufficiently large,

$$N_m \leqq m^2\, 2^{m\alpha}\, 2^{-2m}.$$

For each bad cube of Taylor's covering, there will be $[t\,2^m] + 1$ cubes of diameter $2^{-m}\sqrt{4N+1}$ in R^{N+1} in our covering. Thus the contribution to the sum from all bad cubes in Λ_m will almost surely be no larger than

$$(t\,2^m + 1)\, N_m\, \varphi(2^{-m}\sqrt{4N+1}) \leqq c\, 2^m\, m^2\, 2^{-2m} \log m \leqq c\, m^2\, 2^{-m} \log m$$

for all m sufficiently large. Since the last estimate is summable, we see that with probability one the contribution from all bad cubes in $\bigcup_{m=n}^{6n} \Lambda_m$ will be small when n is large. Thus the φ-Hausdorff measure of $G(t)$ is finite almost surely. Again the argument of Section 7 in Taylor and Wendel [9] concludes the proof by showing that $\varphi - m(G(t)) = c\,t$ a.s. for some constant c.

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