Probab. Th. Rel. Fields 73, 51–85 (1986)

Probability Theory and Related Fields

Asymptotic Behaviour of Stochastic Flows of Diffeomorphisms: Two Case Studies

Peter H. Baxendale

Department of Mathematics, University of Aberdeen, Aberdeen AB9 2TY, Scotland, U.K.

Summary. The two stochastic flows studied are (i) the canonical stochastic flow on the orthonormal frame bundle of hyperbolic space (which gives stochastic parallel translation along Brownian paths in hyperbolic space) and (ii) a stochastic flow on the sphere S^{n-1} arising from its embedding as the unit sphere in $\mathbb{R}^n$. Both flows are controlled by the same stochastic differential equation in a finite-dimensional Lie group. In each case the Lyapunov exponents are computed and a complete description is given of the local and global stability of the flow.

1. Introduction

This paper studies two examples of stochastic flows on manifolds. The first example is the canonical stochastic flow on the orthonormal frame bundle over a simply-connected manifold of constant negative curvature. It represents stochastic parallel translation along Brownian paths in the manifold. The second example is of a stochastic flow on a sphere which arises naturally from the embedding of the sphere as the unit sphere in Euclidean space. The two examples appear together in this paper as they are derived from the same matrix-valued diffusion process $\{A_t: t \geqq 0\}$ given by a linear stochastic differential equation (s.d.e.). The nature of the coefficients in this s.d.e. ensure that for all $t \geqq 0$, A_t lies in the Lie group $SO_+(n, 1)$, the identity component of the group of all $(n+1)\times(n+1)$ matrices preserving the quadratic form $Q(x, u) = |x|^2 - u^2$ for $(x, u) \in \mathbb{R}^n \times \mathbb{R}$. When we think of $SO_+(n, 1)$ as a subset of the orthonormal frame bundle of a simply-connected manifold of constant negative curvature we obtain the first example. When we think of $SO_+(n, 1)$ as the group of all orientation preserving conformal diffeomorphisms of the sphere S^{n-1} we obtain the second example.

Existence and uniqueness results for the stochastic flow of diffeomorphisms $\{\xi_t: t \geqq 0\}$ of a manifold M corresponding to a s.d.e. on M are now well established. See for example [4, 6, 8]. Methods providing a description of the geometrical nature of the flow and of its sensitivity to perturbations of the initial conditions are less well established. Of interest in this context is the limit as $t \to \infty$ of $\frac{1}{t} \log |D\xi_t(x)(v)|$ for $v \in T_x M$. It describes the limiting rate of exponential growth or decay of the derivative of ξ_t at x in the direction v. The multiplicative ergodic theorem of Oseledec [15] guarantees the existence of such limits, called Lyapunov characteristic numbers or Lyapunov exponents, when $D\xi_t(x)$ is replaced by a family of random matrices with independent increments on the left (under appropriate growth conditions on the increments). Ruelle [18] has extended Oseledec's theorem to apply to (deterministic) dynamical systems on compact manifolds and recently Carverhill [5] has extended these results to stochastic flows on compact manifolds. In both of our main examples, and also for the process $\{A_t^{-1}: t \geqq 0\}$ acting directly on $\mathbb{R}^{n+1}$, we shall compute the set of all possible Lyapunov exponents (i.e. the Lyapunov spectrum) and also the filtration of $T_x M$ which determines which Lyapunov exponent corresponds to a given $v \in T_x M$. We shall also give results on the behaviour of the distance apart of $\xi_t(x)$ and $\xi_t(y)$ as $t \to \infty$ for $x, y \in M$. That is, we obtain "global" as well as "local" stability results. Notice that the (local) stable manifold theorems of Ruelle and Carverhill provide a partial connection between local and global stability of the flow.

Section 2 contains the basic information about $\{A_t: t \geqq 0\}$. The main idea is to use a polar decomposition $A_t = D_t R_t$ where R_t is a rotation fixing $(0, \ldots, 0, 1)$ and D_t is a self-adjoint matrix in $SO_+(n, 1)$ parametrised by $y_t \in D^n = \{x \in \mathbb{R}^n: |x| = 1\}$. It turns out that it is the behaviour of the process $\{y_t: t \geqq 0\}$ which is significant for our purposes. This is described in Theorem 2.3. In particular y_t converges to a limit $\theta_\infty \in S^{n-1} = \{x \in \mathbb{R}^n: |x| = 1\}$ with probability 1 (wp1). The process $\{R_t: t \geqq 0\}$ plays a minor role; the only time we need any information about it is in Sect. 6. We complete Sect. 2 with the first of our stability results.

In Sect. 3 we establish $SO_+(n, 1)$ as a subset of the orthonormal frame bundle $O(D^n)$ over the disc D^n provided with a Riemannian metric ρ of constant curvature $-\lambda^2$ $(\lambda \neq 0)$. If we take $\lambda = 1$ we have the Poincaré disc model of n-dimensional hyperbolic space. The canonical stochastic flow on $O(D^n)$ is the flow of the Stratonovich s.d.e. on $O(D^n)$ determined by the n canonical horizontal vector fields on $O(D^n)$. (See Elworthy [6] or Ikeda and Watanabe [8] for details of stochastic parallel translation and the canonical s.d.e. on the orthonormal frame bundle of a Riemannian manifold.) In our case these vector fields correspond to the (matrix-valued) coefficients in the equation for $\{A_t: t \geqq 0\}$, so we may use the results of Sect. 2 to describe the flow on $O(D^n)$. We see that $\{y_t: t \geqq 0\}$ is a Brownian motion process on D^n with its metric ρ and so some of the results in Sect. 2 turn out to be known results about the limiting behaviour of Brownian motion on a negatively curved manifold.

Section 4 is devoted to a complete description of the local and global stability of the canonical stochastic flow on $O(D^n)$. The Lyapunov exponents are non-random and the filtrations of the tangent spaces depend only on $\theta_\infty = \lim_{t \to \infty} y_t \in S^{n-1}$. The filtrations and the stable manifolds are identical with those arising in the case of the deterministic geodesic flow on $O(D^n)$ in the direction θ_∞.

For any manifold M embedded in Euclidean space $\mathbb{R}^k$ there is a natural stochastic flow on M which may be considered as the projection onto M of the rigid flow $x \mapsto W_t + x$ in $\mathbb{R}^k$ (where W_t is a k-dimensional Brownian motion). The vector fields which determine the flow on M are the gradients of the restrictions to M of the coordinate functions in $\mathbb{R}^k$. In Sect. 5 we study the flow $\{\xi_t : t \geqq 0\}$ on S^{n-1} arising from its embedding as the unit sphere in $\mathbb{R}^n$. The vector fields are infinitesimal conformal transformations of S^{n-1}, so the flow takes place in $SO_+(n, 1)$. For this example we use $\{A_t^{-1} : t \geqq 0\}$ so as to obtain a process with independent left increments. The random limit θ_∞ again plays a major role in the asymptotics of the flow. It turns out that θ_∞ is the unique point in S^{n-1} with the property that for any neighbourhood U of θ_∞ in S^{n-1}, $\mathrm{vol}(\xi_t(U)) \to \mathrm{vol}(S^{n-1})$ as $t \to \infty$. That is, θ_∞ acts like a (random) source point for the stochastic flow.

For the flow on S^{n-1} considered in Sect. 5, for each t the random diffeomorphisms ξ_t and ξ_t^{-1} are identically distributed. So any difference between the processes $\{\xi_t : t \geqq 0\}$ and $\{\xi_t^{-1} : t \geqq 0\}$ arises from the difference between left and right composition in $\mathrm{Diff}(S^{n-1})$. We study this difference in Sect. 6. The one-point motion $\{\xi_t^{-1}(x) : t \geqq 0\}$ is not Brownian motion on S^{n-1} and is not even a diffusion. Instead it converges to θ_∞ (though for $n = 2, 3$ it converges only in an average sense; see Theorems 6.4 and 6.5 for details.) The results on local stability show that in general "backwards" Lyapunov exponents do not exist. Finally, study of $\mathrm{vol}(\xi_t^{-1}(U))$ for $U \subset S^{n-1}$ leads us to a study of the Markov process on the space $M(S^{n-1})$ of Borel probability measures on S^{n-1} induced by the flow $\{\xi_t : t \geqq 0\}$. This Markov process is asymptotically stationary and the limiting stationary process is the image in $M(S^{n-1})$ of Brownian motion in S^{n-1} under the map taking $x \in S^{n-1}$ to the unit mass $\delta(x)$ at x.

Let us establish some conventions. For any Riemannian manifold M, $BM(M)$ will be a Brownian motion process in M, i.e. a diffusion process in M whose infinitesimal generator is $\frac{1}{2}\Delta$, where Δ denotes the Laplace-Beltrami operator on M. For $x \in M$, $BM_x(M)$ will denote $BM(M)$ started at $x \in M$. (Some authors omit the factor $\frac{1}{2}$ in this definition. This accounts for some seemingly contradictory results on the rate of escape to infinity of $BM(M)$. Compare (3.4) with results in [16, 17].) We shall use dW_t (respectively $\circ dW_t$) to denote Itô (respectively Stratonovich) stochastic differentials. All non-Euclidean metrics, whether on S^{n-1}, D^n or $SO_+(n, 1)$ will be denoted by d; we rely on the context to avoid confusion.

The author wishes to thank A.P. Carverhill, K.D. Elworthy and T.E. Harris for many helpful comments on the material in this paper.

2. The Stochastic Differential Equation on $SO_+(n, 1)$

Consider the space $M_{n+1}(\mathbb{R})$ of all $(n+1) \times (n+1)$ real matrices $(n \geqq 2)$ with the block decomposition

$$A = \begin{bmatrix} B & g \\ f^* & u \end{bmatrix}$$

where $B \in M_n(\mathbb{R})$, $f, g \in \mathbb{R}^n$ and $u \in \mathbb{R}$. We identify $f \in \mathbb{R}^n$ with the column vector representing it with respect to the standard basis $\{e_1, e_2, \ldots, e_n\}$ of $\mathbb{R}^n$. f^* denotes the transpose of f, so that $f^* g = \langle f, g \rangle \in \mathbb{R}$ and $g f^* \in M_n(\mathbb{R})$. B^* denotes the transpose of B. Define

$$E_i = \begin{bmatrix} 0 & e_i \\ e_i^* & 0 \end{bmatrix} \quad 1 \leqq i \leqq n$$

and

$$E_{ij} = \begin{bmatrix} e_i e_j^* - e_j e_i^* & 0 \\ 0 & 0 \end{bmatrix} \quad 1 \leqq i,\ j \leqq n.$$

In this section we shall consider the following Stratonovich s.d.e. in $M_{n+1}(\mathbb{R})$.

$$\begin{aligned} dA_t &= \lambda \sum_{i=1}^{n} A_t E_i \circ dW_t^i + \mu \sum_{1 \leqq i < j \leqq n} A_t E_{ij} \circ dW_t^{ij} \\ A_0 &= I \end{aligned} \tag{2.1}$$

where $W_t^1, \ldots, W_t^n, W_t^{12}, \ldots, W_t^{n-1,n}$ are independent $BM_0(\mathbb{R})$ processes and $\lambda, \mu \in \mathbb{R}$ with $\lambda \neq 0$. This is a linear constant coefficient equation for $A_t \in M_{n+1}(\mathbb{R})$ so there exists a unique strong solution $\{A_t : t \geqq 0\}$ to (2.1) with continuous sample paths.

The matrices E_i and E_{ij} lie in the Lie subalgebra $o(n, 1)$ of $M_{n+1}(\mathbb{R})$, where

$$o(n, 1) = \left\{ \begin{bmatrix} B & f \\ f^* & 0 \end{bmatrix} : -B^* = B \in M_n(\mathbb{R}), f \in \mathbb{R}^n \right\}.$$

Therefore the vector fields $A \mapsto A E_i$ and $A \mapsto A E_{ij}$ on $M_{n+1}(\mathbb{R})$ are tangent to the Lie group

$$O(n, 1) = \{A \in M_{n+1}(\mathbb{R}) : A^* K A = K\}$$

where $K = \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix}$. It follows that the solution A_t of (2.1) lies in the identity component $SO_+(n, 1)$ of $O(n, 1)$, where

$$SO_+(n, 1) = \left\{ A = \begin{bmatrix} B & g \\ f^* & u \end{bmatrix} \in O(n, 1) : \det A = 1, u \geqq 1 \right\}.$$

In fact $\{A_t : t \geqq 0\}$ is the most general time homogeneous diffusion process in $SO_+(n, 1)$ with $A_0 = I$, independent increments on the right and whose law is invariant under conjugation by members of $SO(n)$ $\left(\text{where we identify } R \in SO(n) \text{ with } \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} \in SO_+(n, 1)\right)$. Equivalently, if ν_t denotes the distribution of A_t then $\{\nu_t : t \geqq 0\}$ is the most general convolution semigroup of Borel probability measures on $SO_+(n, 1)$ satisfying

(i) $\frac{1}{t} \nu_t(SO_+(n, 1) \setminus U) \to 0$ as $t \to 0$ for all neighbourhoods U of I in $SO_+(n, 1)$.

(ii) ν_t is invariant under conjugation by members of $SO(n)$ for all $t \geqq 0$.

When $\lambda=1$ and $\mu=0$ the process $\{A_t: t\geqq 0\}$ is the horizontal diffusion on the Lie group $SO_+(n,1)$. See Malliavin [14] for very general results on limit laws for horizontal diffusions on semi-simple Lie groups.

Given $A=\begin{bmatrix} B & g \\ f^* & u \end{bmatrix}\in SO_+(n,1)$ we have $B^*B-ff^*=I$, $B^*g=uf$ and $|g|^2-u^2=-1$. If $R=B-(1+u)^{-1}gf^*$ then $R^*R=I$ and $g^*R=f^*$. Therefore

$$A=\begin{bmatrix} R+(1+u)^{-1}gg^*R & g \\ g^*R & u \end{bmatrix}=\begin{bmatrix} I+(1+u)^{-1}gg^* & g \\ g^* & u \end{bmatrix}\begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}$$

with $R\in SO(n)$. Let $y=(1+u)^{-1}g$. Then since $|g|^2-u^2=-1$ we have

$$|y|^2=\frac{u-1}{u+1},\qquad u=\frac{1+|y|^2}{1-|y|^2}\quad\text{and}\quad g=\frac{2y}{1-|y|^2}.$$

In particular $y\in D^n=\{x\in R^n: |x|<1\}$.

Definition. For $y\in D^n$ let

$$D_y=\begin{bmatrix} I+\dfrac{2yy^*}{1-|y|^2} & \dfrac{2y}{1-|y|^2} \\ \dfrac{2y^*}{1-|y|^2} & \dfrac{1+|y|^2}{1-|y|^2} \end{bmatrix}.$$

Proposition 2.1. (i) $D_y\in SO_+(n)$, $D_y^*=D_y$, $(D_y)^{-1}=D_{-y}$.

(ii) *If* $A\in SO_+(n,1)$ *there exist unique* $y\in D^n$ and $R\in SO(n)$ *such that* $A=D_yR$.

(iii) $\Phi: D^n\to M_{n+1}(\mathbb{R})$ *given by* $\Phi(y)=D_y$ *is* C^∞ *and*

$$(D_y)^{-1}D\Phi(y)(u)=\frac{2}{1-|y|^2}\begin{bmatrix} uy^*-yu^* & u \\ u^* & 0 \end{bmatrix} \tag{2.2}$$

for $y\in D^n$, $u\in\mathbb{R}^n$.

(iv) *For* $y\neq 0$,

$$D_y=\exp\left(r\begin{bmatrix} 0 & \theta \\ \theta^* & 0 \end{bmatrix}\right) \tag{2.3}$$

where $\theta=\frac{y}{|y|}\in S^{n-1}$ *and* $r=\log\left(\frac{1+|y|}{1-|y|}\right)=2\tanh^{-1}(|y|)$.

Proof. (i) may be checked directly.

(ii) The existence is proved above; the uniqueness comes from

$$A\begin{bmatrix} 0 \\ 1 \end{bmatrix}=D_yR\begin{bmatrix} 0 \\ 1 \end{bmatrix}=D_y\begin{bmatrix} 0 \\ 1 \end{bmatrix}=\frac{1}{1-|y|^2}\begin{bmatrix} 2y \\ 1+|y|^2 \end{bmatrix}.$$

(iii) and (iv) may be checked directly. For (iii) it is convenient to write $D_y=K+\frac{2}{1-|y|^2}\begin{bmatrix} y \\ 1 \end{bmatrix}[y^*\ 1]$. □

We may now write $A_t = D_t R_t$ where $R_t \in SO(n)$ and $D_t = D_{y_t}$ for $y_t \in D^n$. We obtain from (2.1) a pair of equations for the processes $\{y_t : t \geqq 0\}$ in D^n and $\{R_t : t \geqq 0\}$ in $SO(n) \subset M_n(\mathbb{R})$.

Theorem 2.2. *Equation* (2.1) *has a solution of the form* $A_t = D_{y_t} R_t$ *where*

$$dy_t = \frac{\lambda}{2} \sum_{i=1}^{n} (1 - |y_t|^2) R_t e_i \circ dW_t^i \tag{2.4}$$

$$y_0 = 0$$

$$dR_t = \lambda \sum_{i=1}^{n} (y_t (R_t e_i)^* - R_t e_i y_t^*) R_t \circ dW_t^i + \mu \sum_{i<j} R_t E_{ij} \circ dW_t^{ij} \tag{2.5}$$

$$R_0 = I.$$

Note. In (2.4) we have $E_{ij} = e_i e_j^* - e_j e_i^* \in o(n) \subset M_n(\mathbb{R})$. This is essentially the same identification as used when writing $SO(n) \subset SO_+(n, 1)$.

Proof. $y_t = (1 + u_t)^{-1} g_t$ where

$$\begin{bmatrix} g_t \\ u_t \end{bmatrix} = D_{y_t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = A_t \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Now

$$A_t E_i \begin{bmatrix} 0 \\ 1 \end{bmatrix} = A_t \begin{bmatrix} e_i \\ 0 \end{bmatrix} = D_{y_t} \begin{bmatrix} R_t e_i \\ 0 \end{bmatrix} = \begin{bmatrix} R_t e_i + (1 + u_t)^{-1} \langle g_t, R_t e_i \rangle g_t \\ \langle g_t, R_t e_i \rangle \end{bmatrix}$$

and

$$A_t E_{ij} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = A_t \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore, evaluating (2.1) on the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we obtain

$$dg_t = \lambda \sum_i (R_t e_i + (1 + u_t)^{-1} \langle g_t, R_t e_i \rangle g_t) \circ dW_t^i$$

$$du_t = \lambda \sum_i \langle g_t, R_t e_i \rangle \circ dW_t^i.$$

But $y_t = (1 + u_t)^{-1} g_t$ so

$$\begin{aligned} dy_t &= (1 + u_t)^{-1} \circ dg_t - (1 + u_t)^{-2} g_t \circ du_t \\ &= \lambda (1 + u_t)^{-1} \sum_i (R_t e_i + (1 + u_t)^{-1} \langle g_t, R_t e_i \rangle g_t) \circ dW_t^i \\ &\quad - \lambda (1 + u_t)^{-2} \sum_i \langle g_t, R_t e_i \rangle g_t \circ dW_t^i \\ &= \lambda (1 + u_t)^{-1} \sum_i R_t e_i \circ dW_t^i \\ &= \tfrac{1}{2} \lambda \sum_i (1 - |y_t|^2) R_t e_i \circ dW_t^i. \end{aligned}$$

We have $R_t = D_t^{-1} A_t$ where dA_t is given by (2.1) and $d(D_t)$ is given by (2.4) together with Proposition (2.1)(iii). So in principle we may compute dR_t.

Carrying out this plan we obtain

$$\begin{aligned} dR_t &= -D_t^{-1}(\circ\, d(D_t))\, D_t^{-1} A_t + D_t^{-1} \circ dA_t \\ &= -D_t^{-1}(D\Phi(y_t)(\circ\, dy_t))\, R_t + D_t^{-1} \circ dA_t \\ &= -\lambda \sum_i \begin{bmatrix} R_t e_i y_t - y_t^* (R_t e_i)^* & R_t e_i \\ (R_t e_i)^* & 0 \end{bmatrix} R_t \circ dW_t^i \\ &\quad + \lambda \sum_i R_t E_i \circ dW_t^i + \mu \sum_{i<j} R_t E_{ij} \circ dW_t^{ij} \\ &= \lambda \sum_i \begin{bmatrix} y_t (R_t e_i)^* - R_t e_i y_t^* & 0 \\ 0 & 0 \end{bmatrix} R_t \circ dW_t^i + \mu \sum_{i<j} R_t E_{ij} \circ dW_t^{ij} \end{aligned}$$

as required $\left(\text{recalling the identification of } R \text{ and } \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}\right)$. □

Equations (2.4) and (2.5) are Stratonovich s.d.e.'s. We may convert them into Itô s.d.e.'s.

Theorem 2.2′. *Equation* (2.1) *has a solution of the form* $A_t = D_{y_t} R_t$ *where*

$$\begin{aligned} dy_t &= \tfrac{1}{2}\lambda \sum_{i=1}^{n} (1 - |y_t|^2)\, R_t e_i\, dW_t^i + \tfrac{1}{4}\lambda^2 (1 - |y_t|^2)(n-2)\, y_t\, dt \\ y_0 &= 0 \end{aligned} \tag{2.6}$$

$$\begin{aligned} dR_t &= \lambda \sum_{i=1}^{n} (y_t (R_t e_i)^* - R_t e_i y_t^*)\, R_t\, dW_t^i + \mu \sum_{i<j} R_t E_{ij}\, dW_t^{ij} \\ &\quad - \tfrac{1}{2}\lambda^2 (|y_t|^2 + (n-2)\, y_t y_t^*)\, R_t\, dt - \tfrac{1}{2}\mu^2 (n-1)\, R_t\, dt \\ R_0 &= I. \end{aligned} \tag{2.7}$$

Proof. In order to avoid confusion with the inner product in $\mathbb{R}^n$ we shall use $[\,,\,]$ to denote the martingale quadratic variation function. We have

$$\begin{aligned} \frac{d}{dt} [(1 - |y_t|^2)\, R_t e_i,\, W_t^i] &= -2\langle y_t, \tfrac{1}{2}\lambda (1 - |y_t|^2)\, R_t e_i \rangle\, R_t e_i \\ &\quad + (1 - |y_t|^2)\, \lambda (y_t (R_t e_i)^* - R_t e_i y_t^*)\, R_t e_i \\ &= \lambda (1 - |y_t|^2)(y_t - 2\langle y_t, R_t e_i \rangle\, R_t e_i). \end{aligned}$$

Therefore the Itô correction term for Eq. (2.4) is

$$\begin{aligned} \frac{1}{2} \sum_i \frac{d}{dt} \left[\frac{\lambda}{2} (1 - |y_t|^2)\, R_t e_i,\, W_t^i \right] &= \frac{1}{4} \lambda^2 (1 - |y_t|^2) \sum_i (y_t - 2\langle y_t, R_t e_i \rangle\, R_t e_i) \\ &= \frac{1}{4} \lambda^2 (1 - |y_t|^2)(n-2)\, y_t. \end{aligned}$$

The proof for the other equation is similar. □

Our main interest is in the process $\{y_t : t \geq 0\}$. As a first step let us replace Eq. (2.6) by an equivalent one not involving the process $\{R_t : t \geq 0\}$. Since $W_t^1, \ldots, W_t^n$ are independent $BM_0(\mathbb{R})$ then $W_t = \sum_{i=1}^{n} e_i W_t^i$ is a $BM_0(\mathbb{R}^n)$. Define a process $\{U_t: t \geqq 0\}$ in $\mathbb{R}^n$ by $U_0 = 0$ and

$$\begin{aligned} dU_t &= R_t\, dW_t \\ &= \sum_{i=1}^{n} R_t e_i\, dW_t^i. \end{aligned} \tag{2.8}$$

Since $R_t \in SO(n)$ for all $t \geqq 0$ it follows that U_t is a $BM_0(\mathbb{R}^n)$ (see for example [8], p. 75). Substituting for $W_t^1, \ldots, W_t^n$ in (2.6) we obtain the following equation for $\{y_t: t \geqq 0\}$

$$dy_t = \tfrac{1}{2}\lambda(1-|y_t|^2)\, dU_t + \tfrac{1}{4}\lambda^2(1-|y_t|^2)(n-2)\, y_t\, dt. \tag{2.9}$$

Notice that the law of the process $\{y_t: t \geqq 0\}$ in D^n is independent of μ. We shall see in Sect. 3 that $\{y_t: t \geqq 0\}$ is Brownian motion on D^n when D^n is given a Riemannian structure of constant curvature $-\lambda^2$.

Theorem 2.3. *The process* $\{y_t: t \geqq 0\}$ *is a diffusion process on* D^n, *and* $y_t \neq 0$ *for* $t > 0$ *with probability* 1. *Let*

$$r_t = \log\left(\frac{1+|y_t|}{1-|y_t|}\right) \quad \text{and} \quad \theta_t = \frac{y_t}{|y_t|} \in S^{n-1}, \qquad t > 0.$$

Then

(i) r_t *is a diffusion on* $[0, \infty)$ *satisfying the s.d.e.*

$$dr_t = \lambda\, dV_t + \tfrac{1}{2}\lambda^2(n-1)(\coth r_t)\, dt$$

where V_t *is a* $BM_0(\mathbb{R})$.

(ii) $\frac{1}{t} r_t \to \frac{1}{2}\lambda^2(n-1)$ *as* $t \to \infty$ wp1.

(iii) $\theta_t = X_{\tau(t)}$ *where* $\{X_t: t \geqq 0\}$ *is a* $BM(S^{n-1})$ *independent of* $\{r_t: t \geqq 0\}$ *and the clock* τ *is given by* $\tau'(t) = \lambda^2(\sinh r_t)^{-2}$.

(iv) $\lim_{t\to\infty} \theta_t = \theta_\infty$ *exists and*

$$\limsup_{t\to\infty} \frac{1}{t} \log d(\theta_t, \theta_\infty) = -\tfrac{1}{2}\lambda^2(n-1) \text{ wp1}$$

(*where* d *denotes geodesic distance, i.e. angular separation, in* S^{n-1}).

Proof. In (r, θ) coordinates on D^n the diffusion $\{y_t: t \geqq 0\}$ has generator

$$\frac{1}{2}\lambda^2 \frac{\partial^2}{\partial r^2} + \frac{1}{2}\lambda^2(n-1)(\coth r)\frac{\partial}{\partial r} + \frac{1}{2}\lambda^2(\sinh r)^{-2}\Delta_\theta$$

where Δ_θ denotes the Laplace-Beltrami operator on S^{n-1} acting on the θ variable. Parts (i) and (iii) now follow from the skew product decomposition (see Itô and McKean [9], p. 269). Part (ii) follows immediately from (i) (see Prat [17] and Pinsky [16] for similar but more general results). Since $\frac{1}{t} r_t \to \lambda^2(n-1)$ as $t \to \infty$ it follows that $\tau(\infty) = \int^{\infty} \lambda^2(\sinh r_s)^{-2}\, ds < \infty$ and so θ_t

$=X_{\tau(t)} \to X_{\tau(\infty)} = \theta_\infty$, say, as $t \to \infty$. Further since $BM(S^{n-1})$ is reversible and invariant under rotations the processes $\{d(X_{\tau(t)}, X_{\tau(\infty)}): t>0\}$ and $\{d(X_0, X_{\tau(\infty)-\tau(t)}): t>0\}$ are identically distributed. Notice that

$$\frac{1}{t}\log(\tau(\infty)-\tau(t)) = \frac{1}{t}\log\left(\int_t^\infty \lambda^2 (\sinh r_s)^{-2}\, ds\right) \to -\lambda^2(n-1) \quad \text{as} \quad t \to \infty.$$

The last result is now obtained by replacing t by $\tau(\infty)-\tau(t)$ in the following lemma. □

Lemma 2.4. *Let X_t be a $BM(S^m)$, $m \geqq 1$, and let d* denote geodesic distance in *S^m. Then*

$$\limsup_{t\to 0} \frac{\log d(X_t, X_0)}{\log t} = \frac{1}{2} \text{ wp } 1.$$

Proof. It follows from the law of the iterated logarithm that

$$\limsup_{t\to 0} \frac{\log \rho(t)}{\log t} = \frac{1}{2} \text{ wp } 1$$

where $\rho(t)$ is a $BES(k)$ process, i.e. $\rho(t)$ is the modulus of a $BM_0(\mathbb{R}^k)$, for any $k \geqq 1$. If $m=1$ then $d(X_t, X_0)$ is a $BES(1)$ until it first leaves $[0, \pi)$, so the result holds. For $m \geqq 2$, $r(t) = d(X_t, X_0)$ satisfies $r(t) > 0$ for $t > 0$ and

$$dr_t = dW_t + \frac{m-1}{2}(\cot r_t)\, dt$$

until it first leaves $[0, \pi)$, where W_t is a $BM_0(R)$. There exists $\delta > 0$ such that if $0 < r < \delta$ then

$$\frac{m-2}{2r} < \frac{m-1}{2}(\cot r) < \frac{m-1}{2r}.$$

By the comparison theorem there exist $\rho_1(t)$ in $BES(m-1)$ and $\rho_2(t)$ in $BES(m)$ such that

$$\rho_1(t) \leqq r(t) \leqq \rho_2(t)$$

until $\rho_2(t)$ first leaves $[0, \delta)$. The result now follows. □

Note. The comparison technique above may be sharpened so that the conclusion of the lemma is valid for any non-degenerate diffusion on a Riemannian manifold M.

Corollary 2.5. *With probability* 1, $y_t \to \theta_\infty \in S^{n-1}$ *as* $t \to \infty$ *and*

(i) $\lim_{t\to\infty} \frac{1}{t} \log(1-|y_t|) = -\frac{1}{2}\lambda^2(n-1)$.

(ii) $\lim_{t\to\infty} \frac{1}{t} \log|\theta_\infty - y_t| = -\frac{1}{2}\lambda^2(n-1)$.

Proof. (i) is immediate from Theorem 2.3(ii). To prove (ii), notice

$$|\theta_\infty - y_t|^2 = (1-|y_t|)^2 + 4|y_t| \sin^2(\tfrac{1}{2} d(\theta_t, \theta_\infty)). \quad \square$$

For the majority of our purposes all we need to know about $\{R_t: t \geqq 0\}$ is that it is in $SO(n)$. However, notice that we may extend the idea behind Eqs. (2.8) and (2.9) to obtain

$$dR_t = \lambda \sum_{i=1}^{n} (y_t e_i^* - e_i y_t^*) R_t \, dU_t^i + \mu \sum_{i<j} E_{ij} R_t \, dU_t^{ij} - \tfrac{1}{2}\lambda^2 (|y_t|^2 + (n-2) y_t y_t^*) R_t \, dt - \tfrac{1}{2}\mu^2 (n-1) R_t \, dt \tag{2.10}$$

where $U_t^1, \ldots, U_t^n$ $U_t^{12}, \ldots, U_t^{n-1,n}$ are independent $BM_0(\mathbb{R})$.

Therefore $\{R_t: t \geqq 0\}$ is a process in $SO(n)$ with left increments depending in law only on $\{y_t: t \geqq 0\}$. For any $y \neq 0$, the matrices $y e_i^* - e_i y^*$, $1 \leqq i \leqq n$, generate $o(n)$ as a Lie algebra, so that even when $\mu = 0$ the process $\{R_t: t \geqq 0\}$ is an indecomposable process in $SO(n)$. We refer the reader to Ichihara and Kunita [7] for results on the ergodicity of the limiting form of Eq. (2.10) when the variable y_t is replaced by the limit $\theta_\infty \in S^{n-1}$.

We complete this section by giving our first stability result. So as to obtain independent increments on the left we consider the process $\{A_t^{-1}: t \geqq 0\}$ rather than $\{A_t: t \geqq 0\}$ as a linear stochastic flow on $\mathbb{R}^{n+1}$.

Theorem 2.6. *With probability* 1, $\theta_\infty = \lim_{t\to\infty} y_t$ exists and for $z = (x, u) \in \mathbb{R}^n \times \mathbb{R}$, $z \neq (0, 0)$,

$$\lim_{t\to\infty} \frac{1}{t} \log |A_t^{-1}(z)| = \begin{cases} -\frac{1}{2}\lambda^2 (n-1) & \text{if } x = u\theta_\infty \\ 0 & \text{if } x \neq u\theta_\infty \text{ and } u = \langle x, \theta_\infty \rangle \\ \frac{1}{2}\lambda^2 (n-1) & \text{if } u \neq \langle x, \theta_\infty \rangle. \end{cases}$$

Proof.

$$\begin{aligned} (A_t^{-1})^* A_t^{-1} &= D_{y_t}^{-1} R_t R_t^{-1} D_{y_t}^{-1} \\ &= D_{y_t}^{-2} \\ &= \exp\left(-2r_t \begin{bmatrix} 0 & \theta_t \\ \theta_t^* & 0 \end{bmatrix}\right), \end{aligned}$$

so

$$\begin{aligned} [(A_t^{-1})^* A_t^{-1}]^{1/2t} &= \exp\left(-\frac{1}{t} r_t \begin{bmatrix} 0 & \theta_t \\ \theta_t^* & 0 \end{bmatrix}\right) \\ &\to \exp\left(-\tfrac{1}{2}\lambda^2 (n-1) \begin{bmatrix} 0 & \theta_\infty \\ \theta_\infty^* & 0 \end{bmatrix}\right) \\ &= \Lambda, \text{ say, as } t \to \infty, \end{aligned} \tag{2.11}$$

by Theorem 2.3. Now $\begin{bmatrix} 0 & \theta_\infty \\ \theta_\infty^* & 0 \end{bmatrix}$ has eigenspaces

$$\begin{gathered} \{(x, u): x = u\theta_\infty\} \\ \{(x, 0): \langle x, \theta_\infty \rangle = 0\} \\ \{(x, u): x = -u\theta_\infty\} \end{gathered}$$

corresponding to eigenvalues $1, 0, -1$ respectively, so Λ has the same eigenspaces with eigenvalues $\exp(-\frac{1}{2}\lambda^2(n-1))$, 1, $\exp(\frac{1}{2}\lambda^2(n-1))$ respectively. Suppose

$z=z_-+z_0+z_+$ is the decomposition of z as a sum of vectors in the three eigenspaces above. Then using the convergence (2.11) and the positivity of $(A_t^{-1})^* A_t^{-1}$ and Λ we obtain

$$\begin{aligned}\lim_{t\to\infty} |A_t^{-1}(z)|^{1/t} &= \lim_{t\to\infty} \langle z, (A_t^{-1})^* A_t^{-1}(z)\rangle^{1/2t} \\ &= \lim_{t\to\infty} \langle z, \Lambda^{2t} z\rangle^{1/2t} \\ &= \lim_{t\to\infty} \{\exp(-t\lambda^2(n-1))|z_-|^2+|z_0|^2 \\ &\quad +\exp(t\lambda^2(n-1))|z_+|^2\}^{1/2t} \\ &= \begin{cases} \exp(-\frac{1}{2}\lambda^2(n-1)) & \text{if } z_-\neq 0,\ z_0=z_+=0 \\ 1 & \text{if } z_0\neq 0,\ z_+=0 \\ \exp(\frac{1}{2}\lambda^2(n-1)) & \text{if } z_+\neq 0. \quad \square \end{cases}\end{aligned}$$

The proof is similar in style to the version of the multiplicative ergodic theorem given in Ruelle [18]. Notice that the multiplicative ergodic theorem asserts the existence of the limit of $[(A_t^{-1})^* A_t^{-1}]^{1/2t}$ for a general class of processes whereas in this case we obtained the limit by direct calculation.

We have obtained a Lyapunov spectrum consisting of $-\frac{1}{2}\lambda^2(n-1), 0$ (with multiplicity $n-1$) and $\frac{1}{2}\lambda^2(n-1)$. It is fixed (i.e. non-random). The filtration of $\mathbb{R}^{n+1}$ is random but depends only on the limiting value θ_∞. This phenomenon will be repeated later. Since $\{A_t^{-1}: t\geqq 0\}$ consists of linear maps of $\mathbb{R}^{n+1}$ there is no distinction between local and global behaviour of the flow.

3. The Canonical Stochastic Differential Equation on a Space of Constant Negative Curvature

The most general inner product in $o(n,1)$ which is invariant under the adjoint action of $SO(n)$ is

$$\left\langle \begin{bmatrix} B & f \\ f^* & 0 \end{bmatrix}, \begin{bmatrix} C & g \\ g^* & 0 \end{bmatrix} \right\rangle = \alpha^2 \operatorname{tr}(B^* C) + \beta^2 \langle f, g\rangle \tag{3.1}$$

for $\alpha, \beta > 0$. We shall identify elements of $o(n,1)$ with left invariant vector fields on $SO_+(n,1)$; then (3.1) gives a left invariant Riemannian structure on $SO_+(n,1)$ with a corresponding left invariant metric d. Denote the norm on $TSO_+(n,1)$ by $\|\ \|$. Since $\langle\ ,\ \rangle$ is invariant under $\operatorname{Ad}(SO(n))$ then d is invariant under right translations by elements of $SO(n)$. Thus

$$\begin{aligned} d(A,B) &= d(CA, CB), \quad \text{any } C\in SO_+(n,1) \\ &= d(AR, BR), \quad \text{any } R\in SO(n). \end{aligned}$$

Define $\pi: SO_+(n,1)\to D^n$ by $\pi(D_y R)=y$. This induces an action of $SO_+(n,1)$ on D^n as follows. If $A\in SO_+(n,1)$ and $y\in D^n$ define $\tilde{A}y=\pi(AD_y)$ (or more generally $\tilde{A}y=\pi(AB)$ for any $B\in\pi^{-1}(y)$; it is easy to check this is well defined).

From the formula for D_y we obtain $\tilde{A}y = x/(1+u)$ where

$$\begin{bmatrix} x \\ u \end{bmatrix} = \frac{1}{1-|y|^2} A \begin{bmatrix} 2y \\ 1+|y|^2 \end{bmatrix}.$$

Notice that if $R \in SO(n) \subset SO_+(n, 1)$ then $\tilde{R} = R$. An equivalent definition would be to say $\tilde{A}$ is the conjugation of the natural action of A on $H^n = \{(x, u) \in \mathbb{R}^n \times \mathbb{R} : |x|^2 - |u|^2 = -1\}$ by the diffeomorphism $\phi: H^n \to D^n$ given by $\phi(x, u) = x/(1+u)$. For future reference we notice that the action can be extended continuously to the closed ball $\bar{D}_n$. For $y \in S^{n-1} = \partial \bar{D}_n$ we have $\tilde{A}y = x/u$ where $\begin{bmatrix} x \\ u \end{bmatrix} = A \begin{bmatrix} y \\ 1 \end{bmatrix}$.

Define a Riemannian structure ρ on D^n by

$$\rho_y(f, g) = \frac{4\beta^2}{(1-|y|^2)^2} \langle f, g \rangle, \qquad f, g \in T_y D^n \cong \mathbb{R}^n \tag{3.2}$$

where $\langle f, g \rangle$ denotes the standard Euclidean inner product on $\mathbb{R}^n$. We shall use (D^n, ρ) at times to emphasize the metric ρ (as opposed to the Euclidean metric). We use the same symbol d to denote the metric on (D^n, ρ) as well as on $SO_+(n, 1)$.

Lemma 3.1. (i) *For all* $A \in SO_+(n, 1)$

$$D\pi(A): T_A SO_+(n, 1) \to T_{\pi(A)} D^n$$

is a partial isometry.

(ii) $SO_+(n, 1)$ *acts on* (D^n, ρ) *by isometries.*

(iii) (D^n, ρ) *has constant negative curvature* $-1/\beta^2$.

Proof. (i) Suppose $B = \begin{bmatrix} C & e \\ e^* & 0 \end{bmatrix} \in o(n, 1)$. Then

$$(\exp tB) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} C & e \\ e^* & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + O(t^2) = \begin{bmatrix} te \\ 1 \end{bmatrix} + O(t^2)$$

so $\pi(\exp tB) = \frac{1}{2} te + O(t^2)$ as $t \to 0$. Therefore $D\pi(I)(B) = \frac{1}{2} e$. Also for $x, y \in D^n$,

$$\tilde{D}_y(x) = \frac{(1-|y|^2)x + (1+|x|^2 + 2\langle x, y \rangle) y}{1 + 2\langle x, y \rangle + |x|^2 |y|^2},$$

so for $u \in T_0 D^n$ we have

$$D(\tilde{D}_y)(0)(u) = (1-|y|^2) u \in T_y D^n.$$

Together, if B is as above and $A = D_y R \in SO_+(n, 1)$ then

$$\begin{aligned} D\pi(A)(AB) &= \frac{d}{dt}\,(\pi(A(\exp tB)))|_{t=0} \\ &= \frac{d}{dt}\,(\tilde{D}_y R\pi(\exp tB))|_{t=0} \\ &= D(\tilde{D}_y)(0)(\tfrac{1}{2}Re) \\ &= \tfrac{1}{2}(1-|y|^2)\,Re. \end{aligned}$$

The result follows.

(ii) is now immediate as left translation in $SO_+(n,1)$ is an isometry of $SO_+(n,1)$.

(iii) In geodesic polar coordinates (r,θ) centred at $0\in D^n$ the Riemannian metric ρ may be written

$$ds^2 = dr^2 + \beta^2(\sinh(r/\beta))^2\,d\theta^2$$

where $r=\beta\log\left(\frac{1+|y|}{1-|y|}\right)=2\beta\tanh^{-1}(|y|)$. The result follows. □

We now identify $SO_+(n,1)$ as a subset of the orthonormal frame bundle $O(D^n)$ of (D^n,ρ).

Let
$$O_+(n,1)=\left\{A=\begin{bmatrix} B & g \\ f^* & u\end{bmatrix}\in O(n,1)\colon u\geqq 1\right\}.$$

Repeating the work leading up to Proposition 2.1(ii) we see that any $A\in O_+(n,1)$ may be written uniquely as $A=D_yR$ where $y\in D^n$ and $R\in O(n)$ (but now $\det(R)=-1$ is a possibility). We extend the definition of π to $\pi\colon O_+(n,1)\to D^n$. Corresponding to $A=D_yR\in O_+(n,1)$ we have $y=\pi(A)\in D^n$ and the orthonormal frame at y given by $\frac{1}{2\beta}(1-|y|^2)R\colon \mathbb{R}^n\to T_yD^n\cong\mathbb{R}^n$. The right action of $O(n)$ on $O_+(n,1)$ completes the identification of $\pi\colon O_+(n,1)\to D^n$ as the orthonormal frame bundle $\pi\colon O(D^n)\to D^n$. Notice that the metric on $O(D^n)$ induced by (3.1) on $o(n,1)$ is the most general metric on $O(D^n)$ which is invariant under the action of all isometries of D^n. $O(D^n)$ splits into two components according to the orientation of the frame, and $SO_+(n,1)$ corresponds to the component $SO(D^n)$, say, consisting of frames with positive orientation relative to the natural orientation of $D^n\subset\mathbb{R}^n$. (For an alternative treatment see Kobayashi and Nomizu [11], pp. 204–209, in which it is also shown that $O_+(n,1)$ is the group of all isometries of (D^n,ρ).)

At $I\in O_+(n,1)$ the horizontal subspace of $T_IO_+(n,1)=o(n,1)$ corresponding to the Riemannian connection on D^n is the subspace

$$\left\{\begin{bmatrix} 0 & e \\ e^* & 0\end{bmatrix}\colon e\in\mathbb{R}^n\right\}\subset o(n,1).$$

Recalling that $D\pi(I)(E_i)=\frac{1}{2}e_i$ and that $\frac{1}{2\beta}e_i$ is the i^{th} vector in the frame I we see that $\frac{1}{\beta}E_i$ is the value at I of the i^{th} canonical horizontal vector field on

$O(D^n)=O_+(n,1)$. By left translation we obtain $A \mapsto \frac{1}{\beta} AE_i$ as the i^{th} canonical horizontal vector field on $O(D^n)$. Therefore the canonical s.d.e. on $O(D^n)$ is

$$dA_t=\frac{1}{\beta}\sum_{i=1}^{n} A_t E_i \circ dW_t^i. \tag{3.3}$$

If we restrict A_t to denote the solution of (3.3) with $A_0=I$ then the flow $\{F_t: t\geqq 0\}$ on $O(D^n)$ of the canonical s.d.e. is given by $F_t(A)=AA_t$. The flow decomposes naturally into flows on the two components of $O(D^n)$ and left multiplication by an element of $O_+(n,1)\setminus SO_+(n,1)$ sends the flow on one component into the flow on the other component. Henceforth we shall restrict our attention to the flow on $SO_+(n,1)=SO(D^n)$ the bundle of positively oriented frames, although we shall abuse terminology by continuing to refer to it as the canonical stochastic flow on $O(D^n)$.

Notice that (3.3) with $A_0=I$ is (2.1) with $\lambda=1/\beta$ and $\mu=0$, so we can apply the results of Sect. 2. In particular $y_t=\pi(A_t)$ is Brownian motion on (D^n,ρ) so we may interpret Theorem 2.3 as a result about the asymptotic behaviour of this Brownian motion process. Since $r_t=\lambda d(0,y_t)$ we obtain

$$\frac{1}{t}\, d(0,y_t)\to\tfrac{1}{2}\lambda(n-1) \quad \text{as} \quad t\to\infty \ \text{wp}1 \tag{3.4}$$

for Brownian motion in an n-dimensional space of constant curvature $-\lambda^2$.

4. Asymptotic Behaviour of the Canonical Stochastic Flow on $O(D^n)$

We restrict attention to the flow $F_t(A)=AA_t$ on $SO_+(n,1)=SO(D^n)$, the bundle of positively oriented frames. Results for the flow on the full orthonormal frame bundle follow automatically. Recall that with its Riemannian structure ρ, D^n becomes a space of constant curvature $-\lambda^2$. Throughout this section we shall omit the qualifying phrase "with probability 1". All results will be valid on the set of probability 1 on which the estimates of Theorem 2.3(ii) and (iv) hold true. In particular the set of probability 1 will not depend on any initial condition $F_0(A)=A\in SO_+(n,1)$ or any tangent $AB\in T_A SO_+(n,1)$ for $B\in o(n,1)$.

Definition. For $\theta\in S^{n-1}$ let

$$V_\theta^-=\left\{\begin{bmatrix}\theta e^*-e\theta^* & e\\ e^* & 0\end{bmatrix}: \langle e,\theta\rangle=0\right\}\subset o(n,1)$$

$$V_\theta^0=\left\{\begin{bmatrix}C & e\\ e^* & 0\end{bmatrix}: C\theta+e=\langle e,\theta\rangle\,\theta\right\}\subset o(n,1).$$

Theorem 4.1. *Let* $\theta=\theta_\infty=\lim_{t\to\infty} y_t\in S^{n-1}$. *Then for* $A\in SO_+(n,1)$ *and* $B\in o(n,1)$,

$$\lim_{t\to\infty}\frac{1}{t}\log\|DF_t(A)(AB)\|=\begin{cases}-\frac{1}{2}\lambda^2(n-1) & \text{if } B\in V_\theta^-\setminus\{0\}\\ 0 & \text{if } B\in V_\theta^0\setminus V_\theta^-\\ \frac{1}{2}\lambda^2(n-1) & \text{if } B\notin V_\theta^0.\end{cases}$$

Proof. $F_t(A)=AA_t$, so $DF_t(A)(AB)=ABA_t\in T_{AA_t}SO_+(n,1)$ and

$$\begin{aligned}\|DF_t(A)(AB)\| &= \|ABA_t\|\\ &= \|(AA_t)^{-1}ABA_t\|\\ &= \|A_t^{-1}BA_t\|\\ &= \|\mathrm{Ad}(A_t^{-1})B\|\end{aligned}$$

where Ad denotes the adjoint action of $SO_+(n,1)$ on $o(n,1)$. Since the result is independent of the choice of inner product on $o(n,1)$ we choose $\alpha=\beta/\sqrt{2}$ in (3.1), which ensures that $\mathrm{ad}(E_i)$ is self-adjoint for $1\leqq i\leqq n$. Therefore $\mathrm{ad}\begin{bmatrix}0 & \theta_t\\ \theta_t^* & 0\end{bmatrix}$ is self-adjoint, and so is

$$\begin{aligned}\mathrm{Ad}(D_t^{-1}) &= \mathrm{Ad}\left(\exp\left(-r_t\begin{bmatrix}0 & \theta_t\\ \theta_t^* & 0\end{bmatrix}\right)\right)\\ &= \exp\left(-r_t\left(\mathrm{ad}\begin{bmatrix}0 & \theta_t\\ \theta_t^* & 0\end{bmatrix}\right)\right).\end{aligned}$$

Also, since the inner product on $o(n,1)$ is $\mathrm{Ad}(SO(n))$ invariant, we have $(\mathrm{Ad}(R_t^{-1}))^*=\mathrm{Ad}(R_t)$. So

$$\begin{aligned}(\mathrm{Ad}(A_t^{-1}))^*(\mathrm{Ad}(A_t^{-1})) &= (\mathrm{Ad}(R_t^{-1})\,\mathrm{Ad}(D_t^{-1}))^*\,\mathrm{Ad}(R_t^{-1})\,\mathrm{Ad}(D_t^{-1})\\ &= (\mathrm{Ad}(D_t^{-1}))^2\\ &= \exp\left(-2r_t\left(\mathrm{ad}\begin{bmatrix}0 & \theta_t\\ \theta_t^* & 0\end{bmatrix}\right)\right).\end{aligned}$$

Therefore

$$\begin{aligned}[(\mathrm{Ad}(A_t^{-1}))^*(\mathrm{Ad}(A_t^{-1}))]^{1/2t} &= \exp\left(-\frac{1}{t}r_t\left(\mathrm{ad}\begin{bmatrix}0 & \theta_t\\ \theta_t^* & 0\end{bmatrix}\right)\right)\\ &\to \exp\left(-\tfrac{1}{2}\lambda^2(n-1)\,\mathrm{ad}\begin{bmatrix}0 & \theta\\ \theta^* & 0\end{bmatrix}\right)\\ &= \Lambda,\ \text{say, as } t\to\infty.\end{aligned}$$

Now $\mathrm{ad}\begin{bmatrix}0 & \theta\\ \theta^* & 0\end{bmatrix}$ acting on $o(n,1)$ has eigenspaces

$$U_\theta^-=\left\{\begin{bmatrix}\theta e^*-e\theta^* & e\\ e^* & 0\end{bmatrix}:\langle e,\theta\rangle=0\right\}$$

$$U_\theta^0=\left\{\begin{bmatrix}C & \gamma\theta\\ \gamma\theta^* & 0\end{bmatrix}: C\theta=0,\ \gamma\in\mathbb{R}\right\}$$

$$U_\theta^+=\left\{\begin{bmatrix}\theta e^*-e\theta^* & -e\\ -e^* & 0\end{bmatrix}:\langle e,\theta\rangle=0\right\}$$

corresponding to eigenvalues $1, 0, -1$ (respectively). Notice that $V_\theta^-=U_\theta^-$ and $V_\theta^0=U_\theta^-+U_\theta^0$. The result now follows using the same argument as in the proof of Theorem 2.6. □

Note. If we put $T_t = \mathrm{Ad}(A_t^{-1})$, then it follows from (3.3) that

$$dT_t = -\lambda \sum_{i=1}^{n} \mathrm{ad}(E_i)\, T_t \circ dW_t^i$$

and the multiplicative ergodic theorem applies to the process $\{T_t: t \geqq 0\}$.

In the language of Lyapunov exponents we have a (non-random) Lyapunov spectrum consisting of $-\frac{1}{2}\lambda^2(n-1), 0, \frac{1}{2}\lambda^2(n-1)$ with multiplicities $n-1$, $\frac{1}{2}(n-1)(n-2)+1, n-1$ respectively and associated filtration of $T_A SO_+(n,1)$:

$$\begin{aligned}\{0\} &\subset A V_\theta^- = \{AB: B \in V_\theta^-\}\\ &\subset A V_\theta^0 = \{AB: B \in V_\theta^0\}\\ &\subset T_A SO_+(n,1).\end{aligned}$$

The filtration is random but depends only on $\theta = \theta_\infty = \lim_{t\to\infty} y_t$.

For any $\theta \in S^{n-1}$ both V_θ^- and V_θ^0 are Lie subalgebras of $o(n,1)$, so the distributions $\{AV_\theta^-: A \in SO_+(n,1)\}$ and $\{AV_\theta^0: A \in SO_+(n,1)\}$ are both integrable. Thus we obtain foliations $\mathscr{F}_\theta^-$ and $\mathscr{F}_\theta^0$ of $SO_+(n,1)$ determined by these distributions. We shall identify the leaves of these foliations and then show the role they play in the global stability of the flow.

Definition. For $\theta \in S^{n-1}$ let

$$\begin{aligned}H_\theta^- &= \{A \in SO_+(n,1): \tilde{A}\theta = \theta \text{ and } D(\tilde{A}|_{S^{n-1}})(\theta) = I|_{T_\theta S^{n-1}}\}\\ H_\theta^0 &= \{A \in SO_+(n,1): \tilde{A}\theta = \theta\}.\end{aligned}$$

Lemma 4.2. *If $\tilde{A}\theta = \theta$ and $\tilde{A}(-\theta) = -\theta$ then $A = D_y R$ where $y = s\theta$ for some $s \in (-1,1)$ and $R\theta = \theta$. In particular if $A \in H_\theta^-$ and $\tilde{A}(-\theta) = -\theta$ then $A = I$.*

Proof. If $A = \begin{bmatrix} B & g \\ f^* & u \end{bmatrix} = D_y R$ then $R = B - (1+u)^{-1} g f^*$, $y = (1+u)^{-1} g$ and for $\psi \in S^{n-1}$ we have $\tilde{A}\psi = \dfrac{B\psi + g}{\langle f, \psi\rangle + u}$. The assumptions on A imply $B\theta = u\theta$ and $g = \langle f, \theta\rangle\,\theta$. The first result now follows (recalling $|g|^2 = u^2 - 1$). For $A = D_y R$ as above, a direct computation shows that

$$D(\tilde{A}|_{S^{n-1}})(\theta) = \left(\frac{1-|y|}{1+|y|}\right) R|_{T_\theta S^{n-1}}.$$

Therefore if $A \in H_\theta^-$ we have $s = 0$, $R = I$ as required. □

Lemma 4.3. (i) *H_θ^- is a closed connected Abelian subgroup of $SO_+(n,1)$ with Lie algebra V_θ^-. In particular $H_\theta^- = \{\exp B: B \in V_\theta^-\}$.*

(ii) *H_θ^0 is a closed connected subgroup of $SO_+(n,1)$ with Lie algebra V_θ^0. Any $A \in H_\theta^0$ is of the form $A = C D_y R$ where $C \in H_\theta^-$, $y = s\theta$ with $s \in (-1,1)$ and $R\theta = \theta$.*

Proof. (i) Since V_θ^- is an Abelian subalgebra of $o(n,1)$ then $\{\exp B: B \in V_\theta^-\}$ is a connected Abelian subgroup of $SO_+(n,1)$. Since H_θ^- is clearly closed it suffices to prove $H_\theta^- = \{\exp B: B \in V_\theta^-\}$.

If $B=\begin{bmatrix}\theta e^*-e\theta^* & e\\ e^* & 0\end{bmatrix}\in V_\theta^-$ then

$$C=\exp B=\begin{bmatrix}I+\theta e^*-e\theta^*-\frac{1}{2}|e|^2\,\theta\theta^* & e+\frac{1}{2}|e|^2\,\theta\\ e^*-\frac{1}{2}|e|^2\,\theta^* & 1+\frac{1}{2}|e|^2\end{bmatrix}, \tag{4.1}$$

and we may check explicitly that $C\in H_\theta^-$. Notice

$$\tilde{C}(-\theta)=\frac{-(1-|e|^2)\,\theta+2e}{1+|e|^2}.$$

So if $A\in H_\theta^-$ and $\tilde{A}(-\theta)=\psi$ (where necessarily $\psi\neq\theta$) there exists $e\in\theta^\perp\subset\mathbb{R}^n$ such that $\tilde{A}(-\theta)=\tilde{C}(-\theta)$ for C as above. Then $C^{-1}A\in H_\theta^-$ and $(C^{-1}A)^\sim(-\theta)=-\theta$, so by Lemma 4.2 $C^{-1}A=I$, i.e. $A=C=\exp B$ as required.

(ii) Clearly H_θ^0 is a closed subgroup of $SO_+(n,1)$. The same method as above shows that if $A\in H_\theta^0$ then there exists $C\in H_\theta^-$ such that $A=CD_yR$, where $y=s\theta$ with $s\in(-1,1)$ and $R\theta=\theta$. It follows that H_θ^0 is connected. Finally for any $B=\begin{bmatrix}C & e\\ e^* & 0\end{bmatrix}\in o(n,1)$,

$$\exp tB=\begin{bmatrix}I+tC & te\\ te^* & 1\end{bmatrix}+O(t^2)$$

so

$$(\exp tB)^\sim(\theta)=\frac{\theta+tC\theta+te}{t\langle e,\theta\rangle+1}+O(t^2)$$
$$=\theta+t(C\theta+e-\langle e,\theta\rangle\,\theta)+O(t^2)$$

as $t\to 0$. Hence B is in the Lie algebra of H_θ^0 if and only if $C\theta+e-\langle e,\theta\rangle\,\theta=0$. □

Corollary 4.4. *The leaf through A of the foliation $\mathscr{F}_\theta^-$ (respectively $\mathscr{F}_\theta^0$) is the coset AH_θ^- (respectively AH_θ^0).*

Theorem 4.5. *Let $A\neq B\in SO_+(n,1)$. Let $\theta=\theta_\infty=\lim_{t\to\infty} y_t\in S^{n-1}$. Then*

(i) $\frac{1}{t}\log d(F_t(A),F_t(B))\to -\frac{1}{2}\lambda^2(n-1)$ *as $t\to\infty$ if $B\in AH_\theta^-$ (i.e. if A and B are in the same leaf of $\mathscr{F}_\theta^-$) and the convergence is uniform for $B^{-1}A$ in compact subsets of $H_\theta^-\setminus\{I\}$.*

(ii) $\liminf_{t\to\infty} d(F_t(A),F_t(B))>0$ *and*

$$\frac{1}{t}\,d(F_t(A),F_t(B))\to 0 \quad \textit{as} \quad t\to\infty$$

if $B\in AH_\theta^0\setminus AH_\theta^-$ (i.e. if A and B are in the same leaf of $\mathscr{F}_\theta^0$ and different leaves of $\mathscr{F}_\theta^-$) and the convergence is uniform for $B^{-1}A$ in compact subsets of $H_\theta^0\setminus H_\theta^-$.

(iii) $\frac{1}{t}\,d(F_t(A),F_t(B))\to\lambda(n-1)$ *as $t\to\infty$ if $B\notin AH_\theta^0$ (i.e. if A and B are in different leaves of $\mathscr{F}_\theta^0$) and the convergence is uniform for $B^{-1}A$ in compact subsets of $SO_+(n,1)\setminus H_\theta^0$.*

The result in (i) is the obvious analogue of the first case in Theorem 4.1. It reflected in the behaviour of $d(F_t(A), F_t(B))$. However in cases (ii) and (iii) the global nature of $SO_+(n,1)$ and its metric d become significant and the strict analogy with Theorem 4.1 breaks down. Indeed in both cases (ii) and (iii) we have the weaker result $\frac{1}{t}\log d(F_t(A), F_t(B)) \to 0$ as $t\to\infty$. Before we prove the theorem we obtain some results on the metrics on $SO_+(n,1)$ and D^n (both denoted by d).

Lemma 4.6. *Let $y, z\in D^n$, $R, S\in SO(n)$, $\theta\in S^{n-1}$ and $s\in(-1,1)$.*

(i) $d(y,z)\leqq d(D_y R, D_z S)\leqq d(y,z)+K$ *where* $K=\sup\{d(I,Q)\colon Q\in SO(n)\}$.

(ii) $d(D_y, D_z)\leqq \frac{1}{\beta}\sqrt{2\alpha^2+\beta^2}\, d(y,z)$.

(iii) $d(RA, A)\geqq \left(1+\frac{1}{\beta}\sqrt{2\alpha^2+\beta^2}\right)^{-1} d(R, I)$.

(iv)
$$\cosh\left(\frac{1}{\beta} d(\tilde{D}_{s\theta}(Ry), y)\right)=\frac{1+s^2}{1-s^2}+\frac{4(|y|^2-\langle y, Ry\rangle)}{(1-|y|^2)^2}+\frac{8s^2(|y|^2-\langle y,\theta\rangle\langle Ry,\theta\rangle)}{(1-s^2)(1-|y|^2)^2}+\frac{4s(1+|y|^2)\langle\theta, y-Ry\rangle}{(1-s^2)(1-|y|^2)^2}.$$

Proof. Recall $D_y^{-1}(D\Phi(y)(u))=\frac{2}{1-|y|^2}\begin{bmatrix} uy^*-yu^* & u\\ u^* & 0\end{bmatrix}$ so

$$\|D\Phi(y)(u)\|^2=\frac{4}{(1-|y|^2)^2}\{2\alpha^2(|u|^2|y|^2-\langle u,y\rangle^2)+\beta^2|u|^2\}.$$

(i) Since $\pi(D_y R)=y$ and π is a partial isometry, then $d(y,z)\leqq d(D_y R, D_z S)$. Now $p(s)=D_{sy}$, $0\leqq s\leqq 1$ is a path in $SO_+(n,1)$ from I to D_y and $p'(s)=D\Phi(sy)(y)$. So

$$\begin{aligned} d(I, D_y) &\leqq \int_0^1 \|D\Phi(sy)(y)\|\, ds\\ &=\int_0^1 \frac{2\beta|y|}{1-s^2|y|^2}\, ds\\ &=\beta\log\left(\frac{1+|y|}{1-|y|}\right)\\ &=d(0,y). \end{aligned}$$

But $d(0,y)\leqq d(I, D_y)$ by the above, so $d(0,y)=d(I,D_y)$. Therefore, letting $D_y^{-1}D_z=D_x T$ with $T\in SO(n)$ (so that $x=\tilde{D}_y^{-1}(z)$),

$$\begin{aligned} d(D_y R, D_z S) &= d(I, D_y^{-1} D_z S R^{-1})\\ &= d(I, D_x TSR^{-1})\\ &\leqq d(I, D_x)+d(D_x, D_x TSR^{-1})\\ &= d(0,x)+d(I, TSR^{-1})\\ &\leqq d(\tilde{D}_y(0), \tilde{D}_y(x))+K\\ &= d(y,z)+K \text{ as required.} \end{aligned}$$

(ii) This follows directly from the estimate

$$\|D\Phi(y)(u)\|^2 \leqq \frac{4}{(1-|y|^2)^2}(2\alpha^2+\beta^2)|u|^2$$
$$= \frac{2\alpha^2+\beta^2}{\beta^2}\rho_y(u,u).$$

(iii) Suppose $A=D_y S$, so $RA=RD_y S=D_{Ry}RS$. Then

$$\begin{aligned} d(R,I) &= d(D_{Ry}RS, D_{Ry}S) \\ &\leqq d(D_{Ry}RS, D_y S)+d(D_y S, D_{Ry}S) \\ &= d(RA,A)+d(D_y, D_{Ry}) \\ &\leqq d(RA,A)+\frac{1}{\beta}\sqrt{2\alpha^2+\beta^2}\, d(y,Ry) \\ &\leqq d(RA,A)+\frac{1}{\beta}\sqrt{2\alpha^2+\beta^2}\, d(A,RA) \end{aligned}$$

using (i) and (ii) in the last two inequalities.

(iv) Suppose

$$D_{-y}D_{s\theta}\begin{bmatrix} 2Ry(1-|y|^2)^{-1} \\ (1+|y|^2)(1-|y|^2)^{-1}\end{bmatrix}=\begin{bmatrix} x \\ u\end{bmatrix}$$

and $z=(1+u)^{-1}x$. Then $z=\tilde{D}_{-y}\tilde{D}_{s\theta}(Ry)$, so

$$\begin{aligned} d(\tilde{D}_{s\theta}(Ry), y) &= d(\tilde{D}_{-y}\tilde{D}_{s\theta}(Ry), 0) \\ &= d(z,0) \\ &= \beta\log\left(\frac{1+|z|}{1-|z|}\right) \\ &= \beta\cosh^{-1}(u). \end{aligned}$$

The result follows by computing u in the matrix product above. □

Proof of Theorem 4.5. Since

$$d(F_t(A), F_t(B)) = d(AA_t, BA_t) = d(B^{-1}AA_t, A_t)$$

we may assume in each case that $B=I$. The uniformity of the limits in each case will be a natural consequence of the method of proof.

(i) If $A\in H_\theta^-\setminus\{I\}$ then $A=\exp B$ for some $B\in V_\theta^-\setminus\{0\}$. Let $p(s)=(\exp sB)A_t$, $0\leqq s\leqq 1$. Then

$$\begin{aligned} d(F_t(A), F_t(I)) &\leqq \int_0^1 \|p'(s)\|\, ds \\ &= \int_0^1 \|(\exp sB)BA_t\|\, ds \\ &= \int_0^1 \|A_t^{-1}BA_t\|\, ds \\ &= \|A_t^{-1}BA_t\|. \end{aligned}$$

So by Theorem 4.1

$$\limsup_{t\to\infty} \frac{1}{t} \log d(F_t(A), F_t(I)) \leqq -\tfrac{1}{2}\lambda^2(n-1). \tag{4.2}$$

Conversely for any $A \in SO_+(n,1)$ and $B \in o(n,1)$ $\|DF_t(A)(AB)\| = \|\mathrm{Ad}(D_t^{-1})(B)\|$. Assuming without loss of generality that $\alpha = \beta/\sqrt{2}$ we have that

$$\mathrm{Ad}(D_t^{-1}) = \exp\left(-r_t\left(\mathrm{ad}\begin{bmatrix} 0 & \theta_t \\ \theta_t^* & 0 \end{bmatrix}\right)\right) \quad \text{and} \quad \mathrm{ad}\begin{bmatrix} 0 & \theta_t \\ \theta_t^* & 0 \end{bmatrix}$$

is self-adjoint with eigenvalues $1, 0, -1$. Then $\mathrm{Ad}(D_t^{-1})$ is positive self-adjoint with smallest eigenvalue $\exp(-r_t)$. Therefore

$$\|DF_t(A)(AB)\| \geqq \exp(-r_t)\|AB\|,$$

so that $d(F_t(A), F_t(I)) \geqq \exp(-r_t)\, d(A, I)$. Therefore

$$\liminf_{t\to\infty} \frac{1}{t} \log d(F_t(A), F_t(I)) \geqq -\lim_{t\to\infty} \frac{1}{t} r_t = -\tfrac{1}{2}\lambda^2(n-1). \tag{4.3}$$

Together (4.2) and (4.3) give the required result.

(ii) By Lemma 4.3(ii) we may put $A = CD_{s\theta}R$ where $C \in H_\theta^-$, $s \in (-1,1)$, $R\theta = \theta$ and at least one of $s \neq 0$, $R \neq I$ occurs. Since

$$|d(AA_t, A_t) - d(D_{s\theta}RA_t, A_t)| = |d(AA_t, A_t) - d(AA_t, CA_t)| \leqq d(A_t, CA_t),$$

it suffices to consider $d(D_{s\theta}RA_t, A_t)$. If $s = 0$ then $R \neq I$ so $d(D_{s\theta}RA_t, A_t) = d(RA_t, A_t)$ which is bounded away from 0 by Lemma 4.6(iii). Otherwise, using Lemma 4.6(i) and (iv), and observing $\langle \theta, y - Ry\rangle = \langle \theta - R^{-1}\theta, y\rangle = 0$, we obtain

$$\begin{aligned} d(D_{s\theta}RA_t, A_t) &\geqq d(\tilde{D}_{s\theta}(Ry_t), y_t) \\ &= \beta \cosh^{-1}\left\{\frac{1+s^2}{1-s^2} + \frac{4(|y_t|^2 - \langle y_t, Ry_t\rangle)}{(1-|y_t|^2)^2}\right. \\ &\quad \left. + \frac{8s^2(|y_t|^2 - \langle y_t, \theta\rangle^2)}{(1-s^2)(1-|y_t|^2)^2}\right\} \\ &\geqq \beta \cosh^{-1}\left(\frac{1+s^2}{1-s^2}\right). \end{aligned} \tag{4.4}$$

This proves the first assertion. For the second assertion, by Lemma 4.6(i) it suffices to consider $d(\tilde{D}_{s\theta}(Ry_t), y_t)$. Now

$$\begin{aligned} \frac{4(|y_t|^2 - \langle y_t, \theta\rangle^2)}{(1-|y_t|^2)^2} &= (\sinh r_t)^2 (\sin d(\theta_t, \theta))^2 \\ &\leqq (\sinh r_t)^2 (d(\theta_t, \theta))^2 \end{aligned} \tag{4.5}$$

where $d(\theta_t, \theta)$ is geodesic distance on S^{n-1} between θ_t and θ. Similarly

$$\begin{aligned}\frac{4(|y_t|^2-\langle y_t, Ry_t\rangle)}{(1-|y_t|^2)^2} &= (\sinh r_t)^2(1-\cos d(\theta_t, R\theta_t)) \\ &= 2(\sinh r_t)^2(\sin\tfrac{1}{2}d(\theta_t, R\theta_t))^2 \\ &\leq 2(\sinh r_t)^2(d(\theta_t, \theta))^2 \end{aligned} \tag{4.6}$$

since

$$\begin{aligned} d(\theta_t, R\theta_t) &\leq d(\theta_t, \theta)+d(R\theta_t, \theta) \\ &= 2d(\theta_t, \theta). \end{aligned}$$

The result now follows from (4.4), (4.5), (4.6) and the estimate

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{t} \log((\sinh r_t)^2(d(\theta_t, \theta))^2) \\ &\quad = 2\lim_{t\to\infty} \frac{1}{t} r_t + 2\limsup_{t\to\infty} \frac{1}{t} \log d(\theta_t, \theta) \\ &\quad = \lambda^2(n-1) - \lambda^2(n-1) \\ &\quad = 0 \text{ using Theorem 2.4.} \end{aligned}$$

(iii) Choose $S\in SO(n)$ such that $S\theta=\tilde{A}\theta$. Then $S^{-1}A\in H_\theta^0$ so $A=SB$ for some $B\in H_\theta^0$. By a similar argument to that commencing the proof of (ii), we may assume $A=S$ with $S\theta\neq\theta$. Further using Lemma 4.6(i) it suffices to consider $d(Sy_t, y_t)$. By Lemma 4.6(iv) we obtain

$$d(Sy_t, y_t)=\beta\cosh^{-1}\{(\sinh r_t)^2(1-\cos d(\theta_t, S\theta_t))\}.$$

Now $\theta_t\to\theta$ and $S\theta_t\to S\theta\neq\theta$, so

$$1-\cos d(\theta_t, S\theta_t)\to 1-\cos d(\theta, S\theta)\neq 0.$$

Therefore

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{t} d(SA_t, A_t) &= \lim_{t\to\infty} \frac{1}{t} d(Sy_t, y_t) \\ &= \beta\lim_{t\to\infty} \frac{1}{t}(2r_t) \\ &= \beta\lambda^2(n-1) \\ &= \lambda(n-1). \quad \square \end{aligned}$$

Finally in this section we shall describe the leaves AH_θ^- and AH_θ^0 geometrically. This will emphasize the strong connection between our stochastic flow on $O(D^n)$ and the deterministic geodesic flow. See Klingenberg [10] or Arnold and Avez [1] for more information on the geodesic flow on negatively curved manifolds. For any $y\in D^n$ and $v\in T_yD^n\cong\mathbb{R}^n$ there is a unique geodesic $p(t)$, $t\in\mathbb{R}$, in D^n with $p(0)=y$ and $p'(y)=v$. As $t\to\infty$, $p(t)$ converges in the Euclidean metric to some limiting direction θ, say, in S^{n-1}.

Lemma 4.7. *The geodesic started at y with velocity v has limiting direction $\tilde{D}_y(v/|v|)$.*

Proof. Let $\psi = v/|v| \in S^{n-1}$. Then $q(t) = \pi\left(\exp t\begin{bmatrix} 0 & \psi \\ \psi^* & 0 \end{bmatrix}\right)$ is the geodesic started at 0 with velocity $\frac{1}{2}\psi$. It has limiting direction ψ. Then $p(t) = \tilde{D}_y(q(t))$ is the geodesic started at y with velocity $D(\tilde{D}_y(0)(\frac{1}{2}\psi)) = \frac{1}{2}(1-|y|^2)\psi = \gamma v$ for some $\gamma > 0$. It has limiting direction $\tilde{D}_y(\psi)$. □

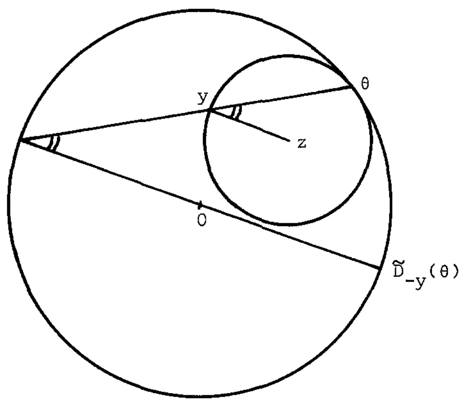
Definition. A *horosphere* at θ is a hypersurface in D^n orthogonal to the family of geodesics in D^n with limiting direction θ. Denote by $H(y, \theta)$ the horosphere at θ passing through $y \in D^n$.

Notice that Lemma 4.7 can be restated to show that a geodesic started at y with limiting direction θ must have initial velocity $\gamma(\tilde{D}_y)^{-1}(\theta) = \gamma(\tilde{D}_{-y})(\theta)$ for some $\gamma > 0$. Thus a horosphere is a leaf of the foliation of D^n given by the distribution orthogonal to $\{\tilde{D}_{-y}(\theta) \colon y \in D^n\}$.

Lemma 4.8. (i) *For $A \in SO_+(n, 1)$, $\tilde{A}$ maps the set of horospheres at θ onto the set of horospheres at $\tilde{A}\theta$.*

(ii) *A horosphere at θ is the set of points in D^n lying on a Euclidean sphere tangent to S^{n-1} at θ.*

Proof. (i) is immediate from the definition as $\tilde{A}$ is an isometry of D^n. (ii) follows from the restatement of Lemma 4.7 together with the diagram showing the geometrical interpretation of the map $\tilde{D}_{-y} \colon S^{n-1} \to S^{n-1}$ for $y \in D^n$ (see (5.5)). □



It follows from the diagram that the horosphere (r, θ) at θ through y has (Euclidean) radius

$$r = \frac{|\theta - y|}{|\theta + \tilde{D}_{-y}(\theta)|} = \frac{|\theta - y|^2}{1 - |y|^2 + |\theta - y|^2}$$

and centre $z = (1-r)\theta$.

The following theorem is given in terms of the interpretation of $\pi \colon SO_+(n, 1) \to D^n$ as the bundle of positively oriented orthonormal frames on D^n.

Theorem 4.9. (i) *For $A \in SO_+(n, 1)$, $\pi(AH_\theta^-)$ is the horosphere $H(y, \tilde{A}\theta)$ at $\tilde{A}\theta$ passing through $y = \pi(A)$. Every horosphere is a flat submanifold of (D^n, ρ).*

$B \in AH_\theta^-$ if and only if B may be obtained from A by parallel translation within the horosphere $H(y, \tilde{A}\theta)$.

(ii) *For $A = D_y R$ and $B = D_z S$, $B \in AH_\theta^0$ if and only if the geodesics started at y with velocity $R\theta$ and z with velocity $S\theta$ have the same limiting direction.*

Proof. (i) It suffices to take $A = I$. Consider $\Psi = \pi \circ \exp \circ i$ where $i: \theta^\perp \to V_\theta^-$ is given by

$$i(e) = \frac{1}{\beta}\begin{bmatrix} \theta e^* - e\theta^* & e \\ e^* & 0 \end{bmatrix} \quad \text{and} \quad \theta^\perp = \{e \in \mathbb{R}^n : \langle e, \theta \rangle = 0\}.$$

Then using (4.1) we obtain $\Psi(e) = \dfrac{2\beta e + |e|^2 \theta}{4\beta^2 + |e|^2}$ and the first part follows from Lemma 4.8(ii). Suppose now $B = D_y R = \exp(i(e))$ for some $e \in \theta^\perp$. Then

$$\begin{aligned} D\Psi(e)(f) &= D\pi(B)(B(i(f))) \\ &= \frac{1-|y|^2}{2\beta} Rf \quad \text{for } f \in \theta^\perp \end{aligned}$$

by the calculation in Lemma 3.1(i). Hence Ψ is an isometry of $\theta^\perp$ onto $H(0, \theta) \subset D^n$, so that $H(0, \theta)$ is flat. Further we see that $D\Psi(e)$ is the restriction to $\theta^\perp$ of the frame B. It follows that B is obtained by parallel translation in $H(0, \theta)$ from the frame $I = \exp(i(0))$ and we are done.

(ii) By Lemma 4.7 the geodesics have limiting direction $\tilde{D}_y(R\theta) = \tilde{A}(\theta)$ and $\tilde{D}_z(S\theta) = \tilde{B}(\theta)$ and we know from the definition of H_θ^0 that

$$\begin{aligned} \tilde{A}(\theta) = \tilde{B}(\theta) &\Leftrightarrow (\tilde{B})^{-1} \tilde{A}(\theta) = \theta \\ &\Leftrightarrow B^{-1} A \in H_\theta^0. \quad \square \end{aligned}$$

We see that the description of the stochastic flow is given in terms of the geometry of the deterministic geodesic flow. All we need to know is $\theta = \theta_\infty = \lim_{t \to \infty} y_t$, the limiting direction of the Brownian motion process $\{y_t : t \geqq 0\}$ on (D^n, ρ). In fact if we replace the random $\sum_{i=1}^n E_i \circ dW_t^i$ in Eq. (3.3) by the (deterministic) $\frac{1}{2}\lambda^2(n-1)\begin{bmatrix} 0 & \theta \\ \theta^* & 0 \end{bmatrix} dt$ for $\theta \in S^{n-1}$ we recover the equation for geodesic flow in the frame bundle $O(D^n)$. The stability results in Theorems 4.1 and 4.5 remain valid. We replace $A_t = \exp\left(r_t \begin{bmatrix} 0 & \theta_t \\ \theta_t^* & 0 \end{bmatrix}\right) R_t$ by $\exp\left(\frac{1}{2}\lambda^2(n-1)\, t \begin{bmatrix} 0 & \theta \\ \theta^* & 0 \end{bmatrix}\right)$ and the proofs go through. In the deterministic case all terms involving $d(\theta_t, \theta)$ vanish and we obtain the sharper estimate that $d(F_t(A), F_t(B))$ is bounded away from 0 and ∞ in Theorem 4.5(ii).

5. The Canonical Stochastic Flow on the Unit Sphere in $\mathbb{R}^n$

Let M be a smooth compact submanifold of $\mathbb{R}^k$. For $x \in M$ let $P(x): \mathbb{R}^k \to T_x M$ be the orthogonal projection and $h(x)$: $T_x M \times T_x M \to T_x^\perp M$ be the second fundamental form for the embedding. If $\{W_t: t \geqq 0\}$ is a $BM_0(\mathbb{R}^k)$ then $\{P(x) W_t: t \geqq 0\}$ is a $BM_0(T_x M)$. Let $j: M \to T_x^\perp M$ be given by

$$\begin{aligned} j(x) &= \tfrac{1}{2} E(h(x)(P(x) W_1, P(x) W_1)) \\ &= \tfrac{1}{2} \operatorname{tr}(h(x)). \end{aligned}$$

We consider the (Itô) s.d.e. on M given by

$$\begin{aligned} d\xi_t(x) &= P(\xi_t(x))\, dW_t + j(\xi_t(x))\, dt \\ \xi_0(x) &= x. \end{aligned} \tag{5.1}$$

Formally we should extend the definition of P and j to some neighbourhood of M in $\mathbb{R}^k$ so as to consider (5.1) as an Itô s.d.e. in $\mathbb{R}^k$, and then check that if $x \in M$ then $\xi_t(x) \in M$ for all $t \geqq 0$. See [2], Sect. 4.6, for details. For x fixed, $\{\xi_t(x): x \geqq 0\}$ is a $BM_x(M)$ (where M is given the Riemannian structure inherited from $\mathbb{R}^k$). It may be thought of as the projection onto the submanifold M of the original $\{W_t: t \geqq 0\}$ in $BM_0(\mathbb{R}^k)$. More exactly, the increments of $\{\xi_t(x): x \geqq 0\}$ are the orthogonal projections of the increments of $\{W_t: t \geqq 0\}$. In a similar manner we may consider the stochastic flow on M given by (5.1) as the projection onto M of the rigid flow on $\mathbb{R}^k$ given by $y \mapsto y + W_t$, $y \in \mathbb{R}^k$.

An alternative construction of a stochastic flow on M is as follows. For $1 \leqq i \leqq k$ let $f_i: M \to \mathbb{R}$ denote the restriction to M of the i^{th} coordinate function, and let $V_i = \operatorname{grad} f_i$. Consider the (Stratonovich) s.d.e.

$$\begin{aligned} d\xi_t(x) &= \sum_{i=1}^k V_i(\xi_t(x)) \circ dW_t^i \\ \xi_0(x) &= x \end{aligned} \tag{5.2}$$

where $W_t^1, \ldots, W_t^k$ are independent $BM_0(\mathbb{R})$.

Theorem 5.1. *If* $W_t = (W_t^1, \ldots, W_t^k)$ *the stochastic flows given by* (5.1) *and* (5.2) *are the same.*

Proof. We show that (5.1) is the Itô version of (5.2). Let D and ∇ denote covariant derivatives in $\mathbb{R}^k$ and M respectively. Notice first that $V_i(x) = P(x)(e_i)$ where e_i denotes the i^{th} standard basis vector in $\mathbb{R}^k$. Therefore (5.2) becomes

$$\begin{aligned} d\xi_t(x) &= \sum_{i=1}^k V_i(\xi_t(x))\, dW_t^i + \tfrac{1}{2} \sum_{i=1}^k DV_i(\xi_t(x))(V_i(\xi_t(x)))\, dt \\ &= \sum_{i=1}^k P(\xi_t(x))(e_i)\, dW_t^i + \tfrac{1}{2} \sum_{i=1}^k DV_i(\xi_t(x))(V_i(\xi_t(x)))\, dt \\ &= P(\xi_t(x)(dW_t) + \tfrac{1}{2} \sum_{i=1}^k DV_i(\xi_t(x))(V_i(\xi_t(x)))\, dt. \end{aligned}$$

Now $DV_i(x)(V_i(x)) = \nabla V_i(x)(V_i(x)) + h(x)(V_i(x), V_i(x))$, so it suffices to prove $\sum_{i=1}^k \nabla V_i(x)(V_i(x)) = 0$. For any vector field X on M we have

$$\langle V_i(x), X(x) \rangle = \langle e_i, X(x) \rangle, \qquad x \in M.$$

Taking the derivative in direction $V_i(x)$ (and noting that D and ∇ agree on functions),

$$\begin{aligned}\langle \nabla V_i(x)(V_i(x)), X(x)\rangle + \langle V_i(x), \nabla X(x)(V_i(x))\rangle \\ = \langle e_i, DX(x)(V_i(x))\rangle \\ = \langle e_i, \nabla X(x)(V_i(x)) + h(x)(X(x), V_i(x))\rangle.\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{i=1}^{k} \langle \nabla V_i(x)(V_i(x)), X(x)\rangle &= \sum_{i=1}^{k} \langle e_i, h(x)(X(x), V_i(x))\rangle \\ &= \sum_{i=1}^{k} \langle e_i - P(x)(e_i), h(x)(X(x), P(x)\, e_i)\rangle \\ &= \operatorname{tr}((I-P(x))\, Q(X(x))\, P(x)) \\ &= 0\end{aligned}$$

where $Q(X(x)) = h(x)(X(x), \cdot): T_x M \to T_x^{\perp} M$. $\square$

We may refer to the flow given by (5.1) (or equivalently (5.2)) as the stochastic flow on M determined by the embedding $M \subset \mathbb{R}^k$. In this section we shall be concerned with the stochastic flow on S^{n-1} determined by its embedding as the unit sphere in $\mathbb{R}^n$. We consider Eq. (5.2) where

$$V_i(x) = e_i - \langle x, e_i\rangle x, \qquad i = 1, 2, \dots, n.$$

Theorem 5.2. *The equation*

$$\begin{aligned} d\xi_t(x) &= \sum_{i=1}^{n} (e_i - \langle \xi_t(x), e_i\rangle\, \xi_t(x)) \circ dW_t^i \\ \xi_0(x) &= x \end{aligned} \tag{5.3}$$

on S^{n-1} has solution of the form

$$\xi_t = R_t^{-1} \tilde{D}_{-y_t}$$

where y_t and R_t are as in Theorem 2.2 with $\lambda = -1$, $\mu = 0$.

Proof. Suppose

$$dA_t = -\sum_{i=1}^{n} A_t E_i \circ dW_t^i. \tag{5.4}$$

Then

$$\begin{aligned} d(A_t^{-1}) &= -A_t^{-1}(\circ\, dA_t)\, A_t^{-1} \\ &= \sum_{i=1}^{n} A_t^{-1}(A_t E_i)\, A_t^{-1} \circ dW_t^i \\ &= \sum_{i=1}^{n} E_i A_t^{-1} \circ dW_t^i. \end{aligned}$$

Suppose $A_t^{-1}\begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} z_t \\ u_t \end{bmatrix}$ and $x_t = \tilde{A}_t^{-1}(x) = z_t / u_t$. Then

$$d\begin{bmatrix} z_t \\ u_t \end{bmatrix} = \sum_{i=1}^{n} E_i \begin{bmatrix} z_t \\ u_t \end{bmatrix} \circ dW_t^i = \sum_{i=1}^{n} \begin{bmatrix} u_t e_i \\ \langle z_t, e_i\rangle \end{bmatrix} \circ dW_t^i$$

and so

$$dx_t = (u_t)^{-1} \circ dz_t - (u_t)^{-2} z_t \circ du_t$$
$$= \sum_{i=1}^{n} (e_i - \langle x_t, e_i \rangle x_t) \circ dW_t^i.$$

Therefore (5.3) has a solution with $\xi_t(x) = \tilde{A}_t^{-1}(x)$ where A_t satisfies (5.4). The result now follows from Theorem 2.2. □

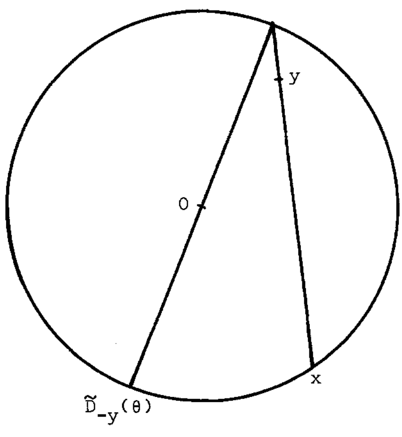
Note. If we put $\lambda = -1/r$, $\mu = 0$, and $\xi_t(x) = rR_t^{-1}\tilde{D}_{-y_t}(x/r)$ then $\{\xi_t: t \geqq 0\}$ is the natural stochastic flow on the sphere of radius r in $\mathbb{R}^n$.

The appearance of A_t^{-1} rather than A_t ensures that ξ_t has independent increments on the left. Hidden in the calculation above is the fact that for the inclusion of $SO_+(n, 1)$ in $\mathrm{Diff}(S^{n-1})$ given by $A \to \tilde{A}$, the vector field V_i corresponds to $E_i \in o(n, 1)$. In fact each V_i is an infinitesimal conformal transformation of S^{n-1} and together $V_1, \ldots, V_n$ generate the group $SO_+(n, 1)$ of orientation preserving conformal diffeomorphisms of S^{n-1}. It follows that ξ_t is a conformal diffeomorphism of S^{n-1} for all $t \geqq 0$.

For $y \in D^n$ and $x \in S^{n-1}$,

$$\tilde{D}_{-y}(x) = \frac{(1-|y|^2)x + 2(\langle x, y \rangle - 1)y}{-2\langle x, y \rangle + 1 + |y|^2}$$
$$= -\left(y + \frac{(1-|y|^2)(y-x)}{|x-y|^2}\right) \tag{5.5}$$

so geometrically $\tilde{D}_{-y}$ is as shown. Thus if y is very close to the boundary of D^n then $\tilde{D}_{-y}$ sends most of S^{n-1} to a small neighbourhood of $-y/|y|$. Therefore for large t, ξ_t consists of $\tilde{D}_{-y_t}$, which compresses most of S^{n-1} into a small neighbourhood of $-\theta_t$, followed by a rotation R_t^{-1}. The equation for R_t ensures that the rotation R_t^{-1} is exactly what is required so that $\xi_t(x)$ is $BM_x(S^{n-1})$ for each $x \in S^{n-1}$. In the limit as $t \to \infty$ there is one (random) unstable point $\theta = \lim_{t\to\infty} \theta_t = \lim_{t\to\infty} y_t$. Away from θ the flow is stable, in the following local and global senses. d denotes geodesic distance in S^{n-1}.



Theorem 5.3. *Let* $\{\xi_t: t \geqq 0\}$ *be the flow of Theorem* 5.2. *With probability* 1, $\theta = \theta_\infty = \lim_{t\to\infty} y_t$ *exists and*

(i) *for* $v \in T_x S^{n-1} \setminus \{0\}$

$$\lim_{t\to\infty} \frac{1}{t} \log |D\xi_t(x)(v)| = \begin{cases} -\frac{1}{2}(n-1) & \text{if } x \neq \theta \\ \frac{1}{2}(n-1) & \text{if } x = \theta \end{cases}$$

(ii) *for* $x_1 \neq x_2$ *and* $x_i \neq \theta$, $i = 1, 2$,

$$\lim_{t\to\infty} \frac{1}{t} \log d(\xi_t(x_1), \xi_t(x_2)) = -\tfrac{1}{2}(n-1)$$

and the convergence is uniform for $d(x_1, x_2)$, $d(x_1, \theta)$ *and* $d(x_2, \theta)$ *all bounded away from zero.*

Proof. By Theorem 2.3 and Corollary 2.5, θ exists and

$$\lim_{t\to\infty} \frac{1}{t} \log(1 - |y_t|) = -\tfrac{1}{2}(n-1) \tag{5.6}$$

$$\lim_{t\to\infty} \frac{1}{t} \log(|\theta - y_t|) = -\tfrac{1}{2}(n-1) \tag{5.7}$$

with probability 1. From (5.5)

$$D(\tilde{D}_{-y})(x)(v) = \frac{(1-|y|^2)\, v}{|x-y|^2} + \frac{2\langle v, y\rangle (1-|y|^2)(x-y)}{|x-y|^4},$$

so

$$|D(\tilde{D}_{-y})(x)(v)| = \frac{(1-|y|^2)\,|v|}{|x-y|^2} \tag{5.8}$$

and

$$\begin{aligned} |D\xi_t(x)(v)| &= |R_t^{-1} D(\tilde{D}_{-y_t})(x)(v)| \\ &= \frac{(1-|y_t|^2)\,|v|}{|x-y_t|^2} \end{aligned}$$

(exhibiting the fact that ξ_t is conformal). If $x \neq \theta$ then $|x - y_t|^2$ is bounded away from zero and infinity and the first result follows using (5.6). If $x = \theta$ we use both (5.6) and (5.7). For (ii) notice that $d(x_1, x_2)$ and $|x_1 - x_2|$ are equivalent metrics on S^{n-1}. We have

$$\begin{aligned} |\xi_t(x_1) - \xi_t(x_2)| &= |\tilde{D}_{-y_t}(x_1) - \tilde{D}_{-y_t}(x_2)| \\ &= (1-|y_t|^2) \left| \frac{x_1 - y_t}{|x_1 - y_t|^2} - \frac{x_2 - y_t}{|x_2 - y_t|^2} \right| \end{aligned}$$

and the result follows as above. □

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be the underlying probability space for the flow, and let m denote normalised Lebesgue measure on S^{n-1}. $\theta = \theta_\infty$ is a $\mathbb{P}$-almost surely defined random variable on S^{n-1} with distribution m. Therefore on the product space $\Omega \times S^{n-1}$ the event $\{\theta \text{ exists}, x \neq \theta\}$ has $\mathbb{P} \times m$ measure 1. It follows that the Lyapunov spectrum for the flow $\{\xi_t\colon t \geqq 0\}$ consists of the single value $-\frac{1}{2}(n-1)$ and that the corresponding filtration is trivial. The stable manifold of $x \in S^{n-1}$ is the random submanifold $S^{n-1} \setminus \{\theta\}$ defined on the set $\{\theta \neq x\}$.

It is interesting to observe what happens to volumes of images of subsets of S^{n-1} under the flow $\{\xi_t : t \geqq 0\}$. Let U be a subset of S^{n-1}. If $\theta \in \operatorname{int}(U)$ then $\theta_t \in \operatorname{int}(U)$ for all sufficiently large t and so $\tilde{D}_{-y_t}(U)$ expands to fill up most of S^{n-1}. Conversely if $\theta \notin \bar{U}$ then $\theta_t \notin \bar{U}$ for all sufficiently large t and so $\tilde{D}_{-y_t}(U)$ shrinks to a small neighbourhood of $-\theta_t$. Thus

$$\begin{aligned} \mathbb{P}\{m(\xi_t(U)) \to 1\} &= \mathbb{P}\{m(\tilde{D}_{-y_t}(U)) \to 1\} \\ &\geqq \mathbb{P}\{\theta \in \operatorname{int}(U)\} \\ &= m(\operatorname{int}(U)) \end{aligned}$$

and

$$\begin{aligned} \mathbb{P}\{m(\xi_t(U)) \to 0\} &= \mathbb{P}\{m(\tilde{D}_{-y_t}(U)) \to 0\} \\ &\geqq \mathbb{P}\{\theta \notin \bar{U}\} \\ &= 1 - m(\bar{U}). \end{aligned}$$

Together we obtain the following result.

Theorem 5.4. (i) *Let U be a Borel subset of S^{n-1} with $m(\partial U) = 0$. Then with probability* 1 $m(\xi_t(U)) \to 0$ *or* 1 *as* $t \to \infty$ *and*

$$\mathbb{E}(\lim_{t \to \infty} m(\xi_t(U))) = m(U). \tag{5.9}$$

(ii) *Let $\{U_n : n \geqq 1\}$ be any countable collection of Borel subsets of S^{n-1} with $m(\partial U_n) = 0$ for all n and which separate points in S^{n-1} (i.e. if $x \neq y$ then there exists n such that $x \in U_n$, $y \notin U_n$). Then with probability* 1,

$$\bigcap_{n \in I} U_n = \{\theta\}$$

where $I = \{n \geqq 1 : m(\xi_t(U_n)) \to 1$ as $t \to \infty\}$.

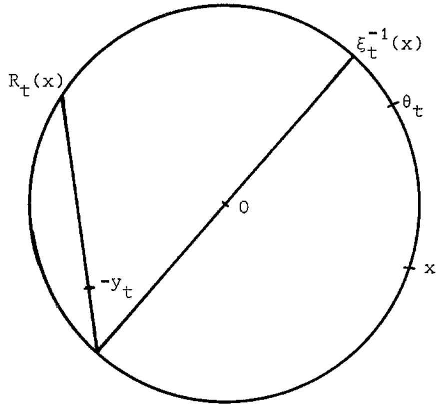
Proof. (i) If $m(\partial U) = 0$ then $m(\operatorname{int}(U)) = m(\bar{U})$ and the result follows from above.

(ii) On the set $\{\theta$ exists, $\theta \notin \bigcup_{n \geqq 1} \partial U_n\}$ we have $\theta \in \bigcap_{n \in I} A_n$. If ϕ is any other point in S^{n-1} then there exists n such that $\theta \in A_n$ (so that $n \in I$) and $\phi \notin A_n$. Therefore $\phi \notin \bigcap_{n \in I} A_n$. □

It may be shown that for any Borel subset U of S^{n-1}, $m(\xi_t(U))$ is a martingale and hence convergent, thus strengthening (5.9). This fact remains true when we consider the stochastic flow determined by an arbitrary compact submanifold M of $\mathbb{R}^k$. It may not however always be the case that there are only two possible values for the limit.

6. The Inverse Flow on S^{n-1}

Since the s.d.e. (5.3) has zero Stratonovich drift it follows that for each fixed $t \geqq 0$ the distributions of the random diffeomorphisms ξ_t and ξ_t^{-1} are the same.



(This may be seen from the approximation to ξ_t obtained by replacing $\{W_s: 0 \leqq s \leqq t\}$ by a polygonal approximation. See, for example, Wong and Zakai [19] for the approximation scheme. The same sequence of approximations converges in distribution to ξ_t^{-1}.) Therefore the difference in behaviour of the processes $\{\xi_t: t \geqq 0\}$ and $\{\xi_t^{-1}: t \geqq 0\}$ is caused solely by the difference between left and right multiplication in the diffeomorphism group.

Alternatively, we may consider the process $\{\xi_{su}: -\infty < s \leqq u < \infty\}$ in $\mathrm{Diff}(S^{n-1})$ given by

$$\begin{aligned} d\xi_{su}(x) &= \sum_{i=1}^{n} (e_i - \langle \xi_{su}(x), e_i \rangle \xi_{su}(x)) \circ dW_u^i \\ \xi_{ss}(x) &= x \end{aligned} \tag{6.1}$$

where each $\{W_u^i: u \in \mathbb{R}\}$ is a process in $\mathbb{R}$ with independent increments distributed like the increments of $BM(\mathbb{R})$. For fixed s, the process $\{\xi_{s,s+t}: t \geqq 0\}$ is distributed like the solution $\{\xi_t: t \geqq 0\}$ of (5.3). This is the usual "forward" process. But we may also consider the "backward" process $\{\xi_{u-t,u}: t \geqq 0\}$ for fixed u. This starts at the identity and has increments on the right. In fact we may check that the backward process $\{\xi_{u-t,u}: t \geqq 0\}$ is distributed like $\{\xi_t^{-1}: t \geqq 0\}$. See Kunita [12] for more details on backward stochastic flows of diffeomorphisms.

For $t > 0$, $\xi_t^{-1} = \tilde{D}_{y_t} R_t$ so ξ_t^{-1} consists of a rotation R_t followed by $\tilde{D}_{y_t}$ which for large t compresses most of S^{n-1} into a small neighbourhood of θ_t. So for large t, ξ_t^{-1} sends most of S^{n-1} very close to θ_t and hence very close to θ_∞.

Consider $\xi_t^{-1}(x) = \tilde{D}_{y_t}(R_t x)$ for some fixed $x \in S^{n-1}$. For large t, $\tilde{D}_{y_t}$ sends all but a small neighbourhood of $-\theta_t$ into a small neighbourhood of θ_t. So we need to consider whether the random rotation R_t sends x into that small neighbourhood of $-\theta_t$ which does not get mapped into a given small neighbourhood of θ_t. More precisely, for any $\varepsilon \in (0, \pi)$, $d(\xi_t^{-1}(x), \theta_t) < \varepsilon$ if and only if $d(R_t x, -\theta_t) > \delta = \delta_t^\varepsilon$ where $\delta_t^\varepsilon \in (0, \pi)$ is given by

$$\cos \delta_t^\varepsilon = \frac{2|y_t| - (1 + |y_t|^2) \cos \varepsilon}{1 + |y_t|^2 - 2|y_t| \cos \varepsilon}. \tag{6.2}$$

This follows from (5.5) since $R_t x = \tilde{D}_{-y_t}(\xi_t^{-1}(x))$. For each $\varepsilon > 0$, δ_t^ε is a random function of t and $\delta_t^\varepsilon \to 0$ as $t \to \infty$ since $|y_t| \to 1$ as $t \to \infty$. We have $d(\xi_t^{-1}(x), \theta_t) \to 0$ as $t \to \infty$ if and only if for all $\varepsilon > 0$, $d(-x, R_t^{-1}\theta_t) = d(R_t x, -\theta_t) > \delta_t^\varepsilon$ for all sufficiently large t. So the question of convergence depends on whether the process $R_t^{-1}\theta_t$ can get inside the shrinking ball of radius δ_t^ε centred at $-x$ infinitely often as $t \to \infty$. In what follows we abbreviate "for all sufficiently large t" to "eventually" and "infinitely often as $t \to \infty$" to "i.o.".

Proposition 6.1. *$R_t^{-1}\theta_t = Y_{\sigma(t)}$ where $\{Y_s : s \geqq 0\}$ is a $BM(S^{n-1})$ independent of $\{|y_t| : t \geqq 0\}$ and the clock $\sigma(t)$ is given by* $\sigma'(t) = \left(\frac{1+|y_t|^2}{2|y_t|}\right)^2$.

Proof. Let $z_t = R_t^{-1} y_t$, so $|z_t| = |y_t|$. We have

$$dz_t = -R_t^{-1}(\circ dR_t) R_t^{-1} y_t + R_t^{-1} \circ dy_t$$
$$= \sum_{i=1}^{n} (z_t \langle z_t, e_i \rangle - \tfrac{1}{2}(1+|z_t|^2) e_i) \circ dW_t^i$$

from Eqs. (2.4), (2.5) with $\lambda = -1$, $\mu = 0$. Write $\phi_t = R_t^{-1}\theta_t = z_t/|z_t|$. Then

$$d|z_t| = \frac{\langle z_t, \circ dz_t \rangle}{|z_t|}$$
$$= -\tfrac{1}{2} \sum_{i=1}^{n} (1-|z_t|^2) \langle \phi_t, e_i \rangle \circ dW_t^i$$

and

$$d\phi_t = |z_t|^{-1} \circ dz_t - |z_t|^2 \circ d|z_t|$$
$$= -\sum_{i=1}^{n} \frac{1+|z_t|^2}{2|z_t|} (e_i - \langle \phi_t, e_i \rangle \phi_t) \circ dW_t^i.$$

The result now follows (c.f. Theorem 2.3(iii)). □

Observe that $-R_t^{-1}\theta_t$ is the centre of the clump formed by ξ_t, i.e. it is the point of maximum density of the induced measure $m\xi_t^{-1}$. The proposition shows that this point moves at faster rate than does $\xi_t(x)$ for any fixed $x \in S^{n-1}$ (although the limiting rate as $t \to \infty$ is the same).

Proposition 6.2. *Fix $x \in S^{n-1}$.*

(i) *If $n = 2$, $\mathbb{P}\{d(x, R_t^{-1}\theta_t) = 0$ i.o.$\} = 1$.*

(ii) *If $n = 3$, $\mathbb{P}\{d(x, R_t^{-1}\theta_t) < \exp(-\gamma t)$ i.o.$\} = 1$ for all $\gamma > 0$.*

(iii) *If $n \geqq 4$, $\mathbb{P}\{d(x, R_t^{-1}\theta_t) > t^{-\gamma}$ eventually$\} = 1$ for all $\gamma > 1/(n-3)$.*

Proof. By the previous proposition, $R_t^{-1}\theta_t = Y_{\sigma(t)}$ where $\sigma'(t) \to 1$ as $t \to \infty$ and in particular $\sigma(t) \to \infty$ as $t \to \infty$ with probability 1. (i) now follows directly from the corresponding result for $BM(S^1)$. For $n \geqq 3$ (i.e. $\dim S^{n-1} \geqq 2$) we use Theorem 1 in [3]. Define $\phi_2(s) = |\log s|^{-1} \wedge 1$ and $\phi_d(s) = s^{d-2}$ for $d \geqq 3$. Then if $f: [0, \infty) \to (0, \infty)$ is monotone decreasing

$$\begin{aligned}&\mathbb{P}\{d(x, R_t^{-1}\theta_t) < f(t) \text{ i.o.}\}\\ &= \mathbb{P}\{d(x, Y_{\sigma(t)}) < f(t) \text{ i.o.} \mid \sigma'(t) \to 1\}\\ &= \mathbb{P}\{\int^{\infty} \phi_{n-1}(f(\sigma^{-1}(s)))\, ds = \infty \mid \sigma'(t) \to 1\}\\ &= \mathbb{P}\{\int^{\infty} \phi_{n-1}(f(t))\, \sigma'(t)\, dt = \infty \mid \sigma'(t) \to 1\}\\ &= 1 \text{ or } 0\end{aligned}$$

according as $\int^{\infty} \phi_{n-1}(f(t))\, dt$ diverges or converges. The result follows upon taking $f(t) = \exp(-\gamma t)$ if $n=3$ and $f(t) = t^{-\gamma}$ if $n \geqq 4$. □

Since $1 - \langle x, y\rangle = 2\sin^2(\frac{1}{2}d(x, y))$ for $x, y \in S^{n-1}$ we obtain the following.

Corollary 6.3. *Fix* $x \in S^{n-1}$.

(i) *If* $n=2$, $\mathbb{P}\{\langle x, R_t^{-1}\theta_t\rangle = 1$ *i.o.*$\} = 1$.

(ii) *If* $n=3$, $\mathbb{P}\{\langle x, R_t^{-1}\theta_t\rangle > 1 - \exp(-\gamma t)$ *i.o.*$\} = 1$ *for all* $\gamma > 0$.

(iii) *If* $n \geqq 4$, $\mathbb{P}\{\langle x, R_t^{-1}\theta_t\rangle < 1 - t^{-\gamma}$ *eventually*$\} = 1$ *for all* $\gamma > 2/(n-3)$.

Theorem 6.4. *Let* $\{\xi_t : t \geqq 0\}$ *be the stochastic flow of Theorem* 5.2 *and let* $\theta = \theta_\infty = \lim_{t\to\infty} y_t$. *Fix* $x \in S^{n-1}$.

(i) *If* $n=2$ *then* $\xi_t^{-1}(x)$ *hits each of the points* θ *and* $-\theta$ *infinitely often as* $t \to \infty$ wp1.

(ii) *If* $n=3$ *then* $\xi_t^{-1}(x)$ *approaches within* $\exp(-\gamma t)$ *of each of the points* θ *and* $-\theta$ *infinitely often as* $t\to\infty$ wp1, *for* $0<\gamma<1$.

(iii) *If* $n \geqq 4$, $\limsup_{t\to\infty} \frac{1}{t} \log d(\xi_t^{-1}(x), \theta) \leqq -\frac{1}{2}(n-1)$ wp1.

Proof. Recall that $\limsup_{t\to\infty} \frac{1}{t} \log d(\theta_t, \theta) = -\frac{1}{2}(n-1)$ wp1. In addition for $n=2$ we may use the intermediate value property for segments of S^1. Therefore it suffices to prove each of the statements in the theorem with θ replaced by θ_t. Notice that we can now drop the restriction $\gamma<1$ in part (ii). From the formula for $\tilde{D}_y$ we obtain

$$1 - \langle \xi_t^{-1}(x), \theta_t\rangle = \frac{(1-|y_t|)^2(1-\langle x, R_t^{-1}\theta_t\rangle)}{(1-|y_t|)^2 + 2|y_t|(1+\langle x, R_t^{-1}\theta_t\rangle)} \tag{6.3}$$

and

$$1 + \langle \xi_t^{-1}(x), \theta_t\rangle = \frac{(1+|y_t|)^2(1+\langle x, R_t^{-1}\theta_t\rangle)}{(1-|y_t|)^2 + 2|y_t|(1+\langle x, R_t^{-1}\theta_t\rangle)}. \tag{6.4}$$

We now obtain (i) by using Corollary 6.3(i) in Eq. (6.3) and Corollary 6.3(i) with x replaced by $-x$ in (6.4). Similarly to prove (ii) observe that if $1 - \langle x, R_t^{-1}\theta_t\rangle < \exp(-2\gamma t)$ then by (6.3) we have $1 - \langle \xi_t^{-1}(x), \theta_t\rangle < \exp(-2\gamma t)$. Also for any $\delta > 0$ if $1 + \langle x, R_t^{-1}\theta_t\rangle < \exp(-(2\gamma+2+\delta)t)$ then by (6.4) we have

$$1 + \langle \xi_t^{-1}(x), \theta_t\rangle < \frac{4\exp(-(2\gamma+2+\delta)t)}{(1-|y_t|)^2}$$

and the result follows since $\frac{1}{t}\log(1-|y_t|)\to -1$ wp1 (for $n=3$). Finally to prove (iii) observe that if $\langle -x, R_t^{-1}\theta_t\rangle < 1-t^{-\gamma}$ then

$$1-\langle \xi_t^{-1}(x), \theta_t\rangle \leqq \frac{(1-|y_t|)^2(2-t^{-\gamma})}{(1-|y_t|)^2+2|y_t|t^{-\gamma}}$$

so

$$\limsup_{t\to\infty}\frac{1}{t}\log d(\xi_t^{-1}(x),\theta_t)=\tfrac{1}{2}\limsup_{t\to\infty}\frac{1}{t}\log(1-\langle\xi_t^{-1}(x),\theta_t\rangle)$$
$$\leqq -\tfrac{1}{2}(n-1). \quad \square$$

We see that in S^1 and S^2 the process $\{\xi_t^{-1}(x): t\geqq 0\}$ does not converge to θ_∞. However it does spend most of its time near θ_∞. The following result is true for all $n\geqq 2$ although for $n\geqq 4$ it follows trivially from the previous theorem. We denote by I_F the indicator function of an event F.

Theorem 6.5. *Let $\{\xi_t: t\geqq 0\}$, θ_∞ and x be as above. For any neighbourhood U of θ_∞ in S^{n-1}, the process $\{\xi_t^{-1}(x): t\geqq 0\}$ spends only a finite time outside U with probability* 1, *i.e.*

$$\int_0^\infty I_{\{\xi_t^{-1}(x)\notin U\}}\,dt<\infty \qquad \text{wp1}.$$

Proof. Since $\theta_t\to\theta_\infty$ wp1 it suffices to prove the result with $d(\xi_t^{-1}(x),\theta_t)>\varepsilon$ replacing the condition $\xi_t^{-1}(x)\notin U$. Given $\varepsilon>0$ there exists $k<\infty$ such that $d(\xi_t^{-1}(x),\theta_t)>\varepsilon$ implies $d(R_t^{-1}\theta_t, -x)<k(1-|y_t|)$. (This follows from (6.2)). Recall from Proposition 6.1 that, conditional on $\mathscr{A}=\sigma\{|y_s|: s\geqq 0\}$, $R_t^{-1}\theta_t$ is a time changed $BM(S^{n-1})$. Therefore

$$\mathbb{E}\left(\int_0^\infty I_{\{d(\xi_t^{-1}(x),\,\theta_t)>\varepsilon\}}\,dt\,|\,\mathscr{A}\right)$$
$$\leqq \mathbb{E}\left(\int_0^\infty I_{\{d(R_t^{-1}\theta_t,\,-x)<k(1-|y_t|)\}}\,dt\,|\,\mathscr{A}\right)$$
$$=\int_0^\infty \mathbb{P}\{d(R_t^{-1}\theta_t, -x)<k(1-|y_t|)\,|\,\mathscr{A}\}\,dt$$
$$\leqq k_1\int_0^\infty (1-|y_t|)^{n-1}\,dt$$
$$<\infty \text{ wp1}$$

where the constant k_1 depends only on k and n. The result now follows immediately. $\square$

Corollary 6.6. *For any continuous $f: S^{n-1}\to\mathbb{R}$ and any $x\in S^{n-1}$*

$$\frac{1}{t}\int_0^t f(\xi_s^{-1}(x))\,ds\to f(\theta_\infty) \quad \text{wp1}.$$

Note. This result may be compared with the ergodic theorem for $\{\xi_t(x): t \geq 0\}$:

$$\frac{1}{t} \int_0^t f(\xi_s(x))\, ds \to \int_{S^{n-1}} f(y)\, dm(y) \quad \text{wp 1.}$$

The following result describes the local stability of the inverse flow $\{\xi_t^{-1}: t \geq 0\}$.

Theorem 6.7. *Let* $\{\xi_t: t \geq 0\}$, θ_∞ *and* x *be as above. Then with probability* 1, *for all* $v \in T_x S^{n-1} \setminus \{0\}$

$$\liminf_{t \to \infty} \frac{1}{t} \log |D\xi_t^{-1}(x)(v)| = -\tfrac{1}{2}(n-1)$$

and

$$\limsup_{t \to \infty} \frac{1}{t} \log |D\xi_t^{-1}(x)(v)| = \begin{cases} -\tfrac{1}{2}(n-1) & \text{if } n \geq 4 \\ \tfrac{1}{2}(n-1) & \text{if } 2 \leq n \leq 3. \end{cases}$$

Proof.

$$\begin{aligned} |D\xi_t^{-1}(x)(v)| &= |D(\tilde{D}_{y_t})(R_t x)(R_t v)| \\ &= \frac{(1-|y_t|^2)\,|v|}{|R_t x + y_t|^2} \\ &= \frac{(1-|y_t|^2)\,|v|}{(1-|y_t|)^2 + 2|y_t|(1+\langle x, R_t^{-1}\theta_t\rangle)}. \end{aligned}$$

The proof now follows from Corollary 6.3 and Corollary 2.5. □

Notice that if $n=2$ or 3 then $\lim_{t\to\infty} \frac{1}{t} \log |D\xi_t^{-1}(x)(v)|$ does not exist. This shows the non-existence in general of "backward" Lyapunov exponents (i.e. Lyapunov exponents for the flow obtained from a backward s.d.e.).

Finally for any Borel subset U of S^{n-1} we consider $m(\xi_t^{-1}(U)) = m_t(U)$ say. This more properly belongs in Sect. 5 as the measure m_t is the image of the measure m under the mapping ξ_t. In general let ν be any Borel probability measure on S^{n-1} and let $\nu_t(U) = \nu(\xi_t^{-1}(U))$ for all Borel subsets U of S^{n-1}. Then $\{\nu_t: t \geq 0\}$ is a Markov process in the space $M(S^{n-1})$ of Borel probability measures on S^{n-1}. The following result describes the limiting behaviour of this Markov process. For $x \in S^{n-1}$ let $\delta(x) \in M(S^{n-1})$ denote the unit mass at x. Let $C(S^{n-1})$ be the set of continuous functions from S^{n-1} to $\mathbb{R}$ and $B(x, \varepsilon)$ the open ball with centre x and radius ε in S^{n-1}.

Theorem 6.8. *Let* ν *be any Borel probability measure on* S^{n-1} *and* $\nu_t = \nu \xi_t^{-1}$. *Let* $z \in S^{n-1}$. *Then* $\nu_t - \delta(\xi_t(z)) \to 0$ *weakly as* $t \to \infty$ *with probability* 1, *i.e.*

$$\int_{S^{n-1}} f(y)\, d\nu_t(y) - f(\xi_t(z)) \to 0 \quad \text{as} \quad t \to \infty$$

for all $f \in C(S^{n-1})$ *with probability* 1.

Proof. $d(\xi_t(z), -R_t^{-1}\theta_t) = d(\tilde{D}_{-y_t}(z), -\theta_t) \to 0$ as $t \to \infty$ wp 1. Therefore it suffices to prove $\nu_t - \delta(-R_t^{-1}\theta_t) \to 0$ weakly as $t \to \infty$ wp 1. Since the distribution of θ_∞

is non-atomic we have $\nu(\{\theta_\infty\})=0$ wp1. From now on we restrict to a fixed set of probability 1 on which $\theta_t \to \theta_\infty$, $|y_t| \to 1$ and $\nu(\{\theta_\infty\})=0$. Given $f \in C(S^{n-1})$ and $\varepsilon>0$ there exists $\delta_1>0$ such that $\nu(B(\theta_\infty, 2\delta_1))<\varepsilon/4\|f\|$. Therefore there exists T_0 such that if $t \geqq T_0$ then $\nu(B(\theta_t, \delta_1))<\varepsilon/4\|f\|$. Also

$$d(\xi_t(x), -R_t^{-1}\theta_t)=d(\tilde{D}_{-y_t}(x), -\theta_t) \to 0$$

as $t \to \infty$ uniformly for $x \notin B(\theta_t, \delta_1)$. Since f is uniformly continuous there exists $\delta_2>0$ such that $d(y_1, y_2)<\delta_2$ implies $|f(y_1)-f(y_2)|<\frac{1}{2}\varepsilon$. Choose T_1 such that if $t \geqq T_1$ then $d(\xi_t(x), -R_t^{-1}\theta_t)<\delta_2$ for all $x \notin B(\theta_t, \delta_1)$. Then for $t \geqq \max(T_1, T_0)$ we have

$$\begin{aligned}
&\Big|\int_{S^{n-1}} f(y)\,d\nu_t(y) - f(-R_t^{-1}\theta_t)\Big| \\
&= \Big|\int_{S^{n-1}} (f(\xi_t(x)) - f(-R_t^{-1}\theta_t))\,d\nu(x)\Big| \\
&\leqq 2\|f\|\,\nu(B(\theta_t, \delta_1)) \\
&\quad + \sup\{|f(\xi_t(x)) - f(-R_t^{-1}\theta_t)| : x \notin B(\theta_t, \delta_1)\} \\
&< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon \text{ as required. } \square
\end{aligned}$$

The theorem shows that the Markov process $\{\nu_t: t \geqq 0\}$ on $M(S^{n-1})$ is asymptotically stationary. In the limit we obtain the stationary process $\{\delta(X_t): t \geqq 0\}$ in $M(S^{n-1})$ where $\{X(t): t \geqq 0\}$ is a $BM(S^{n-1})$ with initial distribution m. Notice that in this example the invariant measure on $M(S^{n-1})$ for the process is supported on atomic measures. See Le Jan [13] for examples of stochastic flows where the invariant measure has no atomic measures in its support.

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Received September 9, 1984; in revised form February 28, 1986