Prob. Th. Rel. Fields 74, 141–159 (1987)

Probability Theory and Related Fields

An Infinite Dimensional Stochastic Differential Equation with State Space $C(\mathbb{R})$

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Summary. We consider a time evolution of unbounded continuous spins on the real line. The evolution is described by an infinite dimensional stochastic differential equation with local interaction. Introducing a condition which controls the growth of paths at infinity, we can construct a diffusion process taking values in $C(\mathbb{R})$. In view of quantum field theory, this is a time dependent model of $P(\phi)_1$ field in Parisi and Wu's scheme.

§ 1. Introduction

Let $\mathscr{C}=C(\mathbb{R},\mathbb{R}^d)$ be the space of all $\mathbb{R}^d$-valued continuous functions defined on $\mathbb{R}$ equipped with the compact uniform topology. $\mathscr{D}(\mathbb{R},\mathbb{R}^d)$ denotes Schwartz's space of $\mathbb{R}^d$-valued C^∞-functions on $\mathbb{R}$ with compact support. The topological dual space $\mathscr{D}'(\mathbb{R},\mathbb{R}^d)$ is the d-fold direct product of the space of Schwartz's distributions $\mathscr{D}'(\mathbb{R})$. For given continuous functions $a(x)=(a_{ij}(x))\colon \mathbb{R}^d\to\mathbb{R}^d\otimes\mathbb{R}^d$ and $b(x)=(b_i(x))\colon \mathbb{R}^d\to\mathbb{R}^d$ we consider the following infinite dimensional stochastic differential equation for a $\mathscr{C}$-valued process $X=\{X_t^i(u)\}$:

$$(1.1)\qquad dX_t(u)=a(X_t(u))\,dB_t(u)+b(X_t(u))\,dt+\tfrac{1}{2}\Delta X_t(u)\,dt,\qquad u\in\mathbb{R},$$

where $B_t=\{B_t^1,\ldots,B_t^d\}$ is a system of independent $\mathscr{D}'(\mathbb{R})$-valued standard Brownian motions (simply called a $\mathscr{D}'(\mathbb{R},\mathbb{R}^d)$-valued standard Brownian motion, cf. Itô [4]) and $\Delta=d^2/du^2$. We interpret the equation in $\mathscr{D}'(\mathbb{R},\mathbb{R}^d)$, since neither the first term of the right hand side of (1.1) nor the last one defines a $\mathscr{C}$-valued process any more. A precise meaning of (1.1) will be given in § 3.

This type of equation describes a diffusion process associated with $P(\phi)_1$ model in the sense of Parisi and Wu [8], where $a(x)$ is the identity matrix and $b(x)=-\frac{1}{2}\operatorname{grad} U(x)$ for some potential function $U(x)$.

Funaki [2] discussed (1.1) as an equation describing a random motion of an elastic string, where parameter u runs over a bounded interval. Marcus [5] also studied (1.1) in a restricted situation to carry out Parisi and Wu's program. However they actually treated the following equation instead of (1.1):

(1.2) $$X_t(u)=Y_t^\infty(u;X_0)+\int_0^t\int_{\mathbb{R}} q(t-s;u,v)\,a(X_s(v))\,dB_s(v)\,dv$$
$$+\int_0^t\int_{\mathbb{R}} q(t-s;u,v)\,b(X_s(v))\,ds\,dv,$$

where $q(t;u,v)=(2\pi t)^{-1/2}\exp(-(u-v)^2/2t)$ is the heat kernel and $Y_t^\infty(u;X_0) = \int_{\mathbb{R}} q(t;u,v)\,X_0(v)\,dv$.

Let us introduce a martingale problem associated with (1.1). $W=C([0,\infty),\mathscr{C})$ stands for the continuous path space taking values in $\mathscr{C}$ and let $\theta_t\colon W\to\mathscr{C}$ be the canonical projection at $t\geqq 0$. $\mathscr{G}$ (resp. $\mathscr{G}_t$) denotes the σ-field on W generated by the canonical projections (up to time t). For each open set $G\subset\mathbb{R}$ we define

$$\mathbb{D}_G=\{f:\mathscr{C}\to\mathbb{R};\,f(X)=\tilde{f}(\langle X,\xi_1\rangle,\dots,\langle X,\xi_n\rangle)\text{ for some }n\geqq 1,$$
$$\xi_1,\dots,\xi_n\in\mathscr{D}(G,\mathbb{R}^d)\text{ and }\tilde{f}\in\mathscr{D}(\mathbb{R}^n)\},$$

where $\mathscr{D}(G,\mathbb{R}^d)$ stands for the subspace of $\mathscr{D}(\mathbb{R},\mathbb{R}^d)$ consisting of those elements with support in G, $\mathscr{D}(\mathbb{R}^n)$ is the totality of real C^∞-functions on $\mathbb{R}^n$ with compact support and $\langle X,\xi\rangle=\int_{\mathbb{R}} X(u)\cdot\xi(u)\,du=\int_{\mathbb{R}}\sum_{i=1}^d X^i(u)\,\xi^i(u)\,du$. We denote by Df the Fréchet derivative of $f\in\mathbb{D}=\mathbb{D}_{\mathbb{R}}$ and define the operator L with the domain $\mathbb{D}$ as follows:

(1.3) $$Lf(X)=\tfrac{1}{2}\operatorname{trace}(a(X)\,a^*(X)\,D^2f(X))+\langle b(X),Df(X)\rangle+\tfrac{1}{2}\langle X,\Delta Df(X)\rangle.$$

Namely if $f(X)=\tilde{f}(\langle X,\xi_1\rangle,\dots,\langle X,\xi_n\rangle)\in\mathbb{D}$, then

$$Lf(X)=\tfrac{1}{2}\sum_{k,l=1}^n\langle a^*(X(\cdot))\,\xi_k,\,a^*(X(\cdot))\,\xi_l\rangle\frac{\partial^2}{\partial x_k\,\partial x_l}\tilde{f}(\langle X,\xi_1\rangle,\dots,\langle X,\xi_n\rangle)$$
$$+\sum_{k=1}^n\{\langle b(X(\cdot)),\xi_k\rangle+\tfrac{1}{2}\langle X,\Delta\xi_k\rangle\}\frac{\partial}{\partial x_k}\tilde{f}(\langle X,\xi_1\rangle,\dots,\langle X,\xi_n\rangle),$$

where $a^*(x)=(a^*_{ij}(x))=(a_{ji}(x))$. A probability measure P on $(W,\mathscr{G})$ is called a solution of the $(\mathscr{C},L)$-martingale problem starting from $X\in\mathscr{C}$ if

(1.4) (i) $P(\theta_0=X)=1$,

(ii) $f(\theta_t)-\int_0^t Lf(\theta_s)\,ds$ is a local martingale relative to $(P,\{\mathscr{G}_t\})$ for all $f\in\mathbb{D}$.

In the present paper we will first establish in § 3 the equivalence between the stochastic differential Eq. (1.1), the stochastic integral Eq. (1.2) and the $(\mathscr{C},L)$-martingale problem (1.4). We will next discuss the existence and the uniqueness of solutions for Eq. (1.1). In order to guarantee the uniqueness we need to restrict the state space to a smaller one rather than $\mathscr{C}$, because even if both $a(x)$ and $b(x)$ are vanishing (then (1.1) is just a heat equation) the uniqueness fails in $\mathscr{C}$. Therefore we introduce a tempered subspace of $\mathscr{C}$:

$$E=\{X\in\mathscr{C};\ \lim_{|u|\to\infty}X(u)\,e^{-\lambda|u|}=0\text{ for every }\lambda>0\}.$$

Note that E is a Fréchet space with the topology induced by a system of norms $\{|\cdot|_\lambda\}_{\lambda>0}$ defined by $|X|_\lambda = \sup_{u\in\mathbb{R}} |X(u)|\, e^{-\lambda|u|}$. We will prove in § 4 the existence and the uniqueness of E-valued solutions for (1.1) under the assumption that both $a(x)$ and $b(x)$ are Lipschitz continuous. However, since our motivation consists in studying Parisi and Wu's stochastic quantization, we need to relax the global Lipschitz condition on $b(x)$. Therefore in § 5 we will discuss this problem and give another sufficient condition for the existence and the uniqueness of E-valued solutions of (1.1), which is applicable to polynomial interaction model ($(P(\phi))_1$ model).

The author wishes to thank Professor H. Ezawa for leading him to this problem. He is also grateful to Professors K. Itô, T. Shiga and T. Funaki for valuable suggestions and kind encouragements.

§ 2. Stochastic Integrals

In order to formulate the Eqs. (1.1) and (1.2) we will here define stochastic integrals with respect to a class of $\mathscr{D}'$-valued martingales. We mention the case $\mathscr{D}' = \mathscr{D}'(\mathbb{R})$.

Let $(\Omega, \mathscr{F}, P)$ be a probability space with a reference family $\{\mathscr{F}_t\}_{t\geq 0}$. $\mathscr{S}$ denotes the predictable σ-field on $[0,\infty)\times\Omega$ relative to $\{\mathscr{F}_t\}$, which is generated by all $\{\mathscr{F}_t\}$-adapted left continuous processes. Let $\mathscr{M}$ (resp. $\mathscr{M}_{\mathrm{loc}}$) stands for the totality of continuous square integrable martingales (resp. continuous local martingales) on $(\Omega, \mathscr{F}, P, \{\mathscr{F}_t\})$. For $M\in\mathscr{M}_{\mathrm{loc}}$, $[M]=\{[M]_t\}$ denotes the quadratic variational process.

Definition 2.1. A $\mathscr{D}'$-valued continuous process $M=\{M_t\}$ is called a $\mathscr{D}'$-valued continuous local martingale with quadratic variational measure (abbr. $\mathscr{D}'$-c.l.m. with q.v.m.), if

(M-1) $M_0 = 0$ a.s. and $\langle M_\cdot, \xi\rangle \in \mathscr{M}_{\mathrm{loc}}$ for every $\xi\in\mathscr{D}=\mathscr{D}(\mathbb{R})$,

(M-2) there exists a nonnegative random Radon measure $[M]$ on $[0,\infty)\times\mathbb{R}$ such that $[M]([0,t]\times A)$ is $\{\mathscr{F}_t\}$-predictable for every $A\in\mathscr{B}(\mathbb{R})$ (the topological σ-field on $\mathbb{R}$) and for every $\xi\in\mathscr{D}$

$$P([\langle M_\cdot, \xi\rangle]_t = \int_{\mathbb{R}} |\xi(u)|^2 [M]([0,t], du) \text{ for all } t\geq 0) = 1.$$

$[M]$ is called the quadratic variational measure of M.

Remark. (i) $[M]([0,t]\times A)$ is, in fact, a continuous process for each $A\in\mathscr{B}(\mathbb{R})$, since $[\langle M_\cdot, \xi\rangle]_t$ is continuous in t for every $\xi\in\mathscr{D}$.

(ii) A $\mathscr{D}'$-valued standard $\{\mathscr{F}_t\}$-Brownian motion $B=\{B_t\}$ is a $\mathscr{D}'$-c.l.m. with q.v.m. $[B](dt,du)=dt\times du$.

Now we define stochastic integrals. Let M be a $\mathscr{D}'$-c.l.m. with q.v.m. We introduce the following classes of random functionals:

$$\mathscr{L}_0 = \Big\{\phi\colon [0,\infty)\times\mathbb{R}\times\Omega\to\mathbb{R};\ \phi(t,u,\omega)=\sum_{i=1}^n f_i(t,\omega)\, g_i(u) \text{ for some } n\geq 1,$$
$$f_1,\dots,f_n \text{ bounded and } \{\mathscr{F}_t\}\text{-predictable and } g_1,\dots,g_n\in\mathscr{D}(\mathbb{R})\Big\},$$

$$\mathscr{L}(M)=\Big\{\phi:[0,\infty)\times\mathbb{R}\times\Omega\to\mathbb{R};\ \{\mathscr{F}_t\}\text{-predictable (i.e., }\mathscr{S}\times\mathscr{B}(\mathbb{R})\text{-measurable)}$$
$$\text{and for every } T>0 \text{ and } N>0\ P\left(\int_0^T\int_{-N}^N \phi(t,u)^2[M](dt,du)<\infty\right)=1\Big\}.$$

Lemma 2.1. *Every* $\phi\in\mathscr{L}(M)$ *defines uniquely a* $\mathscr{D}'$-c.l.m. *with* q.v.m. $I_M(\phi)$ *such that*

(I-1) *if* $\phi(t,u,\omega)=\sum_{i=1}^n f_i(t,\omega)\,g_i(u)\in\mathscr{L}_0$, *then*

$$\langle I_M(\phi)_t,\xi\rangle=\sum_{i=1}^n\int_0^t f_i(s)\,d\langle M_s,g_i\xi\rangle \text{ a.s.} \qquad \textit{for every } \xi\in\mathscr{D},$$

where the integrals in the right hand side are one dimensional stochastic integrals.

(I-2) $I_M(\alpha\phi+\beta\psi)=\alpha I_M(\phi)+\beta I_M(\psi)$ a.s. *for* $\alpha,\beta\in\mathbb{R}$, *and* $\phi,\psi\in\mathscr{L}(M)$,

(I-3) $[I_M(\phi)](dt,du)=\phi(t,u)^2[M](dt,du)$,

(I-4) $I_{M^\sigma}(\phi^\sigma)_t=I_M(\phi)_{t\wedge\sigma}$ *for every stopping time* σ, *where* $M_t^\sigma=M_{t\wedge\sigma}$, $\phi^\sigma(t,u)=1_{[0,\sigma]}(t)\,\phi(t,u)$ *and* $1_{[0,\sigma]}$ *is the indicator function of* $[0,\sigma]$.

Proof. It is a routine work to construct a family $\{\langle I_M(\phi),\xi\rangle_t\}_{\phi,\xi}\subset\mathscr{M}_{\mathrm{loc}}$ that satisfies (I-1)$\sim$(I-4), so that we omit the details. Regarding as a process of linear functionals on $\mathscr{D}$, we see that $\{\langle I_M(\phi),\cdot\rangle_t\}$ satisfies (M-1) and (M-2) in Definition 2.1 for each ϕ.

To realize $I_M(\phi)$ as a $\mathscr{D}'$-valued process we mention a regularization theorem by S. Nakao [7] (cf. Itô [4] and Mitoma [6]).

Lemma 2.2. *Let* $\mathscr{D}$ *be a real vector space with a multi-Hilbertian topology* τ. *Suppose* $M=\{\langle M,\cdot\rangle_t\}$ *is a process of linear functionals on* $\mathscr{D}$ *such that*

(i) $\langle M,\xi\rangle_0=0$ a.s. *and* $\langle M,\xi\rangle_\cdot\in\mathscr{M}_{\mathrm{loc}}$ *for every* $\xi\in\mathscr{D}$,

(ii) *there exists a sequence of stopping times* $\{\sigma_n\}$ *such that* $\sigma_n\nearrow\infty$ a.s. *and Hilbertian semi-norms* $\{\|\cdot\|_n\}$ *defined by* $\|\xi\|_n^2=E\,|\langle M,\xi\rangle_{t\wedge\sigma_n}|^2$ *are Hilbert-Schmidt weaker than* τ *for every* $t>0$. *Then* M *has a unique* σ-*concentrated* $\mathscr{D}'_\tau$-*valued continuous version, where* $\mathscr{D}'_\tau$ *is the dual space of* $\mathscr{D}$ *with respect to* τ.

Therefore we have a $\mathscr{D}'$-c.l.m. with q.v.m. $I_M(\phi)$ for every $\phi\in\mathscr{L}(M)$. It is immediately checked that the conditions (I-1)$\sim$(I-4) characterize $I_M(\cdot)$ uniquely. This completes the proof. □

$I_M(\phi)$ is called the stochastic integral of $\phi\in\mathscr{L}(M)$ with respect to M and is denoted by $\int_0^t\phi(s,\cdot)\,dM_s$. *If* $\phi\in\mathscr{L}(M)$ satisfies

$$E\left[\int_0^T\int_{\mathbb{R}}\phi(t,u)^2[M](dt,du)\right]<\infty \qquad \text{for all } T>0,$$

then we can define $X_{\cdot}\in\mathcal{M}$ as follows: for any $\{\xi_n\}\subset\mathcal{D}$ such that $0\leqq\xi_n(u)\nearrow 1$

$$\lim_{n\to\infty} E\Big[\sup_{0\leqq t\leqq T}\Big|\Big\langle\int_0^t \phi(s,\cdot)\,dM_s,\xi_n\Big\rangle - X_t\Big|^2\Big]=0.$$

Then X_t is denoted by $\int_0^t\int_{\mathbb{R}}\phi(s,u)\,dM_s(u)\,du$. Similarly we shall use the following notation.

$$\int_0^t\int_l^r \phi(s,u)\,dM_s(u)\,du=\int_0^t\int_{\mathbb{R}}\phi(s,u)\,1_{[l,r]}(u)\,dM_s(u)\,du.$$

Finally we prepare some lemmas.

Lemma 2.3. *Let $B=\{B_t\}$ be a $\mathcal{D}'$-valued standard $\{\mathcal{F}_t\}$-Brownian motion. Suppose $\phi\in\mathcal{L}(B)$ satisfies $E\Big[\int_0^T|\int_{\mathbb{R}}\phi(t,u)^2\,du|^p\,dt\Big]<\infty$, $p=1,2,\ldots$. Then*

$$\text{(2.1)}\quad E\Big|\int_0^T\int_{\mathbb{R}}\phi(t,u)\,dB_t(u)\,du\Big|^{2p}\leqq(p(2p-1))^p\Big[\int_0^T\{E\,|\int_{\mathbb{R}}\phi(t,u)^2\,du|^p\}^{1/p}\,dt\Big]^p.$$

Proof. It suffices to show (2.1) for $\phi(t,u,\omega)=f(t,\omega)\,\xi(u)\in\mathcal{L}_0$. Then (2.1) immediately follows, since

$$\int_0^t\int_{\mathbb{R}}\phi(t,u)\,dB_s(u)\,du=\int_0^t f(s)d\langle B_s,\xi\rangle. \quad \square$$

Lemma 2.4. *Let $(U,\mathcal{B},m)$ be a finite measure space and let $X\in\mathcal{M}_{\mathrm{loc}}$. Suppose $\psi(t,u,\omega)$: $[0,\infty)\times U\times\Omega\to\mathbb{R}$ is $\{\mathcal{F}_t\}$-predictable (i.e., $\mathcal{S}\times\mathcal{B}$-measurable) and*

$$\text{(2.2)}\qquad \int_0^T\int_U \psi(t,u)^2\,m(du)\,d[X]_t<\infty \qquad \textit{for all } T>0,\ P\text{-a.s.}$$

Then $\int_0^t\psi(s,u)\,dX_s$ has an $\{\mathcal{F}_t\}$-predictable modification and

$$\text{(2.3)}\qquad \int_0^t\Big(\int_U\psi(s,u)\,m(du)\Big)\,dX_s=\int_U\Big(\int_0^t\psi(s,u)\,dX_s\Big)\,m(du),\qquad t\geqq 0,\ P\text{-a.s.}$$

We omit the proof. But we note that the same result holds for stochastic integrals with respect to a $\mathcal{D}'$-c.l.m. with q.v.m.

§ 3. On the Equivalence Between (1.1), (1.2) and (1.4)

Our first task is to formulate the Eq. (1.1) as a stochastic differential equation in $\mathcal{D}'=\mathcal{D}'(\mathbb{R},\mathbb{R}^d)$. Next we will show the equivalence between the stochastic differential Eq. (1.1), the stochastic integral Eq. (1.2) and the $(\mathcal{C},L)$-martingale problem (1.4).

Definition 3.1. By a solution of (1.1) we mean a $\mathscr{C}$-valued continuous process $X=\{X_t(u)\}$ defined on a probability space $(\Omega, \mathscr{F}, P)$ with a reference family $\{\mathscr{F}_t\}$ such that

(i) there exists a $\mathscr{D}'$-valued standard $\{\mathscr{F}_t\}$-Brownian motion $B=\{B_t\}$,
(ii) $\{X_t(\cdot)\}$ is $\{\mathscr{F}_t\}$-adapted,
(iii) with probability one $X=\{X_t(u)\}$ and $B=\{B_t\}$ satisfy

$$X_t = X_0 + \int_0^t a(X_s(\cdot))\, dB_s + \int_0^t (b(X_s(\cdot)) + \tfrac{1}{2}\Delta X_s)\, ds, \qquad t \geqq 0, \tag{3.1}$$

in $\mathscr{D}'$, where $\int_0^t a(X_s(\cdot))\, dB_s = \left(\sum_{j=1}^{d} \int_0^t a_{ij}(X_s(\cdot))\, dB_s^j \right)$.

Now we introduce a family of integral equations. Given l, $r \in \mathbb{R}$ $(l<r)$ and $w \in W$, $Y_t^{l,r}(u; w)$ stands for the solution of the following initial boundary problem:

$$\frac{\partial}{\partial t} Y_t(u) = \tfrac{1}{2}\Delta Y_t(u), \qquad l<u<r, \tag{3.2}$$

$$Y_0(u) = w_0(u), \quad l<u<r, \qquad Y_t(l) = w_t(l), \qquad Y_t(r) = w_t(r), \quad t \geqq 0.$$

Let $q^{l,r}(t; u, v)$ be the fundamental solution of $\partial/\partial t - 1/2\Delta$ with Dirichlet boundary condition at $u=l$ and $u=r$. Then $Y_t^{l,r}(u; w)$ can be represented as follows (cf. Friedman [1], Chap. 3, Problem 4):

(3.3)
$$\int_l^r q^{l,r}(t; u, v)\, w_0(v)\, dv + \tfrac{1}{2} \int_0^t \left(\frac{\partial}{\partial v} q^{l,r}(t-s; u, l)\, w_s(l) - \frac{\partial}{\partial v} q^{l,r}(t-s; u, r)\, w_s(r) \right) ds.$$

Consider the following family of integral equations:

$$\begin{aligned} X_t(u) = {} & Y_t^{l,r}(u; X.) + \int_0^t \int_l^r q^{l,r}(t-s; u, v)\, a(X_s(v))\, dB_s(v)\, dv \\ & + \int_0^t \int_l^r q^{l,r}(t-s; u, v)\, b(x_s(v))\, ds\, dv, \qquad l \leqq u \leqq r, \\ & -\infty < l < r < \infty. \end{aligned} \tag{3.4}$$

The meaning of the Eqs. (3.4) is similar to that of (1.1) in Definition 3.1. Note that the above equations are regarded as those for a $\mathscr{C}$-valued continuous process, although the stochastic integrals are defined for each fixed $t>0$ and $u \in \mathbb{R}$. Indeed let $X=\{X_t(u)\}$ be an $\{\mathscr{F}_t\}$-adapted $\mathscr{C}$-valued continuous process. Fix l, $r \in \mathbb{R}$ and denote by $X_{2,t}(u)$ the term defined by stochastic integration. We define a sequence of stopping times by

$$\tau_n = \inf\{t>0; \max_{l \leqq u \leqq r} |X_t(u)| > n\}.$$

Next fix $0<t$, t' and $l \leqq u$, $u' \leqq r$. An elementary calculation shows

(3.5) $$\int_0^{t\vee t'}\left(\int_l^r f(s,v)^2\,dv\right)ds \leqq 3/\sqrt{2\pi}\,|t-t'|^{1/2}+|u-u'|,$$

where $f(s,v)=q^{l,r}(t-s;u,v)\,1_{[0,t]}(s)-q^{l,r}(t'-s;u',v)\,1_{[0,t']}(s)$. Thus using Lemma 2.3, we see that for $m=1,2,\ldots,$

(3.6) $$\begin{aligned} &E[|X_{2,t}(u)-X_{2,t'}(u')|^{2m};t,t'\leqq\tau_n]\\ &=E\left[\left|\int_0^{t_1}\int_l^r f(s,v)\,a(X_s(v))\,dB_s(v)\,dv\right|^{2m};t_1\leqq\tau_n\right]\Bigg|_{t_1=t\vee t'}\\ &\leqq E\left[\left|\int_0^{t_1}\int_l^r f(s,v)\,1_{[0,\tau_n]}(s)\,a(X_s(v))\,dB_s(v)\,dv\right|^{2m}\right]\Bigg|_{t_1=t\vee t'}\\ &\leqq(m(2m-1))^m\sup_{|x|\leqq n}|a(x)|^{2m}(3/\sqrt{2\pi}\,|t-t'|^{1/2}+|u-u'|)^m,\end{aligned}$$

where $|a|^2=\sum_{i,j=1}^d|a_{ij}|^2$ for $a=(a_{ij})\in\mathbb{R}^d\otimes\mathbb{R}^d$. Here we recall Kolmogorov's criterion (a version for multi-parameter case, cf. Totoki [10]). Since $\tau_n\nearrow\infty$ a.s., we can choose a jointly continuous $\{X_{2,t}(u)\}$.

Theorem 3.1. *If there exists a solution of the stochastic differential Eq.* (1.1), *it is also a solution of the family of integral Eqs.* (3.4), *and vice versa.*

Proof. Suppose $\{X_t(u)\}$ is a solution of (3.4). Let $\xi\in\mathscr{D}(\mathbb{R},\mathbb{R}^d)$ be fixed. Choose a bounded interval (l,r) that contains the support of ξ. From the definition of $Y_t(u)=Y_t^{l,r}(u;X.)$ we see

$$\tfrac{1}{2}\int_0^t\langle Y_s,\Delta\xi\rangle\,ds=\langle Y_t,\xi\rangle-\langle X_0,\xi\rangle.$$

Denote by $X_{i,t}(u)$ the ith term of the right hand side of (3.4) ($i=2,3$). Changing the order of integration repeatedly (cf. Lemma 2.4), we get

$$\begin{aligned}\tfrac{1}{2}\int_0^t\langle X_{2,s},\Delta\xi\rangle\,ds&=\int_0^t\int_l^r a^*(X_{s'}(v))\\ &\quad\cdot\left\{\int_0^t\left(\int_l^r q^{l,r}(s-s';u,v)\,\Delta\xi(u)\,du\right)1_{[s',\infty)}(s)\,ds\right\}dB_{s'}(v)\,dv\\ &=\langle X_{2,t},\xi\rangle-\left\langle\int_0^t a(X_s(\cdot))\,dB_s,\xi\right\rangle,\end{aligned}$$

and

$$\tfrac{1}{2}\int_0^t\langle X_{3,s},\Delta\xi\rangle\,ds=\langle X_{3,s},\xi\rangle-\int_0^t\langle b(X_s(\cdot)),\xi\rangle\,ds.$$

Since $X_t(u)=Y_t(u)+X_{2,t}(u)+X_{3,t}(u)$, $l<u<r$, it follows that

$$\langle X_t,\xi\rangle=\langle X_0,\xi\rangle+\left\langle\int_0^t a(X_s(\cdot))\,dB_s,\xi\right\rangle+\int_0^t(\langle b(X_s),\xi\rangle+\tfrac{1}{2}\langle X_s,\Delta\xi\rangle)\,ds.$$

Therefore $\{X_t(u)\}$ is a solution of (1.1).

Conversely let $\{X_t(u)\}$ be a solution of (1.1). Fix l, $r \in \mathbb{R}$, then we have

$$\langle X_t - Y_t, \xi\rangle = \int_0^t \int_{\mathbb{R}} a^*(X_s(u))\, \xi(u)\, dB_s(u)\, du + \int_0^t \left(\langle b(X_s), \xi\rangle + \tfrac{1}{2}\langle X_s - Y_s, \Delta\xi\rangle\right) ds, \qquad \xi \in \mathscr{D}((l, r), \mathbb{R}^d),$$

where $Y_t(u) = Y_t^{l,r}(u; X.)$. Denote by $\{\phi_n\}$ the totality of eigenfunctions of $1/2\Delta$ with Dirichlet boundary condition at $u = l$ and $u = r$ ($1/2\Delta\phi_n = -\lambda_n\phi_n$, $\phi_n(u) = 0$, $u \notin (l, r)$). Since Y_s coincides with X_s at $u = l$ and $u = r$, it follows that

$$\langle X_s - Y_s, \phi_n\rangle = \int_0^t \int_{\mathbb{R}} a^*(X_s(u))\, \phi_n(u)\, dB_s(u)\, du + \int_0^t (\langle b(X_s), \phi_n\rangle - \lambda_n\langle X_s - Y_s, \phi_n\rangle)\, ds,$$

and consequently

$$\begin{aligned}\langle X_t - Y_t, \phi_n\rangle &= \int_0^t \int_{\mathbb{R}} e^{-\lambda_n(t-s)} a^*(X_s(u))\, \phi_n(u)\, dB_s(u)\, du + \int_0^t e^{-\lambda_n(t-s)}\langle b(X_s), \phi_n\rangle\, ds \\ &= \langle X_{2,t}, \phi_n\rangle + \langle X_{3,t}, \phi_n\rangle, \qquad \text{for } n = 1, 2, \ldots.\end{aligned}$$

Therefore $\{X_t(u)\}$ is a solution of (3.4). □

If we restrict the state space into the space E, we can use a more convenient Eq. (1.2) in place of (3.4). Here we note that the continuity of solutions is considered with respect to the topology of E and that the corresponding path space $C([0, \infty), E)$ is equipped with the compact uniform topology.

Theorem 3.2. *Suppose $a(x)$ is bounded and $|b(x)| \leqq K(1 + |x|^p)$, $x \in \mathbb{R}^d$ for some $K > 0$ and $p > 0$. If there exists an E-valued solution of the stochastic differential Eq.* (1.1), *it is also a solution of the stochastic integral Eq.* (1.2), *and the converse is true.*

Proof. Because of Theorem 3.1, it suffices to prove that any solution of (3.4) satisfies (1.2) and this is verified by letting $(l, r) \nearrow \mathbb{R}$ in (3.4). To this end we use the following.

Lemma 3.1. *For $w \in C([0, \infty), E)$ extend $Y_t^{l,r}(u; w)$ to $u \in \mathbb{R}$ such that it coincides with w outside (l, r). Then $Y_t^{l,r}(u; w)$ converges to $Y_t^\infty(u; w_0) = \int_{\mathbb{R}} q(t; u, v)\, w_0(v)\, dv$ as $(l, r) \nearrow \mathbb{R}$ in $C([0, \infty), E)$.*

Proof. We keep the expression (3.3) in mind. First using the method of image, we have

$$0 \leqq q(t; u, v) - q^{l,r}(t; u, v) \leqq q(t; u, 2l - v) + q(t; u, 2r - v), \qquad l \leqq u, \ v \leqq r.$$

Hence it follows that

$$I=\left|\int_l^r q^{l,r}(t;u,v)\,w_0(v)\,dv-\int_{\mathbb{R}} q(t;u,v)\,w_0(v)\,dv\right|$$

$$\leqq \int_r^\infty q(t;u,v)(|w_0(2r-v)|+|w_0(v)|)\,dv$$

$$+\int_{-\infty}^{l} q(t;u,v)(|w_0(2l-v)|+|w_0(v)|)\,dv, \qquad u\in(l,r).$$

We may assume $l<0<r$, then for $\lambda>0$ the first term is bounded by:

$$\text{(3.7)}\qquad 2\,|w_0|_\lambda \int_r^\infty q(t;u,v)\,e^{\lambda v}\,dv$$

$$\leqq 2\,|w_0|_\lambda\, e^{-\lambda r}\int_{\mathbb{R}} q(t;u,v)\,e^{2\lambda v}\,dv=2\,|w_0|_\lambda\, e^{2\lambda^2 t}\,e^{-\lambda r}\,e^{2\lambda u}.$$

Next noting $q^{l,r}(t;u,v)\geqq 0$ and $q^{l,r}(t;u,l)=q^{l,r}(t,u,r)=0$ and applying Fatou's lemma, we have

$$J=\left|\frac{1}{2}\int_0^t\left(\frac{\partial}{\partial v}q^{l,r}(t-s;u,l)\,w_s(l)-\frac{\partial}{\partial v}q^{l,r}(t-s;u,r)\,w_s(r)\right)ds\right|$$

$$\leqq \tfrac{1}{2}\lim_{\delta\downarrow 0}\left\{\sup_{0\leqq s\leqq t}|w_s(l)|\int_0^t q^{l,r}(t-s;u,l+\delta)/\delta\,ds\right.$$

$$\left.+\sup_{0\leqq s\leqq t}|w_s(r)|\int_0^t q^{l,r}(t-s;u,r-\delta)/\delta\,ds\right\}.$$

Let $\lambda>0$, then

$$\text{(3.8)}\qquad \lim_{\delta\downarrow 0} 1/\delta\int_0^t q^{l,r}(t-s;u,l+\delta)\,ds\leqq e^{\lambda^2 t/2}\,\frac{\partial}{\partial v}\int_0^\infty q^{l,r}(s;u,v)\,e^{-\lambda^2 s/2}\,ds\bigg|_{v=l}$$

$$\leqq 2e^{\lambda^2 t/2}(1-e^{-2\lambda(r-l)})^{-1}\,e^{\lambda l}\,e^{-\lambda u}, \qquad u\in(l,r).$$

From (3.7) and (3.8) it follows that $\sup_{0\leqq t\leqq T}|I+J|_{2\lambda}$ tends to 0 as $(l,r)\nearrow\mathbb{R}$ for every $\lambda>0$ and $T>0$. Thus the lemma is proved. □

We now let $(l,r)\nearrow\mathbb{R}$, then each term in the right hand side of (3.4) tends to the corresponding term in (1.2). To see the convergence of the third term, we have only to note that $\{b(X_t(u))\}$ is again an E-valued process since $|b(x)|\leqq K(1+|x|^p)$. The proof is complete. □

Finally we show the equivalence between the stochastic differential Eq. (1.1) and the $(\mathscr{C}, L)$-martingale problem (1.4).

Theorem 3.3. *If P is a solution of the $(\mathscr{C}, L)$-martingale problem* (1.4), *then $\{X_t = \theta_t\}$ is a solution of the stochastic differential Eq.* (1.1) *on some extension of $(W, \mathscr{G}, P, \{\mathscr{G}_t\})$. Conversely let $\{X_t(u)\}$ be a solution of* (1.1), *then its probability law on $(W, \mathscr{G})$ solves* (1.4).

Proof. Let P be a solution of (1.4), then

$$M_t = \theta_t - \theta_0 - \int_0^t (b(\theta_s) + \tfrac{1}{2} \Delta \theta_s)\, ds$$

defines a $\mathscr{D}'$-c.l.m. with q.v.m. on $(W, \mathscr{G}, P, \{\mathscr{G}_t\})$ and

$$[M](dt, du) = \left(\sum_{k=1}^{d} a_{ik}(\theta_t(u))\, a_{jk}(\theta_t(u))\, dt \times du \right).$$

Suppose $\det(a(x)) \neq 0$ for all $x \in \mathbb{R}^d$ and denote by $\alpha(x) = (\alpha_{ij}(x))$ the inverse matrix of $a(x)$ for $x \in \mathbb{R}^d$. Then

$$B_t = \int_0^t \alpha(\theta_s(\cdot))\, dM_s$$

is a $\mathscr{D}'$-valued standard $\{\mathscr{G}_t\}$-Brownian motion and

$$M_t = \int_0^t a(\theta_s(\cdot))\, dB_s.$$

This means $\{X_t = \theta_t\}$ is a solution of (1.1). If $\det(a(x)) = 0$ for some x's, we can prove it by applying the same method used in finite dimensional cases (cf. Ikeda and Watanabe [3]).

The converse is an immediate consequence of Itô's formula. □

§ 4. Existence and Uniqueness – Lipschitz Condition

We show here that the Lipschitz conditions on $a(x)$ and $b(x)$ guarantee the existence and the uniqueness of solutions for the Eq. (1.1).

Theorem 4.1. *Suppose $a(x)$ and $b(x)$ are Lipschitz continuous, i.e., there exists a constant $K > 0$ such that*

$$|a(x) - a(y)| + |b(x) - b(y)| \leqq K\, |x - y| \quad \text{for } x, y \in \mathbb{R}^d,$$

and further suppose $a(x)$ is bounded. Then for each $X_0 \in E$ the Eq. (1.1) *has an E-valued solution starting from X_0. Moreover the pathwise uniqueness of solutions holds on E.*

Proof. By virtue of Theorem 3.2 it is sufficient to prove the theorem for the integral Eq. (1.2).

Let $\mathscr{A}$ denote the class of all predictable processes $X_t(u)$ such that for every $\lambda > 0$ and $T > 0$,

$$(4.1) \qquad |||X|||_{\lambda, T}^2 = \sup_{0 \leqq t \leqq T} E\left[\int_{\mathbb{R}} |X_t(u)|^2\, e^{-\lambda |u|}\, du\right] < \infty.$$

We define the mapping Φ on $\mathscr{A}$ by

$$(4.2)\qquad \Phi(X)(t,u)=Y_t^\infty(u;X_0)+\int_0^t\int_{\mathbb{R}} q(t-s;u,v)\,a(X_s(v))\,dB_s(v)\,dv$$
$$+\int_0^t\int_{\mathbb{R}} q(t-s;u,v)\,b(x_s(v))\,ds\,dv$$
$$=Y_t^\infty(u;X_0)+\Phi_2(X)(t,u)+\Phi_3(X)(t,u).$$

Since $a(x)$ is bounded, we have by (3.6),

$$(4.3)\qquad E\,|\Phi_2(X)(t,u)-\Phi_2(X)(t',u')|^{2m}$$
$$\leqq (m(2m-1))^m \sup_x |a(x)|^{2m}(3/\sqrt{2\pi}\,|t-t'|^{1/2}+|u-u'|)^m.$$

Modifying the proof of Kolmogorov's criterion, we can deduce from (4.3) that $\Phi_2(X)$ defines an E-valued continuous process. Next we choose a constant $K'>0$ such that $|b(x)|^2\leqq K'(1+|x|^2)$ for $x\in\mathbb{R}^d$. Then we have

$$(4.4)\quad |\Phi_3(X)(t,u)|^2\leqq \int_0^t\int_{\mathbb{R}} q(t-s;u,v)^2\,e^{\lambda|v|}\,ds\,dv \int_0^t\int_{\mathbb{R}} |b(X_s(v))|^2\,e^{-\lambda|v|}\,ds\,dv$$
$$\leqq \sqrt{t/\pi}\,e^{\lambda^2 t/2}(e^{\lambda u}+e^{-\lambda u})\,K'\int_0^t\Big\{\int_{\mathbb{R}}(1+|X_s(v)|^2)\,e^{-\lambda|v|}\,dv\Big\}\,ds.$$

Hence $\Phi_3(X)$ also defines an E-valued process and so does $\Phi(X)$. Furthermore Φ maps $\mathscr{A}$ into itself.

Let $X^1, X^2\in\mathscr{A}$. We observe

$$E\Big[\int_{\mathbb{R}} |\Phi_2(X^1)(t,u)-\Phi_2(X^2)(t,u)|^2\,e^{-\lambda|u|}\,du\Big]$$
$$=\int_{\mathbb{R}}\Big\{\int_0^t\int_{\mathbb{R}} q(t-s;u,v)^2\,E\,|a(X_s^1(v))-a(X_s^2(v))|^2\,ds\,dv\Big\}e^{-\lambda|u|}\,du$$
$$\leqq \sqrt{2\pi}^{-1}\,e^{\lambda^2 T/2}\,K^2\int_0^t \sqrt{t-s}^{-1}\,E\Big[\int_{\mathbb{R}} |X_s^1(v)-X_s^2(v)|^2\,e^{-\lambda|v|}\,dv\Big]\,ds,$$

and

$$E\Big[\int_{\mathbb{R}} |\Phi_3(X^1)(t,u)-\Phi_3(X^2)(t,u)|^2\,e^{-\lambda|u|}\,du\Big]$$
$$\leqq E\Big[\int_{\mathbb{R}}\Big\{\int_0^t\int_{\mathbb{R}} q(t-s;u,v)\,ds\,dv$$
$$\cdot\int_0^t\int_{\mathbb{R}} q(t-s;u,v)\,|b(X_s^1(v))-b(X_s^2(v))|^2\,ds\,dv\Big\}e^{-\lambda|u|}\,du\Big]$$
$$\leqq T e^{\lambda^2 T/2}\,K^2\int_0^t E\Big[\int_{\mathbb{R}} |X_s^1(v)-X_s^2(v)|^2\,e^{-\lambda|v|}\,dv\Big]\,ds,\qquad 0\leqq t\leqq T,$$

where we used the estimate:

$$\int_{\mathbb{R}} q(t;u,v)\,e^{-\lambda|u|}\,du \leqq \min_{\alpha=\pm\lambda} \int_{\mathbb{R}} q(t;u,v)\,e^{\alpha u}\,du = e^{\lambda^2 t/2}\,e^{-\lambda|v|}.$$

Therefore, if we set $C=2(T^{3/2}+(2\pi)^{-1/2})\,e^{\lambda^2 T/2}\,K^2$, then we have

$$(4.5)\qquad |||\Phi(X^1)-\Phi(X^2)|||^2_{\lambda,t} \leqq C\int_0^t \sqrt{t-s}^{\,-1}\,|||X^1-X^2|||^2_{\lambda,s}\,ds, \qquad 0\leqq t\leqq T.$$

Now we use a successive approximation. Defining X^{n+1} by $\Phi(X^n)$ inductively ($X^0=Y^\infty_\cdot(\cdot\,;X_0)$), we see that there exists a constant $M>0$ such that

$$|||X^{n+1}-X^n|||^2_{\lambda,T} \leqq M\,C^n\,\Gamma(1/2)^n\,\Gamma((n+1)/2)^{-1}\,T^{n/2}.$$

This means that $\{X^n\}$ is a Cauchy sequence in $\mathscr{A}$. Let $X^\infty_t(\cdot)=\lim_{n\to\infty} X^n_t(\cdot)$ (L^2-limit for each t), then $X^\infty_t(\cdot)=\Phi(X^\infty)(t,\cdot)$ a.s. for each t. $\Phi(X^\infty)$ is a desired solution, since $\Phi(X)$ defines an E-valued continuous process for each $X\in\mathscr{A}$.

We can immediately see that X^∞ is a unique solution of the equation $X=\Phi(X)$, $X\in\mathscr{A}$ and $\mathscr{A}$ contains any E-valued solution of (1.2). Thus the pathwise uniqueness of solutions holds for (1.2) on E. □

We can make a step forward in the case of Funaki's string model.

Theorem 4.2. *For each l, $r\in\mathbb{R}$ ($l<r$) and $w\in W$ consider the following stochastic differential equation:*

$$(4.6)\qquad dX_t(u)=a(X_t(u))\,dB_t(u)+b(X_t(u))\,dt+\tfrac{1}{2}\Delta X_t(u)\,dt, \qquad u\in(l,r),$$
$$X_0(u)=w_0(u),\quad u\in(l,r),\qquad X_t(l)=w_t(l),\qquad X_t(r)=w_t(r),\quad t\geqq 0.$$

Suppose $a(x)$ and $b(x)$ are locally Lipschitz continuous, i.e., for every $N>0$ there exists a constant $K_N>0$ such that

$$|a(x)-a(y)|+|b(x)-b(y)|\leqq K_N\,|x-y| \quad \textit{for } x,y\in\{x\in\mathbb{R}^d;\,|x|\leqq N\}.$$

Further suppose $a(x)$ is bounded and there exists a constant $C>0$ such that

$$(4.7)\qquad (b(x)-b(y))\cdot(x-y)\leqq C(|x-y|^2+|x-y|), \qquad x,y\in\mathbb{R}^d.$$

Then the existence and the uniqueness of solutions hold for (4.6).

Proof. It suffices to show the theorem for the following integral equation (cf. Theorem 3.1):

$$(4.8)\qquad X_t(u)=Y^{l,r}_t(u,w)+\int_0^t\int_l^r q^{l,r}(t-s;u,v)\,a(X_s(v))\,dB_s(v)\,dv$$
$$+\int_0^t\int_l^r q^{l,r}(t-s;u,v)\,b(X_s(v))\,ds\,dv$$
$$=Y_t(u)+X_{2,t}(u)+X_{3,t}(u), \qquad u\in[l,r].$$

Using a standard truncation method we see that the above equation has a temporally local solution. Let $\{X_t(u)\}$ be such a solution. We define a sequence of stopping times by $\tau_n = \inf\{t>0; \max_{u\in[l,r]} |X_t(u)|>n\}$ and set $\tau = \lim_{n\to\infty} \tau_n$. The local Lipschitz continuity of $a(x)$ and $b(x)$ implies the pathwise uniqueness of solutions until the explosion time τ.

We will show $\tau=\infty$ a.s. Note the estimate (3.6). Since $a(x)$ is bounded, we see by Fatou's lemma,

$$(4.9)\qquad \begin{aligned} &E[|X_{2,t}(u)-X_{2,t'}(u')|^{2m}; t, t'<\tau] \\ &\leqq (m(2m-1))^m \sup_x |a(x)|^{2m} (3/\sqrt{2\pi}\,|t-t'|^{1/2}+|u-u'|)^m. \end{aligned}$$

Hence, just as in the proof of Kolmogorov's criterion, it follows that

$$(4.10)\qquad \sup_{0\leqq t<\tau\wedge T} \max_{u\in[l,r]} |X_{2,t}(u)|<\infty \quad \text{for any } T>0 \text{ a.s.}$$

Because $b(x)$ is locally Lipschitz continuous, we may assume that $b(X_s(u))$ is jointly Hölder continuous. Then $X_{3,t}(u)$ is continuously differentiable in t and

$$\frac{\partial}{\partial t} X_{3,t}(u) = \tfrac{1}{2}\Delta X_{3,t}(u) + b(X_t(u))$$

(cf. Friedman [1], Chap. 1). Since $X_{3,t}(l)=X_{3,t}(r)=0$, integration by parts yields the following:

$$\begin{aligned} \frac{1}{2m}\frac{d}{dt}\int_l^r |X_{3,t}(u)|^{2m}\,du &= \int_l^r |X_{3,t}(u)|^{2m-2} X_{3,t}(u)\cdot\{\tfrac{1}{2}\Delta X_{3,t}(u)+b(X_t(u))\}\,du \\ &\leqq \int_l^r |X_{3,t}(u)|^{2m-2} X_{3,t}(u)\cdot\{b(X_t(u))-b(X_t(u)-X_{3,t}(u)) \\ &\qquad + b(X_t(u)-X_{3,t}(u))\}\,du \\ &\leqq C\,\|X_{3,t}\|^{2m} + \|X_{3,t}\|^{2m-1}\,\||b(X_t(\cdot)-X_{3,t}(\cdot))|+C\|, \end{aligned}$$

where $\|X\|$ is the $L^{2m}((l,r),\mathbb{R}^d)$-norm of X and we used the condition (4.7) and Hölder's inequality. Therefore,

$$\frac{d}{dt}\|X_{3,t}\| \leqq C\,\|X_{3,t}\| + \||b(X_t(\cdot)-X_{3,t}(\cdot))|+C\|.$$

Noting that $X_{3,0}=0$ and $X=Y+X_2+X_3$ and applying Gronwall's lemma, we have

$$(4.11)\qquad \|X_{3,t}\| \leqq e^{Ct}\int_0^t \||b(Y_s(\cdot)+X_{2,s}(\cdot))|+C\|\,ds$$

and, in particular,

$$(4.12)\qquad \max_{u\in[l,r]} |X_{3,t}(u)| \leqq e^{Ct}\int_0^t (\max_u |b(Y_s(u)+X_{2,s}(u))|+C)\,ds.$$

From (4.10) and (4.12) we obtain $\tau=\infty$ a.s. This completes the proof. □

Remark. (i) Suppose $d=1$ for simplicity. If $b(x)$ can be written as a sum of a nonincreasing function $b_1(x)$, a Lipschitz continuous function $b_2(x)$ and a bounded function $b_3(x)$, i.e., $b(x)=b_1(x)+b_2(x)+b_3(x)$, then $b(x)$ satisfies the condition (4.7).

(ii) The inequalities (4.11) and (4.12) are still valid, if $b(x)$ is just continuous and satisfies (4.7).

§ 5. Existence and Uniqueness – Another Condition

In the previous section we have shown that the boundedness and the Lipschitz continuity of $a(x)$ and $b(x)$ imply the existence and the uniqueness of solutions for the Eq. (1.1). However, as we have mentioned in the introduction, it is not sufficient for the study of stochastic quantization which is our motivation. So that we here give another sufficient condition.

Theorem 5.1. *Suppose $a(x)$ is bounded, $|b(x)|\leqq K(1+|x|^p)$, $x\in\mathbb{R}^d$ for some constants $K>0$, $p>0$ and there exists a constant $C>0$ such that*

$$(b(x)-b(y))\cdot(x-y)\leqq C(|x-y|^2+|x-y|), \qquad x,y\in\mathbb{R}^d. \tag{5.1}$$

Then for each $X\in E$ the Eq. (1.1) *has an E-valued continuous solution starting from X.*

In order to prove Theorem 5.1 first, following Funaki [2], we derive the string model (cf. (4.6)) by a polygonal approximation and then we take a limit $(l,r)\nearrow\mathbb{R}$. In the following we always assume the boundedness of $a(x)$ and the condition (5.1).

For a while we fix two points x_0, $x_1\in\mathbb{R}^d$ and we use $\mathscr{C}$ to denote

$$\{X\in C([0,1],\mathbb{R}^d);\, X(0)=x_0,\, X(1)=x_1\}.$$

Let us introduce a stochastic differential equation on $\mathbb{R}^{d(N-1)}$, $N=2,3,\ldots$.

$$\begin{aligned} dX_t\left(\frac{k}{N}\right) &= \sqrt{N}\, a\left(X_t\left(\frac{k}{N}\right)\right) dB_t(k) + b\left(X_t\left(\frac{k}{N}\right)\right) dt \\ &\quad + \frac{1}{2}\Delta_N X_t\left(\frac{k}{N}\right) dt, \qquad 1\leqq k\leqq N-1, \\ X_t\left(\frac{0}{N}\right) &= x_0, \qquad X_t\left(\frac{N}{N}\right)=x_1, \quad t\geqq 0, \end{aligned} \tag{5.2}$$

where $\{B_t(k)\}$ is a system of independent d-dimensional Brownian motions and

$$\Delta_N X\left(\frac{k}{N}\right)=N^2\left(X\left(\frac{k-1}{N}\right)-2X\left(\frac{k}{N}\right)+X\left(\frac{k+1}{N}\right)\right), \qquad \left\{X\left(\frac{k}{N}\right)\right\}\in\mathbb{R}^{d(N-1)}.$$

Note that the Eq. (5.2) has a non-explosive solution (cf. Theorem 4.2). Let $X\in\mathscr{C}$ and let $\left\{X_t^N\left(\frac{k}{N}\right)\right\}$ be a solution of (5.2) starting from $\left\{X\left(\frac{k}{N}\right)\right\}$. Then a $\mathscr{C}$-valued process is defined by

$$(5.3)\quad X_t^N(u)=X_t^N\left(\frac{k-1}{N}\right)(k-Nu)+X_t^N\left(\frac{k}{N}\right)(Nu-k+1),\qquad u\in\left[\frac{k-1}{N},\frac{k}{N}\right].$$

Lemma 5.1. *Let P_N be the probability law on $C([0,\infty),\mathscr{C})$ induced by $\{X_t^N(u)\}$, $N=2,3,\ldots$, then the family $\{P_N\}$ is tight.*

Proof. Let $q^N(t;k/N,j/N)$ be the fundamental solution of $\partial/\partial t-1/2\Delta_N$ with

$$q^N(t;0/N,j/N)=q^N(t;N/N,j/N)=0$$

and set

$$Y_t^N\left(\frac{k}{N}\right)=\sum_{j=1}^{N-1}q^N\left(t;\frac{k}{N},\frac{j}{N}\right)\left(X\left(\frac{j}{N}\right)-\tilde{X}\left(\frac{j}{N}\right)\right)\frac{1}{N}+\tilde{X}\left(\frac{k}{N}\right),$$

where $\tilde{X}(u)=x_0(1-u)+x_1u$. Then $\left\{X_t^N\left(\frac{k}{N}\right)\right\}$ satisfies the following integral equation:

$$\begin{aligned}(5.4)\quad X_t^N\left(\frac{k}{N}\right)&=Y_t^N\left(\frac{k}{N}\right)+\sum_{j=1}^{N-1}\int_0^t q^N\left(t-s;\frac{k}{N},\frac{j}{N}\right)\sqrt{N}\,a\left(X_s^N\left(\frac{j}{N}\right)\right)dB_s(j)\frac{1}{N}\\&\quad+\sum_{j=1}^{N-1}\int_0^t q^N\left(t-s;\frac{k}{N},\frac{j}{N}\right)b\left(X_s^N\left(\frac{j}{N}\right)\right)ds\frac{1}{N}\\&=Y_t^N\left(\frac{k}{N}\right)+X_{2,t}^N\left(\frac{k}{N}\right)+X_{3,t}^N\left(\frac{k}{N}\right),\qquad 1\leqq k\leqq N-1.\end{aligned}$$

We define $Y_t^N(u)$, $X_{2,t}^N(u)$ and $X_{3,t}^N(u)$ for $u\in[0,1]$ as in (5.3). First we see that $Y_t^N(u)$ converges to the solution of the heat equation with the fixed boundary condition. Just as in the proof of Theorem 4.2, we get

$$\max_{1\leqq k\leqq N-1}\left|X_{3,t}^N\left(\frac{k}{N}\right)\right|\leqq e^{Ct}\int_0^t\left(\max_k\left|b\left(Y_s^N\left(\frac{k}{N}\right)+X_{2,s}^N\left(\frac{k}{N}\right)\right)\right|+C\right)ds.$$

Furthermore an estimate similar to (4.9) holds (cf. Funaki [2]), i.e., for every $m=1,2,\ldots$, there exists a constant $M>0$ such that

$$E|X_{2,t}^N(u)-X_{2,t'}^N(u')|^{2m}\leqq M(|t-t'|^{m/2}+|u-u'|^m),\qquad N\geqq 2.$$

Thus the argument in the proof of Kolmogorov's criterion tells us that for any $T>0$ there exists a sequence $K_n>0$, $n=1,2,\ldots$, such that

$$\lim_{n\to\infty}\inf_{N\geqq 2}P(|X_t^N(u)|\leqq K_n\text{ for }0\leqq t\leqq T\text{ and }u\in[0,1])=1.$$

Therefore we obtain easily the tightness of $\{P_N\}$. □

Let P be any weak limit of $\{P_N\}$. Then following the methods in Stroock and Varadhan's book ([9], Chap. 11), we can show that $P(\theta_0=X,\ \theta_t(0)=X(0),\ \theta_t(1)=X(1),\ t\geqq 0)=1$ and $f(\theta_t)-\int_0^t Lf(\theta_s)\,ds$ is a local martingale relative to

$(P, \{\mathscr{G}_t\})$ for every $f \in \mathbb{D}_{(0,1)}$. In view of Theorem 3.3, this means that for every $l, r \in \mathbb{R}$ $(l<r)$ and $X \in E$ the following stochastic differential equation has a solution.

$$(5.5) \quad dX_t^{l,r}(u) = a(X_t^{l,r}(u))\, dB_t(u) + b(X_t^{l,r}(u))\, dt + \tfrac{1}{2}\Delta X_t^{l,r}(u)\, dt, \qquad u \in (l, r),$$
$$X_0^{l,r}(u) = X(u), \quad u \in (l, r), \qquad X_t^{l,r}(l) = X(l), \qquad X_t^{l,r}(r) = X(r), \quad t \geqq 0.$$

Choose a solution of (5.5) $\{X_t^{l,r}(u)\}$ and extend it to $u \in \mathbb{R}$ such that it coincides with $X(u)$ outside (l, r). Let $P_X^{l,r}$ be its probability law on $C([0, \infty), E)$. To prove the tightness of the family $\{P_X^{l,r}\}$ we further assume that $|b(x)| \leqq K(1+|x|^p)$, $x \in \mathbb{R}^d$. Note that $\{X_t^{l,r}(u)\}$ satisfies the following integral equation:

$$X_t^{l,r}(u) = Y_t^{l,r}(u; \tilde{X}) + \int_0^t \int_l^r q^{l,r}(t-s; u, v)\, a(X_s^{l,r}(v))\, dB_s(v)\, dv$$
$$+ \int_0^t \int_l^r q^{l,r}(t-s; u, v)\, b(X_s^{l,r}(v))\, ds\, dv$$
$$= Y_t^{l,r}(u) + X_{2,t}^{l,r}(u) + X_{3,t}^{l,r}(u), \qquad u \in (l, r),$$

where $\tilde{X} \in C([0, \infty), E)$ is defined by $\theta_t(\tilde{X}) = X$ for $t \geqq 0$.

Lemma 5.2. *For every $T>0$, $\lambda>0$ and $m=1, 2, \ldots,$*

$$(5.6) \qquad \sup_{l, r \in \mathbb{R}, r-l \geqq 1} \; \sup_{0 \leqq t \leqq T} E\left[\int_l^r |X_t^{l,r}(u)|^{2m} e^{-\lambda|u|}\, du\right] < \infty.$$

Proof. Since $X_{2,0}^{l,r} = 0$, (4.9) implies

$$E|X_{2,t}^{l,r}(u)|^{2m} \leqq (m(2m-1))^m \sup_x |a(x)|^{2m} (3\sqrt{t/2\pi})^m.$$

Just as in the proof of Theorem 4.2, we have (though integration by parts yields some extra terms, we can easily deal with them)

$$\frac{1}{2m}\frac{d}{dt}\int_l^r |X_{3,t}^{l,r}(u)|^{2m} e^{-\lambda|u|}\, du \leqq \frac{\lambda^2}{4m}\int_l^r |X_{3,t}^{l,r}(u)|^{2m} e^{-\lambda|u|}\, du$$
$$+ \int_l^r |X_{3,t}^{l,r}(u)|^{2m-2} X_{3,t}^{l,r}(u) \cdot b(X_t^{l,r}(u))\, e^{-\lambda|u|}\, du.$$

Therefore we get

$$\left\{\int_l^r |X_{3,t}^{l,r}(u)|^{2m} e^{-\lambda|u|}\, du\right\}^{1/2m}$$
$$\leqq e^{(C+\lambda^2/4m)t} \int_0^t \left[\int_l^r (|b(Y_s^{l,r}(u) + X_{2,s}^{l,r}(u))| + C)^{2m} e^{-\lambda|u|}\, du\right]^{1/2m} ds.$$

Noting $|b(x)| \leqq K(1+|x|^p)$ and combining with Lemma 3.1, one can complete the proof. □

Lemma 5.3. *For every $i=2, 3$, $T>0$, $\lambda>0$ and $m=1, 2, \ldots$, there is a constant $M>0$ such that for every $l, r \in \mathbb{R}$ $(r-l \geqq 1)$*

$$E|X_{i,t}^{l,r}(u)-X_{i,t'}^{l,r}(u')|^{4m} \leqq M(|t-t'|^{1/2}+|u-u'|)^m e^{\lambda|u|} \tag{5.7}$$

for $0 \leqq t, t' \leqq T$, $l \leqq u, u' \leqq r$: $|u-u'| \leqq 1$.

Proof. The estimate for $i=2$ immediately follows from (4.9). To simplify the notation we set $f(s,v)=q^{l,r}(t-s; u, v)\, 1_{[0,t]}(s) - q^{l,r}(t'-s; u', v)\, 1_{[0,t']}(s)$. Since $|b(x)| \leqq K(1+|x|^p)$, by using Schwarz inequality and Lemma 5.2, we have

$$\begin{aligned} &E|X_{3,t}^{l,r}(u)-X_{3,t'}^{l,r}(u')|^{4m} \\ &\leqq \left(\int_0^{t\vee t'}\int_l^r f(s,v)^2 e^{\lambda|v|}\,ds\,dv\right)^{2m} E\left(\int_0^{t\vee t'}\int_l^r |b(X_s^{l,r}(v))|^2 e^{-\lambda|v|}\,ds\,dv\right)^{2m} \\ &\leqq M'\left(\int_0^{t\vee t'}\int_l^r f(s,v)^2 e^{2\lambda|v|}\,ds\,dv\right)^m \left(\int_0^{t\vee t'}\int_l^r f(s,v)^2\,ds\,dv\right)^m, \qquad 0 \leqq t \leqq T, \end{aligned}$$

where M' is a constant independent of l and r. On the one hand

$$\begin{aligned} &\int_0^{t\vee t'}\int_l^r f(s,v)^2 e^{2\lambda|v|}\,ds\,dv \\ &\leqq \int_0^{t\vee t'}\int_l^r \{q^{l,r}(t-s; u, v)^2\, 1_{[0,t']}(s) \\ &\quad + q^{l,r}(t'-s; u', v)^2\, 1_{[0,t']}(s)\}\, e^{2\lambda|v|}\,ds\,dv \\ &\leqq 2e^{\lambda^2 t}(2t/\pi)^{1/2} e^{2\lambda|u|} + 2e^{\lambda^2 t'}(2t'/\pi)^{1/2} e^{2\lambda|u'|}. \end{aligned}$$

Hence the estimate for $i=3$ follows from (3.5). □

Lemma 5.4. *Let $\{P_N\}$ be a family of probability measures on $C([0,\infty), E)$. Then $\{P_N\}$ is tight if it satisfies the following conditions:*

(i) *there exists a constant $\gamma>0$ such that* $\sup_N \int |\theta_0(0)|^\gamma\, dP_N < \infty$,

(ii) *for each $T>0$ and $\lambda>0$, there exist positive constants α, β and M independent of N such that*

$$\int |\theta_t(u)-\theta_s(v)|^\alpha\, dP_N \leqq M(|t-s|^{2+\beta}+|u-v|^{2+\beta})\, e^{\lambda|u|}$$

for $0 \leqq t, s \leqq T$, $u, v \in \mathbb{R}$: $|u-v| \leqq 1$.

Proof. Note that a relatively compact set A in $C([0,T], E)$ $(T>0)$ is characterized by the following conditions:

(i) for every $K>0$, A is uniformly bounded and equi-continuous on $[0,T]\times[-K,K]$,

(ii) for every $\lambda>0$, $\lim_{K\to\infty} \sup_{w\in A} \sup_{0\leqq t\leqq T, |u|\geqq K} |w_t(u)|\, e^{-\lambda|u|} = 0$.

Hence the lemma is shown by the similar arguments in the proof of Kolmogorov's criterion (cf. [10]). □

We are ready to prove Theorem 5.1.

Proof of Theorem 5.1. A series of Lemmas 5.2 ~ 5.4 and Lemma 3.1 imply the tightness of $\{P_X^{l,r}\}_{r-l\geqq 1}$ on $C([0,\infty), E)$. Clearly any weak limit of $\{P_X^{l,r}\}$ as $(l,r)\nearrow \mathbb{R}$ is a solution of the $(\mathscr{C}, L)$-martingale problem starting from X. Thus we obtain the existence of solutions for the stochastic differential Eq. (1.1) because of Theorem 3.3. □

Finally, we show a uniqueness theorem.

Theorem 5.2. *Suppose $a(x)$ is a constant matrix, $|b(x)|\leqq K(1+|x|^p)$, $x\in\mathbb{R}^d$ for some positive constants K and p, and there exists a constant $C>0$ such that*

$$(5.8)\qquad (b(x)-b(y))\cdot(x-y)\leqq C\,|X-y|^2 \quad \text{for } x, y\in\mathbb{R}^d.$$

Then the pathwise uniqueness of solutions holds for the Eq. (1.1) *on E.*

Proof. Let (X,B) and (X',B') be two solutions of (1.1) on the same probability space with the same reference family such that $X_0=X'_0$ and $B=B'$ a.s. If both X and X' have E-valued continuous sample paths, then Theorem 3.2 tells us that

$$X_t(u)-X'_t(u)=\int_0^t\int_{\mathbb{R}} q(t-s;u,v)(b(X_s(v))-b(X'_s(v))\,ds\,dv.$$

Following the proof of Lemma 5.2 and using the condition (5.8), we have

$$\frac{1}{2}\frac{d}{dt}\int_l^r |Z_t(u)|^2 e^{-\lambda|u|}\,du\leqq (C+\lambda^2/4)\int_l^r |Z_t(u)|^2 e^{-\lambda|u|}\,du + \sum_{u=l,r}|Z_t(u)|\left\{\left|\frac{\partial}{\partial u}Z_t(u)\right|+\frac{\lambda}{4}|Z_t(u)|\right\}e^{-\lambda|u|},$$

where $Z=X-X'$. The boundary terms tend to zero as $(l,r)\nearrow\mathbb{R}$, since for each $T>0$ and $\lambda>0$ we can find a constant $M>0$ such that

$$\int_{\mathbb{R}}\left|\frac{\partial}{\partial u}q(t;u,v)\right|e^{\lambda|v|}\,dv\leqq M t^{-1/2}e^{\lambda|u|},\qquad 0\leqq t\leqq T,\ u\in\mathbb{R}.$$

Therefore

$$\int_{\mathbb{R}}|Z_t(u)|^2 e^{-\lambda|u|}\,du\leqq (2C+\lambda^2/2)\int_0^t\left(\int_{\mathbb{R}}|Z_t(u)|^2 e^{-\lambda|u|}\,du\right)ds.$$

Hence from Gronwall's lemma we obtain $Z_t=0$, i.e., $X=X'$ a.s. □

Because the arguments on the uniqueness of solutions of stochastic differential equations due to Yamada and Watanabe (cf. [3]) are valid for our situation, Theorem 5.1 and 5.2 assert that $(L,\mathbb{D})$ generates uniquely a diffusion process on E under the conditions in Theorem 5.2. In particular let $U(x)$ be a real function on $\mathbb{R}^d$ which can be written as a sum of a convex C^1-function and a C^2-function with compact support (so called double-well potential is described by a function of this type). Then we see that $b(x) = -\frac{1}{2}\operatorname{grad} U(x)$ satisfies the condition (5.8). Thus we have a diffusion process associated with $P(\phi)_1$ model.

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Received November 7, 1985