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Probability Theory and Related Fields

General Wald-type Identities for Exchangeable Sequences and Processes

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Summary. Let $X=(X_1, \ldots, X_d)$ be an $\mathbb{R}^d$-valued Lévy process on $\mathbb{R}_+$ or ergodic exchangeable process on [0, 1], and let $V=(V_1, \ldots, V_d)$ be a predictable process on the same interval. Under suitable moment conditions, it is shown that, if the Lebesgue integrals $\int \prod_{j\in J} V_j$ are a.s. non-random for all $J\subset\{1, \ldots, d\}$ with $\# J\leqq d-1$ or $\# J\leqq d$, respectively, then the product moment $\mathsf{E}\prod\int V_j\,dX_j$ is the same as if X and V were independent. An analogous statement holds in discrete time. The results imply some invariance properties of exchangeable sequences and processes under suitable predictable transformations

1. Background and Motivation

Let $\xi_1, \xi_2, \ldots$ be i.i.d. random variables (r.v.'s) with a finite first moment $\mu_1=\mathsf{E}\,\xi_k$, and consider a predictable sequence of r.v.'s $\eta_1, \eta_2, \ldots$ (Thus each η_k is assumed to be measurable with respect to $\xi_1, \ldots, \xi_{k-1}$.) It is well-known that, under suitable integrability conditions on the η_k,

$$\mathsf{E}\sum_k \xi_k\eta_k=\mu_1\,\mathsf{E}\sum_k\eta_k. \tag{1}$$

If $\mu_1=0$ while $\mu_2=\mathsf{E}\,\xi_k^2<\infty$, it is further known, under somewhat more stringent conditions on the η_k, that

$$\mathsf{E}\{\sum_k \xi_k\eta_k\}^2=\mu_2\,\mathsf{E}\sum_k\eta_k^2. \tag{2}$$

In the special case when $\eta_k\equiv 1\{\tau\geqq k\}$ for some suitable finite stopping time τ, (1) and (2) reduce to the well-known *Wald identities*

$$\mathsf{E}\,X_\tau=\mu_1\,\mathsf{E}\,\tau, \qquad \mathsf{E}\,X_\tau^2=\mu_2\,\mathsf{E}\,\tau, \tag{3}$$

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where $X_n=\xi_1+\ldots+\xi_n$. (Cf. Wald (1945). A modern discussion may be found in, e.g., Chow & Teicher (1978), and some recent results in Franken & Lisek (1982) and in Klass (1988).)

The continuous time analogues of (1) and (2) are the formulas

$$\mathsf{E}\int V dX=\mu_1\,\mathsf{E}\int V, \tag{4}$$

$$\mathsf{E}\{\int V dX\}^2=\mu_2\,\mathsf{E}\int V^2, \tag{5}$$

for the stochastic integral of a suitable predictable process V with respect to a Lévy process X with finite first moment $\mu_1=\mathsf{E}X_1$, or with $\mu_1=0$ and $\mu_2=\mathsf{E}X_1^2<\infty$, respectively. Here (4) states essentially that stochastic integration preserves the martingale property, while (5) is the basic isometry for Itô-type stochastic integrals. (Note that the integrals on the right of (4) and (5) are with respect to Lebesgue measure.) Again one obtains the Wald identities (3) by choosing $V_t=1\{\tau\geqq t\}$ for a suitable stopping time τ. (Cf., e.g., Karatzas & Shreve (1988) for the mentioned results in the special case when X is a Brownian motion.)

Our aim in this paper is to prove some very general versions of formulas (1), (2), (4) and (5). To appreciate these, it is helpful to think of the previous relations in terms of *decoupling*. More precisely, we may introduce two *independent* sequences $(\xi_k')\overset{d}{=}(\xi_k)$ and $(\eta_k')\overset{d}{=}(\eta_k)$, or processes $X'\overset{d}{=}X$ and $V'\overset{d}{=}V$, and note that (1) and (2) are equivalent to

$$\mathsf{E}\{\sum_k \xi_k\eta_k\}^d=\mathsf{E}\{\sum_k \xi_k'\eta_k'\}^d \tag{6}$$

for $d=1$ and 2, while (4) and (5) are equivalent to

$$\mathsf{E}\{\int V dX\}^d=\mathsf{E}\{\int V' dX'\}^d \tag{7}$$

for $d=1$ and 2. Thus informally the first and second moments of the sum $\sum\xi_k\eta_k$ or the integral $\int V dX$ may be computed, under appropriate conditions, as if the two sequences (ξ_k) and (η_k) or the two processes X and V were independent.

In Sect. 3, the identity (6) will be proved for arbitrary $d\in\mathbb{N}$, under suitable moment conditions on the i.i.d. sequence (ξ_j) and the predictable sequence (η_k), and under the additional hypothesis that the sums over $\mathbb{N}$

$$S_m=\sum_k \eta_k^m \tag{8}$$

be a.s. non-random for $m=1,\ldots,d-1$, or at least for $m=2,\ldots,d-1$ when $\mu_1=\mathsf{E}\xi_1=0$. Similarly (7) is proved in Sect. 5 for arbitrary $d\in\mathbb{N}$, under appropriate moment conditions on the Lévy process X and the predictable process V, plus the extra requirement that the Lebesgue integrals over $\mathbb{R}_+$

$$S_m=\int V^m \tag{9}$$

be a.s. non-random for $m=1, \ldots, d-1$, or for $m=2, \ldots, d-1$ when $\mu_1 = \mathsf{E}X_1 = 0$. Note that the conditions on S_m are void for $d=1$, and even for $d=2$ when $\mu_1=0$, so that the above statements generalize the classical results. If $\mu_1=0$, one even gets for $d=3$ the simple formulas

$$\mathsf{E}\{\sum_k \xi_k \eta_k\}^3 = \mu_3 \mathsf{E} S_3 \quad \text{or} \quad \mathsf{E}\{\int V dX\}^3 = \mu_3 \mathsf{E} S_3, \tag{10}$$

with $\mu_3 = \mathsf{E}\xi_1^3$ or $\mu_3 = \mathsf{E}X_1^3$, respectively, in analogy with (1)–(5). The explicit moment formulas for $d \geqq 4$ are more complicated, though still tractable in the continuous time case.

At this point it is natural to ask for extensions of (6) or (7) beyond the case when (ξ_k) is i.i.d. or when X is Lévy. Though the formulas for $d=1$ and 2 were seen to be essentially martingale results, exchangeability theory appears to be the proper context for the general identities. Recall that, by de Finetti's theorem, an infinite sequence of random variables is exchangeable, iff it is a mixture (in the distributional sense) of i.i.d. sequences. Similarly, a process on $\mathbb{R}_+$ is known to be exchangeable (i.e. right-continuous, starting at 0, and with exchangeable increments), iff it is a mixture of Lévy processes (cf. Kallenberg (1973)). Thus (6) and (7) hold, under appropriate conditions, when (ξ_k) or X is an *extreme* (or *ergodic*) exchangeable sequence or process, respectively, indexed by $\mathbb{N}$ or $\mathbb{R}_+$, and it is easily seen (at least formally) how the results could be extended to the non-extreme case.

Much less obvious, and in fact rather surprising, are the corresponding results for finite exchangeable sequences and for exchangeable processes on [0, 1]. Recall that a sequence $\xi_1, \ldots, \xi_n$ is exchangeable, iff it is a mixture of so called *urn sequences*, where the latter are obtainable through successive random sampling without replacement from a set of size n. Thus the extreme or ergodic case is when the measure $\sum \delta_{\xi_k}$ is a.s. non-random. Even the exchangeable processes on [0, 1] are known to have unique representations as mixtures of extreme or ergodic processes, where important examples of the latter are given by the Brownian bridge and the empirical processes. The general exchangeable processes on [0, 1], which we describe later, serve as approximations to summation processes based on finite exchangeable sequences.

In Sect. 3 we shall prove, under a moment condition when $d=1$, that (6) holds when $\xi_1, \ldots, \xi_n$ form an extreme exchangeable sequence, while $\eta_1, \ldots, \eta_n$ are predictable and such that the sums in (8) are a.s. non-random for $m=1, \ldots, d$. Similarly we prove in Sect. 4, again under suitable moment conditions, that (7) holds when X is an extreme exchangeable process on [0, 1], while V is a predictable process on [0, 1], such that the integrals in (9) as a.s. non-random for $m=1, \ldots, d$.

To motivate a reader who might think that moment formulas like those described are dull and of little interest, we insert a simple illustration, phrased in terms of gambling. Imagine an ordinary well-shuffled card-deck, from which you are invited to pick cards at random, one by one. Before you draw card number k, you may bet an amount η_k, and if the card is red you get the double amount back, otherwise nothing. Assume also that you have to decide, before entering the game, on the total amount $\sum \eta_k$ to bet. Then (6) shows that your expected total gain is zero. This is surprising, because you know all through the game the proportion of red cards in the deck, and it would seem to be a good strategy to bet large amounts when this proportion is high. If even $\sum \eta_k^2$ has to be fixed in advance, then the variance of your total gain will also be independent of your strategy, and so on for higher moments.

Our moment identities (6) and (7) are closely related to the *predictable sampling theorem* in Kallenberg (1988), an extension of a result by Doob (1936), which states that if $\xi_1, \xi_2, \ldots$ form a finite or infinite exchangeable sequence and if $\tau_1, \tau_2, \ldots$ are a.s. distinct predictable stopping times w.r.t. the ξ_k, then

$$(\xi_{\tau_1}, \xi_{\tau_2}, \ldots) \stackrel{d}{=} (\xi_1, \xi_2, \ldots). \tag{11}$$

In fact, we shall see in Sect. 6 how the latter result may be easily deduced from the identities in (6). In continuous time, one considers instead an exchangeable process on $I=[0, 1]$ or $\mathbb{R}_+$, and a predictable $\bar{I}$-valued process V on I, such that the paths of V are a.s. Lebesgue measure preserving transformations of I. The claim corresponding to (11) is then that the process

$$(X \circ V^{-1})_t = \int_I 1\{V_s \leqq t\}\, dX_s, \qquad t \in I, \tag{12}$$

has the same distribution as X. Here the proof may be based on the moment identities (7). Thus the present results may be viewed as extensions of those in Kallenberg (1988).

(The methods of proof of the two papers are entirely different, and the present approach seems to provide some better insight into the nature of (11) and its continuous time counterpart. This fact alone would be enough motivation for the present paper, since for finite sequences, formula (11) is just as surprising as (6), cf. Kallenberg (1985).)

By using the present methods, we shall in fact be able to go further, and derive various extensions of the predictable sampling theorem. In particular, we shall prove a multivariate version, where different predictable mappings may be used in the different coordinates. Here the proof rests on certain multivariate extensions of (6) and (7) of the form

$$\mathsf{E} \prod_{j=1}^{d} \sum_k \xi_{jk} \eta_{jk} = \mathsf{E} \prod_{j=1}^{d} \sum_k \xi'_{jk} \eta'_{jk} \tag{13}$$

or

$$\mathsf{E} \prod_{j=1}^{d} \int V_j\, dX_j = \mathsf{E} \prod_{j=1}^{d} \int V'_j\, dX'_j, \tag{14}$$

valid for suitable $\mathbb{R}^d$-valued extreme exchangeable sequences $(\xi_{1k}, \ldots, \xi_{dk})$, $k=1, 2, \ldots$, or processes $(X_1, \ldots, X_d)$, and for predictable sequences $(\eta_{1k}, \ldots, \eta_{dk})$,

$k=1, 2, \ldots$, or processes $(V_1, \ldots, V_d)$, where it is assumed that the sums or Lebesgue integrals

$$S_J=\sum_k \prod_{j\in J} \eta_{jk} \quad \text{or} \quad S_J=\int \prod_{j\in J} V_j \tag{15}$$

are a.s. non-random for all $J\subset\{1, \ldots, d\}$ with $\# J\leqq d$ (in the finite case) or $\# J\leqq d-1$ (in the infinite case). Since (13) and (14) may be proved in the same way as (6) and (7) and without additional effort, we shall actually state and prove all moment identities of this paper in their multivariate versions. With the mentioned application in mind, we shall further incorporate into our main results some other refinements that will make them more powerful, essentially without lengthening their proofs.

We proceed to review some basic facts and to introduce some terminology and notation, which will be used throughout the paper. First we recall the Lévy-Hinchin representation associated with a Lévy process X in $\mathbb{R}^d$, which in case of finite first moments may be written in the form

$$\mathsf{E}\exp(iuX_t)=\exp\{ituy-\tfrac{1}{2}tu\rho u^T+t\int(e^{iux}-1-iux)\,v(dx)\},\ u\in\mathbb{R}^d,\ t\geqq 0, \tag{16}$$

in terms of the *mean vector* $\gamma\in\mathbb{R}^d$, the *covariance matrix* ρ of the continuous component, and the *Lévy measure* v on $\mathbb{R}^d\setminus\{0\}$, where the latter is such that $\int(|x|^2\wedge|x|)\,v(dx)<\infty$. (In (16), u should be regarded as a row vector, and γ, X_t and x as column vectors.) The distribution $\mathsf{P}X^{-1}$ and the triple (γ, ρ, v) determine each other uniquely, and we shall refer to the latter as the *directing triple* of X.

An $\mathbb{R}^d$-valued process X on $[0, 1]$ is known to be ergodic exchangeable, iff it has an a.s. representation (possibly on an extended probability space) of the form

$$X_t=\alpha t+\sigma B_t+\sum_{k=1}^{\infty}\beta_k(1\{\tau_k\leqq t\}-t),\qquad t\in[0, 1], \tag{17}$$

in terms of some vectors $\alpha, \beta_1, \beta_2, \ldots\in\mathbb{R}^d$, some $(d\times d)$-matrix σ, some $\mathbb{R}^d$-valued Brownian bridge B, and some independent set of i.i.d. random variables $\tau_1, \tau_2, \ldots$, each $U(0, 1)$ (uniformly distributed on $[0, 1]$). Note that the series in (17) converges a.s. uniformly in t. (Cf. Kallenberg (1973, 1974b).) Write ρ for the covariance matrix $\sigma\sigma^T$ and β for the counting measure $\sum\delta_{\beta_k}$ restricted to $\mathbb{R}^d\setminus\{0\}$. Here it is clear that $\mathsf{P}X^{-1}$ and (α, ρ, β) determine each other uniquely, and we shall refer to the latter as the *directing triple* of X.

In the previous informal account, the notions of predictable sequences, processes and stopping times, as well as that of stochastic integral, were understood to be with respect to the natural filtration induced by the sequence (ξ_k) or process X. However, all results remain valid and will be stated in the more general setting where (ξ_k) or X is exchangeable with respect to some general filtration $\mathscr{F}$ (or $\mathscr{F}$*-exchangeable*), in the sense of Kallenberg (1982, 1988). In the case of sequences (ξ_k), it is then required that ξ_k be $\mathscr{F}_k$-measurable for each k, while the shifted sequence $\xi_{k+1}, \xi_{k+2}, \ldots$ should be contitionally exchangeable, given $\mathscr{F}_k$. Note that this holds automatically when $\mathscr{F}_k=\sigma(\xi_1, \ldots, \xi_k)$. More generally, we may take $\mathscr{F}_k=\sigma(\xi_1, \ldots, \xi_k, \vartheta_k)$, where the ϑ_k are i.i.d. $U(0, 1)$ and independent of the ξ_k. (In our application to card games, this device allows for a randomized decision at each step.) In the case of infinite ergodic sequences, it is clearly equivalent that (ξ_k) be $\mathscr{F}$*-i.i.d.*, in the sense that each ξ_k is $\mathscr{F}_k$-measurable but independent of $\mathscr{F}_{k-1}$.

The definition of $\mathscr{F}$-exchangeability for a process X on $[0, 1]$ or $\mathbb{R}_+$ is similar, and in case of ergodic exchangeable processes on $\mathbb{R}_+$, it is equivalent that X be $\mathscr{F}$*-Lévy*, in the sense that X_s is $\mathscr{F}_s$-measurable for each s, while the increment X_t-X_s is independent of $\mathscr{F}_s$ for any $t>s$. All filtrations are assumed to be *standard*, in the sense of satisfying the usual conditions of right-continuity and completeness.

For an $\mathscr{F}$-exchangeable process X on $[0, 1]$ as in (17), such that the β_k are distinct and non-zero, the individual jump processes $1\{\tau_k \leqq t\}$ are again $\mathscr{F}$-exchangeable, and have compensators given by $\log(1-t \wedge \tau_k)$ (cf. Kallenberg (1988)). Though the τ_k may not even be $\mathscr{F}_1$-measurable in general, it is easy to see how the basic probability space $(\Omega, \mathscr{O}, \mathsf{P})$ and the associated filtration $\mathscr{F}$ may be extended, in such a way that X will have a representation as in (17) with the stated properties. (See Sect. 5 for a general discussion of such extensions.) Since the definition of a stochastic integral is not affected by an extension of the filtration (cf. Theorem 9.26 in Jacod (1979)), it is enough to prove our results in the extended setting, so we may assume, without loss of generality, that the appropriate extension is already accomplished, and that the τ_k have the stated properties.

Throughout the paper, efficient existence criteria and maximum inequalities will be needed for our stochastic sums and integrals. Such results are provided in Sect. 2. They may be new, and perhaps even of some independent interest, already for i.i.d. sequences and Lévy processes (cf. Propositions 2.1 and 2.2), for which they follow by iterated use of the BDG or Burkholder-Davis-Gundy inequalities (cf. Dellacherie & Meyer (1980)). The corresponding theory for stochastic integrals $\int V dX$ on $[0, 1]$ is somewhat harder, but simplifies when $\int V$ is non-random, since in that case (and under suitable moment conditions) there exists a martingale M and a predictable process U, such that $\int V dX=\int U dM$ (cf. Proposition 2.6). This result plays a key role in the paper, and gives a clue to the understanding of (7) when $d=1$. A similar statement is true in discrete time, but will not be needed in this paper.

Apart from (17) and some other specifically quoted results, no deeper knowledge of exchangeability theory is assumed in this paper, though the reader may wish to consult Aldous' (1985) lecture notes, supplemented by the author's papers Kallenberg (1973, 1974a, 1982, 1988), for some further background and motivation. On the other hand, standard terminology, notation and results from stochastic calculus will be used freely without references, and here the reader may e.g. consult Dellacherie & Meyer (1975/1980) or Jacod (1979) for details. A special convention in this paper is to write Lebesgue integrals as $\int f$, without explicit integrator 'dt' or '$d\lambda$'. If nothing else is stated, L_p norms are defined with respect to the basic probability measure P. We shall often write $a \lesssim b$ instead of $a=0(b)$. When this relation is used in Sect. 2, the implicit constant is assumed to depend on p, but not on any particular random sequences or processes which may occur. In subsequent sections, the dependence on possible parameters should be clear from the context.

2. Preliminaries for Stochastic Sums and Integrals

Here we shall first study predictable summation with respect to i.i.d. sequences $\xi=(\xi_1, \xi_2, \ldots)$, where the underlying filtration $\mathscr{F}=(\mathscr{F}_n)$ is indexed by $\mathbb{Z}_+=\{0, 1, \ldots\}$. Recall that a sequence $\eta=(\eta_1, \eta_2, \ldots)$ is said to be *$\mathscr{F}$-predictable*, if η_n is $\mathscr{F}_{n-1}$-measurable for each $n\in\mathbb{N}=\{1, 2, \ldots\}$.

Proposition 2.1. *Fix $p\geqq 1$, and write $p'=p\wedge 2$ and $p''=p\vee 2$. Then we have, for any filtration $\mathscr{F}$ on $\mathbb{Z}_+$ and for any infinite random sequences ξ and η in $\mathbb{R}$, such that ξ is $\mathscr{F}$-i.i.d. while η is $\mathscr{F}$-predictable,*

$$\mathsf{E}\sup_n \left|\sum_{k=1}^n \xi_k \eta_k\right|^p \lesssim |\mathsf{E}\,\xi_1|^p\,\mathsf{E}\left\{\sum_{k=1}^\infty |\eta_k|\right\}^p + \mathsf{E}|\xi_1|^p\,\mathsf{E}\left\{\sum_{k=1}^\infty |\eta_k|^{p'}\right\}^{p''/2}. \tag{1}$$

When this bound is finite, the sequence

$$\mu_n=\sum_{k=1}^n \xi_k\eta_k-(\mathsf{E}\,\xi_1)\sum_{k=1}^n \eta_k, \qquad n\in\mathbb{Z}_+, \tag{2}$$

converges a.s. and forms an L_p-martingale on $\bar{\mathbb{Z}}_+=\{0, 1, \ldots; \infty\}$.

Proof. We may assume that the right-hand side of (1) is finite, and write $\xi'_k=\xi_k-\mathsf{E}\,\xi_k$. Since ξ'_k and η_k are integrable and independent for each k, the products $\xi'_k\eta_k$ form a martingale difference sequence. Hence we get by the BDG inequality

$$\begin{aligned}\mathsf{E}\sup_n \left|\sum_{k=1}^n \xi_k \eta_k\right|^p &\lesssim \mathsf{E}\left\{\sum_{k=1}^\infty |\mathsf{E}\,\xi_k||\eta_k|\right\}^p + \mathsf{E}\sup_n\left|\sum_{k=1}^n \xi'_k\eta_k\right|^p \\ &\lesssim |\mathsf{E}\,\xi_1|^p\,\mathsf{E}\left\{\sum_{k=1}^\infty |\eta_k|\right\}^p + \mathsf{E}\left\{\sum_{k=1}^\infty \xi'^2_k\eta_k^2\right\}^{p/2}.\end{aligned}$$

Iterating the procedure, we get after m steps, with $2^m\in(p, 2p]$,

$$\begin{aligned}\mathsf{E}\sup_n \left|\sum_{k=1}^n \xi_k \eta_k\right|^p &\lesssim \sum_{r=0}^{m-1} |\mathsf{E}\,\xi_1^{(r)}|^{p2^{-r}}\,\mathsf{E}\left\{\sum_{k=1}^\infty |\eta_k|^{2^r}\right\}^{p2^{-r}} \\ &\quad + \mathsf{E}\left\{\sum_{k=1}^\infty \xi_k^{(m)}\eta_k^{2^m}\right\}^{p2^{-m}},\end{aligned} \tag{3}$$

where we define $\xi_k^{(0)}=\xi_k$, and then recursively

$$\xi_k^{(r+1)}=(\xi_k^{(r)}-\mathsf{E}\,\xi_k^{(r)})^2, \qquad r=0, \ldots, m-1. \tag{4}$$

The above argument is justified by the fact that

$$\mathsf{E}|\xi_k^{(r)}|^{p2^{-r}} \lesssim \mathsf{E}|\xi_k|^p, \qquad r=0, \ldots, m, \tag{5}$$

which follows recursively from (4), if we write for $r=0, \ldots, m-1$

$$\begin{aligned}\mathsf{E}|\xi_k^{(r+1)}|^{p2^{-r-1}} &= \mathsf{E}|\xi_k^{(r)} - \mathsf{E}\,\xi_k^{(r)}|^{p2^{-r}} \\ &\lesssim \mathsf{E}|\xi_k^{(r)}|^{p2^{-r}} + |\mathsf{E}\,\xi_k^{(r)}|^{p2^{-r}} \lesssim \mathsf{E}|\xi_k^{(r)}|^{p2^{-r}}.\end{aligned}$$

Comparing (1) and (3), it is seen that the first terms agree. For the last term in (3), we get by subadditivity and independence

$$\mathsf{E}\left\{\sum_{k=1}^{\infty} \xi_k^{(m)} \eta_k^{2m}\right\}^{p2^{-m}} \lesssim \mathsf{E} \sum_{k=1}^{\infty} |\xi_k^{(m)}|^{p2^{-m}} |\eta_k|^p = \mathsf{E}|\xi_1^{(m)}|^{p2^{-m}} \mathsf{E} \sum_{k=1}^{\infty} |\eta_k|^p. \tag{6}$$

Note also that, by subadditivity,

$$\left\{\sum_{k=1}^{\infty} |\eta_k|^c\right\}^{p/c} \leqq \left\{\sum_{k=1}^{\infty} |\eta_k|^{p'}\right\}^{p''/2}, \qquad c \geqq p'. \tag{7}$$

We now get (1) by combining (3) and (5)–(7). The last assertion follows, since the martingale in (2) is uniformly integrable when the bound in (1) is finite. □

Stochastic integration with respect to Lévy processes was studied extensively in Kallenberg (1975), and the following result extends the simple Corollary 4.1 there.

Proposition 2.2. *Fix $p \geqq 1$, and write $p' = p \wedge 2$ and $p'' = p \vee 2$. Then we have, for any standard filtration $\mathscr{F}$ and any $\mathbb{R}$-valued processes X and V on $\mathbb{R}_+$, such that X is $\mathscr{F}$-Lévy and directed by some (γ, σ^2, ν) while V is $\mathscr{F}$-predictable,*

$$\begin{aligned}\mathsf{E}\sup_t \left|\int_0^t V\,dX\right|^p &\lesssim |\gamma|^p \mathsf{E}\left(\int |V|\right)^p + \sigma^p \mathsf{E}\left(\int V^2\right)^{p/2} \\ &+ \left\{\left(\int |x|^{p'} \nu(dx)\right)^{p''/2} + \int |x|^p \nu(dx)\right\} \mathsf{E}\left[\left(\int |V|^{p'}\right)^{p''/2} + \int |V|^p\right],\end{aligned} \tag{8}$$

in the sense that $\int V\,dX$ exists and satisfies (8) *when the bound is finite. In that case, the process*

$$M_t = \int_0^{t+} V\,dX - \gamma \int_0^t V, \qquad t \geqq 0, \tag{9}$$

converges a.s. as $t \to \infty$ and forms an L_p-martingale on $\bar{\mathbb{R}}_+ = [0, \infty]$.

For our current and future needs, let us record the well-known *norm interpolation formula*

$$\|f\|_q \leqq \|f\|_p \vee \|f\|_r, \qquad 0<p \leqq q \leqq r, \tag{10}$$

valid in arbitrary measure spaces. For a simple proof, note that $\log \|f\|_{1/t}$ is convex in $t>0$ by Hölder's inequality.

Proof. To prove (8), it is clearly enough to decompose X into its drift, diffusion and purely discontinuous components, and to prove (8) for each. Now (8) is trivial if X is linear, and if X is a Brownian motion the integral process $\int V dX$ is a continuous local martingale with quadratic variation $\int V^2$, so (8) follows by the BDG inequality. It thus remains to consider the case when X is purely discontinuous and centered.

Letting $m \in \mathbb{N}$ with $2^m \in (p, 2p]$, and proceeding by iterated *formal* application of the BDG inequality, we then get as in the last proof

$$\begin{aligned} \mathsf{E} \sup_t \left| \int_0^{t+} V dX \right|^p \lesssim \sum_{r=1}^{m-1} \left\{ \mathsf{E} \int_0^1 (dX)^{2^r} \right\}^{p2^{-r}} \mathsf{E} \left\{ \int_0^\infty V^{2^r} \right\}^{p2^{-r}} \\ + \mathsf{E} \left\{ \int_0^\infty V^{2^m} (dX)^{2^m} \right\}^{p2^{-m}}, \end{aligned} \tag{11}$$

where

$$\int_0^{t+} (dX)^{2^r} = \sum_{s \leqq t} (\Delta X_s)^{2^r}, \qquad t \geqq 0, \ r \in \mathbb{N}. \tag{12}$$

To justify (11), we need to show that the stochastic integral processes $\int V dX$ and

$$\int_0^{t+} V^{2^r} \{(dX)^{2^r} - \mathsf{E}(dX)^{2^r}\}, \qquad t \geqq 0, \ r = 1, \ldots, m-1, \tag{13}$$

exist and are local martingales. But this holds by Definition 2.46 in Jacod (1979), provided that the right-hand side of (11) is finite. It is thus enough to show that the latter expression is bounded by the one in (8).

To see this, note that

$$\mathsf{E} \sum_{s \leqq t} |\Delta X_s|^q = t \int |x|^q \nu(dx), \qquad q>0, \ t \geqq 0, \tag{14}$$

and that the jump process J_q on the left of (14) is again $\mathscr{F}$-Lévy, provided that $q \geqq p'$. Thus J_q is compensated, for $q \in [p', p]$, by the function on the right of (14), and we get by the subadditivity of $x^{p2^{-m}}$ and the predictability of $|V|^p$

$$\mathsf{E} \left\{ \int_0^\infty V^{2^m} |dX|^{2^m} \right\}^{p2^{-m}} \leqq \mathsf{E} \sum_{t>0} |V_t \Delta X_t|^p = \mathsf{E} \int_0^\infty |V|^p \int |x|^p \nu(dx). \tag{15}$$

The estimate in (8) now follows from (11) by means of (10), (12), (14) and (15). The last assertion follows from the fact that finiteness in (8) implies uniform integrability of the process $\int V dX$. $\square$

The remainder of this section is devoted to stochastic integration with respect to ergodic exchangeable processes on $[0, 1]$, and we begin with a general integrability condition.

Proposition 2.3. *Let $\mathscr{F}$ be a standard filtration on $[0, 1]$, and let X and V be $\mathbb{R}$-valued processes on $[0, 1]$, such that X is ergodic $\mathscr{F}$-exchangeable and directed by $(\alpha, \sigma^2, \beta)$ while V is $\mathscr{F}$-predictable. Fix $p \in (0, 2]$ and $\varepsilon \geqq 0$ with $\varepsilon > 0$ if $p > 1$, and such that $\sum |\beta_k|^p < \infty$, that $\sigma = 0$ if $p < 2$, and that $\alpha = \sum \beta_k$ if $p < 1$. Then $\int V dX$ exists on $[0, 1]$ provided that*

$$\int_0^1 |V_t|^p (1-t)^{-\varepsilon} dt < \infty \quad \text{a.s.} \tag{16}$$

Note in particular that (16) holds if $\int |V|^r < \infty$ a.s. for some $r > p$. Weaker conditions for integrability on intervals $[0, t]$ with $t < 1$ may be obtained by adaption of the methods in Kallenberg (1975).

Here and below, we shall use the fact from Kallenberg (1988) that X is a special semimartingale on $[0, 1]$ with canonical decomposition of the form

$$X_t = M_t - \int_0^t \frac{X_s - \alpha}{1-s} ds, \qquad t \in [0, 1], \tag{17}$$

where M is an L_2-martingale with associated quadratic variation process

$$[M, M]_t = [X, X]_t = \sigma^2 t + \sum_{k=1}^{\infty} \beta_k^2 \, 1\{\tau_k \leqq t\}, \qquad t \in [0, 1]. \tag{18}$$

Proof. When $p \leqq 1$, we may clearly assume that $\sigma = 0$, $\alpha = \sum \beta_j$, $\sum |\beta_j|^p < \infty$ and $\int |V|^p < \infty$ a.s., and it is then enough to show that $\int V dX$ exists as a Lebesgue-Stieltjes integral, i.e. that

$$\int_0^1 |V| |dX| = \sum_{j=1}^{\infty} |\beta_j V_{\tau_j}| < \infty \quad \text{a.s.} \tag{19}$$

To see this, write

$$Y_t = \sum_{j=1}^{\infty} |\beta_j|^p \, 1\{\tau_j \leqq t\}, \qquad t \in [0, 1], \tag{20}$$

and note that Y is compensated by the process with density

$$N_t = \frac{1}{1-t} \sum_{k=1}^{\infty} |\beta_k|^p \, 1\{\tau_k > t\}, \qquad t \in [0, 1), \tag{21}$$

By subadditivity and dual predictable projection, we get

$$\mathsf{E}\left\{\int_0^1 |V||dX|\right\}^p \leqq \mathsf{E} \sum_{j=1}^{\infty} |\beta_j V_{\tau_j}|^p = \mathsf{E}\int_0^1 |V|^p dY = \mathsf{E}\int_0^1 |V|^p N. \tag{22}$$

The same relation holds with V replaced by the predictable processes

$$V_n(t) = V(t)\cdot 1\{t \leqq \sigma_n\}, \qquad t\in[0,1],\ n\in\mathbb{N}, \tag{23}$$

where $\sigma_1, \sigma_2, \ldots$ denote the $[0,1]$-valued stopping times

$$\sigma_n = \sup\left\{t \leqq 1;\ \int_0^t |V|^p N \leqq n\right\}, \qquad n\in\mathbb{N}, \tag{24}$$

so we get

$$\int_0^{\sigma_n+} |V||dX| < \infty \quad \text{a.s.}, \qquad n\in\mathbb{N}. \tag{25}$$

To obtain (19), it remains to notice that $\sigma_n = 1$ for all sufficiently large n, which holds since N is a positive martingale and therefore a.s. bounded.

Let us next assume that $p\in(1,2]$ and $\sum|\beta_j|^p < \infty$, and that (16) holds for some $\varepsilon > 0$. Then $\int |V| < \infty$ a.s., so we may further take $\alpha = 0$. Starting with the case when $\sigma = 0$, and defining M, Y and N by (17), (20) and (21), we get by (18), Jensen's inequality, subadditivity and dual predictable projection

$$\begin{aligned}\left\{\mathsf{E}\left\{\int_0^1 V^2 d[M,M]\right\}^{1/2}\right\}^p &= \{\mathsf{E}\{\textstyle\sum_j \beta_j^2 V_{\tau_j}^2\}^{1/2}\}^p \leqq \mathsf{E}\{\textstyle\sum_j \beta_j^2 V_{\tau_j}^2\}^{p/2} \\ &\leqq \mathsf{E}\textstyle\sum_j |\beta_j V_{\tau_j}|^p = \mathsf{E}\displaystyle\int_0^1 |V|^p dY = \mathsf{E}\int_0^1 |V|^p N. \end{aligned} \tag{26}$$

Replacing V by the processes V_n in (23), we get from (26)

$$\mathsf{E}\left\{\int_0^{\sigma_n+} V^2 d[M,M]\right\}^{1/2} < \infty, \qquad n\in\mathbb{N}, \tag{27}$$

with the σ_n given by (24). As before, $\sigma_n = 1$ for sufficiently large n, so $\int V dM$ exists by Definition 2.46 in Jacod (1979). If instead $p = 2$ and $\sigma > 0$, we get in place of (26)

$$\left\{\mathsf{E}\left\{\int_0^1 V^2 d[M,M]\right\}^{1/2}\right\}^2 = \mathsf{E}\{\sigma^2 \textstyle\int V^2 + \sum_j \beta_j^2 V_{\tau_j}^2\} \leqq \mathsf{E}\displaystyle\int_0^1 V_t^2(\sigma^2 + N_t)\,dt, \tag{28}$$

and the existence of $\int V dM$ follows as before.

To complete the proof for $p > 1$, it remains to show that V is Lebesgue-Stieltjes integrable with respect to the second component in (17), i.e. that

$V_t X_t/(1-t)$ is Lebesgue integrable over $[0, 1)$. To see this, conclude from Hölder's inequality that

$$\int_0^1 \frac{|V_t X_t|}{1-t}\,dt \leqq \left\{\int_0^1 |V_t|^p (1-t)^{-\varepsilon}\,dt\right\}^{1/p} \left\{\int_0^1 |X_t|^q (1-t)^{-q'}\,dt\right\}^{1/q}, \tag{29}$$

where $p^{-1}+q^{-1}=1$ and $q'=(1-\varepsilon p^{-1})q<q$, and note that the first factor on the right is a.s. finite by (16). To show that even the second factor is finite a.s., we may assume that $1<q'<q$, since $(1-t)^{-q'}$ is non-decreasing in q'. In that case there is a $p'>p$ satisfying $p'^{-1}+q'^{-1}=1$, and by Theorem 2.1 in Kallenberg (1974a) we get $|X_t|^{p'} \lesssim 1-t$ a.s. as $t\to 1$, so

$$|X_t|^q (1-t)^{-q'} \lesssim (1-t)^{-q'+q/p'} \quad \text{a.s.},$$

which is integrable over $[0, 1)$, since $-q'+q/p' > -q'+q'/p' = -1$. □

For the remainder of this section, we assume that $\mathscr{F}$, X, V, $(\alpha, \sigma^2, \beta)$, p and ε are such as in Proposition 2.3. We shall further assume that (1.17) holds for some B and $\tau_1, \tau_2, \ldots$ with the stated properties. Let us write $\bar{V} = \int V$ when the integral exists.

Proposition 2.4. *For $p \leqq 1$, we have*

$$\int_0^1 V\,dX = \sum_{j=1}^{\infty} \beta_j V_{\tau_j} \quad \textit{a.s.}, \tag{30}$$

while for $p \geqq 1$,

$$\int_0^1 V\,dX = \alpha \bar{V} + \sigma \int_0^1 V\,dB + \sum_{j=1}^{\infty} \beta_j (V_{\tau_j} - \bar{V}) \quad \textit{a.s.}, \tag{31}$$

where the series on the right converges in probability.

The proof requires a lemma of some independent interest.

Lemma 2.5. *For fixed $r \in [0, p^{-1} \wedge 1)$ we have*

$$\lim_{n\to\infty} \sup_t\, (t(1-t))^{-r} \left| \sum_{j=n}^{\infty} \beta_j (1\{\tau_j \leqq t\} - t) \right| = 0 \quad \textit{a.s.} \tag{32}$$

If instead $1 \leqq r < p^{-1}$, we get

$$\lim_{n\to\infty} \sup_t\, t^{-r} \left| \sum_{j=n}^{\infty} \beta_j 1\{\tau_j \leqq t\} \right| = 0 \quad \textit{a.s.} \tag{33}$$

The second part is stated here only for completeness and will not be needed in this paper. It follows easily by combination of Theorem 2.1 in Kallenberg (1974a) with Theorem 3 in Kallenberg (1974b). Alternatively, it may be obtained by adaption of the following argument to the case when $r \geqq 1$.

Proof for $r<1$. We may clearly assume that $p>1$. Denote the sum in (32) by X_t, and introduce the martingale $M'_t = X_t/(1-t)$. By the BDG inequality, formula (18), the subadditivity of $x^{p/2}$, and the $\mathscr{F}$-exchangeability of the τ_j, we get for $0 \leqq t \leqq t+h \leqq 1/2$,

$$\begin{aligned} \mathsf{E}[|M'_{t+h}-M'_t|^p \mid \mathscr{F}_t] &\lesssim \mathsf{E}[([M',M']_t^{t+h})^{p/2} \mid \mathscr{F}_t] \lesssim \mathsf{E}[([X,X]_t^{t+h})^{p/2} \mid \mathscr{F}_t] \\ &= \mathsf{E}[(\sum_j \beta_j^2 1\{t<\tau_j<t+h\})^{p/2} \mid \mathscr{F}_t] \\ &\leqq \mathsf{E}[\sum_j |\beta_j|^p 1\{t<\tau_j<t+h\} \mid \mathscr{F}_t] \\ &= \frac{h}{1-t} \sum_j |\beta_j|^p 1\{\tau_j>t\} \leqq 2h \sum_j |\beta_j|^p. \end{aligned}$$

Hence we obtain for any $0=t_0<t_1<\ldots<t_n=t \leqq 1/2$

$$\sum_k \mathsf{E}[|M'_{t_k}-M'_{t_{k-1}}|^p \mid \mathscr{F}_{t_{k-1}}] \lesssim t \sum_j |\beta_j|^p.$$

We may then conclude from Lemma 2.3 in Kallenberg (1975) that, for some constant $c>0$ and for any increasing and continuous function $g: \mathbb{R}_+ \to \mathbb{R}_+$,

$$\mathsf{P}\{\sup_{t \leqq 1/2} |M'_t|^p / g(ct \sum_j |\beta_j|^p) \geqq \varepsilon\} \leqq \frac{2}{\varepsilon} \int_0^\infty \frac{du}{g(u)}, \qquad \varepsilon>0. \tag{34}$$

Here the left-hand side depends only on $g(x)$ for $2x \leqq c \sum |\beta_j|^p$, so we may choose $g(x)=(x/c)^{pr}$ for such x and let $1/g$ be integrable on $(0,\infty)$, to obtain

$$\mathsf{P}\{\sup_{t \leqq 1/2} |X_t| t^{-r} \geqq \varepsilon\} \lesssim \varepsilon^{-p} \sum_{j=1}^\infty |\beta_j|^p, \qquad \varepsilon>0. \tag{35}$$

Applying (35) and the corresponding inequality for $t \geqq 1/2$ to the processes in (32), we obtain as $n \to \infty$

$$\sup_t \left| (t(1-t))^{-r} \sum_{j=n}^\infty \beta_j (1\{\tau_j \leqq t\} - t) \right| \xrightarrow{\mathsf{P}} 0. \tag{36}$$

It follows in particular that the processes on the left have paths in $D[0,1]$. Since the individual terms are independent, it follows by Theorem 3 in Kallenberg (1974b) that the convergence in (36) is in fact a.s. □

Proof of Proposition 2.4. The result for $p \leqq 1$ was established in the proof of Proposition 2.3, so it remains to take $p > 1$. Since (31) is trivial when the sum in (1.17) is finite, it is enough to prove that $\int V dX_n \xrightarrow{\mathsf{P}} 0$, where X_n is the sum in (1.17) for $j \geqq n$. Writing M_n and N_n for the associated martingales M and N in (16) and (21), and introducing the stopping times

$$\sigma_n = \sup\left\{t \leqq 1;\ \int_0^t |V|^p N_n \leqq 1\right\}, \qquad n \in \mathbb{N}, \tag{37}$$

we get as before, by the BDG inequality and dual predictable projection,

$$\mathsf{E}\left|\int_0^{\sigma_n+} V dM_n\right|^p \lesssim \mathsf{E}\left\{\int_0^{\sigma_n+} V^2 d[M_n, M_n]\right\}^{p/2} \leqq \mathsf{E}\left\{1 \wedge \int_0^1 |V|^p N_n\right\} \leqq 1. \tag{38}$$

Now $N_n \downarrow 0$ as $n \to \infty$, so $\int |V|^p N_n \to 0$ a.s. by dominated convergence, and therefore $\sigma_n = 1$ for all sufficiently large n, while the integral on the left of (38) tends to zero in probability. Thus $\int V dM_n \xrightarrow{\mathsf{P}} 0$.

To prove the corresponding result for the compensating term, let $p^{-1} + q^{-1} = 1$, $q' = q(1 - \varepsilon/p)$, and $r + q'^{-1} = 1$, where we may assume that $1 \leqq q' < q$, so that $0 \leqq r < p^{-1} < 1$. Using Hölder's inequality as in (29), we get

$$\left|\int_0^1 \frac{V(t) X_n(t)}{1-t}\right| \leqq \left\{\int_0^1 \frac{|V_t|^p}{(1-t)}\right\}^{1/p} \left\{\int_0^1 (1-t)^{qr-q'}\right\}^{1/q} \sup_t \frac{|X_n(t)|}{(1-t)^r}. \tag{39}$$

Here the first two factors on the right are a.s. finite as before, while the last one tends to zero a.s., by Lemma 2.5. □

The next result gives the fundamental connection to martingales when $\bar{V}$ is constant, and will play a key role in Sect. 4. Recall that M is the martingale in (17).

Proposition 2.6. *Let $p > 1$, and assume that $\bar{V}$ is a.s. non-random. Then*

$$\int_0^1 V_t dX_t = \alpha \bar{V} + \int_0^1 \left(V_t - \frac{1}{1-t} \int_t^1 V_s ds\right) dM_t \quad \text{a.s.} \tag{40}$$

For the proof we shall need the first part of the following lemma. The second part will be needed later.

Lemma 2.7. *Let the function $f\colon \mathbb{R}_+ \to \mathbb{R}$ be locally integrable, and define*

$$g_t = \frac{1}{t} \int_0^t f_s ds, \qquad t > 0.$$

Then we have for any $p>1$ *and* $r\geqq 0$

$$\int_0^\infty |g_t|^p t^{-r}\,dt \leqq \left(\frac{p}{r+p-1}\right)^p \int_0^\infty |f_t|^p t^{-r}\,dt. \tag{41}$$

If f *is square integrable on* $[0,1]$, *we have in addition*

$$\int_0^1 (f_t-g_t)^2\,dt = \int_0^1 (f_t-g_1)^2\,dt. \tag{42}$$

Proof. To prove (41), we may clearly assume that $f\geqq 0$ and $\int f>0$, and by monotone convergence we may further assume the support of f to be compact in $(0,\infty)$, so that the left-hand side of (41) is finite and strictly positive. Writing $F_t=t\,g_t$ and letting $p^{-1}+q^{-1}=1$, we get by partial integration and Hölder's inequality

$$\int_0^\infty g_t^p t^{-r}\,dt = \int_0^\infty F_t^p t^{-r-p}\,dt = \frac{p}{r+p-1}\int_0^\infty F_t^{p-1} f_t r^{-r-p+1}\,dt$$

$$= \frac{p}{r+p-1}\int_0^\infty g_t^{p-1} f_t t^{-r}\,dt \leqq \frac{p}{r+p-1}\left\{\int_0^\infty g_t^p t^{-r}\,dt\right\}^{1/q}\left\{\int_0^\infty f_t^p t^{-r}\,dt\right\}^{1/p},$$

from which (41) follows if we divide by the second factor on the right.

In particular, $f\in L_2[0,1]$ implies $g\in L_2[0,1]$, and in that case we get by repeated use of Fubini's theorem

$$\int_0^1 g_s^2\,ds = \int_0^1 s^{-2} F_s^2\,ds = 2\int_0^1 s^{-2}\,ds \int_0^s f_t F_t\,dt = 2\int_0^1 f_t F_t\,dt \int_t^1 s^{-2}\,ds$$

$$= 2\int_0^1 f_t F_t (t^{-1}-1)\,dt = 2\int_0^1 f_t g_t\,dt - g_1^2.$$

Thus

$$\int (f-g)^2 = \int f^2 - 2\int fg + \int g^2 = \int f^2 - 2\int fg + 2\int fg - g_1^2 = \int (f-g_1)^2. \qquad \square$$

Proof of Proposition 2.6. First note that the stochastic integral in (40) exists by Lemma 2.7 and by the proof of Proposition 2.3. Since M is clearly independent of α, we may assume that $\alpha=0$. Define $M_t'=X_t/(1-t)$ as before, and conclude from Itô's formula and (17) that

$$dX_t = (1-t)\,dM_t' - M_t'\,dt = dM_t - M_t'\,dt. \tag{43}$$

Integrating (stochastically) by parts and using the constancy of $\bar{V}$, we get for $t<1$

$$\int_0^t V_s M_s'\,ds = M_t' \int_0^t V_s\,ds - \int_0^{t+} dM_s' \int_0^s V_r\,dr = \int_0^{t+} dM_s' \int_s^1 V_r\,dr - M_t' \int_t^1 V_s\,ds,$$

so by (43)

$$\int_0^{t+} V_s\,dX_s = \int_0^{t+}\left(V_s - \frac{1}{1-s}\int_s^1 V_r\,dr\right)dM_s + \frac{X_t}{1-t}\int_t^1 V_s\,ds. \tag{44}$$

Thus (40) follows by dominated convergence for stochastic integrals, provided we can show that the last term in (44) tends to zero as $t \to 1$. To see this, use Hölder's inequality with $p^{-1}+q^{-1}=1$ along with formula (18) above, to obtain

$$\begin{aligned}\left|\frac{X_t}{1-t}\int_t^1 V\right| &\leqq \frac{|X_t|}{1-t}\left\{\int_t^1 (1-s)^{\varepsilon q/p}\right\}^{1/q}\left\{\int_t^1 |V_s|^p (1-s)^{-\varepsilon}\right\}^{1/p} \\ &\lesssim |X_t|(1-t)^{-(1-\varepsilon)/p},\end{aligned}$$

and recall that the right-hand side goes to zero a.s., by Theorem 2.1 in Kallenberg (1974a). □

We conclude this section by proving a maximum inequality, similar to those in Propositions 2.1 and 2.2, for the stochastic integral in Proposition 2.6. Let us then denote the integrand by U, i.e.

$$U_t = V_t - \frac{1}{1-t}\int_t^1 V_s\,ds, \qquad t \in [0,1). \tag{45}$$

The constant p may now be different from that in Proposition 2.3.

Proposition 2.8. *Fix constants $p \geqq 1$ and $q > 2p$, and write $p' = p \wedge 2$ and $p'' = p \vee 2$. Then we have, for any X and V as above such that $\bar{V}$ exists and is a.s. non-random,*

$$\mathsf{E}\sup_t\left|\int_0^t U\,dM\right|^p \lesssim \sigma^p\,\mathsf{E}\left\{\int_0^1 V^2\right\}^{p/2} + \{\sum_j |\beta_j|^{p'}\}^{p''/2}\left\{\mathsf{E}\int_0^1 |V|^q\right\}^{p/q}, \tag{46}$$

in the sense that $\int U\,dM$ exists and satisfies (46) *whenever the right-hand side is finite. In that case, $\int U\,dM$ forms a martingale on* $[0,1]$, *and the series in* (31) *converges in L_p.*

Proof. First we conclude from (18) that

$$\begin{aligned}\mathsf{E}\left\{\int_0^1 U^2\,d[M,M]\right\}^{p/2} &= \mathsf{E}\left\{\sigma^2\int_0^1 U^2 + \sum_j \beta_j^2 U_{\tau_j}^2\right\}^{p/2} \\ &\lesssim \sigma^p\,\mathsf{E}\left\{\int_0^1 U^2\right\}^{p/2} + \mathsf{E}\{\sum_j \beta_j^2 U_{\tau_j}^2\}^{p/2}.\end{aligned} \tag{47}$$

We shall show that the right-hand side of (47) is bounded, up to a constant factor, by the expression in (46). If the latter is finite, $\int U\,dM$ will then exist as a local martingale (which we already know from earlier results), and (46) will follow by the BDG inequality. In particular, (46) shows that the process $\int U\,dM$ is uniformly integrable, and the asserted martingale property follows.

To estimate the right-hand side of (47), we note first that $\int U^2 \lesssim \int V^2$ by Lemma 2.7, which takes care of the first term on the right. As for the second term, assume that $p \leqq 2$, and conclude by subadditivity that

$$\mathsf{E}\{\sum_j \beta_j^2 U_{\tau_j}^2\}^{p/2} \leqq \mathsf{E} \sum_j |\beta_j U_{\tau_j}|^p \leqq \sum_j |\beta_j|^p \sup_j \mathsf{E}|U_{\tau_j}|^p.$$

If instead $p \geqq 2$, we obtain by Hölder's inequality

$$\mathsf{E}\{\sum_j \beta_j^2 U_{\tau_j}^2\}^{p/2} \leqq \{\sum_j \beta_j^2\}^{p/2-1} \mathsf{E} \sum_j \beta_j^2 |U_{\tau_j}|^p \leqq \{\sum_j \beta_j^2\}^{p/2} \sup_j \mathsf{E}|U_{\tau_j}|^p.$$

Thus it suffices in both cases to show that

$$\sup_j \mathsf{E}|U_{\tau_j}|^p \lesssim \{\mathsf{E} \int |V|^q\}^{p/q}.$$

To see this, use dual predictable projection, Hölder's inequality with $r = (1-p/q)^{-1}$, Fubini's theorem, and Lemma 2.7, to obtain

$$\begin{aligned}\mathsf{E}|U_{\tau_j}|^p &= \mathsf{E} \int_0^1 |U_t|^p \, d(1\{\tau_j \leqq t\}) = \mathsf{E} \int_0^\tau \frac{|U_t|^p}{1-t} \, dt \\ &\leqq \left\{\mathsf{E} \int_0^1 |U|^q\right\}^{p/q} \left\{\mathsf{E} \int_0^\tau (1-t)^{-r}\right\}^{1/r} \lesssim \left\{\mathsf{E} \int_0^1 |V|^q\right\}^{p/q} \left\{\int_0^1 (1-t)^{1-r}\right\}^{1/r}.\end{aligned}$$

To prove the last assertion, write X_n for the sum in (1.17) over indices $j \geqq n$, let M_n be the corresponding martingale M in (17), and conclude from Propositions 2.4, 2.6 and 2.8 that

$$\begin{aligned}\mathsf{E}\left|\sum_{j=n}^{\infty} \beta_j (V_{\tau_j} - \bar{V})\right|^p &= \mathsf{E}\left|\int_0^1 V\,dX_n\right|^p \\ &= \mathsf{E}\left|\int_0^1 U\,dM_n\right|^p \lesssim \left\{\sum_{j=n}^{\infty} |\beta_j|^{p'}\right\}^{p''/2} \left\{\mathsf{E} \int_0^1 |V|^q\right\}^{p/q},\end{aligned}$$

where the right-hand side tends to zero as $n \to \infty$. □

3. Moment Identities for Predictable Sums

In this section we shall prove moment identities for certain predictable sums w.r.t. exchangeable sequences ξ in $\mathbb{R}^d$. Thus we take the k-th element in ξ to be a random vector $\xi_{\cdot k} = (\xi_{1k}, \ldots, \xi_{dk})$, so that $\xi = (\xi_{jk})$ becomes an array indexed by $j \in \{1, \ldots, d\}$ and $k \in \{1, 2, \ldots\} = \mathbb{N}$ or $k \in \{1, \ldots, n\}$ for some $n \in \mathbb{N}$. The associated filtration $\mathscr{F} = (\mathscr{F}_k)$ is then indexed by $\mathbb{Z}_+ = \{0, 1, \ldots\}$ or $\{0, \ldots, n\}$, respectively. We emphasize that $\mathscr{F}$-exchangeability is to be understood in the *joint* sense in this section, i.e. the random vectors $\xi_{.1}, \xi_{.2}, \ldots$ are assumed to form an exchangeable sequence. Along with ξ we also consider a predictable sequence $\eta=(\eta_{.k})=(\eta_{jk})$ in $\mathbb{R}^d$, where j and k range over the same index sets as before.

We shall first consider the case of finite sequences ξ and η, both of length $n\in\mathbb{N}$. Let us then introduce the sums

$$R_J=\sum_{k=1}^{n}\prod_{j\in J}\xi_{jk}, \qquad S_J=\sum_{k=1}^{n}\prod_{j\in J}\eta_{jk}, \quad \emptyset\neq J\subset\{1,\ldots,d\}. \tag{1}$$

Recall that a finite exchangeable sequence $\xi=(\xi_{jk})$ is *ergodic*, if the counting measure

$$\mu_\xi=\sum_{k=1}^{n}\delta_{\xi,.k}; \qquad \mu_\xi(B)=\#\{k;\xi_{.k}\in B\}, \quad B\subset\mathbb{R}^d, \tag{2}$$

is non-random. In this case each R_J is constant, like any function of ξ which is invariant under permutations in index k. Our basic assumption is that even the sums S_J be non-random. In addition to this we shall need a technical condition, to ensure the existence of moments:

(C_1): There exist constants $p_1,\ldots,p_d\geqq 1$ with $\sum p_j^{-1}\leqq 1$, such that

$$\mathsf{E}\,|\eta_{jk}|^{p_j}<\infty, \qquad j=1,\ldots,d,\ k=1,\ldots,n.$$

Theorem 3.1. *Let $\mathscr{F}$ be a filtration on $\{0,\ldots,n\}$ and let ξ and η be random n-sequences in $\mathbb{R}^d$, such that ξ is ergodic $\mathscr{F}$-exchangeable while η is $\mathscr{F}$-predictable. Assume that (C_1) is fulfilled, and that S_J is a.s. non-random for every $J\subset\{1,\ldots,d\}$. Then*

$$E_d:=\mathsf{E}\prod_{j=1}^{d}\sum_{k=1}^{n}\xi_{jk}\eta_{jk}=P_{nd}\{R_J,S_J\}, \tag{3}$$

for some polynomial P_{nd} in the sums R_J and S_J. In particular $E_1=n^{-1}R_1S_1$, and under the further assumption $R_j\equiv S_j\equiv 0$ we have

$$E_2=\frac{1}{n-1}R_{12}S_{12}, \qquad E_3=\frac{n}{(n-1)(n-2)}R_{123}S_{123}. \tag{4}$$

A slightly more general statement will be proved in Lemma 3.4. But first we need to prove a couple of preliminary results, where the first one will also be useful later on.

Lemma 3.2. *For $x_{jk}\in\mathbb{R}, j=1,\ldots,d, \quad k\in\mathbb{N}$, with*

$$\prod_{j=1}^{d}\sum_{k=1}^{\infty}|x_{jk}|<\infty, \tag{5}$$

define

$$P=\sum_{(k_j)}\prod_{j=1}^{d}x_{j,k_j}, \tag{6}$$

where the summation extends over all choices of distinct $k_1, \ldots, k_d \in \mathbb{N}$. Then P is a polynomial in the sums

$$S_J = \sum_{k=1}^{\infty} \prod_{j \in J} x_{jk}, \qquad \emptyset \neq J \subset \{1, \ldots, d\}. \tag{7}$$

Proof. For $d=1$ we have $P = S_1$, and we shall proceed to general $d>1$ by induction. Thus assume that the corresponding quantity with products over $j \in \{1, \ldots, d-1\}$ is a polynomial P_{d-1} in the sums S_J with $\emptyset \neq J \subset \{1, \ldots, d-1\}$. Then we get, with $k_1, \ldots, k_d$ distinct throughout,

$$\begin{aligned}
\sum_{(k_j)} \prod_{j=1}^{d} x_{j,k_j} &= \sum_{k_1, \ldots, k_{d-1}} \prod_{j=1}^{d-1} x_{j,k_j} \sum_{k_d} x_{d,k_d} \\
&= \sum_{k_1, \ldots, k_{d-1}} \prod_{j=1}^{d-1} x_{j,k_j} (S_d - \sum_{i<d} x_{d,k_i}) \\
&= S_d \sum_{k_1, \ldots, k_{d-1}} \prod_{j=1}^{d-1} x_{j,k_j} - \sum_{i<d} \sum_{k_1, \ldots, k_{d-1}} x_{i,k_i} x_{d,k_i} \prod_{j \neq i,d} x_{j,k_j} \\
&= S_d P_{d-1}(S_J, J \subset \{1, \ldots, d-1\}) - \sum_{i<d} P_{d-1}(S_{J_i}, J \subset \{1, \ldots, d-1\}),
\end{aligned}$$

where

$$J_i = \begin{cases} J, & i \notin J, \\ J \cup \{d\}, & i \in J. \end{cases}$$

Thus the statement remains true in the case of d factors. $\square$

Lemma 3.3. *The assertions of Theorem 3.1 are true when η is non-random.*

Proof. Writing π for an arbitrary partition of the set $\{1, \ldots, d\}$ into at most n subsets J, and (k_J) for an arbitrary collection of distinct indices k_J, $J \in \pi$, we get by the assumptions on ξ

$$\begin{aligned}
\mathsf{E} \prod_{j=1}^{d} \sum_{k=1}^{n} \xi_{jk} \eta_{jk} &= \sum_{\pi} \sum_{(k_J)} \mathsf{E} \prod_{J} \prod_{j \in J} \xi_{j,k_J} \eta_{j,k_J} \\
&= \sum_{\pi} \sum_{(k_J)} \{\mathsf{E} \prod_{J} \prod_{j \in J} \xi_{j,k_J}\} \{\prod_{J} \prod_{j \in J} \eta_{j,k_J}\} \\
&= \sum_{\pi} \frac{(n - \# \pi)!}{n!} \{\sum_{(k_J)} \prod_{J} \prod_{j \in J} \xi_{j,k_J}\} \{\sum_{(k_J)} \prod_{J} \prod_{j \in J} \eta_{j,k_J}\},
\end{aligned}$$

which has the asserted form by Lemma 3.2.

For $d=1$, we get in particular

$$E_1 = \mathsf{E} \sum_{k=1}^{n} \xi_k \eta_k = (\mathsf{E}\, \xi_1) \sum_{k=1}^{n} \eta_k = n^{-1} R_1 S_1.$$

To get E_2 and E_3 without effort when $R_j \equiv S_j \equiv 0$, note that E_d is homogeneous of degree d in both the ξ_{jk} and the η_{jk}, so that necessarily

$$E_2 = c_2 R_{12} S_{12}, \qquad E_3 = c_3 R_{123} S_{123},$$

for some constants c_2 and c_3. The latter may easily be obtained by direct computation in some simple example. We omit the details. $\square$

Lemma 3.4. *Let $\mathscr{F}$, ξ and η be such as in Theorem 3.1, except that the measure μ_ξ and the sums S_J are only assumed to be $\mathscr{F}_0$-measurable, the former with bounded support. Then*

$$\mathsf{E}\left[\prod_{j=1}^{d} \sum_{k=1}^{n} \xi_{jk}\eta_{jk} \middle| \mathscr{F}_0\right] = P_{nd}\{R_J, S_J\} \quad a.s., \tag{8}$$

for some polynomial P_{nd} in the sums R_J and S_J.

Proof. Let us first notice that the product in (8) is integrable by (C_1) and Hölder's inequality, so that the conditional expectations here and below exist. The statement of the lemma is trivially true for $n \wedge d = 0$, if the product over an empty set is taken to be one. We shall proceed to general $n \geqq 1$ by induction, so assume that the statement is true with n replaced by $n-1$ and for all d. Writing J for an arbitrary subset of $\{1, \ldots, d\}$, we get

$$\begin{aligned}\mathsf{E}\left[\prod_{j=1}^{d} \sum_{k=1}^{n} \xi_{jk}\eta_{jk} \middle| \mathscr{F}_0\right] &= \mathsf{E}\left[\sum_J \prod_{i \notin J} \xi_{i1}\eta_{i1} \prod_{j \in J} \sum_{k=2}^{n} \xi_{jk}\eta_{jk} \middle| \mathscr{F}_0\right] \\ &= \sum_J \mathsf{E}\left[\prod_{i \notin J} \xi_{i1}\eta_{i1}\, \mathsf{E}\left[\prod_{j \in J} \sum_{k=2}^{n} \xi_{jk}\eta_{jk} \middle| \mathscr{F}_1\right] \middle| \mathscr{F}_0\right].\end{aligned} \tag{9}$$

Now the sums

$$S'_J = \sum_{k=2}^{n} \prod_{j \in J} \eta_{jk} = S_J - \prod_{j \in J} \eta_{j1}, \qquad \emptyset \neq J \subset \{1, \ldots, d\},$$

are $\mathscr{F}_0$-measurable, while the measure

$$\mu'_\xi = \sum_{k=2}^{n} \delta_{\xi_{\cdot k}} = \mu_\xi - \delta_{\xi_{\cdot 1}}$$

is $\mathscr{F}_1$-measurable, so the induction hypothesis shows that the inner conditional expectations

$$\mathsf{E}\left[\prod_{j \in J} \sum_{k=2}^{n} \xi_{jk}\eta_{jk} \middle| \mathscr{F}_1\right], \qquad J \subset \{1, \ldots, d\},$$

are polynomials in the sums S'_I, $I \subset \{1, \ldots, d\}$, as well as in the variables ξ_{jk} with $j = 1, \ldots, d$ and $k = 2, \ldots, n$. Thus the sum in (9) is a polynomial in the sums S_J, in the random variables $\eta_{11}, \ldots, \eta_{d1}$, and in the conditional product moments of the variables ξ_{jk}, given $\mathscr{F}_0$.

Let us now introduce an array $\xi'=(\xi'_{jk})$, by suitable randomization, such that ξ' is conditionally independent of $\mathscr{F}_n$, given $\mathscr{F}_0$, with the same conditional distribution as ξ. (This amounts to a randomization of the order between the vectors $\xi_{.1}, \ldots, \xi_{.n}$.) Write $\mathscr{F}'_k=\mathscr{F}_k \vee \sigma(\xi'_{.1}, \ldots, \xi'_{.k})$ for $k=0, \ldots, n$, and note that the hypotheses of the lemma remain fulfilled for the triple $(\mathscr{F}', \xi', \eta)$. Repeating the above computation in the new situation, and noting that the result depends only on quantities which are the same in both cases, we get a.s.

$$\mathsf{E}\left[\prod_{j=1}^{d} \sum_{k=1}^{n} \xi_{jk}\eta_{jk}\middle|\mathscr{F}_0\right]=\mathsf{E}\left[\prod_{j=1}^{d} \sum_{k=1}^{n} \xi'_{jk}\eta_{jk}\middle|\mathscr{F}_0\right]$$
$$=\mathsf{E}\left[\mathsf{E}\left[\prod_{j=1}^{d} \sum_{k=1}^{n} \xi'_{jk}\eta_{jk}\middle|\mathscr{F}_n\right]\middle|\mathscr{F}_0\right].$$

But Lemma 3.3 shows that the inner expectation on the right is a polynomial P_{nd} in the sums R_J and S_J. (Note that the sums R_J are the same for ξ and ξ'.) Since the latter are $\mathscr{F}_0$-measurable, this proves the assertion for sequences of length n, and hence completes the induction. □

This also completes the proof of Theorem 3.1, so we may turn to the case of infinite sequences. Here we shall write

$$\mu_J=\mathsf{E}\prod_{j\in J}\xi_{j1}, \qquad \mu'_J=\mathsf{E}\prod_{j\in J}(\xi_{j1}-\mu_j), \qquad S_J=\sum_{k=1}^{\infty}\prod_{j\in J}\eta_{jk}, \qquad \emptyset \neq J\subset\{1, \ldots, d\},$$

whenever these quantities exist. Let us further introduce the condition

(C_2): There exist some constants $p_1, \ldots, p_d \geqq 1$ with $\sum p_j^{-1} \leqq 1$, such that

$$|\mathsf{E}\,\xi_{j1}|\,\mathsf{E}\sum_{k=1}^{\infty}|\eta_{jk}|+\mathsf{E}\,|\xi_{j1}|^{p_j}\left\{\sum_{k=1}^{\infty}|\eta_{jk}|^{p'_j}\right\}^{p''_j/2}<\infty, \qquad j=1, \ldots, d,$$

where $p'_j=p_j \wedge 2$ and $p''_j=p_j \vee 2$.

Theorem 3.5. *Let $\mathscr{F}$ be a filtration on $\mathbb{Z}_+$, and let ξ and η be infinite random sequences in $\mathbb{R}^d$, such that ξ is $\mathscr{F}$-i.i.d. while η is $\mathscr{F}$-predictable. Let $\{K_1, \ldots, K_m\}$ be a partition of $\{1, \ldots, d\}$, such that the corresponding subarrays of ξ are independent. Assume that (C_2) is fulfilled, and that all products $\mu_j S_j$ are a.s. non-random, as well as all sums S_J with $2 \leqq \# J < d$ and $J \subset K_i$ for some i. Then*

$$E_d := \mathsf{E}\prod_{j=1}^{d}\sum_{k=1}^{\infty}\xi_{jk}\eta_{jk}$$
$$=\mathsf{E}\prod_{i=1}^{m}\sum_{\pi_i}\{\prod_J \mu_J\}\,P_{\pi_i}\{S_I\}=\mathsf{E}\prod_{i=1}^{m}\sum_{\pi_i}\{\prod_J S_J\}\,P'_{\pi_i}\{\mu_I\}, \tag{10}$$

where π_i denotes an arbitrary partition of K_i into subsets J, and where each P_{π_i} is a polynomial in the sums S_I indexed by arbitrary unions I of sets in π_i, while each P'_{π_i} is a polynomial in the moments μ_I indexed by arbitrary subsets I of sets in π_i. In particular $E_1 = \mu_1 \mathsf{E} S_1$, and under the further assumption $\mu_j S_j \equiv 0$ we have $E_2 = \mu'_{12} \mathsf{E} S_{12}$ and $E_3 = \mu'_{123} \mathsf{E} S_{123}$.

Weaker versions of this result may be obtained from Theorem 3.1 through a suitable approximation argument. However, a direct proof seems to be required to obtain the above statement in its full strength. A similar remark applies to the corresponding continuous time results in Theorems 4.1 and 5.1.

The theorem follows from Lemma 3.7 below. But first we need the result in a special case.

Lemma 3.6. *The conclusions of Theorem 3.5 are true when η is non-random.*

Proof. First conclude from (C_2), Proposition 2.1 and Hölder's inequality that E_d exists. From (C_2) it is further seen that S_j exists whenever $\mu_j \neq 0$. Finally, it is seen from (C_2) and Hölder's inequality that S_J exists for all $J \subset \{1, \ldots, d\}$ with $\# J \geqq 2$, and that μ_J exists for all J. Similar arguments show that η can be approximated by a sequence with finitely many non-zero elements, so we may assume that already η has this form.

By independence, E_d splits into a product of similar expressions, with the products taken over the sets $K_1, \ldots, K_m$. It is thus enough to consider each factor separately, so we may assume that $m = 1$. Writing π for an arbitrary partition of $\{1, \ldots, d\}$ into subsets J, and (k_J) for a corresponding assignment of arbitrary distinct indices in $\mathbb{N}$, we get

$$\begin{aligned} \mathsf{E} \prod_{j=1}^{d} \sum_{k=1}^{\infty} \xi_{jk} \eta_{jk} &= \mathsf{E} \sum_{\pi} \sum_{(k_J)} \prod_{J} \prod_{j \in J} \xi_{j,k_J} \eta_{j,k_J} \\ &= \sum_{\pi} \{\prod_{J} \mu_J\} \{\sum_{(k_J)} \prod_{J} \prod_{j \in J} \eta_{j,k_J}\}. \end{aligned}$$

By Lemma 3.2, the second factor on the right is a polynomial in the sums S_I with I a union of sets in π. This proves the first representation in (10), and the second one follows if instead we collect the terms corresponding to the different products $\prod S_I$. The explicit formula for E_1, and for E_2 and E_3 when $\mu_j \equiv 0$, are easily obtained in this case by direct computation. If instead $S_j = 0$ for some j, we may reduce to the case $\mu_j = 0$ by subtracting μ_j from each ξ_{jk}, which neither affects the sum $\sum \xi_{jk} \eta_{jk}$ nor the moments μ'_{12} and μ'_{123}. $\square$

Lemma 3.7. *Let $\mathscr{F}$, ξ and η be such as in Theorem 3.5, except that the products $\mu_j S_j$ as well as the sums S_J with $2 \leqq \# J < d$ and $J \subset K_i$ for some i are only assumed to be $\mathscr{F}_0$-measurable. Then*

$$\mathsf{E}\left[\prod_{j=1}^{d} \sum_{k=1}^{\infty} \xi_{jk} \eta_{jk} \middle| \mathscr{F}_0\right] = \mathsf{E}\left[\prod_{i=1}^{m} P_i \middle| \mathscr{F}_0\right] \quad a.s., \tag{11}$$

where the P_i are polynomials of the stated form.

Proof. First we note as before that the quantities involved in (11) exist because of (C_2). To prove (11), we shall proceed by induction over $d \in \mathbb{Z}_+$, starting with the triviality $1=1$ for $d=0$. Thus we fix a $d \in \mathbb{N}$, and assume that the statement is true for dimensions $<d$. Whenever S_J exists, write

$$U_{J,n}=\sum_{k>n}\prod_{j\in J}\eta_{jk}=S_J-\sum_{k\leq n}\prod_{j\in J}\eta_{jk}, \qquad n\in\mathbb{N},$$

and note that the sequence $(U_{J,n})$ is predictable. By the induction hypothesis, we may conclude that

$$\mathsf{E}[\prod_{j\in J}\sum_{k>n}\xi_{jk}\eta_{jk}|\mathscr{F}_n]=\prod_{i=1}^{m}P_{J\cap K_i}(\mu_I, U_{I,n}) \quad \text{a.s.}, \qquad n\in\mathbb{N},$$

where the factors on the right are polynomials in the products $\mu_j U_{j,n}$ with $j\in J\cap K_i$, in the moments μ_I with $I\subset J\cap K_i$, and in the sums $U_{I,n}$ with $I\subset J\cap K_i$ and $\# I\geqq 2$. Letting J denote an arbitrary proper subset of $\{1, \ldots, d\}$, and conditioning in the n-th term below, first on $\mathscr{F}_n$ and then on $\mathscr{F}_{n-1}$, we obtain

$$\begin{aligned}\mathsf{E}\left[\prod_{j=1}^{d}\sum_{k=1}^{\infty}\xi_{jk}\eta_{jk}\middle|\mathscr{F}_0\right]&=\mathsf{E}\left[\sum_{J}\sum_{n=1}^{\infty}\prod_{i\notin J}\xi_{in}\eta_{in}\prod_{j\in J}\sum_{k>n}\xi_{jk}\eta_{jk}\middle|\mathscr{F}_0\right]\\&=\mathsf{E}\left[\sum_{J}\sum_{n=1}^{\infty}\prod_{i\notin J}\xi_{in}\eta_{in}\prod_{r=1}^{m}P_{J\cap K_r}(\mu_I, U_{I,n})\middle|\mathscr{F}_0\right]\\&=\mathsf{E}\left[\sum_{J}\mu_{J^c}\sum_{n=1}^{\infty}\prod_{i\notin J}\eta_{in}\prod_{r=1}^{m}P_{J\cap K_r}(\mu_I, U_{I,n})\middle|\mathscr{F}_0\right].\end{aligned}$$

The remaining argument is similar to that in Lemma 3.4. Thus we construct some $\xi'\overset{d}{=}\xi$ independent of $\mathscr{F}_\infty=\vee\mathscr{F}_n$, and put $\mathscr{F}'_k=\mathscr{F}_k\vee\sigma(\xi'_{\cdot 1}, \ldots, \xi'_{\cdot k})$ for $k\in\mathbb{Z}_+$. Since the above computation gives the same result for the triple $(\mathscr{F}', \xi', \eta)$, we get

$$\begin{aligned}\mathsf{E}\left[\prod_{j=1}^{d}\sum_{k=1}^{\infty}\xi_{jk}\eta_{jk}\middle|\mathscr{F}_0\right]&=\mathsf{E}\left[\prod_{j=1}^{d}\sum_{k=1}^{\infty}\xi'_{jk}\eta_{jk}\middle|\mathscr{F}_0\right]\\&=\mathsf{E}\left[\mathsf{E}\left[\prod_{j=1}^{d}\sum_{k=1}^{\infty}\xi'_{jk}\eta_{jk}\middle|\mathscr{F}_\infty\right]\middle|\mathscr{F}_0\right] \quad \text{a.s.}\end{aligned}$$

Here the right-hand side has the desired form by Lemma 3.6, which completes the induction. □

4. Moment Identities for Predictable Integrals on [0, 1]

In this section we shall prove moment identities for certain stochastic integrals with respect to exchangeable processes on [0, 1]. Thus we consider $\mathbb{R}^d$-valued processes $X=(X_1, \ldots, X_d)$ and $V=(V_1, \ldots, V_d)$ on [0, 1], where X is ergodic exchangeable and directed by (α, ρ, β). Put $\sigma_j \equiv \rho_{jj}^{1/2}$, and define for non-empty $J \subset \{1, \ldots, d\}$

$$\beta_{Jk} = \prod_{j \in J} \beta_{jk}, \qquad \beta_J = \sum_{k=1}^{\infty} \delta_{\beta_{Jk}}, \qquad B_J = \sum_{k=1}^{\infty} \beta_{Jk}, \tag{1}$$

$$V_J = \prod_{j \in J} V_j, \qquad S_J = \int_0^1 V_J, \qquad V'_J = \prod_{j \in J} (V_j - S_j), \qquad S'_J = \int_0^1 V'_J, \tag{2}$$

whenever these expressions make sense. Note that β_J is to be regarded as a measure on $\mathbb{R} \setminus \{0\}$. We shall need the following condition.

(C_3): There exist some constants $p_j \geqq 1$ and $q_j > 2p_j$, $j = 1, \ldots, d$, with $\sum p_j^{-1} \leqq 1$ and such that for every j

$$|\alpha_j| \mathsf{E} \int_0^1 |V_j| + \sigma_j \mathsf{E} \left\{ \int_0^1 V_j^2 \right\}^{p_j/2} + \sum_{k=1}^{\infty} |\beta_{jk}|^{p_j} \mathsf{E} \int_0^1 |V_j|^{q_j} < \infty.$$

Theorem 4.1. *Let $\mathscr{F}$ be a standard filtration on $[0, 1]$, and let X and V be $\mathbb{R}^d$-valued processes on $[0, 1]$, such that X is ergodic $\mathscr{F}$-exchangeable and directed by (α, ρ, β) while V is $\mathscr{F}$-predictable. Assume that (C_3) is fulfilled, and that all the products $\alpha_j S_j$, $\rho_{ij} S_{ij}$ and $\beta_J S_J$ are a.s. non-random. Then*

$$E_d := \mathsf{E} \prod_{j=1}^{d} \int_0^1 V_j \, dX_j = \sum_{\pi} \left\{ \prod_i \alpha_i S_i \right\} \left\{ \prod_{j,k} \rho_{jk} S'_{jk} \right\} \left\{ \prod_J B_J \right\} P_\pi \{S'_I\}, \tag{3}$$

where the summation extends over all partitions π of $\{1, \ldots, d\}$ into singletons $\{i\}$, pairs $\{j, k\}$, and sets J with $\# J \geqq 2$, and where each P_π is a polynomial in the integrals S'_I indexed by arbitrary subsets I of sets $J \in \pi$. In particular $E_1 = \alpha_1 S_1$, and under the further assumption $\alpha_j S_j \equiv 0$ we have $E_2 = (\rho_{12} + B_{12}) S'_{12}$ and $E_3 = B_{123} S'_{123}$.

As in case of Theorem 3.5, there is also a dual form of (3) with B and S' interchanged in the last two factors, and such that each P_π is a polynomial in the sums B_I indexed by arbitrary unions I of sets $J \in \pi$. When X has finite variation, it is natural to write $\alpha'_j = \alpha_j - B_j$, and to replace (3) by the relation

$$E_d = \mathsf{E} \prod_{j=1}^{d} \int_0^1 V_j \, dX_j = \sum_{\pi} \left\{ \prod_j \alpha'_j S_j \right\} \left\{ \prod_J B_J \right\} P_\pi \{S_I\}, \tag{4}$$

where π denotes an arbitrary partition of $\{1, \ldots, d\}$ into singletons $\{i\}$ and sets J, and where each P_π is a polynomial in the integrals S_I indexed by arbitrary subsets I of sets $J \in \pi$.

For the proof of Theorem 4.1 we shall need two lemmas, where the first one will also be needed later. Let us write $M^* = \sup_t |M_t|$.

Lemma 4.2. *Let $M_1, \ldots, M_d$ be continuous $\mathscr{F}$-martingales starting at* 0, *and such that $\rho_{ij}=[M_i, M_j]_\infty$ is a.s. non-random for $i \neq j$. Further assume that $\|M_j^*\|_{p_j}<\infty$ for some constants $p_j \geqq 1$, $j=1, \ldots, d$, where $p^{-1} \equiv \sum p_j^{-1} \leqq 1$. Then the martingale*

$$M_t=\mathsf{E}\left[\prod_{j=1}^{d} M_j(\infty) \middle| \mathscr{F}_t\right], \quad t \geqq 0, \tag{5}$$

has a continuous version satisfying $\|M^*\|_p<\infty$, and moreover

$$\mathsf{E} M_t=M_0=\sum_{\pi} \prod_{i,j} \rho_{ij} \quad a.s., \tag{6}$$

where the summation extends over all partitions π of the set $\{1, \ldots, d\}$ into pairs $\{i, j\}$. In particular, $\mathsf{E} M_t=M_0=0$ a.s. when d is odd.

Proof. Write

$$V_J(t)=[M_i, M_j]_t, \quad \rho_J=V_J(\infty), \quad J=\{i,j\} \subset \{1, \ldots, d\},$$

and conclude from Itô's formula that

$$\prod_{j=1}^{d} M_j(t)=\sum_{j=1}^{d} \int_0^t \prod_{i \neq j} M_i \, dM_j+\sum_J \int_0^t \prod_{i \notin J} M_i \, dV_J, \quad t \geqq 0, \tag{7}$$

where the last summation extends over all (unordered) pairs $J \subset \{1, \ldots, d\}$. Applying this formula to the integrands in the last sum and proceeding recursively, we get

$$\begin{aligned}\prod_j M_j=&\sum_j \int \prod_i M_i \, dM_j+\sum_{1 \leqq k<d/2} \sum_{J_1} \ldots \sum_{J_k} \sum_j \int dV_{J_1} \int \ldots \int dV_{J_k} \int \prod_i M_i \, dM_j \\ &+\sum_{J_1} \ldots \sum_{J_{d/2}} \int dV_{J_1} \int \ldots \int dV_{J_{d/2}},\end{aligned} \tag{8}$$

where the last sum occurs only when d is even. Here the summations in the k-th term extend over all sequences of disjoint pairs $J_1, \ldots, J_k \subset \{1, \ldots, d\}$ and over remaining indices j, while the product in the integrand extends over all indices $i \neq j$ outside $J_1, \ldots, J_k$. Finally, the integration is taken over the set

$$\{(t_1, \ldots, t_{k+1}) \in \mathbb{R}_+^{k+1};\, t_1 \geqq t_2 \geqq \ldots \geqq t_{k+1} \geqq 0\}.$$

Similar conventions apply to the last sum in (8).

We shall now use the fact that, if $V_1, \ldots, V_k$ are continuous functions of bounded variation starting at 0, then

$$\sum_r \int dV_{r_1} \int \ldots \int dV_{r_k}=\prod_{j=1}^{k} V_j,$$

where the summation extends over all permutations $r=(r_1, \ldots, r_k)$ of $(1, \ldots, k)$. Applying this to (8), we get

$$\prod_j M_j = \sum_j \int \prod_i M_i \, dM_j + \sum_{1 \leqq k < d/2} \sum_{\pi'} \sum_j \int d \prod_J V_J \int \prod_i M_i \, dM_j + \sum_\pi \prod_J V_J, \tag{9}$$

where the inner summations in the k-th term extend over all (unordered) collections π' of k disjoint pairs $J \subset \{1, \ldots, d\}$, and over remaining indices j, while the last product is taken over all other indices i. Moreover, integration is now over the set $\{(t_1, t_2) \in \mathbb{R}_+^2; t_1 \geqq t_2\}$.

Next we integrate by parts in (9) to obtain

$$\prod_j M_j = \sum_j \int \prod_i M_i \, dM_j$$
$$+ \sum_{1 \leqq k < d/2} \sum_{\pi'} \sum_j \{\prod_J V_J \int \prod_i M_i \, dM_j - \int \prod_J V_J \prod_i M_i \, dM_j\} + \sum_\pi \prod_J V_J.$$

Changing the order of summation and noting that the products $\prod V_J(\infty) = \prod \rho_J$ are a.s. constant, we finally obtain

$$\prod_j M_j(\infty) = \sum_j \sum_{\pi'} \int_0^\infty \{\prod_J \rho_J - \prod_J V_J\} \prod_i M_i \, dM_j + \sum_\pi \prod_J \rho_J, \tag{10}$$

where the inner summation in the first term extends over all partitions π' of the set $\{1, \ldots, d\} \setminus \{j\}$ into pairs J plus a remaining set of indices i. The assertions of the lemma will follow immediately from (10), if we can only prove that the integral processes on the right are bounded by random variables in L_p.

To see this, let $p_J^{-1} = p_i^{-1} + p_j^{-1}$ when $J = \{i, j\}$, and note by the BDG and Hölder inequalities that

$$\begin{aligned} \|V_J^*\|_{p_J} &\leqq \|\{[M_i, M_i][M_j, M_j]\}^{1/2}\|_{p_J} \\ &\leqq \|[M_i, M_i]^{1/2}\|_{p_i} \|[M_j, M_j]^{1/2}\|_{p_j} \lesssim \|M_i^*\|_{p_i} \|M_j^*\|_{p_j}. \end{aligned}$$

By the same inequalities, we hence obtain for fixed j and π' as above

$$\begin{aligned} &\left\| \sup_t \left| \int_0^t \{\prod_J \rho_J - \prod_J V_J\} \prod_i M_i \, dM_j \right| \right\|_p \\ &\lesssim \left\| \left\{ \int_0^\infty \{\prod_J \rho_J - \prod_J V_J\}^2 \prod_i M_i^2 \, d[M_j, M_j] \right\}^{1/2} \right\|_p \\ &\leqq \|\prod_J V_J^* \prod_i M_i^* [M_j, M_j]_\infty^{1/2}\|_p \\ &\leqq \prod_J \|V_J^*\|_{p_J} \prod_i \|M_i^*\|_{p_i} \|[M_j, M_j]^{1/2}\|_{p_j} \lesssim \prod_{k=1}^d \|M_k^*\|_{p_k} < \infty. \quad \square \end{aligned}$$

Lemma 4.3. *Let M be a continuous $\mathscr{F}$-martingale on $[0, 1]$, let $V_1, \ldots, V_d$ $(d \geqq 1)$ be $\mathscr{F}$-predictable processes on $[0, 1]$ with $\int V_1 = \ldots = \int V_d = 0$ a.s., and let $\tau_1, \ldots, \tau_d$ be i.i.d. $U(0, 1)$ random variables such that the processes $1\{\tau_j \leqq t\}$ are $\mathscr{F}$-exchangeable. Assume for some constants $p, q_1, \ldots, q_d \geqq 1$ with $p^{-1} + 2\sum q_j^{-1} \leqq 1$ that*

$$\mathsf{E}|M^*|^p < \infty; \qquad \mathsf{E}\int |V_j|^{q_j} < \infty, \quad j = 1, \ldots, d. \tag{11}$$

Then

$$\mathsf{E}\, M_1 \prod_{j=1}^{d} V_j(\tau_j) = 0. \tag{12}$$

Proof. By Proposition 2.6 we have

$$V_j(\tau_j) = \int_0^1 V_j(t)\, d(1\{\tau_j \leqq t\}) = \int_0^1 U_j\, dM_j, \qquad j = 1, \ldots, d, \tag{13}$$

where the martingales M_j and the predictable processes U_j are given by

$$M_j(t) = 1\{\tau_j \leqq t\} - \log(1 - t \wedge \tau_j), \qquad t \in [0, 1], \tag{14}$$

$$U_j(t) = V_j(t) - \frac{1}{1-t} \int_t^1 V_j(s)\, ds, \qquad t \in [0, 1). \tag{15}$$

Since the martingale M and the integral processes $N_j = \int U_j\, dM_j$ are mutually orthogonal, we get by Itô's formula

$$M(t) \prod_{j=1}^{d} N_j(t) = \int_0^t \prod_j N_j\, dM + \sum_j \int_0^{t+} M \prod_{i \neq j} N_i\, dN_j \quad \text{a.s.}, \qquad t \in [0, 1]. \tag{16}$$

Thus (12) will follow if we can show that the integral processes on the right are martingales.

To see this, choose $p_j < q_j/2$, $j = 1, \ldots, d$, such that $p^{-1} + \sum p_j^{-1} = 1$. Using the BDG and Hölder inequalities plus Proposition 2.8, we get from (11)

$$\begin{aligned}\mathsf{E} \sup_t \left| \int_0^t \prod_j N_j\, dM \right| &\lesssim \mathsf{E} \left\{ \int_0^1 \prod_j N_j^2\, d[M, M] \right\}^{1/2} \leqq \mathsf{E}\, [M, M]_1^{1/2} \prod_j N_j^* \\ &\leqq \|[M, M]_1^{1/2}\|_p \prod_j \|N_j^*\|_{p_j} \lesssim \|M^*\|_p \prod_j \left\{ \mathsf{E} \int_0^1 |V_j|^{q_j} \right\}^{1/q_j} < \infty,\end{aligned}$$

so by uniform integrability the integral on the left must be a martingale. In the same way we get for $j = 1, \ldots, d$

$$\mathsf{E} \sup_t \left| \int_0^1 M \prod_{i \neq j} N_i\, dN_j \right| \lesssim \mathsf{E} \left\{ \int_0^1 M^2 \prod_{i \neq j} N_i^2\, d[N_j, N_j] \right\}^{1/2} < \infty,$$

where the finiteness of the second expression shows that the stochastic integral on the left is a local martingale, and hence justifies the use of the BDG inequality in the first step. □

Proof of Theorem 4.1. Note first that E_d exists, in view of (C_2) and Propositions 2.6 and 2.8. By Proposition 2.4 we may integrate termwise in (1.17), and by Proposition 2.8 it is enough to assume that X has finitely many jumps. Writing $X'_j(t) \equiv X_j(t) - \alpha_j t$, so that $X'_j(1) \equiv 0$, we get

$$\mathsf{E} \prod_{j=1}^{d} \int_0^1 V_j \, dX_j = \mathsf{E} \prod_{j=1}^{d} \left\{ \alpha_j S_j + \int_0^1 V'_j \, dX'_j \right\} = \sum_J \prod_{j \notin J} \alpha_j S_j \, \mathsf{E} \prod_{j \in J} \int_0^1 V'_j \, dX'_j,$$

where the summation extends over all subsets $J \subset \{1, \ldots, d\}$. Thus we may further assume that $\alpha_j \equiv S_j \equiv 0$.

For each $j = 1, \ldots, d$ we write M_j for the martingale component of B_j and define U_j by (15), so that

$$\int_0^1 V_j \, dB_j = \int_0^1 U_j \, dM_j = N_j(1), \qquad j = 1, \ldots, d, \tag{17}$$

by Proposition 2.6, where the integral process N_j on the right is a continuous L_{q_j}-martingale. From Lemma 2.7 it is further seen that for $i \neq j$

$$[N_i, N_j]_1 = \int_0^1 U_i U_j \, d[M_i, M_j] = \rho_{ij} \int_0^1 U_i U_j = \rho_{ij} \int_0^1 V_i V_j = \rho_{ij} S_{ij}.$$

By Lemma 4.2 there hence exists for every $J \subset \{1, \ldots, d\}$ some continuous L_{p_J}-martingale M_J $(p_J^{-1} = \sum_J p_j^{-1})$ satisfying

$$M_J(1) = \prod_{j \in J} N_j(1) = \prod_{j \in J} \int_0^1 V_j \, dB_j, \tag{18}$$

and

$$\mathsf{E} M_J(1) = \sum_{\pi_J} \prod_{i,j} [N_i, N_j]_1 = \sum_{\pi_J} \prod_{i,j} \rho_{ij} S_{ij}, \tag{19}$$

where the summations in (19) extend over all partitions π_J of J into pairs $\{i, j\}$.

Let us now write π' for an arbitrary collection of disjoint sets $J \subset \{1, \ldots, d\}$, put $J' = \bigcap J^c$, and let the indices $k_J \in \mathbb{N}$, $J \in \pi'$, be different but otherwise arbitrary. Using Proposition 2.4 and (18), we get

$$\begin{aligned} \mathsf{E} \prod_{j=1}^{d} \int_0^1 V_j \, dX_j &= \mathsf{E} \prod_{j=1}^{d} \left\{ \int_0^1 V_j \, dB_j + \sum_{k=1}^{\infty} \beta_{jk} V_j(\tau_k) \right\} \\ &= \mathsf{E} \sum_{\pi'} M_{J'}(1) \sum_{(k_J)} \prod_J \beta_{J, k_J} V_J(\tau_{k_J}) \\ &= \sum_{\pi'} \sum_{(k_J)} \prod_J \beta_{J, k_J} \, \mathsf{E} M_{J'}(1) \prod_J V_J(\tau_{k_J}). \end{aligned}$$

Writing

$$\prod_J V_J(\tau_{k_J}) = \prod_J \{ S_J + (V_J(\tau_{k_J}) - S_J) \},$$

and expanding the product on the right, it is seen from Lemma 4.3 and (19) that

$$\mathsf{E} M_{J'}(1) \prod_J V_J(\tau_{k_J}) = \mathsf{E} M_{J'}(1) \prod_J S_J = \sum_{\pi_{J'}} \prod_{i,j} \rho_{ij} S_{ij} \prod_J S_J,$$

so we get

$$\mathsf{E} \prod_{j=1}^{d} \int_0^1 V_j \, dX_j = \sum_{\pi} \prod_{i,j} \rho_{ij} S_{ij} \prod_J S_J \sum_{(k_J)} \prod_J \beta_{J,k_J}. \tag{20}$$

By Lemma 3.2, the inner sum on the right is a polynomial in the sums B_K with K a union of sets $J \in \pi'$. If instead we collect the terms involving a given product ΠB_K, it is clear that the coefficient will be a polynomial in the integrals S_J with J a subset of some K. This completes the proof of (3).

The explicit formula for E_1 follows immediately from (3), while those for E_2 and E_3 when $\alpha_j S_j \equiv 0$ are obtained from (20) with S'_J in place of S_J. $\square$

5. Moment Identities for Predictable Lévy Integrals

In this section we shall prove moment identities for certain stochastic integrals with respect to Lévy processes. Thus we consider $\mathbb{R}^d$-valued processes $X = (X_1, \ldots, X_d)$ and $V = (V_1, \ldots, V_d)$ on $\mathbb{R}_+$, where X is Lévy and directed by (γ, ρ, ν). Put $\sigma_j \equiv \rho_{jj}^{1/2}$, and define for non-empty subsets $J \subset \{1, \ldots, d\}$

$$N_J = \int \prod_{j \in J} x_j \, \nu(dx) = \int \prod_{j \in J} x_j \, \nu(dx_1 \ldots dx_d), \qquad V_J = \prod_{j \in J} V_j, \qquad S_J = \int_0^\infty V_J, \tag{1}$$

whenever these expressions make sense. The following condition will be needed.

(C_4): There exist some constants $p_1, \ldots, p_d \geqq 1$ with $\sum p_j^{-1} \leqq 1$, such that for all j

$$|\gamma_j| \, \mathsf{E} \int |V_j| + \sigma_j \mathsf{E} \{\int V_j^2\}^{p_j/2} \\ + \int |x_j|^{p_j} \nu(dx) \, \mathsf{E}[\{\int |V_j|^{p'_j}\}^{p''_j/2} + \int |V_j|^{p_j}] < \infty,$$

where $p'_j = p_j \wedge 2$ and $p''_j = p_j \vee 2$.

Theorem 5.1. *Let $\mathscr{F}$ be a standard filtration on $\mathbb{R}_+$, and let X and V be $\mathbb{R}^d$-valued processes on $\mathbb{R}_+$, such that X is $\mathscr{F}$-Lévy and directed by (γ, ρ, ν) while V is $\mathscr{F}$-predictable. Assume that (C_4) is fulfilled, and that the products $\gamma_i S_j$ (for $d > 1$), $\rho_{ij} S_{ij}$ (for $d > 2$), and $N_J S_J$ (for $2 \leqq \# J < d$) are a.s. non-random. Then*

$$\mathsf{E} \prod_{j=1}^{d} \int_0^\infty V_j \, dX_j = \mathsf{E} \sum_{\pi} \prod_i \gamma_i S_i \prod_{j,k} (\rho_{jk} + N_{jk}) S_{jk} \prod_J N_J S_J, \tag{2}$$

where the summation extends over all partitions π of $\{1, \ldots, d\}$ into singletons $\{i\}$, pairs $\{j, k\}$, and subsets J with $\# J \geqq 3$.

Note that N_{jk} can be omitted from the second product on the right, provided that sets J with $\# J=2$ are allowed in π. If X has locally finite variation while $|V|$ is integrable on $\mathbb{R}_+$, one may introduce the constants $\gamma'_j=\gamma_j-N_j$, and write (2) in the form

$$\mathsf{E}\prod_{j=1}^{d}\int_0^\infty V_j\,dX_j=\mathsf{E}\sum_\pi\prod_i\gamma'_i S_i\prod_J N_J S_J, \tag{3}$$

where the summation extends over all partitions π of $\{1,\ldots,d\}$ into singletons $\{i\}$ and subsets J.

The method of proof is similar to that for Theorem 3.5, though technically more complicated. The key step is Lemma 5.8, where we proceed by induction over d to establish a conditional version of (2) (though formally in terms of optional projections). Our proof of Lemma 5.8 requires ν to be bounded, so a reduction to that case is given through the construction in Lemmas 5.4 and 5.5. We shall also need some simple moment estimates, as provided by Lemmas 5.2 and 5.3. The remaining Lemmas 5.6, 5.7 and 5.9 are simple results in real analysis and stochastic calculus, which ought to be known, though we were unable to find references.

Unless otherwise stated, we assume that X and V are such as in Theorem 5.1, and in particular that (C_4) is fulfilled. As before, let p_J be defined for subsets $J\subset\{1,\ldots,d\}$ by $p_J^{-1}=\sum_J p_j^{-1}$.

Lemma 5.2. *For any $J\subset\{1,\ldots,d\}$ with $\# J\geqq 2$, we have*

$$\int\prod_{j\in J}|x_j|^p\,\nu(dx)\,\mathsf{E}\{\int|V_J|^p\}^{p_J/p}<\infty,\qquad 1\leqq p\leqq p_J. \tag{4}$$

Proof. We may assume that

$$\int|x_j|^{p_j}\nu(dx)<\infty,\qquad \mathsf{E}\{\int|V_j|^{p'_j}\}^{p'_j/2}+\mathsf{E}\int|V_j|^{p_j}<\infty,\quad j\in J, \tag{5}$$

since (4) is trivially true if any of these integrals vanishes. Then Hölder's inequality yields

$$\int\prod_{j\in J}|x_j|^{p_J}\nu(dx)<\infty,\qquad \mathsf{E}\int|V_J|^{p_J}<\infty,$$

so by norm interpolation (formula (2.10)) it remains to show that

$$\int\prod_{j\in J}|x_j|\,\nu(dx)<\infty,\qquad \mathsf{E}\{\int|V_J|\}^{p_J}<\infty.$$

To see this, note that $x^{p\wedge 2}\leqq x^2\wedge 1+x^p$ for $x,p>0$, so that by (5)

$$\int|x_j|^{p'_j}\nu(dx)\leqq\int(x_j^2\wedge 1)\,\nu(dx)+\int|x_j|^{p_j}\nu(dx)<\infty,\qquad j\in J. \tag{6}$$

By norm interpolation we get from (5) and (6)

$$\int|x_j|^p\,\nu(dx)<\infty,\qquad \mathsf{E}\{\int|V_j|^p\}^{p_j/p}<\infty,\qquad p'_j\leqq p\leqq p_j,\quad j\in J. \tag{7}$$

Now clearly

$$\sum_{j\in J} p_j^{-1}=p_J^{-1}\leqq 1\leqq \tfrac{1}{2}\,\# J\leqq \sum_{j\in J} p_j'^{-1},$$

so we may choose some $q_j\in[p_j', p_j]$, $j\in J$, satisfying $\sum_J q_j^{-1}=1$. Using (7) with $p=q_j$, we get by Hölder's inequality

$$\int \prod_{j\in J} |x_j|\, \nu(dx)\leqq \prod_{j\in J}\{\int |x_j|^{q_j}\nu(dx)\}^{1/q_j}<\infty,$$

$$\mathsf{E}\{\int |V_J|\}^{p_J}\leqq \mathsf{E}\prod_{j\in J}\{\int |V_j|^{q_j}\}^{p_J/q_j}\leqq \prod_{j\in J}\{\mathsf{E}\{\int |V_j|^{q_j}\}^{p_j/q_j}\}^{p_J/p_j}<\infty,$$

as desired. □

In the special case when $\rho=0$, we introduce the covariation processes X_J and their associated total variation processes $\bar{X}_J$, given for $J\subset\{1,\ldots,d\}$ with $\# J\geqq 2$ and for $t\geqq 0$ by

$$X_J(t)=\sum_{s\leqq t}\prod_{j\in J}\Delta X_j(s),\qquad \bar{X}_J(t)=\sum_{s\leqq t}\prod_{j\in J}|\Delta X_j(s)|=\int_0^t |dX_J|. \tag{8}$$

Note that X_J and $\bar{X}_J$ are again $\mathscr{F}$-Lévy with Lévy measures ν_J and $\bar{\nu}_J$, given for Borel sets $B\subset\mathbb{R}\setminus\{0\}$ by

$$\nu_J(B)=\nu\{x\in\mathbb{R}^d;\ \prod_{j\in J} x_j\in B\},\qquad \bar{\nu}_J(B)=\nu\{x\in\mathbb{R}^d;\ \prod_{j\in J}|x_j|\in B\}. \tag{9}$$

In particular X_J has drift N_J. Recall that $p_J^{-1}=\sum_J p_j^{-1}$.

Lemma 5.3. *If $\rho=0$, we have for any $J\subset\{1,\ldots,d\}$ with $\# J\geqq 2$*

$$\mathsf{E}\{\int |V_J\, dX_J|\}^{p_J}\lesssim \sup_{1\leqq p\leqq p_J}\{\int \prod_{j\in J}|x_j|^p\,\nu(dx)\}^{p_J/p}\sup_{1\leqq p\leqq p_J}\mathsf{E}\{\int |V_J|^p\}^{p_J/p}<\infty. \tag{10}$$

Proof. The expression on the right is finite by Lemma 5.2 plus norm interpolation, so Proposition 2.2 applies with X, V and p replaced by $\bar{X}_J$, $|V_J|$ and p_J, and (10) follows. □

Lemma 5.4. *Fix arbitrary numbers $m_J\in\mathbb{R}$, $\emptyset\neq J\subset\{1,\ldots,d\}$. Then there exists some measure μ on the cube $C=[-1,1]^d$, such that $\mu(C)=\sum|m_J|$ and moreover*

$$\int \prod_{j\in J} x_j\,\mu(dx)=m_J,\qquad \emptyset\neq J\subset\{1,\ldots,d\}. \tag{11}$$

Proof. Suppose we can find some probability measures μ_J^+ and μ_J^- on C satisfying

$$\int \prod_{i\in 1} x_i\,\mu_J^{\pm}(dx)=\pm 1\{I=J\},\qquad \emptyset\neq I, J\subset\{1,\ldots,d\}. \tag{12}$$

Then the measure

$$\mu=\sum_J (m_J\vee 0)\mu_J^+ - \sum_J (m_J\wedge 0)\mu_J^- \tag{13}$$

has clearly the desired properties. To construct $\mu_J^\pm$, fix $k\in J$, and let ξ_j, $j\in J\setminus\{k\}$, be independent random variables (on some probability space) with $\mathsf{P}\{\xi_j=1\}=\mathsf{P}\{\xi_j=-1\}=1/2$. Choose ξ_k such that $\prod_J \xi_j=\pm 1$, and let $\xi_j=0$ for $j\notin J$. Take $\mu_J^\pm$ to be the distribution of $(\xi_1, \ldots, \xi_d)$. Then (12) is trivially fulfilled for $I\setminus J\neq\emptyset$ or $I\subset J\setminus\{k\}$, and if $k\in I\subset J$ we get

$$\int\prod_{i\in I} x_i\,\mu_J^\pm(dx)=\mathsf{E}\prod_{i\in I}\xi_i=\pm\mathsf{E}\prod_{i\in I}\xi_i\prod_{j\in J}\xi_j=\pm\mathsf{E}\prod_{j\in J\setminus I}\xi_j=\pm 1\{I=J\}. \quad \square$$

For the purpose of the next lemma, say that the probability space $(\Omega', \mathcal{O}', \mathsf{P}')$ is an *extension* of $(\Omega, \mathcal{O}, \mathsf{P})$, if P is the image of P' under some $\mathcal{O}'/\mathcal{O}$-measurable mapping $\psi\colon \Omega'\to\Omega$. Note that any random element ξ on Ω then extends, with preserved distributional and path properties, to a random element ξ' on Ω' through the composition $\xi'=\xi\circ\psi$. We shall further say that a filtration $\mathcal{F}'$ on Ω' extends $\mathcal{F}$ on Ω, if ψ is also $\mathcal{F}_t'/\mathcal{F}_t$-measurable for every t. In this case, adaptedness and predictability are automatically preserved by the extension, as is the stopping time property of a random variable. Usually $(\Omega', \mathcal{O}', \mathsf{P}')$ is formed as a product of $(\Omega, \mathcal{O}, \mathsf{P})$ with some other probability space, in which case ψ is always taken to be the natural projection of Ω' onto Ω (cf. Ikeda & Watanabe (1981), p. 89).

Lemma 5.5. *For every $\varepsilon>0$ there exists on some extended standard filtered probability space $(\Omega', \mathcal{O}', \mathcal{F}', \mathsf{P}')$ an $\mathbb{R}^d$-valued $\mathcal{F}'$-Lévy process X' on $\mathbb{R}_+$, such that X' is directed by (γ, ρ, ν') for some bounded and boundedly supported Lévy measure ν' with the same moments N_J $(\# J\geqq 2)$ as ν, and such that moreover*

$$\mathsf{E}\left|\prod_{j=1}^d \int_0^\infty V_j\,dX_j' - \prod_{j=1}^d \int_0^\infty V_j\,dX_j\right| < \varepsilon. \tag{14}$$

Proof. For each $n\in\mathbb{N}$, form a process Y_n on $\mathbb{R}_+$ by adding to the drift and diffusion components of X the centered sum of jumps in X with size between n^{-1} and n. Note that both Y_n and $X-Y_n$ are again $\mathcal{F}$-Lévy, and directed by (γ, ρ, κ_n) and $(0, 0, \kappa_n')$, respectively, where κ_n is the restriction of ν to the set $\{x\in\mathbb{R}^d\colon n^{-1}<|x|<n\}$, while $\kappa_n'=\nu-\kappa_n$. For $J\subset\{1, \ldots, d\}$ with $\# J\geqq 2$, put

$$m_{nJ}=\int\prod_{j\in J} x_j\,\kappa_n'(dx)=N_J-\int\prod_{j\in J} x_j\,\kappa_n(dx), \tag{15}$$

and form a measure μ_n as in Lemma 5.4. Next define on the Lebesgue unit interval $(I, \mathcal{B}, \lambda)$ a centered compound Poisson process Z_n with Lévy measure μ_n, and consider all random processes as functions on the product space $(\Omega', \mathcal{O}', \mathsf{P}')=(\Omega\times I, \mathcal{O}\times\mathcal{B}, \mathsf{P}\times\lambda)$. Let $\mathcal{F}_n$ be the $(\mathcal{O}', \mathsf{P}')$-completed filtration on Ω' generated by $\mathcal{F}$ and Z_n, and note that $\mathcal{F}_n$ is automatically right-continuous. It is easy to check that the pair (Y_n, Z_n) and hence also the sum $U_n=Y_n+Z_n$

are $\mathscr{F}_n'$-Lévy. Note also that U_n is directed by (γ, ρ, ν_n), where $\nu_n = \kappa_n + \mu_n$, and that ν_n gives the same values as ν to the moments N_J with $\# J \geqq 2$.

It remains to show that (14) is fulfilled for $X' = U_n$ when n is large. We may then assume by (C_4) and Lemma 5.2 that (with $p_j' = p_j \wedge 2$)

$$\int |x_j|^p \nu(dx) < \infty, \qquad p \in [p_j', p_j],\ j = 1, \ldots, d, \tag{16}$$

$$\int \prod_{j \in J} |x_j| \, \nu(dx) < \infty, \qquad J \subset \{1, \ldots, d\},\ \# J \geqq 2. \tag{17}$$

From (15)–(17) we get by dominated convergence as $n \to \infty$

$$\int |x_j|^p \kappa_n'(dx) \to 0, \qquad p \in [p_j', p_j],\ j = 1, \ldots, d, \tag{18}$$

$$\int |x_j|^p \mu_n(dx) \leqq \mu_n(\mathbb{R}^d) = \sum_J |m_{nJ}| \leqq \int \prod_{j \in J} |x_j| \, \kappa_n'(dx) \to 0. \tag{19}$$

Hence, by (C_4) and Proposition 2.2,

$$\left\| \int_0^\infty V_j \, d(X_j - U_{nj}) \right\|_{p_j} \leqq \left\| \int_0^\infty V_j \, d(X_j - Y_{nj}) \right\|_{p_j} + \left\| \int_0^\infty V_j \, dZ_{nj} \right\|_{p_j} \to 0. \tag{20}$$

Since also $\| \int V_j \, dX_j \|_{p_j} < \infty$ for each j, we may conclude by Hölder's inequality that the left hand side of (14) tends to zero as $n \to \infty$. $\square$

Lemma 5.6. *Let $F_1, \ldots, F_d$ be right-continuous functions of locally bounded variation, and define for $J \subset \{1, \ldots, d\}$ with $\# J \geqq 2$*

$$F_J(t) - F_J(s) = \sum_{u \in (s,t]} \prod_{j \in J} \Delta F_j(u), \qquad -\infty < s \leqq t < \infty. \tag{21}$$

Then

$$\prod_{j=1}^d F_j(t) - \prod_{j=1}^d F_j(s) = \sum_J \int_{s+}^{t+} dF_J(u) \prod_{j \notin J} F_j(u-), \qquad -\infty < s \leqq t < \infty, \tag{22}$$

where the summation extends over all non-empty subsets $J \subset \{1, \ldots, d\}$.

Proof. This is obvious for $d = 1$, and for $d = 2$ it reduces to the formula for integration by parts. The assertion for general d follows easily by induction. $\square$

Lemma 5.7. *Let X, Y and A be random processes on $\mathbb{R}_+$, such that X is measurable with $\mathsf{E} X^* < \infty$, while Y is optional and A is adapted and right-continuous with locally bounded variation. Assume that $\mathsf{E}[X_\tau; \tau < \infty] = \mathsf{E}[Y_\tau; \tau < \infty]$ for every stopping time τ, and that $\mathsf{E} \int |X| |dA| < \infty$. Then $\mathsf{E} \int X \, dA = \mathsf{E} \int Y dA$.*

Proof. For any stopping time τ,

$$Y_\tau = \mathsf{E}[X_\tau | \mathscr{F}_\tau] \quad \text{a.s. on} \quad \{\tau < \infty\},$$

so by Jensen's inequality

$$\mathsf{E}[|Y_\tau|; \tau < \infty] \leqq \mathsf{E}[|X_\tau|; \tau < \infty].$$

Assuming without loss that A is non-decreasing, and letting τ_t, $t \geqq 0$, denote the associated random time change, we get

$$\mathsf{E}\int |Y|\, dA = \int \mathsf{E}[|Y_{\tau_t}|; \tau_t < \infty]\, dt \leqq \int \mathsf{E}[|X_{\tau_t}|; \tau_t < \infty]\, dt = \mathsf{E}\int |X|\, dA.$$

Using Fubini's theorem, we thus obtain

$$\mathsf{E}\int Y\, dA = \int \mathsf{E}[Y_{\tau_t}; \tau_t < \infty]\, dt = \int \mathsf{E}[X_{\tau_t}; \tau_t < \infty]\, dt = \mathsf{E}\int X\, dA. \quad \square$$

We are now ready for our key lemma, where we assume again that X and V are such as in Theorem 5.1. For any subset $J \subset \{1, \ldots, d\}$ with $\# J \geqq 2$, we write

$$U_J(t) = \int_t^\infty V_J(s)\, ds, \qquad t \geqq 0. \tag{23}$$

Lemma 5.8. *Assume that $\gamma, \rho = 0$, and that ν is bounded with bounded support. Let M be a continuous martingale with $\|M^*\|_p < \infty$, where $p^{-1} + \sum p_j^{-1} \leqq 1$, and assume that $N_J S_J$ is a.s. non-random even for $J = \{1, \ldots, d\}$, unless M is constant. Then we have for any stopping time τ*

$$\mathsf{E} M_\infty \prod_{j=1}^d \int_{\tau+}^\infty V_j\, dX_j = \mathsf{E} M_\tau \sum_\pi \prod_J N_J U_J(\tau), \tag{24}$$

where the summation extends over all partitions π of $\{1, \ldots, d\}$ into sets J with $\# J \geqq 2$.

Proof. We shall proceed by induction over d, starting for $d = 0$ with the fact that M_∞ has optional projection M_t. Let us thus fix a $d \in \mathbb{N}$, and assume that (24) is true with d replaced by $1, \ldots, d-1$. To extend (24) to d, fix $T > 0$, and proceed as follows, where each step will be explained in detail below:

$$\begin{aligned}
&\mathsf{E} M_\infty \prod_{j=1}^d \int_{\tau+}^\infty V_j\, dX_j - \mathsf{E} M_\infty \prod_{j=1}^d \int_{\tau \vee T+}^\infty V_j\, dX_j \\
&= \mathsf{E} M_\infty \sum_J \int_{\tau+}^{\tau \vee T+} V_J(t)\, dX_J(t) \prod_{j \notin J} \int_{t+}^\infty V_j\, dX_j \qquad \text{(integration by parts)} \\
&= \mathsf{E} \sum_J \int_{\tau+}^{\tau \vee T+} V_J(t)\, dX_J(t) M_t \sum_{\pi'} \prod_I N_I U_I(t) \qquad \text{(optional projection)} \\
&= \mathsf{E} \sum_J N_J \int_\tau^{\tau \vee T} V_J(t)\, dt\, M_t \sum_{\pi'} \prod_I N_I U_I(t) \qquad \text{(dual predictable projection)} \\
&= \mathsf{E} M_\infty \sum_J N_J \int_\tau^{\tau \vee T} V_J(t)\, dt \sum_{\pi'} \prod_I N_I U_I(t) \qquad \text{(optional projection)} \\
&= \mathsf{E} M_\infty \sum_\pi \prod_J N_J U_J(\tau) - \mathsf{E} M_\infty \sum_\pi \prod_J N_J U_J(\tau \vee T) \qquad \text{(integration by parts)} \\
&= \mathsf{E} M_\tau \sum_\pi \prod_J N_J U_J(\tau) - \mathsf{E} M_\infty \sum_\pi \prod_J N_J U_J(\tau \vee T) \qquad \text{(optional sampling).}
\end{aligned}$$

Here the integration by parts in the first step is according to Lemma 5.6 but in reversed time. The optional projection in the second step is by the induction hypothesis plus Lemma 5.7. Note that the inner summation on the right extends over all partitions π' of the set $\{1, \ldots, d\} \setminus J$ into subsets I of size $\geqq 2$, and that the integrability requirements are fulfilled by Proposition 2.2 and Lemma 5.3. Since the new integrands are continuous adapted, and hence predictable, we may proceed in the next step by a dual predictable projection, where each process $X_J(t)$ is replaced by its compensator, which is $N_J t$ if $\# J \geqq 2$ and vanishes otherwise. Note that the outer summation on the right is restricted to subsets $J \subset \{1, \ldots, d\}$ with $\# J \geqq 2$. The third step is formally justified by Proposition 2.2 with $p=1$, where the integrability condition follows from the fact that, by Hölder's inequality and Lemma 5.2,

$$\mathsf{E} \int_0^T |V_J M \prod_I U_I| \leqq \|M^*\|_p \left\| \int_0^T V_J \right\|_{p_J} \prod_I \|U_I^*\|_{p_I} < \infty. \tag{25}$$

In step number four we are using Lemma 5.7 again to replace M_t by M_∞, where the required integrability conditions now follow as in (25). Step number five is again by reversed integration by parts, as in Lemma 5.6. The sum in the first term is now $\mathscr{F}_\tau$-measurable, unless M is constant, so we may replace M_∞ by $\mathsf{E}[M_\infty | \mathscr{F}_\tau] = M_\tau$, which yields the final expression.

To complete the proof, it remains to notice that, by Hölder's inequality

$$\left| \mathsf{E} M_\infty \prod_{j=1}^d \int_{\tau \vee T+}^\infty V_j \, dX_j \right| \leqq \|M^*\|_p \prod_{j=1}^d \left\| \int_{\tau \vee T+}^\infty V_j \, dX_j \right\|_{p_j},$$

while

$$\left| \mathsf{E} M_\infty \sum_\pi \prod_J N_J U_J(\tau \vee T) \right| \leqq \|M^*\|_p \sum_\pi \prod_J |N_J| \left\| \int_{\tau \vee T}^\infty |V_J| \right\|_{p_J},$$

where the expressions on the right tend to zero as $T \to \infty$, by Proposition 2.2 and Lemma 5.2 plus dominated convergence. □

The following simple result will be needed to prove Theorem 5.1 when $d=2$.

Lemma 5.9. *Fix $p, q \geqq 1$ with $p^{-1} + q^{-1} \leqq 1$, and let M and N be martingales with $\|M^*\|_p < \infty$ and $\|N^*\|_q < \infty$. Then*

$$\mathsf{E} M_\infty N_\infty = \mathsf{E} M_0 N_0 + \mathsf{E}[M, N]_\infty. \tag{26}$$

Proof. By the BDG and Hölder inequalities,

$$\begin{aligned} \mathsf{E} \sup_t \left| \int_0^t M_- \, dN \right| &\lesssim \mathsf{E} \left\{ \int_0^\infty M_-^2 \, d[N, N] \right\}^{1/2} \leqq \mathsf{E} M^* [N, N]_\infty^{1/2} \leqq \|M^*\|_p \, \|[N, N]_\infty^{1/2}\|_q \\ &\lesssim \|M^*\|_p \, \|N^*\|_q < \infty, \end{aligned}$$

and similarly with M and N interchanged, so the processes $\int M_- \, dN$ and $\int N_- \, dM$ are uniformly integrable martingales, and (26) follows from Itô's formula. □

Proof of Theorem 5.1. Since the assertion holds for $d=1$ by Proposition 2.2, we may assume that $d \geqq 2$. In that case the products $\gamma_j S_j$ are non-random, so we get with $X'_j(t) = X_j(t) - \gamma_j t$

$$\mathsf{E} \prod_{j=1}^{d} \int_0^\infty V_j \, dX_j = \sum_J \prod_{j \in J} \gamma_j S_j \mathsf{E} \prod_{j \notin J} \int_0^\infty V_j \, dX'_j, \tag{27}$$

where the summation on the right extends over all subsets $J \subset \{1, \ldots, d\}$. Thus we may henceforth assume that X is centered. By Lemma 5.8 we may further take v to be bounded and boundedly supported.

Write B and Y for the continuous and purely discontinuous components of X, and denote the integral processes $\int V_j \, dB_j$ by M_j. Then the quantities

$$[M_i, M_j]_\infty = \int_0^\infty V_i V_j \, d[B_i, B_j] = \rho_{ij} S_{ij}, \qquad i \neq j, \tag{28}$$

are non-random when $d \geqq 3$, so in that case there exist by Lemma 4.2 some continuous martingales M_J, $J \subset \{1, \ldots, d\}$, satisfying $M_J(\infty) = \prod_J M_j(\infty)$, $\|M_J^*\|_{p_J} < \infty$, and

$$\mathsf{E} M_J = \sum_{\pi'} \prod_{i,j} \rho_{ij} S_{ij}, \qquad \emptyset \neq J \subset \{1, \ldots, d\}, \tag{29}$$

where the summation on the right extends over all partitions π' of J into pairs $\{i, j\}$. Putting $M_J \equiv 1$ when $J = \emptyset$, we get by Lemma 5.8 with $\tau = 0$

$$\mathsf{E} \prod_{j=1}^{d} \int_0^\infty V_j \, dX_j = \mathsf{E} \sum_J M_J(\infty) \prod_{j \notin J} \int_0^\infty V_j \, dY_j = \mathsf{E} \sum_\pi \prod_{i,j} \rho_{ij} S_{ij} \prod_J N_J S_J, \tag{30}$$

with summations first over arbitrary subsets $J \subset \{1, \ldots, d\}$, and next over partitions π of $\{1, \ldots, d\}$ into pairs $\{i, j\}$ and subsets J with $\# J \geqq 2$. This completes the proof for $d \geqq 3$, so it remains only to take $d = 2$. But in that case the first equality in (30) holds with $M_{12} = M_1 M_2$, as does the second one, since by (28) and Lemma 5.9

$$\mathsf{E} M_1(\infty) M_2(\infty) = \mathsf{E} [M_1, M_2]_\infty = \mathsf{E} \rho_{12} S_{12}. \qquad \square$$

6. Invariance Under Predictable Transformations

Our aim in this section is to demonstrate the power of the moment identities of the previous sections, by giving new and simple proofs of certain invariance theorems involving predictable transformations of exchangeable sequences or processes. First we shall show how the predictable sampling theorem of Kallenberg (1988) and its continuous time analogue follow easily from the appropriate moment formulas.

We begin with the claim that (1.11) holds for any finite or infinite $\mathscr{F}$-exchangeable sequence $\xi_1, \xi_2, \ldots$, and any sequence of a.s. distinct $\mathscr{F}$-predictable stopping times $\tau_1, \tau_2, \ldots$. By conditioning on the (permutation) invariant σ-field (cf. Aldous (1985)), we may reduce to the case when (ξ_k) is ergodic. By a suitable transformation of the state space, we may further assume that the ξ_k are bounded by a constant.

As in Kallenberg (1988), we introduce the *allocation sequence*

$$\alpha_k = \inf\{j;\, \tau_j = k\}, \qquad k = 1, 2, \ldots, \tag{1}$$

associated with the τ_j. Fixing arbitrary constants $c_1, c_2, \ldots \in \mathbb{R}$ such that at most finitely many are distinct from 0, we get

$$\sum_j c_j \xi_{\tau_j} = \sum_k c_{\alpha_k} \xi_k, \tag{2}$$

where $c_\infty = 0$ by convention. Since the sequence (c_{α_k}) is predictable, and since

$$\sum_k c_{\alpha_k}^m = \sum_j c_j^m \tag{3}$$

is a constant for each $m \in \mathbb{N}$, it follows by Theorem 3.1 or 3.5 that

$$\mathsf{E}\{\sum_j c_j \xi_{\tau_j}\}^m = \mathsf{E}\{\sum_j c_j \xi_j\}^m, \qquad m \in \mathbb{N}. \tag{4}$$

Here the sums within brackets are bounded, so the moments determine their distributions, and therefore

$$\sum_j c_j \xi_{\tau_j} \stackrel{d}{=} \sum_j c_j \xi_j. \tag{5}$$

Thus (1.11) follows by the Cramér-Wold theorem.

The proof of the continuous time version is similar. Assume for the sake of simplicity that the exchangeable process X is ergodic, i.e. a Lévy process or a process of the form (1.17). Given any predictable and a.s. measure preserving transformation V of the time scale, it is required to show that the process $X \circ V^{-1}$ defined by (1.12) has the same finite-dimensional distributions as X.

Let us then fix a finite set of times $t_1, t_2, \ldots$ and associated real constants $c_1, c_2, \ldots$, and write

$$\sum_k c_k (X \circ V^{-1})_{t_k} = \int \{\sum_k c_k 1\{V_s \leqq t_k\}\}\, dX_s. \tag{6}$$

By the hypothesis on V, we have

$$\int \{\sum_k c_k 1\{V_s \leqq t_k\}\}^m ds = \int \{\sum_k c_k 1\{s \leqq t_k\}\}^m ds, \qquad m \in \mathbb{N}, \tag{7}$$

so if the appropriate moment conditions are fulfilled, we may conclude from Theorem 4.1 or 5.1 that

$$\mathsf{E}\{\sum_k c_k (X \circ V^{-1})_{t_k}\}^m = \mathsf{E}\{\sum_k c_k X_{t_k}\}^m, \qquad m \in \mathbb{N}. \tag{8}$$

If the moments determine the distributions, we obtain

$$\sum_k c_k (X \circ V^{-1})_{t_k} \stackrel{d}{=} \sum_k c_k X_{t_k}, \tag{9}$$

and the assertion follows as before.

The required moment conditions are automatically fulfilled for the processes in (1.17), and also when X is a Lévy process with bounded jumps. For a general Lévy process X, we may introduce the processes X_n obtained by deleting all jumps of modules $>n$. Then

$$((X_n \circ V^{-1})_{t_1}, (X_n \circ V^{-1})_{t_2}, \ldots) \stackrel{d}{=} (X_n(t_1), X_n(t_2), \ldots), \tag{10}$$

and as $n \to \infty$, the right-hand side tends to $(X_{t_1}, X_{t_2}, \ldots)$. Applying the statement for bounded jumps to the Poisson process of jump times for $X - X_n$, it is further seen that the left-hand side of (10) tends in distribution to the corresponding sequence for X. Hence the assertion is generally true.

The predictable sampling theorem and its continuous time counterpart have non-trivial extensions to the multivariate case. To state these, say that a finite or infinite sequence of $\mathbb{R}^d$-valued random vectors $\xi = (\xi_{\cdot 1}, \xi_{\cdot 2}, \ldots)$ with $\xi_{\cdot k} = (\xi_{1k}, \ldots, \xi_{dk})$ is *separately exchangeable*, if the distribution of ξ is invariant under arbitrary, *possibly different* permutations in the d coordinates. (To distinguish this from the original notion of exchangeability, where the same permutation is being used for all coordinates, we may refer to the latter as *joint exchangeability*.) Similarly, an $\mathbb{R}^d$-valued process X on $[0, 1]$ or $\mathbb{R}_+$ is said to be separately exchangeable, if X starts at 0, is right-continuous, and has separately exchangeable increments. In each case, the definition extends in an obvious way to the context of arbitrary filtrations $\mathscr{F}$, defined on the appropriate index set.

The next result shows that the distribution of ξ or X remains invariant under possibly different predictable permutations or measure preserving transformations in the d coordinates.

Proposition 6.1. *Let $\xi = (\xi_{\cdot 1}, \xi_{\cdot 2}, \ldots)$ be a finite or infinite, $\mathbb{R}^d$-valued, separately $\mathscr{F}$-exchangeable sequence indexed by I, and let τ_{jk}, $j = 1, \ldots, d$, $k \in I$, be I-valued, $\mathscr{F}$-predictable stopping times, a.s. distinct for fixed k. Then $(\xi_{j, \tau_{jk}}) \stackrel{d}{=} \xi$. If instead X is an $\mathbb{R}^d$-valued, separately $\mathscr{F}$-exchangeable process on $I = [0, 1]$ or $\mathbb{R}_+$, while $U_1, \ldots, U_d$ are $\mathscr{F}$-predictable, a.s. measure preserving transformations of I, then $(X_1 \circ U_1^{-1}, \ldots, X_d \circ U_d^{-1}) \stackrel{d}{=} X$.*

To see how this follows from previous results, we need a simple lemma of some independent interest (along with its corollary).

Lemma 6.2. *A finite or infinite $\mathbb{R}^d$-valued random sequence $\xi = (\xi_{\cdot 1}, \xi_{\cdot 2}, \ldots)$ is extreme separately exchangeable, iff the $\mathbb{R}$-valued sequences $\xi_{j\cdot} = (\xi_{j1}, \xi_{j2}, \ldots)$, $j = 1, \ldots, d$, are mutually independent and ergodic exchangeable. Similarly, an*

$\mathbb{R}^d$-valued process $X=(X_1, \ldots, X_d)$ on $[0, 1]$ or $\mathbb{R}_+$ is extreme separately exchangeable, iff the $\mathbb{R}$-valued processes $X_1, \ldots, X_d$ are mutually independent and ergodic exchangeable.

Proof. We may e.g. consider the case of processes $X=(X_1, \ldots, X_d)$. Directly from the definitions it is clear that, if X is extreme separately exchangeable, then each component process X_j is ergodic exchangeable. Moreover, each X_j is seen to be conditionally exchangeable, given the processes X_i with $i \neq j$, and since X_j is extreme, the conditional distribution must be a.s. independent of X_i, $i \neq j$. This shows that $X_1, \ldots, X_d$ are independent.

Conversely, it is clear that any set of independent ergodic exchangeable processes $X_1, \ldots, X_d$ gives rise to a separately exchangeable process $X = (X_1, \ldots, X_d)$ in $\mathbb{R}^d$. To see that X is extreme, write its distribution as a mixture of extreme distributions, and note as before that each of these is a product measure $\mu_1 \times \ldots \times \mu_d$, with the μ_j ergodic exchangeable. But since $X_1, \ldots, X_d$ are extreme, the measures $\mu_1, \ldots, \mu_d$ must be a.s. unique, and the extremality of X follows. □

Corollary 6.3. *Any $\mathbb{R}^d$-valued extreme separately exchangeable process on $[0, 1]$ or $\mathbb{R}_+$ is also jointly ergodic exchangeable. The corresponding statement holds for infinite (but not for finite) separately exchangeable sequences.*

Proof. For processes on $\mathbb{R}_+$ we note that, if $X_1, \ldots, X_d$ are independent Lévy processes in $\mathbb{R}$, then $X=(X_1, \ldots, X_d)$ is a Lévy process in $\mathbb{R}^d$, and is therefore jointly ergodic exchangeable. A similar argument applies to infinite sequences. For processes on $[0, 1]$ we note that, if $X_1, \ldots, X_d$ are independent processes in $\mathbb{R}$ of the form (1.17), then $X=(X_1, \ldots, X_d)$ is an $\mathbb{R}^d$-valued process of the same form. □

It is now clear, in the three cases covered by Corollary 6.3, how the assertion of Proposition 6.1 may be obtained through an application of the multivariate moment identities of Theorems 3.5, 4.1 and 5.1. Though the original methods of Kallenberg (1988) in the one-dimensional context could be extended, with some effort, to cover the present more general situation, the approach via moment identities seems to be easier and more natural in this case.

For finite sequences, extremality in the sense of separate exchangeability does not imply extremality in the joint sense. Nevertheless, the moment method can still be adapted to this case, via the following slightly extended version of Theorem 3.1, which may be proved by similar arguments. We omit the details.

Lemma 6.4. *Let $\mathscr{F}$ be a filtration on $\{0, \ldots, n\}$, and consider two random n-sequences ξ and η in $\mathbb{R}^d$, such that ξ is $\mathscr{F}$-exchangeable while η is $\mathscr{F}$-predictable. Let $\{K_1, \ldots, K_m\}$ be a partition of $\{1, \ldots, d\}$ which splits ξ into independent ergodic sequences. Assume that (C_1) is fulfilled, and that for each $i \in \{1, \ldots, m\}$, the sums S_J with $J \subset K_i$ are a.s. non-random. Then*

$$\mathsf{E} \prod_{j=1}^{d} \sum_{k=1}^{n} \xi_{jk} \eta_{jk} = \prod_{i=1}^{m} P_{n,K_i}\{R_J, S_J\}, \tag{11}$$

where each $P_{n,K}$ is a polynomial in the sums R_J and S_J with $J \subset K$.

The invariance theorems in continuous time admit vast improvements, in the special case when the exchangeable process X is assumed to be continuous. From our present point of view, this merely reflects the fact that only the first and second order integrals $S_j = \int V_j$ and $S_{ij} = \int V_i V_j$ enter into the hypotheses and conclusions of Theorems 4.1 and 5.1. The situation is particularly nice when $S_1 = \ldots = S_d$ a.s., since in that case both (4.3) and (5.2) reduce to

$$\mathsf{E} \prod_{j=1}^{d} \int V_j \, dX_j = \sum_{\pi} \prod_{j,k} \rho_{jk} S_{jk}, \tag{12}$$

where π ranges over all partitions of the set $\{1, \ldots, d\}$ into pairs $\{j, k\}$. Thus in particular, all moments of odd order equal zero in this case.

The following simple result, which in the case of Brownian motion could also be obtained directly from elementary properties of the Itô integral, illustrates the power of formula (12). Say that a Brownian motion or bridge B is defined *with respect to a filtration* $\mathscr{F}$, if B is $\mathscr{F}$-exchangeable.

Proposition 6.5. *Let B be a Brownian motion on $I = \mathbb{R}_+$ or a Brownian bridge on $I = [0, 1]$, each defined with respect to some standard filtration $\mathscr{F}$, and let $V_t \colon \Omega \times I \to \mathbb{R}$, $t \in I$, be a family of $\mathscr{F}$-predictable processes with $\int V_t = 0$ a.s. for each t, and such that*

$$\int_0^\infty V_s V_t = s \wedge t \quad \text{a.s.}, \qquad s, t \in \mathbb{R}_+, \tag{13}$$

or

$$\int_0^1 V_s V_t = s \wedge t - st \quad \text{a.s.}, \qquad s, t \in [0, 1], \tag{14}$$

respectively. Then the process $Y_t = \int V_t \, dB$, $t \in I$, is another Brownian motion or bridge.

Proof. If $I = [0, 1]$, we get by Theorem 4.1, for any $d \in \mathbb{N}$ and $t_1, \ldots, t_d \in [0, 1]$,

$$\mathsf{E} \prod_{j=1}^{d} Y_{t_j} = \sum_{\pi} \prod_{i,j} (t_i \wedge t_j - t_i t_j) = \mathsf{E} \prod_{j=1}^{d} B_{t_j}, \tag{15}$$

where π ranges over all partitions of the set $\{1, \ldots, d\}$ into pairs $\{i, j\}$. Since any Gaussian distribution is determined by its moments, the assertion follows. The proof for $I = \mathbb{R}_+$ is similar. □

As a non-trivial example, we may take $V = (V_t)$ to be an integrable and ergodic exchangeable process on $I = \mathbb{R}_+$ or $[0, 1]$, directed by some triple (γ, σ^2, ν) or $(\alpha, \sigma^2, \beta)$, respectively, satisfying

$$\gamma = 0 \quad \text{and} \quad \sigma^2 + \int x^2 \, \nu(dx) = 1 \tag{16}$$

or

$$\alpha = 0 \quad \text{and} \quad \sigma^2 + \sum \beta_j^2 = 1. \tag{17}$$

Choosing the Lebesgue unit interval as our probability space, we may regard each random variable V_t as a deterministic process on $[0, 1]$, which as such is trivially predictable. If $I=\mathbb{R}_+$, we extend V_t to a process on $\mathbb{R}_+$ by putting $V_t(u)=0$ for $u>1$. Then (13) and (14) reduce to obvious moment properties of the process V. Using Proposition 6.3, we may hence conclude that the process $Y_t=\int V_t\,dB$, $t\in I$, is another Brownian motion or bridge on I. This is surprising, if we think of the integral $Y=\int V dB$ as a stochastic average over the paths of V, in the sense of integration with respect to Gaussian white noise on Ω with control measure P. Note that integration with respect to P instead would yield the expected value of V, which is identically zero.

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