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Probability Theory and Related Fields

Strong Approximations for Partial Sums of I.I.D. *B*-valued R.V.'s in the Domain of Attraction of a Gaussian Law

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Summary. We obtain a strong approximation theorem for partial sums of i.i.d. d-dimensional r.v.'s with possibly infinite second moments. Using this result, we can extend Philipp's strong invariance principle for partial sums of i.i.d. B-valued r.v.'s satisfying the central limit theorem to B-valued r.v.'s which are only in the domain of attraction of a Gaussian law. This new strong invariance principle implies a compact as well as a functional law of the iterated logarithm which improve some recent results of Kuelbs (1985).

1. Introduction and Main Results

Let B denote a real separable Banach space with norm $\|\cdot\|$. Let $X\colon \Omega \to B$ be a random variable, defined on a p-space $(\Omega, \mathscr{A}, P)$. Suppose that X satisfies the Central Limit Theorem (CLT). This means that there exists a (nondegenerate) Gaussian mean zero r.v. Y such that

$$\mathscr{L}\left(\sum_1^n X_k/\sqrt{n}\right) \text{ converges weakly to } \mu = \mathscr{L}(Y), \tag{1.1}$$

whenever $\{X_n\}$ is a sequence of i.i.d. r.v.'s with common law $\mathscr{L}(X)$.

Let $H_\mu \subseteq B$ be the reproducing kernel Hilbert space determined by the covariance structure of μ, and denote by K its unit ball which is known to be a compact subset of B. For detailed definitions see Kuelbs (1976), Lemma 2.1. The following compact law of the iterated logarithm (CLIL) was obtained independently of each other by Heinkel (1979) and Goodman et al. (1981).

Theorem A. *Assume* (1.1). *Let* $\{X_n\}$ *be a sequence of i.i.d. r.v.'s with the same distribution as X. With probability one,* $\left\{\sum_1^n X_k/\sqrt{2nL_2 n}\colon n \in \mathbb{N}\right\}$ *is relatively compact in B, and its limit set coincides with K, iff*

$$E[\|X\|^2/L_2\|X\|] < \infty. \tag{1.2}$$

Here and in the following Lx denotes $\log(\max(x, e))$, whereas $L_2 x$ stands for $L(Lx)$, $x \geqq 0$.

Moreover, Philipp (1979) showed that CLT plus CLIL implies a strong invariance principle with error term $o(\sqrt{n L_2 n})$.

Combining this result with Theorem A, one obtains the following basic strong invariance principle for partial sums of i.i.d. B-valued r.v.'s satisfying CLT.

Theorem B. *Assume* (1.1). *One can construct a p-space* $(\Omega_0, \mathscr{A}_0, P_0)$ *and two sequences of i.i.d. r.v.'s* $\{X_n\}$, $\{Y_n\}$ *with the same distribution as* X *and* Y, *respectively, such that*

$$\sum_1^n X_k - \sum_1^n Y_k = o(\sqrt{n L_2 n}) \quad a.s. \tag{1.3}$$

iff (1.2) *holds.*

Since $\{Y_n\}$ is a sequence of i.i.d. Gaussian r.v.'s, $\{Y_n\}$ satisfies a functional law of the iterated logarithm. Therefore, it immediately follows from (1.3) that $\{X_n\}$ satisfies not only the CLIL but also the functional law of the iterated logarithm.

It is now of interest to find analogous results if X is only in the domain of attraction of Y, i.e., if

$$\mathscr{L}\left(\sum_1^n X_k/a_n\right) \text{ converges weakly to } \mu = \mathscr{L}(Y) \tag{1.4}$$

for some sequence $a_n \uparrow \infty$.

Kuelbs (1985) obtained a compact as well as a functional law of the iterated logarithm under the assumption (1.4). The main purpose of the present paper is to show that even more is possible: We give an extension of the strong invariance principle (1.3) to B-valued r.v.'s in the domain of attraction of a Gaussian law. Using this general theorem, we obtain a compact and a functional law of the iterated logarithm which improve somewhat upon the above mentioned results of Kuelbs (1985).

Theorem 1. *Let* X *be a* B*-valued mean zero r.v. satisfying* (1.4). *One can construct a p-space* $(\Omega_0, \mathscr{A}_0, P_0)$ *and two sequences of independent r.v.'s* $\{X_n\}$, $\{Y_n\}$ *such that* $\mathscr{L}(X_n) = \mathscr{L}(X)$, $\mathscr{L}(Y_n) = \mathscr{L}(\sigma_n Y)$, *where* $\sigma_n^2 := b_{[nL_2 n]}$, $b_n := a_n^2/n$, $n \in \mathbb{N}$, *and*

$$\sum_1^n X_k - \sum_1^n Y_k = o(a_{[nL_2 n]}) \quad a.s. \tag{1.5}$$

iff

$$\sum_1^\infty P(\|X\| > a_{[nL_2 n]}) < \infty. \tag{1.6}$$

Corollary 1. *Let* X *be as in Theorem* 1. *Assume* (1.6). *If* $\{X_n\}$ *is a sequence of i.i.d. r.v.'s with the same distribution as* X, *we have with probability one:*

$\left\{\frac{1}{a_{[2nL_2n]}}\sum_1^n X_k\colon n\in\mathbb{N}\right\}$ *is relatively compact in B, and the set of its limit points coincides with K.*

Before we state our functional law of the iterated logarithm we need still some further notation. We denote by $C_B[0, 1]$ the space of all continuous B-valued functions on $[0, 1]$ endowed with the sup-norm. Let $\mathscr{K}$ be the canonical cluster set determined by the covariance structure of Y (cf. Kuelbs (1976), (5.1)). We denote by $\eta_n\colon \Omega\to C_B[0, 1]$ the random polygon defined by

$$\eta_n(t):=\begin{cases}\sum_1^m X_k, & t=m/n,\ 0\leqq m\leqq n\\ \text{linearly interpolated elsewhere for } 0<t<1.\end{cases} \tag{1.7}$$

where $\{X_n\}$ is a sequence of independent copies of X.

Corollary 2. *Let X be as in Theorem* 1. *Assume* (1.6). *If* $\{\eta_n\}$ *is defined by* (1.7), *we have with probability one:* $\{\eta_n/a_{[2nL_2n]}\colon n\in\mathbb{N}\}$ *is relatively compact in* $C_B[0, 1]$, *and the set of its limit points coincides with* $\mathscr{K}$.

We now show that our condition (1.6) is weaker than the moment assumption (2.5) of Kuelbs (1985). Since by the convergence of types theorem all norming sequences $\{a_n\}$ in (1.4) are asymptotically equivalent, we have

$$a_{[2nL_2n]}\sim d(2nL_2n)\quad\text{as}\quad n\to\infty, \tag{1.8}$$

where $d(t)$, $t\geqq 0$ is defined as in Proposition 1 of Kuelbs (1985). Moreover, we have for the norming sequence $\{\gamma_n\}$ in the compact (functional) LIL of Kuelbs (1985):

$$\gamma_n\sim d(2n/L_2n)\,L_2n\quad\text{as}\quad n\to\infty. \tag{1.9}$$

Since $d(t)/\sqrt{t}$ is non-decreasing, we can conclude:

$$\varliminf_n\,(a_{[2nL_2n]}/\gamma_n)\geqq 1. \tag{1.10}$$

Thus the assumption $\sum_1^\infty P(\|X\|>\gamma_n)<\infty$, which is equivalent to condition (2.5) of Kuelbs (1985), implies (1.6).

An example given in Sect. 5 below shows that the converse implication does not hold in general.

Basic tool of the proof of our Theorem 1 is Theorem 2 below which is of independent interest.

We denote by $(\mathbb{R}^d, |\cdot|)$ the d-dimensional euclidean space. Σ is always a non-degenerate positive semidefinite matrix.

Theorem 2. *Let* $X\colon \Omega\to\mathbb{R}^d$ *be a mean zero random vector satisfying* (1.4) *with* $\mu=N(0, \Sigma)$. *Let* $\{c_n\}$ *be a sequence of positive real numbers such that* $c_n/\sqrt{n}$ *is non-decreasing, and* $\varliminf_n\,(c_n/a_n)>0$. *Assume that* $\sum_1^\infty P(|X|>c_n)<\infty$. *Then one can* *construct a p-space* $(\Omega_0, \mathscr{A}_0, P_0)$ *and two sequences of independent random vectors* $\{X_n\}$, $\{Y_n\}$ *with* $\mathscr{L}(X_n)=\mathscr{L}(X)$, $\mathscr{L}(Y_n)=N(0, \Sigma_n)$, *where* $\Sigma_n := \operatorname{cov}(X 1\{|X| \leqq c_n\})$, $n \in \mathbb{N}$, *such that*

$$\sum_1^n X_k - \sum_1^n Y_k = o(c_n) \quad a.s. \tag{1.11}$$

An analogous result has been obtained by Einmahl (1987a) for sequences $\{c_n\}$ such that $c_n/\sqrt{n}$ is non-increasing and $c_n/n^{1/3}$ is non-decreasing (see [6, Theorem 1]).

The first strong approximation result for partial sums of i.i.d. real valued random variables with infinite variances, being still in the domain of attraction of $N(0, 1)$, is due to Mijnheer (1980). Unlike Theorem 2 this result is restricted to symmetric r.v.'s and sequences $\{c_n\}$ such that $c_n/\sqrt{n}$ is slowly varying. Similarly as in (1.11), Mijnheer obtains under his assumptions an approximation with error term $o(c_n)$. But the approximating sequence $\{Y_n\}$ of normally distributed random variables in his Theorem 2.1 is different, and it does not appear very helpful when trying to obtain general LIL results like Corollaries 1 and 2 above. That is the main reason why we prefer to prove the strong approximation theorem (1.11) instead of simply extending Mijnheer's Theorem 2.1 to the multi-dimensional case.

Moreover, it will turn out when specializing the subsequent proof of Theorem 2 to real valued random variables that we have in this case no need for the assumption of X being in the domain of attraction of $N(0, 1)$. We obtain in Sect. 2 as a byproduct a general strong approximation theorem being valid for *arbitrary* real valued random variables (see Theorem 3).

In Sect. 3 we infer Theorem 1 from Theorem 2, which is proved in Sect. 2. Corollaries 1 and 2 are proved in Sect. 4. We finally give in Sect. 5 the above announced example for a random variable satisfying the assumptions of our Theorem 1 but not those ones of Kuelbs (1985). Our example also shows that the known LIL of Feller (1968) contains an error.

2. Proof of Theorem 2

2.1. A Preliminary Result

We need the following

Proposition 1. *Let* $X: \Omega \to \mathbb{R}^d$ *be a mean zero r.v. in the domain of attraction of* $N(0, \Sigma)$. *Denote by* $\Sigma(t)$ *the covariance matrix of* $X 1\{|X| \leqq t\}$, $t \geqq 0$. *Let the function* $G(t)$, $t \geqq 0$ *be defined by*

$$G(t) := E[\langle X, y\rangle^2 1\{|\langle X, y\rangle| \leqq t\}], \tag{2.1}$$

where $y \in \mathbb{R}^d$ *is a fixed vector such that* $\langle y, \Sigma y\rangle = 1$. *Then:* $\frac{1}{G(t)} \Sigma(t) \to \Sigma$ *as* $t \to \infty$.

Proof of the Proposition. (i) Let $\{X_n\}$ be a sequence of independent copies of X. According to (1.4) we have:

$$\mathscr{L}\left(\frac{1}{a_n}\sum_1^n X_k\right) \text{ converges weakly to } N(0, \Sigma). \tag{2.2}$$

From (2.2) we easily infer:

$$\mathscr{L}\left(\frac{1}{a_n}\sum_1^n \langle X_k, y\rangle\right) \text{ converges weakly to } N(0, 1). \tag{2.3}$$

Therefore, we can conclude from the general 1-dimensional central limit theorem (cf., Chow and Teicher (1978), Chapter 9, Theorem 4) and the convergence of types theorem:

$$a_n \sim \bar{a}(n) \quad \text{as} \quad n \to \infty, \tag{2.4}$$

where the function $\bar{a}$ is defined by

$$\bar{a}(t) := \sup\{s > 0 : s^2/G(s) \leqq t\}, \qquad t > 0. \tag{2.5}$$

Since $G(t)$, $t > 0$ is non-decreasing, we easily obtain from the above definition:

$$\bar{a}(t)^2 = t\, G(\bar{a}(t)), \qquad t > 0. \tag{2.6}$$

Because of (2.3) G is slowly varying at infinity, and we obtain from (2.4) and (2.6):

$$a_n^2/n \sim G(a_n) \quad \text{as} \quad n \to \infty. \tag{2.7}$$

(ii) Since $a_n \uparrow \infty$, we have for $\varepsilon > 0$:

$$P(|X| > \varepsilon \cdot a_n) \to 0 \quad \text{as} \quad n \to \infty. \tag{2.8}$$

We set $X_{nk} := X_k/a_n$, $1 \leqq k \leqq n$, $n \in \mathbb{N}$.

Since the X_k's are i.i.d. r.v.'s with the same distribution as X, we obtain from (2.8):

$$\max_{1 \leqq k \leqq n} P(|X_{nk}| > \varepsilon) \to 0 \quad \text{as} \quad n \to \infty \ (\varepsilon > 0). \tag{2.9}$$

Thus $\{X_{nk}\}$ is an infinitesimal triangular array such that

$$\mathscr{L}\left(\sum_{k=1}^n X_{nk}\right) \text{ converges weakly to } N(0, \Sigma). \tag{2.10}$$

(Recall (2.2).)

Applying Corollary 2.12(2), De Acosta, Araujo and Giné (1978) to $f(x) = \langle z, x\rangle$, $x \in \mathbb{R}^d$, we obtain:

$$\sum_{k=1}^n \langle z, \operatorname{cov}(X_{nk1})\, z\rangle \to \langle z, \Sigma z\rangle \qquad (z \in \mathbb{R}^d). \tag{2.11}$$

Since $\operatorname{cov}(X_{nk1}) = \frac{1}{a_n^2} \Sigma(a_n)$, we have:

(2.12)
$$\frac{n}{a_n^2} \Sigma(a_n) \to \Sigma \quad \text{as} \quad n \to \infty.$$

Hence we infer from (2.7):

(2.13)
$$\frac{1}{G(a_n)} \Sigma(a_n) \to \Sigma \quad \text{as} \quad n \to \infty.$$

(iii) Let now $\{t_n\}$ be a sequence such that $a_n \leqq t_n \leqq a_{n+1}$, $n \in \mathbb{N}$. Since $a_{n+1}/a_n \to 1$ as $n \to \infty$, we have $t_n \sim a_n$ as $n \to \infty$. Therefore, we obtain from the convergence of types theorem:

(2.14)
$$\mathscr{L}\left(\frac{1}{t_n} \sum_1^n X_k\right) \text{ converges weakly to } N(0, \Sigma).$$

Replacing $\{a_n\}$ by $\{t_n\}$ in part (ii) of the proof, we obtain:

(2.15)
$$\frac{1}{G(t_n)} \Sigma(t_n) \to \Sigma \quad \text{as} \quad n \to \infty.$$

Since (2.15) holds true for all sequences $\{t_n\}$ such that $a_n \leqq t_n \leqq a_{n+1}$, $n \in \mathbb{N}$, it can easily be seen that it must hold for any sequence $t_n \uparrow \infty$. This proves the assertion.

2.2. Conclusion of the Proof of Theorem 2

W.l.o.g. we assume $\Sigma = I$ ($= d$-dimensional unit matrix). We show that there exist a p-space $(\Omega_1, \mathscr{A}_1, P_1)$ and two sequences of independent r.v.'s $\{\bar{X}_n\}$, $\{Y_n'\}$ such that

$$\mathscr{L}(\bar{X}_n) = \mathscr{L}(X 1\{|X| \leqq c_n\}), \qquad \mathscr{L}(Y_n') = N(0, \Sigma_n), \quad n \in \mathbb{N}$$

and

(2.16)
$$\sum_1^n (\bar{X}_k - E[\bar{X}_k]) - \sum_1^n Y_k' = o(c_n) \quad \text{a.s.}$$

To prove (2.16), we now apply Theorem 2, Einmahl (1987b). According to this result it is possible to obtain a construction which yields (2.16) if the following condition is fulfilled:

(2.17)
$$\sum_1^\infty c_k^{-3} E[|\bar{X}_k - E[\bar{X}_k]|^3] < \infty.$$

Though this result is only formulated for sequences of independent random vectors $\{\xi_n\}$ such that $\operatorname{cov}(\xi_n) = \sigma_n^2 \Gamma$, $n \in \mathbb{N}$, the following considerations show that it is also applicable in the present situation. Using remark (a) following Proposition 1, and the proof of Theorem 2, Einmahl (1987b), one can easily show that this result remains valid provided that the following condition is fulfilled:

(2.18) $$\overline{\lim_n}(\Lambda(\xi_n)/\lambda(\xi_n))<\infty,$$

where $\Lambda(\xi_n)(\lambda(\xi_n))$ denotes the largest (smallest) eigenvalue of $\operatorname{cov}(\xi_n)$, $n\in\mathbb{N}$.

It now follows from Proposition 1 that

$$\lambda(\overline{X}_n-E[\overline{X}_n])\sim G(c_n)\sim\Lambda(\overline{X}_n-E[\overline{X}_n]) \quad \text{as} \quad n\to\infty,$$

whence (2.18) holds.

Since $E[|\bar{X}_k-E[\bar{X}_k]|^3]\leqq 8E[|\bar{X}_k|^3]$, $k\in\mathbb{N}$, we obtain (2.17) from the subsequent Lemma 1, applied with $Z=|X|$.

Lemma 1. *Let $Z\colon \Omega\to[0,\infty)$ be a r.v. such that $\sum_1^\infty P(Z>c_n)<\infty$, where $\{c_n\}$ is a positive sequence such that $c_n/\sqrt{n}$ is non-decreasing. Then we have:*

$$\sum_1^\infty c_n^{-3}\,E[Z^3\,1\{Z\leqq c_n\}]<\infty.$$

Proof. We set $p_n:=P(c_{n-1}<Z\leqq c_n)$, $n\in\mathbb{N}$. ($c_0:=0$). Then:

$$\begin{aligned}\sum_1^\infty c_n^{-3}\,E[Z^3\,1\{Z\leqq c_n\}]&\leqq\sum_1^\infty c_n^{-3}\sum_{k=1}^n p_k c_k^3\\ &=\sum_{k=1}^\infty\sum_{n=k}^\infty (c_k/c_n)^3 p_k\leqq\sum_{k=1}^\infty\left(\sum_{n=k}^\infty (k/n)^{3/2}\right)p_k\\ &\leqq\sum_{k=1}^\infty k^{3/2}\left(k^{-3/2}+\int_k^\infty x^{-3/2}\,dx\right)p_k\\ &\leqq 1+\sum_{k=1}^\infty 2k\,p_k=1+2\sum_{k=1}^\infty P(Z>c_{k-1})<\infty.\quad\square\end{aligned}$$

Let $\{\xi_n\}$ be a sequence of i.i.d. random vectors with the same distribution as X. Put $\bar{\xi}_n:=\xi_n\,1\{|\xi_n|\leqq c_n\}$, $n\in\mathbb{N}$. Employing the Borel-Cantelli lemma, we easily obtain that

$$\sum_1^n\xi_k-\sum_1^n\bar{\xi}_k=O(1)\quad\text{a.s.}$$

Using Lemma A.1, Berkes and Philipp (1979), and (2.16), we can construct a p-space $(\Omega_0,\mathscr{A}_0,P_0)$ and two sequences of independent random vectors $\{X_n\}$, $\{Y_n\}$ with the desired distributions such that

(2.19) $$\sum_1^n(X_k-E[X\,1\{|X|\leqq c_k\}])-\sum_1^n Y_k=o(c_n)\quad\text{a.s.}$$

Thus, it remains to show that

$$\sum_{1}^{n} E[X1\{|X| \leqq c_k\}] = o(c_n). \tag{2.20}$$

For the sake of later reference we now prove (2.20) directly for random variables taking values in an arbitrary separable Banach space.

Proposition 2. *Let X be a B-valued r.v. satisfying* (1.4). *Then we have for any sequence $\{c_n\}$ such that $n^{-\alpha} c_n$ is non-decreasing for some $\alpha > 0$, and $\underline{\lim}_n (c_n/a_n) > 0$:*

$$\left\|\sum_{1}^{n} E[X1\{\|X\| \leqq c_k\}]\right\| = o(c_n).$$

Proof. Let a: $[0, \infty) \to [0, \infty)$ be an increasing continuous function such that $a(n) = a_n$, $n \in \mathbb{N}$, $a(0) = 0$. It is well known that (1.4) implies $nP(\|X\| > a_n) \to 0$ as $n \to \infty$.

By means of interpolation we obtain:

$$\lim_{u \to \infty} (a^{-1}(u) P(\|X\| > u)) = 0. \tag{2.21}$$

Furthermore, (1.4) implies that a is a regularly varying function at infinity with exponent 1/2, whence the inverse function a^{-1} of a is regularly varying at infinity with exponent 2. Using the Karamata representation of a^{-1} (see Seneta (1976), Theorem 1.2), it is easy to see that a^{-1} satisfies condition (2.24) of Lemma 3 for any $\gamma \in (1, 2)$. We conclude:

$$\lim_{u \to \infty} \left(\frac{a^{-1}(u)}{u} E[\|X\| 1\{\|X\| > u\}]\right) = 0. \tag{2.22}$$

Noticing that $E[X] = 0$, which is a consequence of (1.4), we infer:

$$\lim_{u \to \infty} \left(\frac{a^{-1}(u)}{u} \|E[X1\{\|X\| \leqq u\}]\|\right) = 0. \tag{2.23}$$

Combining (2.23) with the subsequent Lemma 2 yields Proposition 2. □

Lemma 2. *Let X be a B-valued r.v. such that*

$$\lim_{u \to \infty} \left(\frac{g(u)}{u} \|E[X1\{\|X\| \leqq u\}]\|\right) = 0,$$

where $g: (u_0, \infty) \to (0, \infty)$ is an increasing function. Then, we have for any sequence $\{c_n\}$ such that $n^{-\alpha} c_n$ is non-decreasing for some $\alpha > 0$, and $\underline{\lim}_n (g(c_n)/n) > 0$:

$$\left\|\sum_{1}^{n} E[X1\{\|X\| \leqq c_k\}]\right\| = o(c_n).$$

Proof. Let $\varepsilon>0$ be fixed. From the above assumptions it easily follows that we have for $k \geqq k_0(\varepsilon)$ (say):

$$\|E[X1\{\|X\| \leqq c_k\}]\| \leqq \varepsilon c_k/k.$$

We get for $n \geqq k_0$:

$$c_n^{-1}\left\|\sum_1^n E[X1\{\|X\| \leqq c_k\}]\right\| \leqq k_0\, c_{k_0}/c_n + \sum_{k=k_0}^n \varepsilon c_k/k\, c_n$$

$$\leqq k_0\, c_{k_0}/c_n + \sum_{k=k_0}^n n^{-1}\varepsilon (k/n)^{-1+\alpha}$$

$$\leqq k_0\, c_{k_0}/c_n + \varepsilon \int_0^1 x^{-1+\alpha}\, dx \leqq 2\varepsilon/\alpha \qquad \text{for } n \geqq k_1(\varepsilon) \text{ (say)}. \quad \square$$

Lemma 3. *Let $\xi\colon \Omega \to [0, \infty)$ be a random variable such that $\overline{\lim}_{u\to\infty}(h(u)\, P(\xi>u)) \leqq C$, where $h\colon [u_0, \infty) \to (0, \infty)$ is a function such that the following condition is fulfilled for some $K \geqq 1$ and some $\gamma>1$:*

(2.24) $$h(u)/h(v) \leqq K(u/v)^\gamma, \qquad u_0 \leqq u \leqq v.$$

Then we have

$$\overline{\lim_{u\to\infty}}\left(\frac{h(u)}{u} E[\xi 1\{\xi>u\}]\right) \leqq 2CK/(\gamma-1).$$

Proof. By means of partial integration we obtain:

$$E[\xi 1\{\xi>u\}] \leqq u P(\xi>u) + \int_u^\infty P(\xi>v)\, dv.$$

Let $\delta>0$ be fixed. Then we obtain for $u \geqq u_1(\delta)$ (say):

$$\frac{h(u)}{u} E[\xi 1\{\xi>u\}] \leqq (C+\delta)\left(1+u^{-1}\int_u^\infty (h(u)/h(v))\, dv\right)$$

$$\leqq K(C+\delta)\left(1+u^{\gamma-1}\int_u^\infty v^{-\gamma}\, dv\right) \leqq (2C+2\delta)\, K/(\gamma-1). \quad \square$$

2.3. A General Strong Approximation Theorem for Partial Sums of I.I.D. Real Valued Random Variables

The only place in the proof of (2.19), where we have used the assumption that X is in the domain of attraction of $N(0, \Sigma)$ is (2.18). But if $d=1$ (2.18) is automatically fulfilled. Thus, we have implicitly proved:

Theorem 3. *Let X be a real valued random variable. Assume that $\sum_1^\infty P(|X|>c_n) <\infty$, where $\{c_n\}$ is a sequence of positive real numbers such that $c_n/\sqrt{n}$ is non-decreasing. Then one can construct a p-space $(\Omega_0, \mathscr{A}_0, P_0)$ and two sequences of independent random variables $\{X_n\}$, $\{Y_n\}$ with $\mathscr{L}(X_n)=\mathscr{L}(X)$, $\mathscr{L}(Y_n)=N(0, \sigma_n^2)$, where $\sigma_n^2=E[X^2 1\{|X|\leqq c_n\}]-(E[X 1\{|X|\leqq c_n\}])^2$, $n\in\mathbb{N}$, such that*

$$\sum_1^n (X_k - E[X 1\{|X|\leqq c_k\}]) - \sum_1^n Y_k = o(c_n) \quad a.s. \tag{2.25}$$

When X has a symmetric distribution, (2.25) reduces to (1.11). Thus, Theorem 2 holds true for *any* real valued symmetric random variable.

The following Lemma 4 shows that (1.11) also remains valid for arbitrary non-symmetric mean zero random variables under additional assumptions on the sequence $\{c_n\}$.

Lemma 4. *Let $X: \Omega\to\mathbb{R}$ be a mean zero r.v. such that $\sum_1^\infty P(|X|>c_n)<\infty$, where $\{c_n\}$ is a sequence of positive real numbers such that $c_n/\sqrt{n}$ is non-decreasing, and*

$$c_n/n^\gamma \text{ is eventually non-increasing for some } \gamma<1 \tag{2.26}$$

or

$$c_n=c(n),\ n\in\mathbb{N}, \text{ where } c: [1, \infty)\to(0, \infty) \text{ is regularly varying at infinity with exponent } \gamma<1. \tag{2.27}$$

Then we have:

$$\sum_1^n E[X 1\{|X|\leqq c_k\}] = o(c_n) \quad as \quad n\to\infty.$$

Proof. Lemma 4 follows by an obvious modification of the proof of Proposition 2. Notice that $\sum_1^\infty P(|X|>c_n)<\infty$ implies: $nP(|X|>c_n)\to 0$, since $\{c_n\}$ is non-decreasing. □

3. Proof of Theorem 1

To prove Theorem 1, we proceed similarly as Philipp (1979) and Kuelbs and Philipp (1980): We first prove Theorem 1 for finite-dimensional r.v.'s (cf. 3.1). Combining this special case of Theorem 1 with an appropriate law of the iterated logarithm, we obtain Theorem 1 for r.v.'s taking values in an arbitrary separable Banach space (cf. 3.2).

Since $Y_n/a_{[nL_2n]} \to 0$ a.s., we easily see that (1.6) is necessary for (1.5). Thus, it suffices to show that under condition (1.6) a construction yielding (1.5) is possible.

3.1. The Finite-Dimensional Case

W.l.o.g. we assume that X is in the domain of attraction of $N(0, I)$.

Since we have:

$$a_{[nL_2n]} \sim \bar{a}(nL_2 n) \quad \text{as} \quad n \to \infty, \tag{3.1}$$

where the function $\bar{a}$ is defined as in (2.5), we obtain from (1.6):

$$\sum_1^\infty P(|X| > \bar{a}(nL_2 n)) < \infty. \tag{3.2}$$

Applying Theorem 2 with $c_n = \bar{a}(nL_2 n)$, we obtain a p-space $(\Omega_0, \mathscr{A}_0, P_0)$ and two sequences of independent random vectors $\{X_n\}$, $\{\tilde{Y}_n\}$ such that $\mathscr{L}(X_n) = \mathscr{L}(X)$, $\mathscr{L}(\tilde{Y}_n) = N(0, \Sigma_n)$, where $\Sigma_n = \operatorname{cov}(X 1\{|X| \leqq \bar{a}(nL_2 n)\})$, $n \in \mathbb{N}$, and

$$\sum_1^n X_k - \sum_1^n \tilde{Y}_k = o(\bar{a}(nL_2 n)) = o(a_{[nL_2n]}) \quad \text{a.s.} \tag{3.3}$$

(Notice that relation (2.6) shows that $\bar{a}(nL_2 n)/\sqrt{nL_2 n}$ is non-decreasing.)

From Proposition 1 we infer:

$$G(\bar{a}(nL_2 n))^{-1} \Sigma_n \to I \quad \text{as} \quad n \to \infty. \tag{3.4}$$

Therefore, we can w.l.o.g. assume that Σ_n is positive definite for all $n \in \mathbb{N}$.

We set:

$$Y_n = \sigma_n \Sigma_n^{-1/2} \tilde{Y}_n, \qquad n \in \mathbb{N}. \tag{3.5}$$

Since $\{Y_n - \tilde{Y}_n\}$ is a sequence of independent d-dimensional r.v.'s such that

$$\mathscr{L}(Y_n - \tilde{Y}_n) = N(0, (\sigma_n I - \Sigma_n^{1/2})^2), \qquad n \in \mathbb{N}, \tag{3.6}$$

we obtain from (3.4), (2.6) and (3.1), using an appropriate LIL for normally distributed random variables:

$$\sum_1^n (Y_k - \tilde{Y}_k) = o((n\sigma_n^2 L_2 n)^{1/2}) = o(a_{[nL_2n]}) \quad \text{a.s.} \tag{3.7}$$

Combining (3.3) and (3.7) yields the assertion.

3.2. The General Case

We need the following law of the iterated logarithm:

Theorem 4. *Let $\{X_n\}$ be a sequence of i.i.d. B-valued mean zero r.v.'s such that*

$$\mathscr{L}\left(\frac{1}{a_n}\sum_1^n X_k\right) \text{ converges weakly to } \mu=\mathscr{L}(Y) \tag{3.8}$$

for some sequence $a_n \uparrow \infty$, where Y is a (non-degenerate) Gaussian mean zero r.v. Assume:

$$\sum_1^\infty P(\|X\| > a_{[nL_2 n]}) < \infty. \tag{3.9}$$

Then we have:

$$d\left(\frac{1}{a_{[2nL_2 n]}}\sum_1^n X_k, K\right) \to 0 \quad a.s.,$$

where K is the unit ball of the Hilbert space H_μ.

Recall that for $A \subseteq B$ $d(\cdot, A)$ is defined by

$$d(x, A) := \inf\{\|y-x\| : y \in A\}, \qquad x \in B.$$

To prove Theorem 4, we first show by a modification of the proof of Corollary 7, Kuelbs and Zinn (1983), that, with probability one, $\left\{\frac{1}{a_{[2nL_2 n]}}\sum_1^n X_k : n \in \mathbb{N}\right\}$ is relatively compact in B. (cf. (i) and (ii).) Applying our Theorem 1 to appropriate *1-dimensional* r.v.'s, we finally obtain that, with probability one, the limit set of $\left\{\frac{1}{a_{[2nL_2 n]}}\sum_1^n X_k : n \in \mathbb{N}\right\}$ is a subset of K (cf. (iii)).

Proof. (i) We first assume that X has a symmetric distribution. Let $f \in B^*$ be a fixed functional such that $E[f(Y)^2]=1$. Set $G(t) := E[f(X)^2\, 1\{|f(X)| \leqq t\}]$, $\bar{a}(t) := \sup\{s>0 : s^2/G(s) \leqq t\}$, $t>0$. Using the same argument as in (2.4), we obtain:

$$a_n \sim \bar{a}(n) \quad \text{as} \quad n \to \infty. \tag{3.10}$$

We set $g(t) := t^2/S(t L_2 t)$, $t>0$, where

$$S(t) := \int_0^t E[|f(X)|\, 1\{|f(X)|>u\}]\, du, \qquad t>0.$$

Since $t/S(t)$, $t>0$ is non-decreasing, we easily see that $g(t)$, $t>0$ is an increasing function. Moreover, it can be shown by obvious changes in the proof of Corollary 7, Kuelbs and Zinn (1983), that our function g fulfills (2.61-i, ii, iii) of Kuelbs and Zinn (1983).

Since $S(t) \sim G(t)$ as $t \to \infty$ (cf. Kuelbs and Zinn (1983), (6.25)), we have:

$$g(\bar{a}(t(L_2 t)^2)/L_2 t) \sim t \quad \text{as} \quad t \to \infty. \tag{3.11}$$

Thus, we obtain for $d(t):=g^{-1}(t)$, $t>0$:

(3.12) $$d(t)\sim \bar{a}(t(L_2 t)^2)/L_2 t \quad \text{as} \quad t\to\infty.$$

(Notice that S is slowly varying at infinity, since $f(X)$ is in the domain of attraction of $N(0, 1)$.)

Since $\bar{a}(t)/\sqrt{t}$, $t>0$ is non-decreasing, we infer from (3.12):

(3.13) $$\underline{\lim}_n (d(n)/\bar{a}(n))\geqq 1.$$

Using (3.8) and (3.13), we obtain that $\left\{\mathscr{L}\left(\frac{1}{d(n)}\sum_1^n X_k\right): n\in\mathbb{N}\right\}$ is tight. Therefore, we can choose a compact convex symmetric set D such that the following holds true for $n\in\mathbb{N}$:

(3.14) $$P\left(\frac{1}{d(n)}\sum_1^n X_k\notin D\right)\leqq 1/(16\,e^2).$$

Since $(\alpha^{-1}\,d\alpha)(n)\sim\bar{a}(n\,L_2\,n)$ as $n\to\infty$, where $\alpha(t)=t/L_2\,t$, $t>0$, we obtain from (3.9) and (3.10):

(3.15) $$\sum_1^\infty P(\|X\|>(\alpha^{-1}\,d\alpha)(n))<\infty.$$

Hence:

(3.16) $$E[\alpha^{-1}\,g\,\alpha(\|X\|)]<\infty.$$

Applying Theorem 5 of Kuelbs and Zinn (1983) to the seminorms $x\to q_\delta(x) :=\inf\{t: x/t\in D^\delta\}$, $\delta>0$, we obtain similarly as in their Corollary 7 that, with probability one, $\left\{\frac{1}{\bar{a}(n\,L_2\,n)}\sum_1^n X_k: n\in\mathbb{N}\right\}$ is relatively compact in B.

(ii) We now show that we have for not necessarily symmetric r.v.'s with probability one:

(3.17) $$\left\{\frac{1}{\bar{a}(n\,L_2\,n)}\sum_1^n X_k: n\in\mathbb{N}\right\} \text{ is relatively compact in } B.$$

To prove (3.17), it now suffices to show that

(3.18) $$\frac{n}{\bar{a}(n\,L_2\,n)}E[\|X\|\,1\{\|X\|>\rho\,\bar{a}(n\,L_2\,n)\}]\to 0 \qquad (\rho>0).$$

(We use the same argument as in the proof of Corollary 7, Kuelbs and Zinn (1983).)

From relation (2.22) above we obtain:

$$\frac{a^{-1}(\rho\,\bar{a}(n\,L_2\,n))}{\bar{a}(n\,L_2\,n)}E[\|X\|\,1\{\|X\|>\rho\,\bar{a}(n\,L_2\,n)\}]\to 0 \qquad (\rho>0),$$

where a: $[0, \infty) \to [0, \infty)$ is an increasing continuous function with $a(n) = a_n$, $n \in \mathbb{N}$, $a(0) = 0$. In general, the function $\bar{a}$ is not continuous, but we have:

$$\bar{a}(n L_2 n) \sim a(n L_2 n) \quad \text{as} \quad n \to \infty.$$

Recalling that a^{-1} is regularly varying at infinity, we conclude:

$$\frac{n L_2 n}{\bar{a}(n L_2 n)} E[\|X\| \, 1\{\|X\| > \rho \, \bar{a}(n L_2 n)\}] \to 0 \qquad (\rho > 0), \tag{3.19}$$

whence (3.18) holds.

Noticing (3.10), we obtain from (3.17) that, with probability one, the following holds:

$$\left\{\frac{1}{a_{[2nL_2n]}} \sum_1^n X_k : n \in \mathbb{N}\right\} \text{ is relatively compact in } B. \tag{3.20}$$

(iii) To finish the proof, it remains to show that with probability one:

$$C\left(\frac{1}{a_{[2nL_2n]}} \sum_1^n X_k : n \in \mathbb{N}\right) \subseteq K, \tag{3.21}$$

where $C(\{z_n\})$ denotes the cluster set of the sequence $\{z_n\} \subseteq B$.

Using analogous arguments as in Kuelbs (1985), (4.6)–(4.12), we can see that (3.21) holds if we have for all functionals $h \in B^*$ such that $E[h(Y)^2] > 0$:

$$\overline{\lim_n} \left(\frac{1}{a_{[2nL_2n]}} \sum_1^n h(X_k)\right) \leqq (E[h(Y)^2])^{1/2} \quad \text{a.s.} \tag{3.22}$$

According to Theorem 1 (the finite-dimensional case-3.1), there exists for every fixed functional $h \in B^*$ a sequence of independent r.v.'s $\{y_n\}$ such that $\mathscr{L}(y_n) = N(0, \sigma_n^2 E[h(Y)^2])$, where $\sigma_n^2 = b_{[nL_2n]}$, $b_n = a_n^2/n$, $n \in \mathbb{N}$, and

$$\sum_1^n h(X_k) - \sum_1^n y_k = o(a_{[nL_2n]}) \quad \text{a.s.} \tag{3.23}$$

Setting $t_k := \sum_1^k \sigma_m^2$, $k \in \mathbb{N}$, we obtain by the same argument as in (3.7):

$$\overline{\lim_n} \frac{1}{(2 t_n L_2 t_n)^{1/2}} \sum_1^n y_k \leqq (E[h(Y)^2])^{1/2} \quad \text{a.s.} \tag{3.24}$$

Since σ_n^2, $n \in \mathbb{N}$ is slowly varying, we have $t_n \sim n \sigma_n^2$ as $n \to \infty$. Using this asymptotic equivalence, we can easily show that

$$2 t_n L_2 t_n \sim (a_{[2nL_2n]})^2 \quad \text{as} \quad n \to \infty. \tag{3.25}$$

(3.22) now follows from (3.23), (3.24) and (3.25). □

We consider the maps Π_N: $B \to B$ obtained from $\mu = \mathscr{L}(Y)$ according to Lemma 2.1 of Kuelbs (1976). Using Theorem 4, we can show:

Corollary 3. *Given $\varepsilon>0$ we have for N sufficiently large:*

$$\overline{\lim_n}\, a_{[nL_2n]}^{-1} \left\| \sum_1^n (X_k - \Pi_N X_k) \right\| \leqq \varepsilon \quad a.s.$$

Proof. Since $x \to x - \Pi_N(x) =: Q_N(x)$ is continuous, we have by Theorem 4:

$$d\left(Q_N\left(a_{[nL_2n]}^{-1} \sum_1^n X_k\right), Q_N(K)\right) \to 0 \quad \text{a.s.} \tag{3.26}$$

By relation (3.7), Kuelbs (1976), there exists $N_0 \in \mathbb{N}$ such that

$$Q_N(K) \subseteq \{x \colon \|x\| \leqq \varepsilon\} \qquad \text{for } N \geqq N_0. \tag{3.27}$$

Combining (3.26) and (3.27), we obtain Corollary 3. □

Let now $\{Y_k\}$ be a sequence of independent r.v.'s such that $\mathscr{L}(Y_k) = \mathscr{L}(\sigma_k Y)$, $k \in \mathbb{N}$, where σ_k^2 is defined as above, i.e. $\sigma_k^2 = b_{[kL_2k]}$, $k \in \mathbb{N}$.

Then we have for any given $\varepsilon > 0$:

$$\overline{\lim_n}\, a_{[nL_2n]}^{-1} \left\| \sum_1^n (Y_k - \Pi_N Y_k) \right\| \leqq \varepsilon \quad \text{a.s.,} \tag{3.28}$$

if N is sufficiently large.

To verify (3.28), we consider a mean zero Brownian motion $\{W_\mu(t) \colon t \geqq 0\}$ with the covariance structure of $\mu = \mathscr{L}(Y)$. From Lemma 4.2, Kuelbs and Philipp (1980), we infer for sufficiently large N:

$$\overline{\lim_n}\, (t_n L_2 t_n)^{-1/2} \| W_\mu(t_n) - \Pi_N W_\mu(t_n) \| \leqq \varepsilon \quad \text{a.s.,} \tag{3.29}$$

where t_n is defined as in (3.24).

Using (3.25), we easily obtain (3.28) from (3.29).

We now consider the r.v. $\Pi_N X \colon \Omega \to B$. Since we have $\dim(\Pi_N B) = \min(N, \dim(H_\mu)) = N$ (w.l.o.g.), we can identify the spaces $(\Pi_N B, \|\cdot\|_\mu)$ and $(\mathbb{R}^N, |\cdot|)$ by

$$t_1 S\alpha_1 + \ldots + t_N S\alpha_N \leftrightarrow (t_1, \ldots, t_N). \tag{3.30}$$

(cf. Lemma 2.1, Kuelbs (1976).)

We denote by $X^{(N)} \colon \Omega \to \mathbb{R}^N$ the r.v. obtained from $\Pi_N X$ according to (3.30). Since Π_N is continuous, we have for all sequences $\{X_k^{(N)}\}$ of i.i.d. r.v.'s with the same distribution as $X^{(N)}$:

$$\mathscr{L}\left(\frac{1}{a_n} \sum_1^n X_k^{(N)}\right) \text{ converges weakly to } \mathscr{L}(Y^{(N)}), \tag{3.31}$$

where $Y^{(N)}$ is the $\mathbb{R}^N$-valued Gaussian r.v. obtained from $\Pi_N Y$ according to (3.30).

Since $|X^{(N)}| = \|\Pi_N X\|_\mu \leqq C_1 \|\Pi_N X\| \leqq C_2 \|X\|$, where C_i, $i=1, 2$ are appropriate positive constants, we obtain from (1.6):

$$\sum_1^\infty P(|X^{(N)}| > a_{[nL_2 n]}) < \infty. \tag{3.32}$$

(Notice that $\|\cdot\|$ and $\|\cdot\|_\mu$ are equivalent norms on the finite-dimensional space $\Pi_N B$.)

Since X and consequently $X^{(N)}$ have mean zero, we can apply Theorem 1 to the finite-dimensional r.v. $X^{(N)}$.

Thus we obtain a p-space $(\Omega_0, \mathscr{A}_0, P_0)$ and two sequences of independent r.v.'s $\{\xi_n\}$, $\{\eta_n\}$ such that $\mathscr{L}(\xi_n) = \mathscr{L}(X^{(N)})$, $\mathscr{L}(\eta_n) = \mathscr{L}(\sigma_n Y^{(N)})$, $n \in \mathbb{N}$, and

$$\sum_1^n \xi_k - \sum_1^n \eta_k = o(a_{[nL_2 n]}) \quad \text{a.s.} \tag{3.33}$$

By means of (3.30) we obtain two sequences $\{\tilde{X}_k\}$, $\{\tilde{Y}_k\}$ of B-valued r.v.'s such that $\mathscr{L}(\tilde{X}_n) = \mathscr{L}(\Pi_N X)$, $\mathscr{L}(\tilde{Y}_n) = \mathscr{L}(\sigma_n \Pi_N Y)$, $n \in \mathbb{N}$ and

$$\sum_1^n \tilde{X}_k - \sum_1^n \tilde{Y}_k = o(a_{[nL_2 n]}) \quad \text{a.s.} \tag{3.34}$$

(with convergence w.r.t. $\|\cdot\|$).

(Notice that we have according to Lemma 2.1, Kuelbs (1976): $\|\cdot\| \leqq c\, \|\cdot\|_\mu$ for some positive constant c.)

Using Lemma A.1, Berkes and Philipp (1979), we can w.l.o.g. assume that $\tilde{X}_k = \Pi_N X_k$, $\tilde{Y}_k = \Pi_N Y_k$ if $\{X_k\}$, $\{Y_k\}$ are the sequences considered in Corollary 3 and (3.28), respectively. Hence we have for sufficiently large N:

$$\overline{\lim_n}\, a_{[nL_2 n]}^{-1} \left\| \sum_1^n X_k - \sum_1^n Y_k \right\| \leqq 2\varepsilon \quad \text{a.s.} \tag{3.35}$$

Thus, we have shown that for any $\varepsilon > 0$ a construction is possible such that (3.35) holds. But this is still too weak to prove Theorem 1 since, as the proof shows, the sequences $\{X_n\}$, $\{Y_n\}$ are depending on N and consequently on ε. Therefore, it remains to show that there exist "universal" sequences $\{X_n\}$, $\{Y_n\}$, which fulfill (3.35) for all $\varepsilon > 0$. But this can be done by a well known argument of Major (1976) (cf. Philipp (1979), p. 187 and Dudley/Philipp (1983), p. 532).

4. Proof of Corollaries 1 and 2

W.l.o.g. we assume that $a_n/\sqrt{n}$ is non-decreasing. Let the p-space $(\Omega_0, \mathscr{A}_0, P_0)$ of Theorem 1 be given. Set $\bar{Y}_k := \sigma_k^{-1} Y_k$, $k \in \mathbb{N}$. In order to prove Corollaries 1 and 2, it suffices to show:

$$\max_{1 \leqq m \leqq n} \left\| \sum_1^m X_k - \sigma_n \sum_1^m \bar{Y}_k \right\| = o(a_{[nL_2 n]}) \quad \text{a.s.,} \tag{4.1}$$

when (1.6) holds.

Then we can immediately infer the functional (compact) LIL for X with respect to the norming sequence $a_{[2nL_2n]} \sim \sqrt{2nL_2n}\,\sigma_n$ from the usual LIL results for the Gaussian r.v. Y (with respect to the norming sequence $\sqrt{2nL_2n}$).

Suppose now that X satisfies condition (1.6). Since (1.5) implies:

(4.2) $$\max_{1 \leq m \leq n} \left\| \sum_1^m X_k - \sum_1^m \sigma_k \bar{Y}_k \right\| = o(a_{[nL_2n]}) \quad \text{a.s.,}$$

it remains to be shown:

(4.3) $$\Delta_n := \max_{1 \leq m \leq n} \left\| \sum_{k=1}^m (\sigma_n - \sigma_k) \bar{Y}_k \right\| = o(a_{[nL_2n]}) \quad \text{a.s.}$$

It follows from

(4.4) $$\Delta_{2^n}/a_{[2^nLn]} \to 0 \quad \text{a.s.}$$

and

(4.5) $$\max_{2^n \leq m \leq 2^{n+1}} (\Delta_m - \Delta_{2^{n+1}})/a_{[2^nLn]} \to 0 \quad \text{a.s.}$$

Since Y is Gaussian, there exists a $\beta > 0$ such that

(4.6) $$E[\exp(\beta \|Y\|^2)] < \infty.$$

(cf., Araujo and Giné (1980), Theorem 6.5, Ch. 3.)

Moreover, we have:

(4.7) $$\mathscr{L}\left(\sum_1^n (\sigma_n - \sigma_k) \bar{Y}_k\right) = \mathscr{L}\left(\left(\sum_1^n (\sigma_n - \sigma_k)^2\right)^{1/2} Y\right).$$

Since $\{\sigma_n^2\}$ is slowly varying (cf. 2.6), it follows

(4.8) $$\sum_1^n \sigma_k^2 \sim n\sigma_n^2 \quad \text{as} \quad n \to \infty.$$

Using (4.6), (4.7) and the Levi inequality, we get for $\varepsilon > 0$:

$$\begin{aligned} P_0(\Delta_{2^n} \geq \varepsilon a_{[2^nLn]}) &\leq 2P_0\left(\left\| \sum_1^{2^n} (\sigma_{2^n} - \sigma_k) \bar{Y}_k \right\| \geq \varepsilon a_{[2^nLn]}\right) \\ &= 2P(\|Y\| \geq \varepsilon a_{[2^nLn]} (\delta_n 2^n \sigma_{2^n}^2)^{-1/2}) \\ &\leq 2E[\exp(\beta \|Y\|^2)] \exp(-\beta \varepsilon^2 a_{[2^nLn]}^2 (\delta_n 2^n \sigma_{2^n}^2)^{-1}) \\ &\leq n^{-2} \text{ for sufficiently large } n, \end{aligned}$$

since $\delta_n := \sum_1^{2^n} (\sigma_{2^n} - \sigma_k)^2 / 2^n \sigma_{2^n}^2 \to 0$ by (4.8). This immediately implies (4.4).

It is easy to see that

$$\Delta_m \leqq \Delta_{2^{n+1}} + (\sigma_{2^{n+1}} - \sigma_{2^n}) \max_{1 \leqq m \leqq 2^{n+1}} \left\| \sum_1^m \bar{Y}_k \right\|, \qquad 2^n \leqq m \leqq 2^{n+1}.$$

Since $\sigma_{2^n} \sim \sigma_{2^{n+1}}$ as $n \to \infty$, we can show by similar arguments as in the proof of (4.4) that

$$(\sigma_{2^{n+1}} - \sigma_{2^n}) \max_{1 \leqq m \leqq 2^{n+1}} \left\| \sum_1^m \bar{Y}_k \right\| = o(a_{[2^n L n]}) \quad \text{a.s.}$$

This proves (4.5) and consequently (4.3).

5. An Example

Main purpose of this section is to show that our compact (functional) law of the iterated logarithm is more general than the compact (functional) LIL proved by Kuelbs (1985).

In Sect. 1 it was shown that the moment assumption (2.5) of Kuelbs (1985) implies our condition (1.6). We give now an example of a r.v. $X: \Omega \to \mathbb{R}$ such that our condition (1.6) holds and, at the same time, the LIL of Kuelbs (1985) is no more applicable.

We first give an equivalent reformulation of (1.6).

Let $X: \Omega \to \mathbb{R}$ be a mean zero r.v. in the domain of attraction of $N(0, 1)$. Put $G(t) := E[X^2 1\{|X| \leqq t\}]$, $\tilde{G}(t) := G(t) \vee 1$, $t \geqq 0$. Then we have:

(5.1) Condition (1.6) holds iff $E[X^2/\tilde{G}(|X|) L_2 |X|] < \infty$.

Proof of (5.1). We put $S(t) := \int_0^t E[|X| 1\{|X| > u\}] \, du$, $t > 0$. Then, it is easy to see that $g(t) := t^2/S(t)$, $t > 0$ is an increasing continuous function. Setting $d(t) := g^{-1}(t)$, $t > 0$, we obtain from Proposition 1, Kuelbs (1985):

$$\mathscr{L}\left(\sum_1^n X_k/d(n)\right) \text{ converges weakly to } N(0, 1), \tag{5.2}$$

whenever $\{X_n\}$ is a sequence of i.i.d. r.v.'s with the same distributin as X.

Therefore, our condition (1.6) is equivalent to

$$\sum_1^\infty P(|X| > d(n L_2 n)) < \infty. \tag{5.3}$$

Since $d(n L_2 n) \sim g^{-1} \alpha^{-1}(n)$, where $\alpha(t) = t/L_2 t$, $t > 0$, we see that (5.3) holds iff

$$E[\alpha g(|X|)] < \infty. \tag{5.4}$$

Since G is slowly varying at infinity, $G(t) \sim S(t)$ as $t \to \infty$, we easily obtain that (5.4) is equivalent to $E[X^2/\tilde{G}(|X|) L_2 |X|] < \infty$. □

Moreover, it can be shown that X satisfies the moment assumption (2.5) of Kuelbs (1985) iff $E[X^2/\tilde{G}(|X|/L_2|X|)L_2|X|]<\infty$ (cf. Kuelbs (1985), (4.4)).

Thus, it suffices to find a r.v. $X: \Omega \to \mathbb{R}$ in the domain of attraction of $N(0,1)$ such that

(5.5) $E[X^2/\tilde{G}(|X|)L_2|X|]<\infty$ and $E[X^2/\tilde{G}(|X|/L_2|X|)L_2|X|]=\infty.$

We show:

Example. *There exists a symmetric r.v.* X: $\Omega \to \mathbb{R}$ *in the domain of attraction of* $N(0,1)$ *such that*

$$E[X^2/\tilde{G}(|X|)L_2|X|]<\infty \quad \text{and} \quad E[X^2/\tilde{G}_\alpha(|X|)L_2|X|]=\infty$$

for all $\alpha>0$, *where* $\tilde{G}_\alpha(t):=\tilde{G}(t/(L_2 t)^\alpha)$, $t \geqq 0$.

Proof. We set $m_n:=\exp(n(Ln)(L_2 n)^2)$, $A_n:=\{\exp(m_n+k): 1 \leqq k \leqq [L_2 n]\}$, $n \in \mathbb{N}$.

We define X in a way such that

$$P\left(|X| \in \bigcup_{n=1}^{\infty} A_n\right)=1. \tag{5.6}$$

Because of (5.6) the distribution of X is determined by the following relations:

$$G(\exp(m_1+1))=c \tag{5.7}$$

$$G(\exp(m_n+k))/G(\exp(m_n+k-1))=\exp(1/\sqrt{L_2 n}), \quad 1 \leqq k \leqq [L_2 n],\ n \geqq 2. \tag{5.8}$$

(From (5.6), (5.7) and (5.8) we obtain that we must have:

$$P(X=\exp(m_n+k))=P(X=-\exp(m_n+k))=\tfrac{1}{2}q_{n,k}, \quad 1 \leqq k \leqq [L_2 n],\ n \in \mathbb{N},$$

where $q_{1,1}=c\exp(-2m_1-2)$ and

$$G(\exp(m_n+k-1))(\exp(1/\sqrt{L_2 n})-1)=q_{n,k}\exp(2m_n+2k), \quad 1 \leqq k \leqq [L_2 n],\ n \geqq 2.$$

It now easily follows that

$$\sum_{n=1}^{\infty} \sum_{k=1}^{[L_2 n]} q_{n,k}=cK<\infty,$$

where K is a positive constant. Setting $c=1/K$, we obtain the distribution of X.)

Since we have by (5.8): $G(et)/G(t) \leqq \exp(1/\sqrt{L_2 n})$ for $t \geqq \exp(m_n)$, we immediately see that G is slowly varying at infinity. This shows that X is in the domain of attraction of $N(0,1)$.

To prove that $E[X^2/\tilde{G}(|X|)L_2|X|]<\infty$, it suffices to show:

$$\sum_{n \geqq m_2} \frac{G(\exp(n))-G(\exp(n-1))}{G(\exp(n-1))} \frac{1}{Ln}<\infty. \tag{5.9}$$

From (5.6) and (5.8) we obtain:

$$\sum_{n\geqq m_2} \frac{G(\exp(n))-G(\exp(n-1))}{G(\exp(n-1))\,Ln} \leqq \sum_{n=1}^{\infty} [L_2 n](\exp(1/\sqrt{L_2 n})-1)\frac{1}{nLn(L_2 n)^2}$$
$$\leqq e(nLn(L_2 n)^{3/2})^{-1} < \infty.$$

Since $L_2 \exp(m_n) \geqq n$, $n \geqq 1$, we have for $n \geqq n_0 = n_0(\alpha)$ (say):

$\tilde{G}_\alpha(t) \leqq G(\exp(m_n))$ if $t \leqq \exp(m_{n+1})$.

Therefore, we have:

$$E\left[\frac{X^2}{G_\alpha(|X|)\,L_2|X|}\right] \geqq \sum_{n=n_0}^{\infty} \frac{G(\exp(m_{n+1}))-G(\exp(m_n))}{G(\exp(m_n))}(nLn)^{-1}(L_2 n)^{-2}$$

$$\geqq \sum_{n_0}^{\infty} \exp(\sqrt{L_2 n}-1)(nLn)^{-1}(L_2 n)^{-2} = \infty. \quad \square$$

Let now $\{X_n\}$ be a sequence of i.i.d. r.v.'s with the same distribution as the r.v. X considered in the above example. Since X is in the domain of attraction of $N(0, 1)$, there exists a sequence $a_n \uparrow \infty$ such that

(5.10) $$\mathscr{L}\left(\sum_1^n X_k/a_n\right) \text{ converges weakly to } N(0, 1).$$

Applying our Theorem 1, we obtain:

(5.11) $$\overline{\lim_n} \sum_1^n X_k/a_{[nL_2 n]} = \sqrt{2} \quad \text{a.s.}$$

Since $E[X^2/\tilde{G}(|X|/L_2|X|)\,L_2|X|] = \infty$, we have:

(5.12) $$\overline{\lim_n} \sum_1^n X_k/\gamma_n = \infty \quad \text{a.s.,}$$

if $\{\gamma_n\}$ is the sequence considered in Theorem 1, Kuelbs (1985). This shows that our compact (functional) law of the iterated logarithm is also applicable in situations when the law of the iterated logarithm of Kuelbs (1985) does not hold.

Moreover, we have if $\{X_n\}$ is as above:

(5.13) $$\overline{\lim_n} \sum_1^n X_k/a_n\sqrt{L_2 a_n} = \infty \quad \text{a.s.}$$

(Notice that $E[X^2/\tilde{G}(|X|/\sqrt{L_2|X|})\,L_2|X|] = \infty$.)

This shows that Theorem 1, Feller (1968), cannot hold true as stated there. It can only be valid if $E[X^2/\tilde{G}(|X|/\sqrt{L_2|X|})\,L_2|X|] < \infty$.

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