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# On $\sigma$-Finite Invariant Measures *

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## 1. Introduction

In this paper we consider the problem of the existence of a $\sigma$-finite measure which is equivalent to a given measure and invariant with respect to a given automorphism of the measure space. (An automorphism of a measure space is a bijective, measurable, non-singular transformation of the space whose inverse has the same properties.) In Section 2 we state the problem in detail and discuss previous results. In Section 3 we prove two theorems. Theorem 1 describes two necessary and sufficient conditions for the existence of a $\sigma$-finite, invariant, equivalent measure. In Theorem 2 we show that for an automorphism defined almost everywhere on the unit interval satisfying certain conditions there does not exist any $\sigma$-finite, invariant measure that is equivalent to Lebesgue measure. In Section 4 we describe three automorphisms which satisfy the conditions in Theorem 2 and hence do not have $\sigma$-finite, invariant measures which are equivalent to Lebesgue measure.

## 2. Statement of the Problem

By a *measure space* we shall mean a triple $(X, \mathscr{B}, m)$ where $X$ is a set, $\mathscr{B}$ is a $\sigma$-algebra of subsets of $X$, and $m$ is a measure, i.e., an extended real valued, countably additive, non-negative function defined on $\mathscr{B}$ such that $m(\emptyset) = 0$. ($\emptyset$ denotes the empty set.) The sets in $\mathscr{B}$ are called *measurable sets*. The space $(X, \mathscr{B}, m)$ and the measure $m$ are called *finite* if $m(X) < \infty$, and $(X, \mathscr{B}, m)$ and $m$ are called *$\sigma$-finite* if there is a countable collection $\{B_i\}$ of measurable sets whose union is $X$ such that $m(B_i) < \infty$ for $i = 1, 2, 3, \ldots$. If $\mu$ is a second measure defined on $\mathscr{B}$, then $\mu$ is said to be *absolutely continuous* with respect to $m$, written $\mu \ll m$, if $\mu(B) = 0$ whenever $m(B) = 0$. $\mu$ and $m$ are said to be *equivalent*, written $\mu \equiv m$, if they have precisely the same sets of measure zero. Clearly, $\mu \equiv m$ if and only if $\mu \ll m$ and $m \ll \mu$.

The following terminology is used by A. TULCEA in [1].

**Definition 1.** Let $(X, \mathscr{B}, m)$ be a measure space. A transformation $\phi: X \to X$ is called an *automorphism* if it satisfies the following conditions:

i) $\phi$ is one-to-one and onto.

ii) $\phi$ and $\phi^{-1}$ preserve measurable sets, i.e., $B \in \mathscr{B}$ implies $\phi(B) \in \mathscr{B}$ and $\phi^{-1}(B) \in \mathscr{B}$.

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iii) $\phi$ and $\phi^{-1}$ are non-singular, i.e.,

$$m(B) = 0 \quad \text{implies} \quad m(\phi(B)) = m(\phi^{-1}(B)) = 0 .$$

**Definition 2.** Let $(X, \mathscr{B}, m)$ be a measure space, and let $\phi\colon X \to X$ be an automorphism. A measure $\mu$ defined on $\mathscr{B}$ is said to be *invariant* if $\mu(\phi(B)) = \mu(B)$ for each measurable set $B$.

Our problem is the following one. Given a $\sigma$-finite measure space $(X, \mathscr{B}, m)$ and an automorphism $\phi\colon X \to X$, when does there exist a $\sigma$-finite measure $\mu$ defined on $\mathscr{B}$ such that $\mu$ is invariant and $m \ll \mu$? The requirements that $\mu$ be $\sigma$-finite and that $m \ll \mu$ are intended to exclude such trivial invariant measures as the counting measure and zero measure. However, the following proposition shows there is no loss of generality in requiring that $\mu \equiv m$.

**Proposition.** *Let $(X, \mathscr{B}, m)$ be a $\sigma$-finite measure space, and let $\phi\colon X \to X$ be an automorphism. Then there is a $\sigma$-finite invariant measure $\mu$ defined on $\mathscr{B}$ such that $m \ll \mu$ if and only if there is a $\sigma$-finite invariant measure $\mu_0$ defined on $\mathscr{B}$ such that $\mu_0 \equiv m$.*

*Proof.* Suppose $\mu$ is a $\sigma$-finite invariant measure defined on $\mathscr{B}$ and $m \ll \mu$. Let $f$ be a Radon-Nikodym derivative of $m$ with respect to $\mu$. Put $A = \{x \in X \mid f(x) = 0\}$, and

$$B = \bigcup_{j=-\infty}^{+\infty} \phi^j(A) .$$

Clearly, $m(A) = 0$, and by non-singularity $m(B) = 0$. Moreover, $\phi(B) = B$. Put $\mu_0(E) = \mu(E - B)$ for each measurable set $E$. It is easy to verify $\mu_0$ is the desired measure. The converse is clear.

The problem of finding an invariant measure has been thoroughly discussed by P. HALMOS [2] and K. JACOBS [4]. Necessary and sufficient conditions for the existence of a finite, invariant, equivalent measure have been given by E. HOPF [5], Y.N. DOWKER [6] and [7], CALDERON [9], and KAKUTANI and HAJIAN [10]. In 1947 HALMOS [3] used the result in [5] to obtain a necessary and sufficient condition for the existence of a $\sigma$-finite, invariant, equivalent measure. In 1951 Y. N. DOWKER [8] obtained a different condition equivalent to the existence of a $\sigma$-finite, invariant, equivalent measure. However, both these results seemed to be rather difficult to apply, and when they were published it was not yet known that there are automorphisms for which no $\sigma$-finite, equivalent, invariant measures exist. It was not until 1960 that D. S. ORNSTEIN [11] gave the first example of an automorphism which has no $\sigma$-finite, equivalent, invariant measure. Since then other examples of such automorphisms have been given by R. V. CHACON [12] and A. BRUNEL [13]. In what follows we obtain a new necessary and sufficient condition for the existence of a $\sigma$-finite, equivalent, invariant measure (Theorem 1) and use it to give a new example of an automorphism that does not have a $\sigma$-finite, equivalent, invariant measure. The proof (Theorem 2) that our example has the desired property also covers the automorphisms of ORNSTEIN and BRUNEL.

## 3. Main Results

Let $(X, \mathscr{B}, m)$ be a $\sigma$-finite measure space, and let $\phi\colon X \to X$ be an automorphism. We first remark that it is no loss of generality to assume $m$ is finite. To see this, suppose $m(X) = \infty$. Since $m$ is $\sigma$-finite, there is a countable collection $\{B_i\}$ of pairwise disjoint measurable sets such that $0 < m(B_i) < \infty$ whose union is $X$. Define a new measure $m_0$ on $\mathscr{B}$ by the formula

$$m_0(B) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{m(B \cap B_n)}{m(B_n)}.$$

Clearly, $m_0$ is finite and $m_0 \equiv m$. If there is a $\sigma$-finite invariant measure $\mu$ defined on $\mathscr{B}$ such that $\mu \equiv m_0$, then $\mu \equiv m$, also. So we can assume $m$ is finite.

**Definition 3.** Given a measure space $(X, \mathscr{B}, m)$ and an automorphism $\phi: X \to X$, we define a new measure $m\phi^p$ on $\mathscr{B}$ for each integer $p$ by the formula

$$m\phi^p(B) = m(\phi^p(B))$$

for each measurable set $B$. If $\mu$ is any other measure defined on $\mathscr{B}$, the measures $\mu\phi^p$ for each integer $p$ are defined similarly.

It is easy to see that the non-singularity of $\phi$ and $\phi^{-1}$ implies $m \equiv m\phi^p$ for each integer $p$.

We also remark that $\mu$ is an invariant measure if and only if $\mu\phi = \mu$. If $\mu$ is invariant, an easy inductive argument yields $\mu\phi^p = \mu$ for every integer $p$.

We now describe a result of Halmos which will be used in the sequel.

**Definition 4.** Let $(X, \mathscr{B}, m)$ be a measure space. By a *decomposition* of a measurable set $B$, we mean a countable collection $\{B_i\}$ of pairwise disjoint measurable sets whose union is $B$. (We allow finite decompositions. In that case, for some integer $i_0 > 0$, $B_i = \emptyset$ for all $i \geqq i_0$.)

**Definition 5.** Let $(X, \mathscr{B}, m)$ be a measure space, and let $\phi: X \to X$ be an automorphism. If $A$ and $B$ are measurable sets, we say $B$ is a *copy* of $A$, written $A \sim B$, if there are decompositions $\{A_i\}$ and $\{B_i\}$ of $A$ and $B$, respectively, and a sequence of integers $\{n_i\}$ such that $\phi^{n_i}(A_i) = B_i$, for $i = 1, 2, \ldots$.

It is easy to see that $\sim$ is an equivalence relation on $\mathscr{B}$.

**Definition 6.** Let $(X, \mathscr{B}, m)$ and $\phi$ be as in Definition 5. We define a measurable set $E$ to be *unbounded* if there is a measurable subset $A$ of $E$ such that $A \sim E$ and $m(E - A) > 0$. We say $E$ is *bounded* if $E$ is not unbounded. $E$ is *$\sigma$-bounded* if $E$ is the countable union of bounded sets.

**Theorem** (Halmos). *Let $(X, \mathscr{B}, m)$ be a finite measure space, and let $\phi: X \to X$ be an automorphism. Then there is a $\sigma$-finite, invariant measure $\mu$ defined on $\mathscr{B}$ which is equivalent to $m$ if and only if $X$ is $\sigma$-bounded.*

*Proof.* See [3].

We shall need the following lemma on unbounded sets.

**Lemma.** *Let $(X, \mathscr{B}, m)$ be a finite measure space, and let $\phi: X \to X$ be an automorphism. If $E$ is an unbounded measurable set, then there exists a measurable set $B \subset E$ with the following properties:*

i) $m(B) > 0$.

ii) *For any $\varepsilon > 0$ there exists a measurable set $C \subset E$ such that $B \sim C$ and $m(C) < \varepsilon$.*

*Proof:* By hypothesis there is a measurable set $A \subset E$ such that $A \sim E$ and $m(E - A) > 0$. This means there are decompositions $\{E_i\}$ and $\{A_i\}$ of $E$ and $A$, respectively, and integers $\{n_i\}$ such that $\phi^{n_i}(E_i) = A_i$. Define a transformation $\tau: E \to A$ by $\tau(x) = \phi^{n_i}(x)$ for all $x$ in $E_i$, $i = 1, 2, 3 \ldots$. Since $A \subset E$, $\tau^n$ is well-defined for all positive integers $n$. Since $\phi$ is an automorphism, if $D$ is any measurable subset of $E$, then $\tau(D)$ is measurable and $m(D) = 0$ if and only if $m(\tau(D)) = 0$. Observe that $D \sim \tau(D)$. An obvious inductive argument shows the same is also true of $\tau^n(D)$. Now let $B = E - A$. Another easy inductive argument shows that $\{\tau^n(B) \mid n = 1, 2 \ldots\}$ is a pairwise disjoint family of sets all contained in $E$. Since $m(E)$ is finite, $m(\tau^n(B)) \to 0$ as $n \to \infty$. Since $\tau^n(B) \sim B$, the lemma follows.

We shall also need the Lebesgue Density Theorem which we state in the following form.

**Theorem** (LEBESGUE). *Let $B$ be a Lebesgue measurable subset of the real line. Let $m$ be Lebesgue measure. Then almost every point in $B$ has the following property: For each $\varepsilon > 0$ there is $\delta > 0$ such that*

$$\frac{m(B \cap I)}{m(I)} \geqq 1 - \varepsilon$$

*whenever $I$ is an interval, $x \in I$, and $0 < m(I) < \delta$.*

*Proof.* See [14].

We can now state our first result.

**Theorem 1.** *Let $(X, \mathscr{B}, m)$ be a finite measure space, and let $\phi: X \to X$ be an automorphism. Then the following statements are equivalent:*

(a) *There exists a $\sigma$-finite, invariant measure $\mu$ defined on $\mathscr{B}$ which is equivalent to $m$.*

(b) *For every $\varepsilon > 0$ there is a decomposition $\{X_i\}$ of $X$ (which depends on $\varepsilon$) with the following property: for each $i$, for each measurable set $B \subset X_i$, and for each integer $p$ such that $\phi^p(B) \subset X_i$, it is true that*

$$m(B)/(1 + \varepsilon) \leqq m\phi^p(B) \leqq (1 + \varepsilon)m(B).$$

(c) *There is a decomposition $\{Y_i\}$ of $X$ and positive constants $k_i$, for $i = 1, 2, \ldots$, with the following property: for each $i$, for each measurable set $B$ such that $B \subset Y_i$, and for each integer $p$ such that $\phi^p(B) \subset Y_i$, it is true that*

$$k_i m(B) \leqq m\phi^p(B).$$

*Proof.* (a) $\to$ (b): Let $\mu$ be a $\sigma$-finite invariant measure defined such that $\mu \equiv m$. Let $d\mu/dm\,[dm/d\mu]$ be a Radon-Nikodym derivative of $\mu$ with respect to $m$ [$m$ with respect to $\mu$]. The fact that $\mu \equiv m$ implies we can choose $d\mu/dm$ and $dm/d\mu$ so that the following equations hold for each $x$ in $X$.

$$0 < \frac{d\mu}{dm}(x) < \infty \tag{1}$$

$$0 < \frac{dm}{d\mu}(x) < \infty \tag{2}$$

$$\frac{d\mu}{dm}(x) = \left(\frac{dm}{d\mu}(x)\right)^{-1}. \tag{3}$$

Given $\varepsilon > 0$, let $a_n = (1 + \varepsilon)^n$ for $n = 0, \pm 1, \pm 2, \ldots$. Then (because of equation (1)) the collection of sets consisting of

$$F_i = \left\{x \,\middle|\, a_i \leqq \frac{d\mu}{dm}(x) < a_{i+1}\right\},$$

for $i = 0, \pm 1, \pm 2, \ldots$, is a decomposition of $X$. Fix $i$. By equation (3) we also have

$$F_i = \left\{x \,\middle|\, \frac{1}{a_{i+1}} < \frac{dm}{d\mu}(x) \leqq \frac{1}{a_i}\right\}.$$

Suppose $B$ is measurable, $B \subset F_i$, and $p$ is an integer such that $\phi^p(B) \subset F_i$. Then

$$\begin{aligned} m\phi^p(B) &= \int\limits_{\phi^p(B)} \frac{dm}{d\mu}(x)\,d\mu \leqq \left(\frac{1}{a_i}\right) \int\limits_{\phi^p(B)} d\mu \leqq \left(\frac{1}{a_i}\right) \mu\phi^p(B) \leqq \\ &\leqq \left(\frac{1}{a_i}\right) \mu(B) \leqq \frac{1}{a_i} \int\limits_{B} \frac{d\mu}{dm}(x)\,dm \leqq \frac{a_{i+1}}{a_i} m(B) \end{aligned} \tag{4}$$

$$m\phi^p(B) \leqq (1+\varepsilon) m(B).$$

Inequality (4) also holds for $E = \phi^p(B)$ and $\phi^{-p}(E) = B$, since $E \subset F_i$ and $\phi^{-p}(E) \subset F_i$. Hence,

$$m(B) \leqq (1+\varepsilon)\, m\phi^p(B)$$

$$m(B)/(1+\varepsilon) \leqq m\phi^p(B) \tag{5}$$

Since this is true for each $i$, we conclude (a) implies (b).

(b) $\rightarrow$ (c): Clear.

(c) $\rightarrow$ (a): Let $\{Y_i\}$ be the decomposition of (c). We assert $Y_i$ is a bounded set for each $i$. To see this, suppose $B$ and $C$ are measurable sets, $B \subset Y_i$, $C \subset Y_i$, and $C$ is a copy of $B$. Then there exist decompositions $\{B_j\}$ and $\{C_j\}$ of $B$ and $C$, respectively, and integers $\{n_j\}$ such that

$$\phi^{n_j}(B_j) = C_j \quad \text{for} \quad j = 1, 2, \ldots.$$

So

$$m(C) = \sum_{j=1}^{\infty} m(C_j) = \sum_{j=1}^{\infty} m\phi^{n_j}(B_j).$$

By hypothesis, $m\phi^{n_j}(B_j) \geqq k_i m(B_j)$ for each $j$, so

$$m(C) \geqq \sum_{j=1}^{\infty} (k_i) m(B_j) \geqq k_i m(B).$$

It now follows from the lemma that $Y_i$ must be bounded for each $i$. Hence, $X$ is $\sigma$-bounded. By the theorem of Halmos quoted above, (a) must hold. This completes the proof.

Remark 1.1. Theorem 1 generalizes a result of G. D. Birkhoff and P. Smith [15] (see also [5]) which deals with the case of finite invariant measures.

Remark 1.2. The sets $Y_i$ and numbers $k_i$ for $i = 1, 2, \ldots$ in (c) of Theorem 1 also satisfy the following: for each $i$, if $B$ is measurable, $B \subset Y_i$, and $p$ is an integer such that $\phi^p(B) \subset Y_i$, then

$$m\phi^p(B) \leqq \frac{1}{k_i} m(B).$$

Conversely, this condition implies the one in (c). This is proved in the same way inequality (5) was obtained from inequality (4).

**Corollary.** *Let $(X, \mathscr{B}, m)$ and $\phi$ be as in Theorem 1. Suppose $m(X) > 0$ and that there is a $\sigma$-finite, invariant measure $\mu$ defined on $\mathscr{B}$ such that $\mu \equiv m$. Then for each $\varepsilon > 0$ there is a measurable set $E$ (depending on $\varepsilon$) such that $m(E) > 0$ and for every integer $p$*

$$\frac{1}{1+\varepsilon} \leqq \frac{dm\phi^p}{dm}(x) \leqq 1+\varepsilon$$

*holds a.e. on $E \cap \phi^{-p}(E)$. ($dm\phi^p/dm$ is the Radon-Nikodym derivative of $m\phi^p$ with respect to $m$).*

*Proof.* Given $\varepsilon > 0$, let $\{X_i\}$ be the decomposition of (b) of Theorem 1. $m(X) > 0$ implies at least one of the sets $X_i$ must satisfy $m(X_i) > 0$. Let $E$ be that set. For any measurable set $B \subset E \cap \phi^{-p}(E)$ we have

$$m(B)/(1+\varepsilon) \leqq m\phi^p(B) \leqq (1+\varepsilon)\, m(B)$$

since $B \subset E$ and $\phi^p(B) \subset E$. This means

$$\int_B \frac{1}{1+\varepsilon}\, dm \leqq \int_B \frac{dm\phi^p}{dm}(x)\, dm \leqq \int_B (1+\varepsilon)\, dm\,.$$

Since this is true for every measurable $B \subset E \cap \phi^{-p}(E)$,

$$\frac{1}{1+\varepsilon} \leqq \frac{dm\phi^p}{dm}(x) \leqq (1+\varepsilon)$$

holds a.e. on $E \cap \phi^{-p}(E)$. This is true for each integer $p$, Q.E.D.

We now use this corollary to obtain a new technique for showing that certain automorphisms do not have a $\sigma$-finite, invariant, equivalent measure.

**Theorem 2.** *Let $X$ be the unit interval $[0, 1]$, let $\mathscr{B}$ be the Lebesgue measurable subsets of $[0, 1]$, and let $m$ be Lebesgue measure on $\mathscr{B}$. Let $\phi\colon X \to X$ be an automorphism. Suppose there exists a family $\mathscr{I}$ of intervals contained in $X$ with the following properties:*

(a) *For almost all $x$ in $X$, and for every $\delta > 0$, there is an $I$ in $\mathscr{I}$ such that $x \in I$ and $0 < m(I) < \delta$.*

(b) *There are positive numbers $\beta_1$, $\beta_2$, and $\alpha$ with $0 < \alpha < 1$ such that for every $I$ in $\mathscr{I}$ there are measurable sets $A, B_1, \ldots, B_n$ contained in $I$ satisfying*

(b1) $B_i \cap B_j = \emptyset$ for $1 \leqq i < j \leqq n$

(b2) *$m(A) \geqq \beta_1 m(I)$ and $m(B) = \beta_2 m(I)$ where $B = \bigcup_{i=1}^{n} B_i$*

(b3) *$m(B_i) \leqq \alpha m(A)$ for $i = 1, \ldots, n$.*

(b4) *For each $i = 1, \ldots, n$, there is an integer $p_i$ such that $\phi^{p_i}(A) = B_i$, and $dm\phi^{p_i}/dm$ is constant on $A$. (The constant is necessarily $m(B_i)/m(A) \leqq \alpha$.)*

*Then there is no $\sigma$-finite invariant measure $\mu$ defined on $\mathscr{B}$ such that $\mu \equiv m$.*

*Proof.* If such a $\mu$ did exist, then by the corollary there would be a measurable set $E$ with $m(E) > 0$ such that for every integer $p$

$$\frac{\alpha+1}{2} \leqq \frac{dm\phi^p}{dm}(x) \leqq \frac{2}{\alpha+1} \tag{6}$$

holds a.e. on $E \cap \phi^{-p}(E)$. (Take $\varepsilon = (1-\alpha)/(1+\alpha)$ in the corollary.) We shall show that for any measurable set $E$ of positive measure there is at least one $p$ such that equation (6) does not hold a.e. on $E \cap \phi^{-p}(E)$.

Choose $\varepsilon$ such that

$$0 < \varepsilon < \frac{\beta_2\beta_1}{\beta_2+1}.$$

Suppose $E$ is measurable and $m(E) > 0$. By the Lebesgue Density Theorem and condition (a) there is an $I$ in $\mathscr{I}$ such that

$$m(E \cap I) \geqq (1-\varepsilon)m(I). \tag{7}$$

This implies

$$m(I - E) \leqq \varepsilon m(I).$$

So then by (b2)

$$m(A \cap E) \geqq (\beta_1 - \varepsilon)m(I)$$
$$m(B \cap E) \geqq (\beta_2 - \varepsilon)m(I),$$

where $A$ and $B$ are the sets referred to in (b2). From (b1) and (b4) we conclude that

$$\begin{aligned} m\left(\bigcup_{i=1}^{n}\phi^{p_i}(E) \cap B_i\right) &= \sum_{i=1}^{n} m(\phi^{p_i}(E) \cap B_i) = \sum_{i=1}^{n} m\phi^{p_i}(E \cap A) \\ &= \sum_{i=1}^{n} \frac{m(B_i)}{m(A)} m(E \cap A) = \frac{m(B)}{m(A)} m(E \cap A) \end{aligned}$$

$$m\left(\bigcup_{i=1}^{n}\phi^{p_i}(E) \cap B_i\right) \geqq \frac{\beta_2 m(I)}{m(I)}(\beta_1 - \varepsilon)m(I) \geqq \beta_2(\beta_1-\varepsilon)m(I).$$

This means

$$\begin{aligned} m\left(\bigcup_{i=1}^{n}\phi^{p_i}(E) \cap B_i\right) + m(E \cap B) &\geqq \beta_2(\beta_1-\varepsilon)m(I) + (\beta_2-\varepsilon)m(I) \\ &\geqq (\beta_2\beta_1 - \varepsilon\beta_2 + \beta_2 - \varepsilon)m(I) > \beta_2 m(I) = m(B) \end{aligned}$$

by our choice of $\varepsilon$. Since $E \cap B$ and $\bigcup_{i=1}^{n}(\phi^{p_i}(E) \cap B_i)$ are both subsets of $B$, this last inequality implies

$$m\left(\left[\bigcup_{i=1}^{n}\phi^{p_i}(E) \cap B_i\right] \cap E \cap B\right) > 0.$$

So for some $j$

$$m(\phi^{p_j}(E) \cap B_j \cap E) > 0.$$

Apply $\phi^{-p_j}$ to this set and use non-singularity to obtain

$$m(E \cap \phi^{-p_j}(E) \cap A) > 0.$$

But $dm\phi^{p_j}/dm(x) \leqq \alpha < (\alpha+1)/2$ on $A$. It follows that inequality (6) does not hold a.e. on $E \cap \phi^{-p}(E)$ when $p = p_j$. This completes the proof.

Remark 2.1. The conclusion of Theorem 2 is still true if the hypothesis is modified in the following way. First, let $X = [0, 1]$-$N$ where $N$ is a set of Lebesgue measure 0, let $\mathscr{B}$ be the Lebesgue measurable subsets of $X$, and let $m$ be Lebesgue measure restricted to $\mathscr{B}$. Second, interpret the word interval to mean a set of the form $I_0 \cap X$, where $I_0$ is an interval in the usual sense. The proof remains the same; in particular, the Lebesgue Density Theorem can still be used to obtain the inequality (7). The details are left to the reader. It is this modified version of the Theorem that we shall use in the next section.

## 4. Applications

It is not obvious how Theorem 2 can be used to show the existence of automorphisms which do not have $\sigma$-finite, invariant, equivalent measures. However, the hypotheses of Theorem 2 were chosen with a specific automorphism (Example 1) in mind. It turned out that the automorphisms of Ornstein and Brunel also satisfy these hypotheses. We shall now describe these three automorphisms.

Example 1. We shall first define a transformation $\phi$ from $[0, 1)$ onto $(0, 1)$. Suppose $0 < \alpha < 1$. We define $x_n = 1 - [\alpha/(\alpha+1)]^{n+1}$ for $n = 1, 2, \ldots$, and $x_{-n} = 1/(\alpha+1)^{n+1}$ for $n = 0, 1, \ldots$. Then $\{x_n \mid -\infty < n < \infty\}$ is a strictly increasing sequence in $[0, 1]$ with $\lim_{n\to\infty} x_n = 1, \lim_{n\to-\infty} x_n = 0$. We define $\phi$ on $[0, x_0) = [0, 1/(\alpha+1))$ by

$$\begin{aligned}\phi(x) &= \left(\frac{1-x_0}{x_0}\right) x + x_0 \\ &= \alpha x + \frac{1}{\alpha+1},\end{aligned}$$

so that $\phi$ maps $[0, x_0)$ onto $[x_0, 1)$. We define $\phi$ on $[x_n, x_{n+1})$ by

$$\begin{aligned}\phi(x) &= \left(\frac{x_{-n} - x_{-(n+1)}}{x_{n+1} - x_n}\right)(x - x_n) + x_{-(n+1)} \\ &= \alpha^{-n}\left(x - 1 + \left(\frac{\alpha}{\alpha+1}\right)^{n+1}\right) + \frac{1}{(\alpha+1)^{n+2}},\end{aligned}$$

so that $\phi$ maps $[x_n, x_{n+1})$ onto $[x_{-(n+1)}, x_{-n})$. It is now easy to see that $\phi$ maps $[0, 1)$ onto $(0, 1)$, that $\phi$ is one-to-one, and that $\phi$ and $\phi^{-1}$ are measurable and non-singular.

Now let

$$X = [0, 1) - \bigcup_{j=0}^{\infty} \phi^j(\{0\}).$$

Let $\mathscr{B}$ be the Lebesgue measurable subsets of $X$, and let $m$ be Lebesgue measure on $\mathscr{B}$. Then $\phi(X) = X$, so the restriction of $\phi$ to $X$, which we also denote by $\phi$, is an automorphism of $(X, \mathscr{B}, m)$.

We now interpret the word interval to mean a set of the form $J \cap X$ where $J$ is an interval in the usual sense. We shall show that $\phi$ has been defined such that for each non-negative integer $k$ the space $X$ is the union of $2^{k+1}$ pairwise disjoint intervals, $I_j^k$, $j = 1, 2, \ldots, 2^{k+1}$, with the property that $\phi(I_j^k) = I_{j+1}^k$ and $dm\phi/dm$ is constant on $I_j^k$ for $j = 1, 2, \ldots, 2^{k+1} - 1$. These intervals will form the set $\mathscr{I}$ of Theorem 2.

Let $I_1^0 = [0, x_0) \cap X$ and $I_2^0 = [x_0, 1) \cap X$. Then $\phi(I_1^0) = I_2^0$. Suppose we have shown that there are $2^{k+1}$ intervals, $I_1^k, \ldots, I_n^k$, (letting $n = 2^{k+1}$ to simplify notation) such that the following are satisfied:

i) $I_1^k = [0, x_{-k}) \cap X$ and $I_n^k = [x_k, 1) \cap X$.

ii) $\{I_j^k \mid j = 1, \ldots, n\}$ is a decomposition of $X$.

iii) $\phi(I_j^k) = I_{j+1}^k$ for $j = 1, \ldots, n-1$.

iv) $I_j^k$ is a subset of $[0, x_0)$ or $[x_i, x_{i+1})$ for some integer $i \geqq 0$, for $j = 1, \ldots, n-1$.

Define $I_{1,1}^k = [0, x_{-(k+1)}) \cap X$ and $I_{1,2}^k = [x_{-(k+1)}, x_{-k}) \cap X$. Define $I_{j,i}^k = \phi^{j-1}(I_{1,i}^k)$ for $j = 1, \ldots, n$ and $i = 1, 2$. It follows from iii) that $\phi^{n-1}(I_1^k) = I_n^k$. It follows from iii), iv), and the definition of $\phi$ that

$$\phi^{n-1}(x) = ax + b \tag{8}$$

for $x$ in $I_1^k$, where $a$ and $b$ are real constants, $a > 0$. Hence $I_{n,1}^k = \phi^{n-1}(I_{1,1}^k) = [x_k, y) \cap X$ and $I_{n,2}^k = \phi^{n-1}(I_{1,2}^k) = [y, 1) \cap X$ for some $y$ such that $x_k < y < 1$. Now $x_{-(k+1)}$ satisfies

$$\frac{x_{-k} - x_{-(k+1)}}{x_{-(k+1)}} = \alpha .$$

Hence

$$\frac{m(I_{1,2}^k)}{m(I_{1,1}^k)} = \alpha .$$

Equation (8) implies $\dfrac{dm\,\phi^{n-1}}{dm}(x) = a$ for $x$ in $I_{1,1}^k$. So

$$\frac{m(I_{n,2}^k)}{m(I_{n,1}^k)} = \frac{m\phi^{n-1}(I_{1,2}^k)}{m\phi^{n-1}(I_{1,1}^k)} = \frac{am(I_{1,2}^k)}{am(I_{1,1}^k)} = \alpha .$$

This implies

$$\frac{1-y}{y-x_k} = \alpha ,$$

which implies $y = x_{k+1}$. So $I_{n,1}^k = [x_k, x_{k+1}) \cap X$ and $I_{n,2}^k = [x_{k+1}, 1) \cap X$. It follows from the definition of $\phi$ that $\phi(I_{n,1}^k) = I_{1,2}^k$. So now, if we define $I_j^{k+1} = I_{j,1}^k$ and $I_{n+j}^{k+1} = I_{j,2}^k$ for $j = 1, \ldots, n$, then $\phi(I_j^{k+1}) = I_{j+1}^{k+1}$ for $j = 1, 2, \ldots, 2n-1$. (Note $2n = 2^{k+2}$.) Hence $\{I_j^{k+1} \mid j = 1, \ldots, 2^{k+2}\}$ is a set of $2^{k+2}$ intervals that satisfy conditions i)—iv) with $k$ replaced by $k+1$. It follows by induction that for each non-negative integer $k$ there is a set $\{I_j^k \mid j = 1, \ldots, 2^{k+1}\}$ of intervals that satisfy conditions i)—iv).

Let $I_j^k$ be one of the intervals we have just constructed. Then $I_j^k = I_{j,1}^k \cup I_{j,2}^k$ and $I_{j,1}^k \cap I_{j,2}^k = \emptyset$. Also, $\phi^p(I_{j,1}^k) = I_{j,2}^k$ where $p = 2^{k+1}$. It follows from conditions iii) and iv) that $dm\,\phi^{j-1}/dm$ is constant on $I_1^k$, and this implies

$$\frac{m(I_{j,2}^k)}{m(I_{j,1}^k)} = \frac{m(I_{1,2}^k)}{m(I_{1,1}^k)} = \alpha .$$

Hence,

$$m(I_{j,1}^k) = \left(\frac{1}{\alpha+1}\right) m(I_j^k) \tag{9}$$

$$m(I_{j,2}^k) = \left(\frac{\alpha}{\alpha+1}\right) m(I_j^k) \tag{10}$$

$$m(I_{j,2}^k) = \alpha m(I_{j,1}^k) . \tag{11}$$

It also follows from conditions iii) and iv) that $dm\,\phi^p/dm$ is constant on $I_{j,1}^k$. Equation (11) implies this constant is $\alpha$. Also, it follows from equations (9) and (10) that

$$\max_j m(I_j^{k+1}) = \frac{1}{\alpha+1} \max_j m(I_j^k) ,$$

and hence

$$\lim_{k\to\infty} \max_j m(I_j^k) = 0 .$$

It is now easy to see that the automorphism $\phi$ satisfies the hypothesis of Theorem 2 as modified in Remark 2.1. Let $\{I_j^k \mid j = 1, \ldots, 2^{k+1}; k = 0, 1, \ldots\}$ be the family $\mathscr{I}$. For each $I_j^k$ in $\mathscr{I}$, let $A = I_{j,1}^k$ and $B = B_1 = I_{j,2}^k$. Let $\alpha$ be the $\alpha$ used in the construction of $\phi$, and let $\beta_1 = 1/(\alpha + 1)$ and let $\beta_2 = \alpha/(\alpha + 1)$. The integer $p_1 = 2^{k+1}$. By Theorem 2 there is no $\sigma$-finite, invariant measure defined on $\mathscr{B}$ which is equivalent to Lebesgue measure.

Example 2. We now describe the automorphism constructed by D. S. ORNSTEIN in [11]. Let $I_1^1 = (1/2, 1]$ and $I_2^1 = (0, 1/2]$. Define $T$ on $I_1^1$ by $T(x) = x - 1/2$, so that $T(I_1^1) = I_2^1$. If $J_1$ and $J_2$ are intervals, we call the map of the form $x \to ax + b$, where $a$ and $b$ are real constants, $a > 0$, which maps $J_1$ onto $J_2$, the *affine* map of $J_1$ onto $J_2$. Suppose we have constructed $K_N$ pairwise disjoint intervals whose union is (0, 1]. Call them $I_1^N, \ldots, I_{K_N}^N$. (All intervals in this example are assumed to be left open and right closed.) Suppose we have defined $T$ on

$$\bigcup_{j=1}^{K_N-1} I_j^N$$

so that the restriction of $T$ to $I_j^N$ is the affine map of $I_j^N$ onto $I_{j+1}^N$, for $j = 1, \ldots, K_N - 1$. We shall now define $T$ on at least half of $I_{K_N}^N$. To do this, divide $I_1^N$ into $K_N^1$ intervals as follows. Let $I_{1,1}^N$ be the left half of $I_1^N$, and let $I_{1,j}^N$, $j = 2, \ldots, K_N^1$, be $K_N^1 - 1$ pairwise disjoint intervals of equal length whose union is $I_1^N - I_{1,1}^N$. The number $K_N^1$ is chosen so that $K_N^1 \geqq 2$ and

$$m(T^{k-1}(I_{1,j}^N)) \leqq \frac{1}{(100)NK_N} \tag{12}$$

for $k = 1, 2, \ldots, K_N$ and $j = 2, 3, \ldots, K_N^1$. ($m$ denotes Lebesgue measure.) Now define $I_{k,j}^N = T^{k-1}(I_{1,j}^N)$ for $k = 1, \ldots, K_N$ and $j = 1, 2, \ldots, K_N^1$. Define $T$ on $I_{K_N,j}^N$ by putting $T$ equal to the affine map of $I_{K_N,j}^N$ onto $I_{1,j+1}^N$ for $j = 1, 2, \ldots, K_N^1 - 1$. Now define $I_{k,j}^N = I_{K_N(j-1)+k}^{N+1}$ for $k = 1, 2, \ldots, K_N^1$ and $j = 1, 2, \ldots, K_N^1$. Then $\{I_i^{N+1} \mid i = 1, 2, \ldots, K_N K_N^1\}$ is a set of $K_{N+1} = K_N K_N^1$ pairwise disjoint intervals whose union is (0, 1] such that $T$ is defined on

$$\bigcup_{j=1}^{K_{N+1}-1} I_j^{N+1}$$

and the restriction of $T$ to $I_j^{N+1}$ is the affine map of $I_j^{N+1}$ onto $I_{j+1}^{N+1}$ for $j = 1$ $2, \ldots, K_{N+1} - 1$. This procedure, after a countable number of steps, defines $T$ on (0, 1] in such a way that the range of $T$ is (0, 1). Note that it is possible to give explicit formulas for $T$ as we did for $\phi$ in Example 1, but the formulas for $T$ would be rather complicated.

Let

$$X = (0, 1] - \bigcup_{k=0}^{\infty} T^k(\{1\}) .$$

Let $\mathscr{B}$ be the Lebesgue measurable subsets of $X$, and let $m$ be Lebesgue measure restricted to $\mathscr{B}$. Then it is easy to verify that $T$ is an automorphism of $(X, \mathscr{B}, m)$.

We remark that inequality (12) implies that $K_N^1 \geqq 3$ for each $N$. To see this, observe that $m(I_1^N) = 1/2^N$ for each $N$. If $K_N^1 = 2$ for some $N$, then $m(I_{1,2}^N) = 1/2^{N+1}$. Then (12) would not be satisfied for $j = 2$ and $k = 1$, since clearly $K_N \geqq 2^N$ for each $N$. So $K_N^1 \geqq 3$ for each $N$. In fact, these remarks show that $K_N^1$ must tend to infinity very rapidly as $N$ tends to infinity. ORNSTEIN used (12) in his proof that there is no $\sigma$-finite, invariant measure for $T$ which is equivalent to Lebesgue measure. However, in order for $T$ to satisfy the hypotheses of Theorem 2, it is only necessary that $K_N^1 \geqq 3$ for each $N$. That is, at each stage of the construction of $T$, when $I_1^N$ is divided into smaller intervals, it is only necessary that the right half of $I_1^N$ be divided into 2 or more parts.

We now show $T$ satisfies the hypotheses of Theorem 2 as modified in Remark 2.1. We let $\mathscr{I}$ be the collection $\{I_j^N \mid j = 1, \ldots, K_N; N = 1, 2, \ldots\}$ of intervals. (Interval is now interpreted as in Remark 2.1.) It follows from the construction of $T$ that for each $N$ and $j$

$$m(I_{j,1}^N) = \tfrac{1}{2} m(I_j^N)$$

$$m\left(\bigcup_{i=2}^{K_N^1} I_j^N\right) = \tfrac{1}{2} m(I_j^N)$$

and since $K_N^1 \geqq 3$

$$m(I_{j,i}^N) \leqq \tfrac{1}{2} m(I_{j,1}^N) \quad \text{for} \quad i = 2, \ldots, K_N^1 .$$

So for each $I_j^N$ we let $A = I_{j,1}^N$ and $B_i = I_{j,i+1}^N$ for $i = 1, \ldots, K_N^1 - 1$. We take $\alpha = \beta_1 = \beta_2 = 1/2$. The integers $p_i = i K_N$ for $i = 1, \ldots, K_N^1 - 1$. Finally, we note that

$$\max_j m(I_j^{N+1}) = \tfrac{1}{2} \max_j m(I_j^N) .$$

Therefore

$$\lim_{N\to\infty} \max_j m(I_j^N) = 0 .$$

It follows from. Theorem 2 that $T$ has no $\sigma$-finite, invariant measure defined on $\mathscr{B}$ which is equivalent to Lebesgue measure.

Example 3. We now describe the automorphism constructed by A. BRUNEL. The method of construction is similar to the one used in the preceding example. Let $I_1^1 = [0, 1/4)$, $I_2^1 = [1/4, 3/4)$, and $I_3^1 = [3/4, 1)$. Define $\psi$ on $I_1^1$ so that $\psi$ is the affine map of $I_1^1$ onto $I_2^1$, and define $\psi$ on $I_2^1$ so that $\psi$ is the affine map of $I_2^1$ onto $I_3^1$. Now suppose we have constructed $3^k$ pairwise disjoint intervals whose union is $[0, 1)$. Call them $I_1^k, \ldots, I_n^k$ (where $n = 3^k$). Suppose we have defined $\psi$ on $\bigcup_{j=1}^{n-1} I_j^k$ in such a way that the restriction of $\psi$ to $I_j^k$ is the affine map of $I_j^k$ onto $I_{j+1}^k$ for $j = 1, \ldots, n-1$. (Each of these intervals is assumed to be left closed and right open.) We shall now define $\psi$ on 3/4 of $I_n^k$. To do this, divide $I_1^k$ into 3 intervals as follows: let $I_{1,1}^k$ be the left quarter of $I_1^k$, let $I_{1,2}^k$ be the middle half of $I_1^k$, and let $I_{1,3}^k$ be the right quarter of $I_1^k$. Now define $I_{j,i}^k = \psi^{j-1}(I_{1,i}^k)$ for $i = 1, 2, 3$, and $j = 1, \ldots, n$. Define $\psi$ on $I_{n,i}^k$ by putting $\psi$ equal to the affine map of $I_{n,i}^k$ onto $I_{1,i+1}^k$ for $i = 1$ and 2. Now define $I_{j,i}^k = I_{n(i-1)+j}^{k+1}$ for $i = 1, 2, 3$, and $j = 1, \ldots, n$. Then $\{I_j^{k+1} \mid j = 1, \ldots, 3n\}$ is a family of $3^{k+1}$ pairwise disjoint intervals whose union is $[0, 1)$, and $\psi$ is defined on

$$\bigcup_{j=1}^{3^{k+1}-1} I_j^{k+1}$$

so that the restriction of $\psi$ to $I_j^{k+1}$ is the affine map of $I_j^{k+1}$ onto $I_{j+1}^{k+1}$ for $j = 1$ $2, \ldots, 3^{k+1} - 1$. This procedure, after a countable number of steps, defines $\psi$ on $[0, 1)$ such that the range of $\psi$ is $(0, 1)$. We remark that Brunel did not define $\psi$ by this procedure. He defined $\psi$ by describing its graph. It is also possible to give explicit formulas for $\psi$ as we did for $\phi$ in Example 1.

Now let

$$X = [0,1) - \bigcup_{j=0}^{\infty} \psi^j(\{0\}) .$$

Let $\mathscr{B}$ be the Lebesgue measurable subsets of $X$, and let $m$ be Lebesgue measure on $\mathscr{B}$. Then $\psi$ is an automorphism of $(X, \mathscr{B}, m)$.

We now show that $\psi$ satisfies the hypotheses of Theorem 2 as modified in Remark 2.1. We let $\mathscr{I} = \{I_j^k \mid j = 1, \ldots, 3^k;\ k = 1, 2, 3, \ldots\}$. (As in Examples 1 and 2, we shall now denote $I_j^k \cap X$ by $I_j^k$.) It follows from the construction of $\psi$ that for each $k$ and $j$ we have

$$m(I_{j,2}^k) = \tfrac{1}{2}\, m(I_j^k)$$
$$m(I_{j,3}^k) = \tfrac{1}{4}\, m(I_j^k)$$

and hence

$$m(I_{j,3}^k) = \tfrac{1}{2} m(I_{j,2}^k) .$$

So for each $I_j^k$ we let $A = I_{j,2}^k$ and $B = B_1 = I_{j,3}^k$. We take $\alpha = 1/2 = \beta_1$ and $\beta_2 = 1/4$. The integer $p_1 = 3^k$. Finally we note that

$$\max_j m(I_j^{k+1}) = \tfrac{1}{2} \max_j m(I_j^k)$$

so that

$$\lim_{k\to\infty} \max_j m(I_j^k) = 0 .$$

It now follows from Theorem 2 that there is no $\sigma$-finite, invariant measure for $\psi$ which is defined on $\mathscr{B}$ which is equivalent to Lebesgue measure.

Remarks. In Examples 1 and 2 we have actually defined classes of automorphisms, since in Example 1 we get a different automorphism for each choice of $\alpha$ in $(0, 1)$, and in Example 2 we get a different automorphism for each choice of the sequence $\{K_N^1\}$. Also, the construction used in Example 3 can be modified so as to produce a class of automorphisms of which $\psi$ is the simplest example. Other more or less technical modifications are possible.

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