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Supplements and Corrections to the Paper:

A Classification of the Second Order Degenerate Elliptic Operators and Its Probabilistic Characterization

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The purpose of this note is two-fold. The first, Theorem 1* is an improvement of Theorem 1 in [2]. The second is a correction and an improvement of Theorem 2 in [2], which are divided to Theorem 2* and 2** below.

1. Local Classification

We quote quickly notations and assumptions from [2]. Let $X_1, X_2, \ldots, X_r, Y$ be C^∞-vector fields over a smooth, paracompact manifold M^d of dimension d, and let L be an elliptic-parabolic operator defined by $\frac{1}{2}\sum_{j=1}^{r} X_j^2 + Y$. The operator L is called parabolic at x of M^d, if there exists a local coordinate $(s, x^1, \ldots, x^{d-1})$ in a coordinate neighborhood of x such that X_j, Y are expressed as

$$X_j = \sum_{i=1}^{d-1} \sigma^{ij}(s, x^1, \ldots, x^{d-1}) \frac{\partial}{\partial x^i},$$

$$Y = \sum_{i=1}^{d-1} f^i(s, x^1, \ldots, x^{d-1}) \frac{\partial}{\partial x^i} + \frac{\partial}{\partial s}.$$

If such a coordinate does not exist at any neighborhood of x, L is called elliptic at x.

We denote by $L(X_1, \ldots, X_r, Y)$ the Lie algebra generated by vector fields $X_1, \ldots, X_1, Y$. The ideal in $L(X_1, \ldots, X_r, Y)$ generated by $X_1, \ldots, X_r$ is denoted by L_0. Then L_0 is expressed as

$$L_0 = \left\{ \sum_{i=1}^{d} \lambda_i X_i + Z;\ Z \in L' \right\},$$

where L' is the linear span of $[X,Z], X,Z \in L(X_1,\dots,X_r,Y)$. The projection of the Lie algebra $L(X_1,\dots,X_r,Y)$ to the point x of M^d is denoted by $L(X_1,\dots,X_r,Y)(x)$. $L_0(x)$ is defined similarly. Then $\tilde{L}_0=\{L_0(x);\ x \in M^d\}$ is an involutive differential system. A vector field X is called an element of $\tilde{L}_0$ if $X(x) \in L_0(x)$ holds for all x.

Throughout this note, we assume that $\dim L(X_1,\dots,X_r,Y)(x)=d$ holds for all x of M^d. Then $\dim L_0(x)=d$ or $d-1$. Theorem 1 and Theorem 1′ in [2] are valid without condition (A). In fact, we have

Theorem 1*. *Let M_p^d be the set of all parabolic points of the operator L. Then*

$$M_p^d=\operatorname{int}\{x;\ \dim L_0(x)=d-1\}.$$

Proof. Set $M'=\operatorname{int}\{x;\ \dim L_0(x)=d-1\}$. The relation $M_p^d \subset M'$ is proved in [2, p. 239]. Before the proof of the converse relation, we require a preliminary fact. Let x be any point of M'. Then there exists a neighborhood U of x included in M' and $Z^1,\dots,Z^{d-1}$ of $\tilde{L}_0$ such that every Z of $\tilde{L}_0$ is represented as $Z=\sum_{i=1}^{d-1} f_i Z^i$ in U with C^∞-functions $f_1,\dots,f_{d-1}$ on U. The fact follows immediately from Frobenius' theorem.

Now let $\varphi_t, t \in (-\varepsilon,\varepsilon)$ be one parameter local group of transformations on U that induces Y. We shall prove that the differential $d\varphi_t$ induces the isomorphism of $\tilde{L}_0$ in U for $|t|<\varepsilon$. Let $Z^1,\dots,Z^{d-1}$ be a basis of $\tilde{L}_0$ in U described above. Set $Z_t^j=d\varphi_t Z^j$ $(|t|<\varepsilon)$. Then it holds $\frac{dZ_t^j}{dt}=d\varphi_t[Z^j,Y]$. Since $[Z^j,Y] \in \tilde{L}_0$ holds for all Z^j, these are represented as $[Z^j,Y]=\sum_{i=1}^{d-1} f_i^j Z^i$. Therefore we have a system of linear differential equation

$$\frac{dZ_t^j}{dt}=\sum_{i=1}^{d-1} f_i^j Z_t^i, \qquad j=1,\dots,d-1,\ |t|<\varepsilon.$$

The solution is then written as $Z_t^j=\sum_{i=1}^{d-1} g_i^j(t)\, Z^i, j=1,\dots,d-1$ with regular matrix $(g_i^j(t))$. This proves that $Z_t^j, j=1,\dots,d-1$ is a basis of $\tilde{L}_0$ in U. We have thus shown that $d\varphi_t$ is the isomorphism of $\tilde{L}_0$ in U for $|t|<\varepsilon$.

We can now apply the discussion in the proof of Theorem 1′ to the present case without any change, and we see that all points of M^1 are parabolic. The proof is complete.

2. Global Classification

In case where $X_1,\dots,X_r,Y$ are analytic vector fields on analytic manifold, we have shown in Theorem 2 of [2] that the set M_p^d is empty or the whole space M^d. We shall show this fact in C^∞-case.

Definition (Hermann [1]). The Lie algebra L_0 is called locally finitely generated, if for any x of M^d, there exists an open neighborhood U and a finite subset $Z^1, \dots, Z^n$ of L_0 (depending on U) such that any Z of L_0 is represented as $Z = \sum_{i=1}^{n} f_i Z^i$ on U with C^∞-functions $f_i, i=1, \dots, n$ on U.

Suppose that L_0 is locally finitely generated then L_0 is integrable, i.e., M^d is written as the disjoint union of maximal integrable manifolds of L_0 (Hermann [1]). Also, it holds $d\varphi_t \tilde{L}_0 = \tilde{L}_0$ for small $|t|$ (The proof is similar as in the proof of Theorem 1*). Making use of these facts, we see easily that Lemma 3.4 in [2] is valid in this case. Therefore we have

Theorem 2*. Assume that L_0 is locally finitely generated. Then it holds $M_p^d = \phi$ or M^d. Further, $M_p^d = \phi$ holds if and only if $\dim L_0(x) \equiv d$ for all x.

Lobry [3] shows that L_0 is locally finitely generated if L_0 consists of analytic vector fields. Thus the above theorem is a generalization of the first half of Theorem 2 in [2].

The latter assertion of Theorem 2 is not correct without additional condition. It should be rectified as follow.

Theorem 2.** *Assume that $X_1, X_2, \dots, Y$ are complete vector fields and that $\dim L_0(x) = d-1$ holds for all x. Assume further one of the following two conditions is satisfied.*

(a) *M^d is simply connected.*

(b) *There exists a regular maximal integral manifold of L_0.*

Then, M^d is diffeomorphic to $T \times M^{d-1}$ where $T = (-\infty, \infty)$ or a circle and M^{d-1} is a maximal integral manifold of L_0.

Before the proof. We have to correct Lemma 3.3 in [2] as

Lemma 3.3*. *Let $A = \{\tau: \varphi_\tau(M^{d-1}) = M^{d-1}\}$. If condition* (a) *or* (b) *of Theorem 2** is satisfied, then $A = \{0\}$ or a discrete subgroup isomorphic to integers.*

Proof. If condition (a) is satisfied, the map F; $R \times M^{d-1} \to M^d$ defined by $F(\tau, x) = \varphi_\tau(x)$ is diffeomorphic, since F is a covering projection. Thus $A = \{0\}$.

Suppose next that condition (b) is satisfied but there exists a subsequence $\tau_n (\neq 0)$ in A converging to 0. Let x be a point in M^{d-1} and let $x_n = \varphi_{\tau_n}^{-1}(x)$. Then $x_n \in M^{d-1}$ and $\{x_n\}$ converges to x in the topology of M^d. Since M^{d-1} is a regular submanifold, $\{x_n\}$ converges to x in the topology of M^{d-1}. (By definition of the regular submanifold, the topology of M^{d-1} coincides with that as the subspace of M^d.) Therefore the map F is not one to one on any neighborhood of x. This implies that F is not locally diffeomorphic, condradicting to Lemma 3.1. We have thus seen that $\sigma = \min\{\tau > 0; \tau \in A\} > 0$. The discussion of Lemma 3.3 then proves Lemma 3.3*.

The assertion of Proposition 3.1 is valid under condition (a) or (b) (but without condition (A)). As a consequence, we remark that M^{d-1} is always a regular submanifold, if M^d is simply connected. Further, the condition (b) is necessary for the assertion of Lemma 3.3* and Proposition 3.1*.

Corollary to Proposition 3.1* is stated as

Corollary*. *Suppose that the universal covering space of M^d is compact. Then the operator L is elliptic.*

In fact, if L is parabolic in M^d, $R \times M^{d-1}$ is a covering space of M^d. This contradicts that the universal covering space of M^d is compact.

Now we can see that the proof of Theorem 2 can be applied without essential change to the present Theorem 2**.

Theorem 3′ requires an obvious modification.

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