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Uniformity in Convergence of Measures

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Assume that a net (μ_α) of measures converges in some sense to a measure μ. Then we investigate whether for a given class $\mathscr{E}$ of functions, we can conclude that

$$\lim_\alpha \sup_{f \in \mathscr{E}} \left| \int f \, d\mu_\alpha - \int f \, d\mu \right| = 0.$$

The present paper offers an axiomatic treatment which allows us to extend the "ζ-criterion", until now only available in the weak convergence case, to the case of setwise convergence as well. Actually this was our principal motivation.

The paper is selfcontained and yet due to the development of new techniques there is only little overlap with previous research.

Some concrete results – mainly refinements of known ones – are derived with the present theory at hand. In particular, we characterize those probability distributions on R^n for which the empirical distributions almost surely converge to the theoretical one, uniformly over the class of convex sets.

1. Introduction

Let X be a set, $\mathscr{B}$ a σ-field of subsets of X and $\mathscr{G}$ a $(\emptyset, X, \cup f, \cap f)$ subpaving of $\mathscr{B}$ (i.e. $\mathscr{G}$ contains $\emptyset$ and X, and $\mathscr{G}$ is closed under finite unions and intersections).

All measures considered are finite non-negative measures on $\mathscr{B}$.

For a measure μ and a net $(\mu_\alpha)_{\alpha \in A}$ of measures, we say that μ_α *$\mathscr{G}$-converges* to μ, and we write $\mu_\alpha \to \mu[\mathscr{G}]$, if

$$\mu_\alpha X \to \mu X$$

and

$$\liminf_\alpha \mu_\alpha G \geqq \mu G \quad \text{for all } G \in \mathscr{G}.$$

Introducing the paving

$$\mathscr{F} = C\mathscr{G}$$

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of complementary $\mathscr{G}$-sets, $\mathscr{G}$-convergence is equivalent to convergence of the total mass and

$$\limsup_{\alpha} \mu_\alpha F \leqq \mu F \qquad \text{for all } F \in \mathscr{F}.$$

If $\mathscr{G}$ is a subalgebra of $\mathscr{B}$, $\mathscr{G}$-convergence is equivalent to setwise convergence on $\mathscr{G}$. In particular, $\mathscr{B}$-convergence is the same as setwise convergence on all of $\mathscr{B}$; this form of convergence will also be referred to simply as *setwise convergence.*

If X is a topological space, $\mathscr{B}$ the Borel σ-field and $\mathscr{G}$ the paving of open sets, $\mathscr{G}$-convergence is equivalent to the familiar notion of *weak convergence* (we do not demand the Hausdorff axiom).

Let (x_α) be a net on X and $x \in X$. Then $\mathscr{G}$-convergence of the corresponding point masses: $\varepsilon_{x_\alpha} \to \varepsilon_x [\mathscr{G}]$ is equivalent to convergence of x_α to x in the topology $\tau(\mathscr{G})$. [We use the convention that $\tau(\cdot)$ means "topology generated by".] We point out that the sets in $\tau(\mathscr{G})$ need not all be measurable, for instance, if $\mathscr{G} = \mathscr{B} =$ Borel-sets on $[0, 1]$, then $\tau(\mathscr{G})$ consists of all subsets of $[0, 1]$.

We remark that when μ_α $\mathscr{G}$-converges to μ, we can use the ideas of the standard "Portmanteau theorem" (cf. Theorem 8.1 of [19]) to deduce convergence properties for functions. For example, it follows that $\int f d\mu_\alpha \to \int f d\mu$ for every bounded f which is $\tau(\mathscr{G})$-continuous and $\mathscr{B}$-measurable.

Besides X, $\mathscr{B}$ and $\mathscr{G}$, we assume that there is given a measure μ and a uniformly bounded class $\mathscr{E}$ of real-valued $\mathscr{B}$-measurable functions.

For two measures μ_1 and μ_2, we put

$$\|\mu_1 - \mu_2\|_{\mathscr{E}} = \sup\{|\int f d\mu_1 - f d\mu_2| : f \in \mathscr{E}\}.$$

By ξ we denote the quantity

$$\xi = \sup_{(\mu_\alpha)} \limsup_{\alpha} \|\mu_\alpha - \mu\|_{\mathscr{E}},$$

where the supremum is taken over all nets (μ_α) which $\mathscr{G}$-converges to μ. If desirable, we write $\xi(\mathscr{E}, \mu)$.

If $\xi = 0$, i.e. if $\|\mu_\alpha - \mu\|_{\mathscr{E}} \to 0$ for every net (μ_α) which $\mathscr{G}$-converges to μ, then $\mathscr{E}$ is called a *μ-uniformity class for $\mathscr{G}$-convergence.* If $\mathscr{G} = \mathscr{B}$ we speak of a *μ-uniformity class for setwise convergence.* Clearly, this is the least restrictive notion as any μ-uniformity class for $\mathscr{G}$-convergence is a μ-uniformity class for setwise convergence. If $\mathscr{E}$ is a μ-uniformity class for setwise convergence for every measure μ, $\mathscr{E}$ is called an *ideal uniformity class for setwise convergence.*

In the other important case, that of a topological space and $\mathscr{G}$ the paving of open sets, we say that $\mathscr{E}$ is a *μ-uniformity class for weak convergence* if $\xi = 0$. If $\mathscr{E}$ is a μ-uniformity class for every $\mathscr{E}$-continuous μ, i.e. for every μ such that each $f \in \mathscr{E}$ is continuous almost everywhere w.r.t. μ, then $\mathscr{E}$ is called an *ideal uniformity class for weak convergence.* We mention, that if $\mathscr{E}$ is a μ-uniformity class for weak convergence, then μ is necessarily $\mathscr{E}$-continuous (cf. the remark following the proof of Theorem 7).

The main aim of this paper is to derive necessary and sufficient conditions for ξ to be 0.

Besides ξ, we introduce ξ^+ and ξ^-, mainly to be thought of as auxiliary quantities, defined by

$$\xi^+ = \sup_{(\mu_\alpha)} \limsup_{\alpha} \sup_{f\in\mathscr{E}} \left(\int f\,d\mu_\alpha - \int f\,d\mu\right),$$

$$\xi^- = \sup_{(\mu_\alpha)} \limsup_{\alpha} \sup_{f\in\mathscr{E}} \left(\int f\,d\mu - \int f\,d\mu_\alpha\right),$$

where the supremum is over all (μ_α) which $\mathscr{G}$-converges to μ. Clearly,

$$\xi = \max(\xi^+, \xi^-).$$

Our results depend on certain topological concepts which we now discuss.

Let π be a topology on X. By $\mathscr{U}(\pi)$ we denote the class of upper, and by $\mathscr{L}(\pi)$ the class of lower semi-continuous functions on X w.r.t. π. Thus $f\in\mathscr{U}(\pi)$ if and only if $\{f \geqq t\} \in C\pi$, the paving of closed sets, for every real t. By $\mathscr{U}_+^1(\pi)$ we denote the class of $f\in\mathscr{U}(\pi)$ with $0 \leqq f \leqq 1$.

For an arbitrary (bounded) function f on X, f_π^+ denotes the upper, and f_π^- the lower semi-continuous envelope of f, i.e.

$$f_\pi^+ = \inf\{h\in\mathscr{U}(\pi) \colon h \geqq f\},$$

$$f_\pi^- = \sup\{g\in\mathscr{L}(\pi) \colon g \leqq f\}.$$

Furthermore, we define the *topological oscillation* of f (w.r.t. π) by

$$\partial_\pi(f) = f_\pi^+ - f_\pi^-.$$

We see that $\partial_\pi(f)\in\mathscr{U}(\pi)$. The usual oscillation of f over a set N is the quantity

$$\omega_f(N) = \sup\{|f(x)-f(y)| \colon x\in N, y\in N\}.$$

By standard results on semi-continuous functions we see that the value of the topological oscillation of f at a point x is given by

$$\partial_\pi f(x) = \inf_{N(x)} \omega_f(N(x)),$$

where the infimum is over all neighbourhoods $N(x)$ of x.

Denoting by $\partial_\pi(A)$ the topological boundary w.r.t. π of the subset A, we have

$$\partial_\pi(1_A) = 1_{\partial_\pi(A)}$$

($1.$ denotes the "indicator function of").

Now return to the given objects X, $\mathscr{B}$, $\mathscr{G}$, μ and $\mathscr{E}$. By $\Pi(\mathscr{G})$ we denote the set of all finite topologies contained in $\mathscr{G}$, directed by inclusion. Equivalently, $\Pi(\mathscr{G})$ is the set of all finite $(\emptyset, X, \cup f, \cap f)$ subpavings of $\mathscr{G}$.

We define η, η^+ and η^- by

$$\eta = \inf_{\pi} \sup_{f\in\mathscr{E}} \int \partial_\pi f\,d\mu,$$

$$\eta^+ = \inf_{\pi} \sup_{f\in\mathscr{E}} \int (f_\pi^+ - f)\,d\mu,$$

$$\eta^- = \inf_{\pi} \sup_{f\in\mathscr{E}} \int (f - f_\pi^-)\,d\mu,$$

where all the infima are over $\pi \in \Pi(\mathscr{G})$. We note that these infima may be replaced by limits over the directed set $\Pi(\mathscr{G})$. If desirable, we may write $\eta(\mathscr{E}, \mu)$, $\eta^+(\mathscr{E}, \mu)$ and $\eta^-(\mathscr{E}, \mu)$.

Lastly, we introduce $\zeta = \zeta(\mathscr{E}, \mu)$ by

$$\zeta = \sup_{(f_\pi)} \int \operatorname{essinf} \partial\mu(f_\pi)\, d\mu,$$

where the supremum is over all subfamilies $(f_\pi)_{\pi \in \Pi(\mathscr{G})}$ of $\mathscr{E}$ indexed by $\Pi(\mathscr{G})$, and the integrand is the essential infimum w.r.t. μ of the functions $\partial_\pi(f_\pi)$; $\pi \in \Pi(\mathscr{G})$.

Our basic result is that the conditions $\xi = 0$, $\eta = 0$ and $\zeta = 0$ are equivalent. For a detailed formulation of this result, together with some further information, we refer to Theorem 5. The preparatory theoretical considerations are to be found in Sections 2 and 3. These sections also contain material of independant interest.

Section 4 contains the main theoretical results. Besides Theorem 5 two further results are developed which express the condition for uniformity with reference to a single topology (compare with Theorem 5 which involves all the topologies in $\Pi(\mathscr{G})$). The topological concepts which are needed for this development seem to be new.

In Section 5 we show how previous results, actually refinements of these, on uniformity classes for weak convergence may be obtained from the present ones.

Section 6 points out a generalization of the theory to cases where $\mathscr{E}$ depends on a parameter belonging to a directed set. Naturally, this more general setting could have been studied right from the beginning, but that would result, we believe, in a loss of clarity.

Some applications are given in Section 7. For other possible applications we refer to [2, 11, 16] and [18].

2. The Relation between ξ and η

As explained in the introduction, the main results are expressed in terms of certain finite topologies on X. In our first lemma we summarize some simple but useful facts related to a fixed finite topology.

Let π be a finite topology on X. Denote by $\alpha(\pi)$ the algebra generated by π – this is the Borel field w.r.t. π, if you wish – and denote by $\alpha_0(\pi)$ the set of all atoms in $\alpha(\pi)$. For each $x \in X$ we define:

Δ_x = the atom (in $\alpha_0(\pi)$) containing x,
G_x = the smallest set in π containing x,
F_x = the smallest set in $C\pi$ containing x.

Furthermore, for an atom Δ we define

G_Δ = the smallest set in π containing Δ.

F_x is the closure of $\{x\}$ in the topology π. As G_x is the smallest open set containing x, a net x converges to x in the topology π if and only if $x_\alpha \in G_x$, eventually. If it is not clear which topology we consider, we write $\Delta_{\pi x}$, $G_{\pi x}$ and $F_{\pi x}$.

For a bounded function f on X we have

$$f_\pi^+(x)=\sup\{f(y)\colon\ y\in G_x\},$$
$$f_\pi^-(x)=\inf\{f(y)\colon\ y\in G_x\},$$
$$\partial_\pi f(x)=\omega_f(G_x).$$

We leave the simple proof of the following lemma to the reader.

Lemma 1. *Let π be a finite topology on X. Then one has:*

(i) $\Delta_x=G_x\cap F_x;\ x\in X$,

(ii) $y\in G_x\Rightarrow F_y\supseteq\Delta_x$,

(iii) $G_x=G_\Delta$ for all $x\in\Delta;\ \Delta\in\alpha_0(\pi)$,

(iv) *The atoms $\Delta\in\alpha_0(\pi)$ can be characterized as the maximal sets on which every upper semi-continuous function w.r.t. π is constant.*

For every bounded function f on X we have:

(v) $f_\pi^+=\sum\sup_{G_\Delta} f\cdot 1_\Delta$,

(vi) $f_\pi^-=\sum\inf_{G_\Delta} f\cdot 1_\Delta$,

(vii) $\partial_\pi f=\sum\omega_f(G_\Delta)\cdot 1_\Delta$,

the sums being over all $\Delta\in\alpha_0(\pi)$.

Lemma 2. *For every $\pi\in\Pi(\mathcal{G})$, $\xi^+(\mathcal{U}_+^1(\pi))=0$.*

Proof. We employ the general inequalities

$$\frac{1}{n}\sum_1^n 1_{\{g\geqq \nu/n\}}\leqq g\leqq\frac{1}{n}+\frac{1}{n}\sum_1^n 1_{\{g\geqq \nu/n\}},$$

valid for $0\leqq g\leqq 1$.

Assume that $\mu_\alpha\to\mu[\mathcal{G}]$. To $\varepsilon>0$ choose α_0 such that

$$\mu_\alpha F\leqq\mu F+\varepsilon\qquad\text{for all }\alpha\geqq\alpha_0\text{ and all }F\in C\pi.$$

Also choose n such that $n^{-1}\mu(X)<\varepsilon$.

For any $f\in\mathcal{U}_+^1(\pi)$ and any $\alpha\geqq\alpha_0$ we then have

$$\begin{aligned}\int f\,d\mu_\alpha-\int f\,d\mu&\leqq\frac{1}{n}\mu_\alpha X+\frac{1}{n}\sum_1^n\mu_\alpha(\{f\geqq\nu/n\})-\int f\,d\mu\\&\leqq\frac{1}{n}\mu X+\frac{1}{n}\varepsilon+\frac{1}{n}\sum_1^n\mu(\{f\geqq\nu/n\})+\varepsilon-\int f\,d\mu\\&\leqq 3\varepsilon.\end{aligned}$$

This shows that $\xi^+(\mathcal{U}_+^1(\pi))=0$. □

Proposition 1. $\xi^+=\eta^+$.

Proof. To prove $\xi^+ \leqq \eta^+$ assume, as we may, that all functions in $\mathscr{E}$ lie between 0 and 1. Let $\mu_\alpha \to \mu[\mathscr{G}]$. To $\varepsilon > 0$, choose $\pi \varepsilon \Pi(\mathscr{G})$ such that

$$\int (f_\pi^+ - f)\, d\mu \leqq \eta^+ + \varepsilon \quad \text{for all } f \in \mathscr{E}.$$

Then choose, applying Lemma 2, α_0 such that

$$\int g\, d\mu_\alpha - \int g\, d\mu \leqq \varepsilon \quad \text{for } \alpha \geqq \alpha_0 \text{ and } g \in \mathscr{U}_+^1(\pi).$$

For $\alpha \geqq \alpha_0$ and $f \in \mathscr{E}$ we then have

$$\begin{aligned}\int f\, d\mu_\alpha - \int f\, d\mu &\leqq \left(\int f_\pi^+\, d\mu_\alpha - \int f_\pi^+\, d\mu\right) + \int (f_\pi^+ - f)\, d\mu \\ &\leqq \eta^+ + 2\varepsilon.\end{aligned}$$

Hence $\xi^+ \leqq \eta^+ + 2\varepsilon$. $\xi^+ \leqq \eta^+$ follows.

To prove $\xi^+ \geqq \eta^+$, let t be any number with $t < \eta^+$. We shall prove that $\xi^+ \geqq t$. Fix, for a while, $\pi \in \Pi(\mathscr{G})$. We can find $f \in \mathscr{E}$ with

$$\int f_\pi^+\, d\mu > \int f\, d\mu + t.$$

According to Lemma 1, (v), we can find, to each $\Delta \in \alpha_0(\pi)$, a point $y_\Delta \in G_\Delta$ such that

$$\sum_\Delta f(y_\Delta)\, \mu(\Delta) \geqq \int f\, d\mu + t,$$

the sum being over all $\Delta \in \alpha_0(\pi)$.

Put

$$\mu_\pi = \sum_\Delta \mu(\Delta) \cdot \varepsilon_{y_\Delta},$$

ε_y denoting a unit mass at y. Then:

$$\mu_\pi X = \mu X, \tag{1}$$

$$\int f\, d\mu_\pi - \int f\, d\mu \geqq t, \tag{2}$$

$$\mu_\pi F \leqq \mu F \quad \text{for all } F \in C\pi. \tag{3}$$

(1) and (2) are obvious and (3) follows from Lemma 1, (iii) in the following way:

$$\mu_\pi F = \sum \{\mu(\Delta) : y_\Delta \in F\} = \sum \{\mu(\Delta \cap F) : y_\Delta \in F\} \leqq \mu F.$$

When we now carry out this construction for every $\pi \in \Pi(\mathscr{G})$, and let π run through the directed set $\Pi(\mathscr{G})$, we see by (1) and (3) that $\mu_\pi \to \mu[\mathscr{G}]$. And by (2), where we must now remember that the function occuring there depends on π, it then follows that $\xi^+ \geqq t$. □

Proposition 1 applied to $-\mathscr{E} = \{-f : f \in \mathscr{E}\}$ shows that $\xi^- = \eta^-$. Then $\xi = \max(\eta^+, \eta^-)$ follows. As

$$\max(\eta^+, \eta^-) \leqq \eta \leqq 2 \max(\eta^+, \eta^-),$$

we obtain:

Theorem 1. *$\xi = 0$ if and only if $\eta = 0$.*

In case $\mathscr{G}$ is an algebra, this is equivalent to a result of Stute (Theorem 1.2 of [14]).

It follows from the proof that Theorem 1 remains true if $\mathscr{B}$ is only assumed to be an algebra and if μ is only assumed to be a finitely additive measure.

The proof also shows that if $\mathscr{G}$ is countable, or more generally if $\Pi(\mathscr{G})$ contains a countable cofinal set, then, whenever $\mathscr{E}$ is not a μ-uniformity class for $\mathscr{G}$-convergence, there exists a sequence (μ_n) which $\mathscr{G}$-converges to μ such that $\|\mu_n - \mu\|_{\mathscr{E}}$ does not converge to 0.

Clearly, if there exists $\mathscr{G}^*$, $a(\emptyset, X, \cup f, \cap f)$ subpaving of $\mathscr{G}$ such that $\mathscr{E}$ is a μ-uniformity class for $\mathscr{G}^*$-convergence, then $\mathscr{E}$ is also a μ-uniformity class for $\mathscr{G}$-convergence. As a corollary to the theorem, we see that conversely, if $\mathscr{E}$ is a μ-uniformity class for $\mathscr{G}$-convergence, then there exists a countable $(\emptyset, X, \cup f, \cap f)$ subpaving $\mathscr{G}^*$ of $\mathscr{G}$ such that $\mathscr{E}$ is even a μ-uniformity class for $\mathscr{G}^*$-convergence. Naturally, it is not true that there need exist a countable $(\emptyset, X, \cup f, \cap f)$ subpaving $\mathscr{G}^*$ of $\mathscr{G}$ such that $\mathscr{G}^*$-convergence implies $\mathscr{G}$-convergence.

From the last remarks it follows, as in [16], Theorem 1.3, that if $\mathscr{E}$ is a μ-uniformity class for setwise convergence with μ a probability measure, then for every net $(\mu_{\alpha\omega})$ of random measures defined on some ω-probability space and obeying the strong law of large numbers with limit measure μ (i.e., for every $E \in \mathscr{B}$, $\mu_{\alpha\omega}(E) \to \mu(E)$ a.e. $[\omega]$), the Glivenko-Cantelli assertion

$$\|\mu_{\alpha\omega} - \mu\|_{\mathscr{E}} \to 0 \qquad \text{a.e. } [\omega]$$

holds.

The theorem also implies a converse to this: If $\mathscr{E}$ is not a μ-uniformity class for setwise convergence, then there exists a net of random measures obeying the strong law of large numbers with limit measure μ, such that the Glivenko-Cantelli assertion fails (we may take the ω-probability space as a one-point space).

As a consequence of the above discussion, we have:

Theorem 2. *A necessary and sufficient condition that the Glivenko-Cantelli assertion*

$$\|\mu_{\alpha\omega} - \mu\|_{\mathscr{E}} \to 0 \qquad \text{a.e. } [\omega]$$

holds for every measure μ and for every net $(\mu_{\alpha\omega})$ of random measures obeying the strong law of large numbers with limit measure μ, is that $\mathscr{E}$ be an ideal uniformity class for setwise convergence.

3. The Relation between η and ζ

We shall establish the equality of η and ζ in a setting more general than really required for our uniformity problem. We believe that the result we arrive at is interesting in its own right.

We start by recalling that a set $\mathscr{D}$ of functions between 0 and 1 defined on X is conditionally compact in $[0, 1]^X$ with the uniform topology, if and only if, for every

$\varepsilon > 0$, there exists a finite decomposition of X: $X = \bigcup_1^n \Delta_i$ such that the oscillation of any function in $\mathscr{D}$ over any of the sets Δ_i is at most ε (cf. [3], Theorem IV.5.6.).

The main idea of this section is contained in the following lemma. In loose terms the lemma shows that, under rather special circumstances, it is possible to modify a net of functions which is not downward filtering in a way which compensates for this missing property.

Lemma 3 *(modification lemma). Let $\Pi = (\Pi, \leqq)$ be a directed set, and let, for each $\pi \in \Pi$, $\mathscr{D}_\pi$ be a subset of $[0,1]^X$ such that the following two conditions are satisfied:*

(a) *$\mathscr{D}_\pi$ is conditionally compact for each $\pi \in \Pi$,*

(b) $\pi_1 \leqq \pi_2,\ f_2 \in \mathscr{D}_{\pi_2} \Rightarrow \exists f_1 \in \mathscr{D}_{\pi_1}: f_1 \geqq f_2$.

Then, for every family $(f_\pi)_{\pi\in\Pi}$ with $f_\pi \in \mathscr{D}_\pi$; $\pi \in \Pi$, and any $\varepsilon > 0$, there exists another family $(h_\pi)_{\pi\in\Pi}$ with $h_\pi \in \mathscr{D}_\pi$; $\pi \in \Pi$ such that, for every finite subset $\Pi_0 \subseteq \Pi$, there exists $\pi_0 \in \Pi$ such that $h_\pi \geqq f_{\pi_0} - \varepsilon$ for every $\pi \in \Pi_0$. In short, the assertion can be written:

$$\forall (f_\pi)\ \forall \varepsilon > 0\ \exists (h_\pi)\ \forall \Pi_0 \text{ finite } \exists \pi_0 \in \Pi: \min_{\pi \in \Pi_0} h_\pi \geqq f_{\pi_0} - \varepsilon. \tag{4}$$

Proof. Let $(f_{\pi_\alpha})_{\alpha\in A}$ be a universal subnet of $(f_\pi)_{\pi\in\Pi}$. Fix $\pi\in\Pi$ and $\varepsilon>0$. For each α with $\pi_\alpha \geqq \pi$, let g_α denote a function in $\mathscr{D}_\pi$ with $g_\alpha \geqq f_{\pi_\alpha}$. By (a) there exists $h_\pi \in \mathscr{D}_\pi$ with $\|g_\alpha - h_\pi\| \leqq \varepsilon$, frequently in α (here we consider the uniform norm). $h_\pi \geqq f_{\pi_\alpha} - \varepsilon$, frequently follows, and as (f_{π_α}) is universal, we even have $h_\pi \geqq f_{\pi_\alpha} - \varepsilon$, eventually. Clearly, varying π, (h_π) has the desired property. □

As a special case of Lemma 3 we note the following result: *If $\mathscr{D}_\pi$ is a finite paving on X for each $\pi \in \Pi$, and if, whenever $A_2 \in \mathscr{D}_{\pi_2}$ with $\pi_2 \geqq \pi_1$, there exists $A_1 \in \mathscr{D}_{\pi_1}$ with $A_1 \supseteq A_2$, then, expressed in the same short way as in* (4), *we have:*

$$\forall (A_\pi)\ \exists (B_\pi)\ \forall \Pi_0 \text{ finite } \exists \pi_0 \in \Pi: \bigcap_{\pi\in\Pi_0} B_\pi \supseteq A_{\pi_0}. \tag{5}$$

Probably, this result is equivalent to the axiom of choice.

As a corollary to Lemma 3 we get:

Theorem 3. *Let $(X, \mathscr{B})$ be a set with a σ-field, let $\Pi = (\Pi, \leqq)$ be a directed set and let, for each $\pi\in\Pi$, $\mathscr{D}_\pi$ be a set of measurable functions in $[0,1]^X$. Assume that* (a) *and* (b) *from Lemma* 3 *hold. Then, for every measure μ on $\mathscr{B}$, we have*

$$\inf_{\pi\in\Pi} \sup_{f\in\mathscr{D}_\pi} \int f\, d\mu = \sup_{(f_\pi)} \inf_{\Pi_0} \int \min_{\pi\in\Pi_0} f_\pi\, d\mu, \tag{6}$$

where, on the right hand side, the supremum is over all $(f_\pi)_{\pi\in\Pi}$ with $f_\pi \in \mathscr{D}_\pi$; $\pi\in\Pi$, and the infimum is over all finite subsets Π_0 of Π.

Actually, in this formulation, the result holds even if $\mathscr{B}$ is only a field and μ is only supposed to be a finitely additive measure. Of course, in the σ-additive case, we may write (6) more naturally in the form

$$\inf_{\pi\in\Pi} \sup_{f\in\mathscr{D}_\pi} \int f\, d\mu = \sup_{(f_\pi)} \int \operatorname{essinf}_{\pi} f_\pi\, d\mu. \tag{7}$$

The simplicity in the proof of Lemma 3, and hence also in the proof of Theorem 3, is to some extent due to the appeal to the axiom of choice. We now show how a result can be obtained without using the axiom of choice. The result is less general in as far as we assume that $\Pi = N$ but, on the other hand, we weaken assumption (a) of Lemma 3.

Proposition 2. *Let $(X, \mathscr{B}, \mu)$ be a finite measure space and, for each natural number n, let $\mathscr{D}_n$ be a class of $\mathscr{B}$-measurable functions between 0 and 1. Assume that:*

(a) *For all $n \geqq 1$ and $\varepsilon > 0$ there exists a finite decomposition $X = \bigcup_{i=0}^{k} \Delta_i$, say, of X in sets from $\mathscr{B}$ such that $\mu(\Delta_0) < \varepsilon$ and such that $\omega_f(\Delta_i) < \varepsilon$ for all $i = 1, \ldots, k$ and all $f \in \mathscr{D}_n$.*

(b) $n \leqq m,\ f \in \mathscr{D}_m \Rightarrow \exists g \in \mathscr{D}_n \colon g \geqq f.$

Then for every sequence $(f_n)_{n \geqq 1}$ with $f_n \in \mathscr{D}_n$; $n \geqq 1$, and any $\varepsilon > 0$, there exists another sequence $(h_n)_{n \geqq 1}$ with $h_n \in \mathscr{D}_n$; $n \geqq 1$ such that, for every $n_0 \geqq 1$ there exists $k \geqq 1$ such that

$$\mu(\{\min_{n \leqq n_0} h_n < f_k - \varepsilon\}) < \varepsilon.$$

In short, this assertion can be written:

$$\forall (f_n)\ \forall \varepsilon > 0\ \exists (h_n)\ \forall_{n_0}\ \exists k\colon\ \mu(\{\min_{n \leqq n_0} h_n < f_k - \varepsilon\}) < \varepsilon. \tag{8}$$

In particular, it follows that

$$\inf_n \sup_{f \in \mathscr{D}_n} \int f \, d\mu = \sup_{(f_n)} \int \inf_n f_n \, d\mu, \tag{9}$$

where, on the right hand side, the supremum is over all sequences $(f_n)_{n \geqq 1}$ with $f_n \in \mathscr{D}_n$; $n \geqq 1$.

Proof. To prove (8), let (f_n) and $\varepsilon > 0$ be given. Put $\varepsilon_m = \frac{1}{4} \varepsilon\, 2^{-m}$; $m \geqq 1$. Determine decompositions

$$X = \bigcup_{i=0}^{N_m} \Delta_{mi}; \qquad m \geqq 1$$

as specified by (a) such that, for each $m \geqq 1$,

$$\mu(\Delta_{m0}) < \varepsilon_m; \qquad \omega_f(\Delta_{mi}) < \varepsilon_m \quad \text{for } i = 1, \ldots, N_m \quad \text{and} \quad f \in \mathscr{D}_m.$$

By the Cantor diagonal procedure, we can find a subsequence $(f_{n_k})_{k \geqq 1}$ of $(f_n)_{n \geqq 1}$ such that, for all $m \geqq 1$ and all $i = 1, \ldots, N_m$, the limit

$$\lim_{k \to \infty} \sup_{x \in \Delta_{mi}} f_{n_k}(x)$$

exists.

Fix, for a while, $m \geqq 1$. For all k such that $n_k \geqq m$, let g_k denote a function in $\mathscr{D}_m$ with $g_k \geqq f_{n_k}$. By an elementary argument, which we shall leave to the reader, it can be proved that there exists a function $h_m \in \mathscr{D}_m$, which may in fact be chosen among the function g_k; $k \geqq 1$, such that

$$\sup_{x \in X \setminus \Delta_{m0}} |h_m(x) - g_k(x)| < 2\varepsilon_m, \qquad \text{frequently } [k].$$

It follows that

$$h_m \geqq f_{n_k} - 2\varepsilon_m \quad \text{on } X \setminus \Delta_{m0}, \qquad \text{frequently } [k].$$

Thus for each $i = 1, \dots, N_m$ and for each $x \in \Delta_{mi}$, we have

$$h_m(x) \geqq \sup_{y \in \Delta_{mi}} f_{n_k}(y) - 3\varepsilon_m, \qquad \text{frequently } [k],$$

hence, by the construction of (f_{n_k}), we have

$$h_m(x) \geqq \sup_{y \in \Delta_{mi}} f_{n_k}(y) - 4\varepsilon_m, \qquad \text{eventually } [k].$$

It follows, that

$$h_m \geqq f_{n_k} - 4\varepsilon_m \quad \text{on } X \setminus \Delta_{m0}, \qquad \text{eventually } [k]. \tag{10}$$

Now we vary m and consider $(h_m)_{m \geqq 1}$. By (10) we see that for each n_0, there exists k such that

$$\{\min_{n \leqq n_0} h_n < f_{n_k} - 4\varepsilon_1\} \subseteq \bigcup_{n=1}^{n_0} \Delta_{n0}.$$

By subadditivity of μ, (8) follows.

(9) is a simple consequence of (8). □

It follows from the proof, that (8) holds if μ is only assumed to be subadditive. For instance, we may take $\mathscr{B} = 2^X$ and $\mu A = 1$ for $A \neq \emptyset$, $\mu\emptyset = 0$; thus the result contains a proof of Lemma 3 not using the axiom of choice in case $\Pi = N$.

(9) depends on σ-additivity, but if the right hand side is changed to the form appearing in (6), the result also holds in the finitely additive case.

We leave the more general discussion and return to our problem of uniformity. So once more, we are faced with the objects $(X, \mathscr{B}, \mu)$, $\mathscr{G}$ and $\mathscr{E}$ and the accompanying concepts defined in the introduction. For each $\pi \in \Pi(\mathscr{G})$ consider the class of topological oscillations of functions in $\mathscr{E}$:

$$\mathscr{D}_\pi = \{\partial_\pi(f) : f \in \mathscr{E}\}.$$

By Lemma 1, these classes are conditionally compact so that condition (a) of Lemma 3 is satisfied. Condition (b) of that lemma is obvious since, for $\pi_1 \leqq \pi_2$ and any bounded function f, we have

$$\partial_{\pi_1}(f) \geqq \partial_{\pi_2}(f).$$

As we may assume that $\mathscr{E} \subseteq [0,1]^X$, Theorem 3 is readily applicable and we obtain the following result:

Theorem 4. *Let η and ζ have the meaning as explained in the introduction. Then $\eta = \zeta$.*

We remark that it does not seem ot be so easy to find a quantity that relates in the same way to η^+ as ζ relates to η.

4. Main Results

Let π be a topology on X, f a function on X and $\varepsilon>0$. By the π, ε-*boundary* of f we understand the subset of X defined by

$$\partial_{\pi\varepsilon}(f)=\{\partial_\pi(f)\geqq\varepsilon\}.$$

$\partial_{\pi\varepsilon}(f)$ is closed in the topology π. For an indicator function and an ε with $0<\varepsilon\leqq 1$, we have

$$\partial_{\pi\varepsilon}(1_A)=\partial_\pi(A).$$

If π is a finite topology, we have

$$\partial_{\pi\varepsilon}(f)=\bigcup\{\Delta\in\alpha_0(\pi):\omega_f(G_\Delta)\geqq\varepsilon\}.$$

Theorem 5. *Let X, $\mathscr{B}$, $\mathscr{G}$, $\mathscr{E}$ and μ be as in the introduction. Then the following conditions are equivalent:*

(a) *$\mathscr{E}$ is a μ-uniformity class for $\mathscr{G}$-convergence,*

(b) $\inf_{\pi\in\Pi(\mathscr{G})}\sup_{f\in\mathscr{E}}\int\partial_\pi(f)\,d\mu=0,$

(b′) $\forall\varepsilon>0\colon\ \inf_{\pi\in\Pi(\mathscr{G})}\sup_{f\in\mathscr{E}}\mu(\partial_{\pi\varepsilon}f)=0,$

(c) $\forall(f_\pi)_{\pi\in\Pi(\mathscr{G})}\colon\ \int\operatorname{essinf}\partial_\pi(f_\pi)\,d\mu=0,$

(c′) $\forall(f_\pi)_{\pi\in\Pi(\mathscr{G})}\ \forall\varepsilon>0\colon\ \mu(\operatorname{essinf}\partial_{\pi\varepsilon}(f_\pi))=0.$

In (c) and in (c′) it is understood that $f_\pi\in\mathscr{E}$ for all $\pi\in\Pi(\mathscr{G})$.

Proof. The equivalence of (a), (b) and (c) follows from Theorems 1 and 4. The equivalence of (b) and (b′) is obvious in view of the inequalities

$$\varepsilon\mu(\partial_{\pi\varepsilon}f)\leqq\int\partial_\pi(f)\,d\mu\leqq\omega_f(X)\cdot\mu(\partial_{\pi\varepsilon}f)+\varepsilon\mu(X).$$

The equivalence of (b′) and (c′) follows by a simple application of Theorem 3. □

As usual, changing the form of (c) and (c′) in a way such that essential infima are avoided, it is seen that only finite additivity of μ is essential.

In accordance with one of the remarks made to Theorem 1, we see that if (a) holds then there exists a countable $(\emptyset, X, \bigcup f, \bigcap f)$ subpaving $\mathscr{G}^*$ of $\mathscr{G}$ such that (b), (b′), (c) and (c′) even hold with $\mathscr{G}$ replaced by $\mathscr{G}^*$.

Corollary. *In the setting of Theorem 5, assume that both $\mathscr{E}_1$ and $\mathscr{E}_2$ are μ-uniformity classes for $\mathscr{G}$-convergence. Then*

$$\mathscr{E}_1\cdot\mathscr{E}_2=\{f_1\cdot f_2\colon f_1\in\mathscr{E}_1, f_2\in\mathscr{E}_2\}$$

is also a μ-uniformity class for $\mathscr{G}$-convergence.

Proof. Let K be a constant such that $|f(x)| \leqq K$ for all $x \in X$ and all $f \in \mathscr{E}_1 \cup \mathscr{E}_2$. From the inequality

$$|f_1 \cdot f_2(x) - f_1 \cdot f_2(z)| \leqq K\,|f_1(x) - f_1(z)| + K\,|f_2(x) - f_2(z)|,$$

we get

$$\partial_\pi(f_1 \cdot f_2) \leqq K(\partial_\pi(f_1) + \partial_\pi(f_2)),$$

and the result follows readily from Theorem 5. □

A similar result can be obtained for the class of maxima or minima of a function in $\mathscr{E}_1$ and a function in $\mathscr{E}_2$.

Even though Theorem 5 seems satisfactory, it is possible to develop another general result which sometimes is advantageous. Note that the conditions of Theorem 5 depend on a whole class of topologies related to $\mathscr{G}$. The main feature of the result we now aim at, is that it only invokes one single topology related to $\mathscr{G}$. In contrast to the situation in Theorem 5, this topology will generally not be finite, but should be thought of as approximating the topology $\tau(\mathscr{G})$. That we can not work with $\tau(\mathscr{G})$ itself is partly because, in general, $\tau(\mathscr{G})$ is not contained in $\mathscr{B}$.

We need some purely topological preparations.

Let (X, ε) be a topological space. Let (f_i) be a uniformly bounded net of functions on X. By the *lower limit oscillation* of (f_i) we understand the function h_* defined by

$$h_*(x) = \inf_{N(x)} \liminf_i \omega_{f_i}(N(x)); \qquad x \in X,$$

the infimum being over all τ-neighbourhoods $N(x)$ of x. The *upper limit oscillation* is the function defined by

$$h^*(x) = \inf_{N(x)} \limsup_i \omega_{f_i}(N(x)); \qquad x \in X.$$

If these functions coincide, say $h_* = h^* = h$, then h is called the *limit oscillation* of (f_i) and we write

$$h = \operatorname{limosc}(f_i).$$

If it is desirable to stress the dependance on τ, we talk of the limit oscillation w.r.t. τ.

If $f_i = f$ for all i, the limit oscillation exists and coincides with the previously introduced topological oscillation $\partial_\tau(f)$.

Note that h_* and h^* can also be expressed by the formula

$$h_* = \inf_{\pi \in \Pi(\tau)} \liminf_i \partial_\pi(f_i),$$

$$h^* = \inf_{\pi \in \Pi(\tau)} \limsup_i \partial_\pi(f_i).$$

For a net (A_i) of subsets of X we define the *lower limit boundary* F_* and the *upper limit boundary* F^* by

$$F_* = \{x : \forall_{N(x)} \ (N(x) \cap A_i \neq \emptyset \text{ and } N(x) \cap \complement A_i \neq \emptyset), \text{ eventually in } i\}, \tag{11}$$

$$F^* = \{x : \forall_{N(x)} \ (N(x) \cap A_i \neq \emptyset \text{ and } N(x) \cap \complement A_i \neq \emptyset), \text{ frequently in } i\}. \tag{12}$$

If $F_* = F^* = F$, say, then F is called the *limit boundary* of (A_i) and we write

$F = \lim \mathrm{bd}(A_i)$.

In case $f_i = 1_{A_i}$ for all i, then, with the above notations, $h_* = 1_{F_*}$ and $h^* = 1_{F^*}$. If (X, τ) is locally connected, (11) and (12) reduce to

$$F_* = \{x \colon \forall_{N(x)} N(x) \cap \partial_\tau A_i \neq \emptyset, \text{ eventually in } i\},$$

$$F^* = \{x \colon \forall_{N(x)} N(x) \cap \partial_\tau A_i \neq \emptyset, \text{ frequently in } i\},$$

hence in this case the limit boundary of (A_i) exists if and only if the topological limit of the net of boundaries $(\partial_\tau A_i)$ exists, and when so, the two sets coincide. We note that this simple relation to the notion of topological limit need not hold if (X, τ) is not locally connected.

Lemma 4. *Let (X, τ) be a topological space and let (f_i) be a uniformly bounded net of real valued functions on X. Then:*

(i) *The lower limit oscillation and the upper limit oscillation of (f_i) are both upper semi-continuous.*

(ii) *If (f_i) is an universal net, then the limit oscillation exists.*

(iii) *If τ has a countable base and if (f_i) is a sequence, then the limit oscillation exists for some subsequence of (f_i).*

The simple verification is left to the reader.

Theorem 6. *Let $\mathscr{G}_0$ be a countable $(\emptyset, X, \bigcup f, \bigcap f)$ subpaving of $\mathscr{G}$.*

(i) *If $\int h\,d\mu = 0$ for every h obtainable as the limit oscillation w.r.t. the topology $\tau(\mathscr{G}_0)$ of a sequence of functions from $\mathscr{E}$, then $\mathscr{E}$ is a μ-uniformity class for $\mathscr{G}$-convergence.*

(ii) *If, furthermore, $\tau(\mathscr{G}_0) = \tau(\mathscr{G})$, then the above condition is also necessary for $\mathscr{E}$ to be a μ-uniformity class for $\mathscr{G}$-convergence.*

Proof. First observe that any limit oscillation w.r.t. $\tau(\mathscr{G}_0)$ is $\mathscr{B}$-measurable; this follows by Lemma 4, (i) since $\mathscr{G}_0$ is countable.

(i) Let $G_1, G_2, \ldots$ be an enumeration of the sets in $\mathscr{G}_0$. Denote by π_n the $(\emptyset, X, \bigcup f, \bigcap f)$-closure of $\{G_1, \ldots, G_n\}$. Let (f_n) be any sequence of functions from $\mathscr{E}$ and put

$$g = \inf_n \partial_{\pi_n}(f_n).$$

We shall prove that $\int g\,d\mu = 0$. By Theorem 5, (c) this will imply that $\mathscr{E}$ is a μ-uniformity class for $\mathscr{G}$-convergence.

By Lemma 4, (iii), the limit oscillation w.r.t. $\tau(\mathscr{G}_0)$ exists for some subsequence of (f_n). Assume for notational convenience, that the limit oscillation of (f_n) itself exists. Put $h = \mathrm{limosc}(f_n)$.

We shall prove that $g \leqq h$. To do this, let $x \in X$ and let $N(x)$ be a $\tau(\mathscr{G}_0)$-neighbourhood of x. Choose n_0 such that $x \in G_{n_0} \subseteq N(x)$. Let $n \geqq n_0$ and denote by H_n

the smallest set in π_n containing x. Then $H_n \subseteq N(x)$, hence

$$\omega_{f_n}(N(x)) \geqq \omega_{f_n}(H_n) = \partial_{\pi_n}(f_n)(x) \geqq g(x).$$

It follows that

$$\liminf_n \omega_{f_n}(N(x)) \geqq g(x)$$

and as this holds for every $\tau(\mathscr{G}_0)$-neighbourhood $N(x)$, $h(x) \geqq g(x)$ follows.

Since $g \leqq h$ and $\int h\,d\mu = 0$, $\int g\,d\mu = 0$ follows.

(ii) Now assume that $\tau(\mathscr{G}_0) = \tau(\mathscr{G})$ and that $\mathscr{E}$ is a μ-uniformity class for $\mathscr{G}$-convergence. Let $h = \text{limosc}(f_n)$ with (f_n) a sequence on $\mathscr{E}$. We are to prove that $\int h\,d\mu = 0$.

Given $t > 0$, we choose by Theorem 5, (b), $\pi \in \Pi(\mathscr{G})$ such that

$$\int \partial_\pi f\,d\mu \leqq t \qquad \text{for all } f \in \mathscr{E}.$$

Let Δ_i; $i = 1, \dots, k$ be the atoms in $\alpha_0(\pi)$. Choose, to each i, $x_i \in \Delta_i$ such that

$$h(x_i) \geqq \sup\{h(x) : x \in \Delta_i\} - t.$$

Let $1 \leqq i \leqq k$ and let G_i be the smallest set in π containing x_i. As $\mathscr{G} \subseteq \tau(\mathscr{G}_0)$, G_i is a $\tau(\mathscr{G}_0)$-neighbourhood of x_i, hence

$$\liminf_n \omega_{f_n}(G_i) \geqq h(x_i).$$

It follows that

$$\omega_{f_n}(G_i) \geqq h(x_i) - t, \qquad \text{eventually in } n,$$

hence, for all $x \in \Delta_i$ we have

$$\partial_\pi(f_n)(x) \geqq h(x) - 2t, \qquad \text{eventually in } n.$$

As this holds for all $i = 1, \dots, k$, there exists an n such that

$$\partial_\pi(f_n) \geqq h - 2t.$$

It follows that

$$\int h\,d\mu \leqq t + 2t\,\mu(X).$$

As t was arbitrary, this shows that $\int h\,d\mu = 0$. ☐

In accordance with previous remarks we note, that even if there exists no countable $\mathscr{G}_0$ such that $\tau(\mathscr{G}_0) = \tau(\mathscr{G})$, it is always possible when $\mathscr{E}$ is a μ-uniformity class for $\mathscr{G}$-convergence, to find a countable $\mathscr{G}$ such that $\int h\,d\mu = 0$ for every limit oscillation h w.r.t. $\tau(\mathscr{G}_0)$ of a sequence on $\mathscr{E}$. However, this class $\mathscr{G}_0$ will usually depend both on $\mathscr{E}$ and on μ.

In case $\mathscr{E}$ is a subclass of $\mathscr{B}$, the condition appearing in Theorem 6 is that $\mu(F) = 0$ for every F obtainable as the limit boundary of a sequence of sets from $\mathscr{E}$. A special instance of this result is the following:

Corollary. *Assume that $\tau(\mathscr{G})$ is locally connected and that $\tau(\mathscr{G})$ has a countable base. For a subclass $\mathscr{E} \subseteq \mathscr{B}$ and a measure μ we then have that $\mathscr{E}$ is a μ-uniformity class for $\mathscr{G}$-convergence if and only if, $\mu(F)=0$ for every F obtainable as the topological limit of a sequence of boundaries of sets in $\mathscr{E}$.*

The topological limit here refers to the topology $\tau(\mathscr{G})$ and the boundaries also refer to $\tau(\mathscr{G})$.

The proof is easily carried out by first noticing that there exists a countable base $\mathscr{G}_0$ contained in $\mathscr{G}$ (cf. Theorem 1.1.7 of [5]).

The result was first proved in a special case in [17] (statement immediately preceeding Theorem 5).

We note that, since $\tau(\mathscr{G})$ is assumed to have a countable base, $\mathscr{G}$-convergence coincides with weak convergence of measures w.r.t. $\tau(\mathscr{G})$.

If the measure μ has extra smootness properties, it is possible to avoid the countability assumptions of Theorem 6. Even though it is possible to formulate a general result (by expressing the conditions in terms of an extension of the measure, cf. Theorem 5.1 of [19]), we restrict ourselves to the weak convergence case. For the notion of τ-smooth measures, see [19].

Theorem 7. *Let $\mathscr{G}$ be a topology on X, $\mathscr{E}$ a uniformly bounded class of Borel-measurable functions and μ a τ-smooth measure.*

Then $\mathscr{E}$ is a μ-uniformity class for weak convergence if and only if $\int h\,d\mu=0$ for every function h obtainable as the limit oscillation of a net of functions from $\mathscr{E}$.

Proof. As the ideas in this proof are more or less the same as those employed in the proof of Theorem 6, we only give an outline.

For the "if" part we employ (c) of Theorem 5, noting that due to the τ-smoothness of μ,

$$\int \operatorname{essinf} \partial_\pi(f_\pi)\,d\mu = \int \inf \partial_\pi(f_\pi)\,d\mu.$$

The essential step is then the proof of the inequality

$$\inf_\pi \partial_\pi(f_\pi) \leqq \operatorname{limosc}_\alpha(f_{\pi_\alpha}),$$

where (f_{π_α}) is a subnet of (f_π) for which the limit oscillation exists. Briefly, this can be proved as follows: To $x \in X$ and $N(x)$, an open neighbourhood of x, choose α_0 such that $N(x) \in \pi_\alpha$ for all $\alpha \geqq \alpha_0$. Then, for $\alpha \geqq \alpha_0$, the oscillation of f_{π_α} over $N(x)$ is $\geqq \inf \partial_\pi(f_\pi)(x)$. This implies the desired result.

The proof of the "only if" part follows closely the corresponding part of the proof of Theorem 6. □

Note that τ-smoothness of μ is not required for the proof of necessity. We show by an example that τ-smoothness of μ is essential for sufficiency.

Example. Let Ω be the first uncountable ordinal and consider $[0, \Omega]$ in the order topology. $[0, \Omega]$ is a compact Hausdorff space. Let $\mathscr{B}$ be the paving of all $A \subseteq [0, \Omega]$ such that either A contains a closed unbounded subset of $[0, \Omega] \setminus \{\Omega\}$ or else the complement of A has this property. We point out that closed here refers to the subspace $[0, \Omega] \setminus \{\Omega\}$. According to [9], $\mathscr{B}$ is the Borel σ-field on $[0, \Omega]$. Let μ

denote the measure on $\mathscr{B}$ defined by

$$\mu(A)=\begin{cases}1 & \text{if } A \text{ contains a closed unbounded subset of } [0,\Omega]\setminus\{\Omega\}\\ 0 & \text{otherwise.}\end{cases}$$

Then μ is a measure which is not τ-smooth. According to [8], this observation is due to Dieudonné. We refer to [8], 52.10 or to [9].

Denote by $\mathscr{E}$ the class of all subsets $[\alpha,\Omega]$ with $\alpha\in[0,\Omega]$. It is easy to see that the net $(\varepsilon_\alpha)_{\alpha\in[0,\Omega[}$ of point masses converges weakly to μ (it also converges to ε_Ω). Hence $\mathscr{E}$ is not a μ-uniformity class for weak convergence.

On the other hand, the condition of Theorem 7 is satisfied since, if $F=\operatorname{limbd}(E_i)$ with all the E_i in $\mathscr{E}$, then either $F=\{\Omega\}$ or F avoids an entire neighbourhood of Ω, hence $\mu F=0$. □

Naturally, Theorem 7 admits a corollary analogous to the corollary to Theorem 6:

Corollary. *For a locally connected topological space, and a τ-smooth measure μ, a class $\mathscr{E}$ of Borel sets is a μ-uniformity class for weak convergence if and only if, $\mu F=0$ for every set F obtainable as the topological limit of a net of boundaries of sets in $\mathscr{E}$.*

5. Uniformity Classes in Uniform Spaces

In previous research ([2] and [19]), conditions for uniformity in weak convergence were developed in metrizable or uniformizable (=completely regular) spaces; these conditions were expressed in terms of a metric or a uniformity. Even though Theorems 5 and 7 are applicable in completely general topological spaces, it is of interest to see how the older results – actually slight refinements of these results – may be deduced from the present results.

Let X be a completely regular topological space (not necessarily Hausdorff) provided with its Borel field. Consider a fixed uniformity on X and denote by $\mathscr{U}$ the class of open and symmetric members of the uniformity. For $U\in\mathscr{U}$ and f a bounded measurable function, we define the *U-oscillation* of f as the function

$$\partial_U(f)(x)=\omega_f(U[x]);\qquad x\in X.$$

For $\varepsilon>0$, the *U,ε-boundary* of f is the set

$$\partial_{U,\varepsilon}(f)=\{x:\omega_f(U[x])>\varepsilon\}=\{\partial_U f>\varepsilon\}.$$

All these sets are open, hence $\partial_U f$ is lower semi-continuous.

Theorem 8. *Let $\mathscr{U}$ be the open and symmetric members of a uniformity on X, let $\mathscr{E}$ be a uniformly bounded class of Borel measurable functions, and let μ be a τ-smooth measure on X. Then the following conditions are equivalent:*

(a) *$\mathscr{E}$ is a μ-uniformity cflass for weak convergence;*

(b) $\inf\limits_{U\in\mathscr{U}}\sup\limits_{f\in\mathscr{E}}\int\partial_U f\,d\mu=0;$

(b′) $\forall\varepsilon>0\colon\ \inf\limits_{U\in\mathscr{U}}\sup\limits_{f\in\mathscr{E}}\mu(\partial_{U,\varepsilon}f)=0;$

(c) $\exists \mathcal{U}_0$ *countable* $\forall (f_U)_{U\in\mathcal{U}_0}$: $\int \inf_{U\in\mathcal{U}_0} \partial_U(f_U)\,d\mu=0$;

(c′) $\exists \mathcal{U}_0$ *countable* $\forall (f_U)_{U\in\mathcal{U}_0}\,\forall \varepsilon>0$: $\mu(\bigcap_{U\in\mathcal{U}_0} \partial_{U,\varepsilon}(f_U))=0$.

In (c) and in (c′) it is understood that $\mathcal{U}_0\subseteq\mathcal{U}$ and that $f_U\in\mathscr{E}$ for every $U\in\mathcal{U}_0$.

Proof. (a) $\Rightarrow$ (b): given $t>0$, choose by Theorem 5, (b) $\pi\in\Pi(\mathscr{G})$, with $\mathscr{G}$ the paving of open sets, such that

$$\int \partial_\pi f\,d\mu\leqq t \qquad \text{for all } f\in\mathscr{E}.$$

Since, for any $G\in\mathscr{G}$, the family of all $G^*\in\mathscr{G}$ for which $U[G^*]\subseteq G$ holds for some $U\in\mathcal{U}$ is upward filtering with union G, we can, by τ-smoothness of μ, find $U\in\mathcal{U}$ and to each $G\in\pi$ a set $G^*\in\mathscr{G}$ such that

$$U[G^*]\subseteq G \qquad \text{for all } G\in\pi,$$

$$\mu(\bigcup\{G\setminus G^*\colon G\in\pi\})<t.$$

We claim that for any f:

$$\partial_U f\leqq\partial_\pi f \text{ on the set } X\setminus\bigcup\{G\setminus G^*\colon G\in\pi\}.$$

Clearly, if we can prove this, (b) will follow. Assume then that x lies in non of the sets $G\setminus G^*$; $G\in\pi$. Let G be smallest set in π containing x. Then, as $x\in G^*$:

$$\partial_U f(x)=\omega_f(U[x])\leqq\omega_f(U[G^*])\leqq\omega_f(G)=\partial_\pi f(x),$$

as desired.

Clearly, (b) $\Rightarrow$ (b′) $\Rightarrow$ (c′) $\Rightarrow$ (c).

(c) $\Rightarrow$ (a): Assume that $h=\operatorname{limosc}(f_i)$ with $f_i\in\mathscr{E}$ for each i. By Theorem 7, it suffices to show that $\int h\,d\mu=0$. Two observations are essential for the proof we shall give of this:

(i) For every $U\in\mathcal{U}$ and every $t>0$, there exists a finite set $\Lambda\subseteq X$ such that $\mu(X\setminus U[\Lambda])<t$.

(ii) For any $U\in\mathcal{U}$, any $U^*\in\mathcal{U}$ with $U^*\circ U^*\subseteq U$, any finite set $\Lambda\subseteq X$, and for any $t>0$ there exists $f\in\mathscr{E}$ such that $h\leqq\partial_U f+t$ on the set $U^*[\Lambda]$.

(i) is obvious by τ-smoothness of μ.

To prove (ii), f is chosen among the functions f_i such that

$$\liminf \omega_{f_i}(U^*[x])\leqq\omega_f(U^*[x])+t \qquad \text{for all } x\in\Lambda.$$

Then, if $y\in U^*[x]$ with $x\in\Lambda$, $U^*[x]\subseteq U[y]$ and we have

$$h(y)\leqq\liminf \omega_{f_i}(U^*[x])\leqq\omega_f(U^*[x])+t\leqq\partial_U f(y)+t,$$

as desired.

Employing (i) and (ii) together with an "$\varepsilon 2^{-n}$-argument", it is easy to deduce that $\int h\,d\mu=0$ from (c). □

One may ask if the conditions

(d) $\forall (f_U)_{U\in\mathscr{U}}: \int \operatorname{ess\,inf} \partial_U(f_U)\, d\mu = 0$;

(d′) $\forall (f_U)_{U\in\mathscr{U}}\ \forall \varepsilon > 0: \mu(\operatorname{ess\,inf} \partial_{U,\varepsilon}(f_U)) = 0$

are necessary and sufficient for $\mathscr{E}$ to be a μ-uniformity class for weak convergence. Note that (d) and (d′) are obtained from (c) and (c′) by reversing the order of the existance and the all-quantor. Necessity of (d) and (d′) is obvious. It is not difficult to prove, using the ideas of the above proof, that a strengthening of (i) from that proof implies sufficiency:

Proposition 3. *In the setting of Theorem* 8, *assume that there exists, to each* $t < \mu(X)$, *a family* $(\Lambda_U)_{U\in\mathscr{U}}$ *of finite subsets of* X *such that*

$$\mu(\operatorname{ess\,inf} U[\Lambda_U]) > t. \tag{13}$$

Then (d) *and* (d′) *are both necessary and sufficient for* $\mathscr{E}$ *to be a* μ*-uniformity class for weak convergence.*

The condition of this result covers the case where $\mathscr{U}$ has a countable base, for instance $\mathscr{U}$ could be induced by a metric, and also the case when, to each $t < \mu X$, there exists a compact and measurable set K such that $\mu K \geqq t$.

We note that the essential infima occuring in (d), (d′) and in (13) may be replaced by ordinary infima (intersections) by changing the functions to their upper semi-continuous envelopes and the sets to their closures.

Example. Let X be the half-open interval $[0, 1)$ and let λ be Lebesgue measure on X. Provide X with the Sorgenfrey topology, i.e. the topology having all intervals of the form $[a, b)$ as a base, cf. [5]. Then X is a Lindelöf space, and λ is τ-smooth. Let $\mathscr{U}$ denote the class of all neighbourhoods of the diagonal in $X \times X$ and let $\mathscr{U}^*$ be the subclass of $\mathscr{U}$ consisting of all sets of the form

$$\bigcup \{J \times J : J \in \mathscr{J}\}$$

where $\mathscr{J}$ is a (necessarily countable) class of pairwise disjoint intervals of the form $[\cdot, \cdot)$ covering X. From the fact that any open covering of X has a refinement $\mathscr{J}$ of pairwise disjoint intervals of the form $[\cdot, \cdot)$, it follows that any set in $\mathscr{U}$ contains a set in $\mathscr{U}^*$. This implies that $\mathscr{U}$ is a uniformity on X, indeed it is the finest uniformity on X compatible with the given topology.

Let $(\Lambda_U)_{U\in\mathscr{U}^*}$ be any family of finite subsets of X indexed by $U \in \mathscr{U}^*$. We claim that

$$F = \bigcap \{U[\Lambda_U] : U \in \mathscr{U}^*\}$$

is countable. We show this by proving that for any $x \in F$ with $x > 0$, there exists $y < x$ such that the interval (y, x) is disjoint with F. To see this, let $(y_n)_{n\geqq 1}$ be chosen so that $0 = y_1 < y_2 < \cdots$ and $x - n^{-1} < y_n < x$; $n \geqq 1$ hold. Let $\mathscr{J}$ be the covering of X consisting of all the intervals $[y_n, y_{n+1})$; $n \geqq 1$ and the interval $[x, 1]$, and denote by U the union of the sets $J \times J$ with $J \in \mathscr{F}$. Then $U \in \mathscr{U}^*$. Since Λ_U is finite, there exists n such that $[y_n, x)$ is disjoint with Λ_U. Then $[y_n, x)$ is also disjoint with $U[\Lambda_U]$, hence also with F.

As F is countable, $\lambda(F)=0$, and as all the sets $U[A_U]$; $U\in\mathscr{U}^*$ are closed, we have, by τ-smoothness of λ, that

$$\lambda(\operatorname*{essinf}_{U\in\mathscr{U}^*} U[A_U])=0.$$

These considerations show that the τ-smooth measure λ does not satisfy the condition of Proposition 3.

If we denote by $\mathscr{E}$ the class of all finite subsets of X, it follows that (d) is satisfied, but clearly, $\mathscr{E}$ is not a λ-uniformity class for weak convergence. Actually, this shows that a claim made in the notes and remarks to Sections 12–14 of [19], concerning a generalisation from Radon measures to τ-smooth measures, is unjustified. □

6. A Refinement to Classes of Functions Depending on a Parameter

Let X, $\mathscr{B}$, $\mathscr{G}$ and μ be as in the introduction. Let $I=(I, \leqq)$ be a directed set and let, for each $i\in I$, $\mathscr{E}_i$ be a non-empty class of real-valued $\mathscr{B}$-measurable functions between 0 and 1.

We say that $(\mathscr{E}_i)$ is a *μ-uniformity system for $\mathscr{G}$-convergence* if

$$\lim_{\alpha} \sup_{f\in\mathscr{E}_{i_\alpha}} |\int f\,d\mu_\alpha-\int f\,d\mu|=0$$

for every net $(\mu_\alpha, i_\alpha)_{\alpha\in D}$ with $\mu_\alpha\to\mu[\mathscr{G}]$ and $(i_\alpha)_{\alpha\in D}$ a subnet of I. When we here speak of I as a net, we have the identity map $I\to I$ in mind with the domain directed by $\leqq$.

Theorem 9. *The following are equivalent:*

(a) *$(\mathscr{E}_i)$ is a μ-uniformity system for $\mathscr{G}$-convergence,*

(b) $\inf_{\pi\in\Pi(\mathscr{G})} \limsup_{i} \sup_{f\in\mathscr{E}_i} \int \partial_\pi f\,d\mu=0,$

(c) $\forall(\pi_\alpha, i_\alpha, f_\alpha)_{\alpha\in D}$: $\int \operatorname*{ess\,inf}_{\alpha\in D} \partial_{\pi_\alpha}(f_\alpha)\,d\mu=0.$

In (c) it is understood that $(\pi_\alpha, i_\alpha, f_\alpha)_{\alpha\in D}$ is a net such that $(\pi_\alpha, i_\alpha)_{\alpha\in D}$ is a subnet of $\Pi(\mathscr{G})\times I$ (i.e., for every (π, i), $(\pi_\alpha, i_\alpha)\geqq(\pi, i)$, eventually in α) and that $f_\alpha\in\mathscr{E}_{i_\alpha}$ for all $\alpha\in D$.

We note that (b) is equivalent to the condition

$$\lim_{(\pi,i)} \sup_{f\in\mathscr{E}_i} \int \partial_\pi f\,d\mu=0,$$

the limit being over the directed set $\Pi(\mathscr{G})\times I$.

The proof of Theorem 9 is easily carried out by generalizing the proof of Theorem 5 in a rather straight forward manner. Perhaps it is not entirely clear how the modification lemma, Lemma 3 should be generalized, so let us discuss that, and leave all the remaining details of the proof to the reader.

To generalize the modification lemma, let Π and I be directed sets, and for each (π, i) let $\mathscr{D}_{\pi,i}$ be a class of functions between 0 and 1. (We shall need this with $\mathscr{D}_{\pi,i}$ the class of all $\partial_\pi(f)$ with $f\in\mathscr{E}_i$.) Assume that, for every i and $\pi_1\leqq\pi_2$, the implication

$$f_2\in\mathscr{D}_{\pi_2,i}\Rightarrow f_1\geqq f_2 \quad \text{for some } f_1\in\mathscr{D}_{\pi_1,i}$$

holds, and that, for every π, $\bigcup\{\mathscr{D}_{\pi,i}: i\in I\}$ is conditionally compact.

Then, for every net $(\pi_\alpha, i_\alpha, f_\alpha)_{\alpha\in D}$ with $(\pi_\alpha, i_\alpha)_{\alpha\in D}$ a subnet of $\Pi\times I$ and with $f_\alpha\in\mathscr{D}_{\pi_\alpha,i_\alpha}$ for all $\alpha\in D$, and for every $\varepsilon>0$, there exists a "modification" $(\pi_\alpha^*, i_\alpha^*, f_\alpha^*)_{\alpha\in D}$ of the same type as the given net such that, for every finite subset $D_0\subseteq D$ there exists $\alpha_0\in D$ such that

$$\min_{\alpha\in D_0} f_\alpha^*\geqq f_{\alpha_0}-\varepsilon.$$

It is not difficult to establish this generalized modification lemma.

It follows from Theorem 9, that if $(\mathscr{E}_i)$ is a μ-uniformity system for $\mathscr{G}$-convergence, there exists $\mathscr{G}_0$, a countable $(\emptyset, X, \bigcup f, \bigcap f)$ subpaving of $\mathscr{G}$ such that $(\mathscr{E}_i)$ is even a μ-uniformity system for $\mathscr{G}_0$-convergence.

Theorem 10. *Let $\mathscr{G}$ be a topology on X, $(\mathscr{E}_i)_{i\in I}$ a net of classes of Borel-measurable functions between 0 and 1, and μ a τ-smooth measure. Then $(\mathscr{E}_i)$ is a μ-uniformity system for weak convergence if and only if, for every net $(i_\alpha, f_\alpha)_{\alpha\in D}$ with (i_α) a subnet of I and with $f_\alpha\in\mathscr{E}_{i_\alpha}$ for all $\alpha\in D$, and for which* limosc (f_α) *exists, we have*

$$\int \operatorname{limosc}(f_\alpha)\,d\mu=0.$$

For necessity, τ-smoothness of μ is not needed.

With a little care, it is also possible to generalize Theorem 8. However, we shall leave this to the interested reader.

7. Applications

The first result of this section arises when investigating the conditions for μ-uniformity to hold for every measure μ. It is necessary, at least convenient, with topological assumptions, and we only formulate a result for weak convergence.

Let $(X,\mathscr{G})$ be a topological space and let $(\mathscr{E}_i)_{i\in I}$ be a family of classes of Borel-measurable functions between 0 and 1, indexed by a directed set I. The family $(\mathscr{E}_i)$ is said to be *equicontinuous in the limit at the point* x if, to every $\varepsilon>0$ there exists a neighbourhood $N(x)$ of x and an index $i_0\in I$ such that, for any $i\geqq i_0$, any $f\in\mathscr{E}_i$ and any $y\in N(x)$, $|fx-fy|<\varepsilon$ holds. If this condition is satisfied for all $x\in X$, $(\mathscr{E}_i)$ is *everywhere equicontinuous in the limit.* If I is a one-point set, these concepts reduce to the usual concepts of equicontinuity.

Note that $(\mathscr{E}_i)$ is equicontinuous in the limit at the point x if and only if, for every net $(x_\alpha, i_\alpha)_{\alpha\in D}$ such that (i_α) is a subnet of I, and $x_\alpha\to x$, we have

$$\lim_\alpha \sup_{f\in\mathscr{E}_{i_\alpha}} |f(x_\alpha)-f(x)|=0.$$

We introduce the set of non-equicontinuity:

$$D((\mathscr{E}_i))=\{x: (\mathscr{E}_i) \text{ is not equicontinuous in the limit at } x\}$$

and also, we introduce a certain function, which could be called the *index of equicontinuity*, by

$$\partial((\mathscr{E}_i)) = \inf_{\pi\in\Pi(\mathscr{G})} \limsup_{i} \sup_{f\in\mathscr{E}_i} \partial_\pi f.$$

Our notation is slightly unprecise, in that the dependance on the topology is supressed.

It is straight forward to check that the sets

$$\{x: \partial((\mathscr{E}_i))(x) \geqq t\}$$

are all closed, hence $\partial((\mathscr{E}_i))$ is always upper semi-continuous. As

$$D((\mathscr{E}_i)) = \{\partial((\mathscr{E}_i)) > 0\},$$

$D((\mathscr{E}_i))$ is always measurable, indeed, it is always an F_σ-set.

Theorem 11. *Let the topological space X and the family* $(\mathscr{E}_i)_{i\in I}$ *be given.*

(i) *If* μ *is a* τ*-smooth measure and if* $(\mathscr{E}_i)$ *is* μ*-almost everywhere equicontinuous in the limit, then* $(\mathscr{E}_i)$ *is a* μ*-uniformity system for weak convergence.*

(ii) *A necessary and sufficient condition that* $(\mathscr{E}_i)$ *is a* μ*-uniformity system for weak convergence for every* τ*-smooth measure, is that* $(\mathscr{E}_i)$ *is everywhere equicontinuous in the limit.*

Proof. (i) We know that $\mu(D(\mathscr{E}_i))=0$. It follows that

$$\int \partial(\mathscr{E}_i)\, d\mu = 0.$$

Consider any net $(\pi_\alpha, i_\alpha, f_\alpha)$ as appearing in (c) of Theorem 9. We have

$$\int \operatorname{essinf} \partial_{\pi_\alpha}(f_\alpha)\, d\mu = \int \inf \partial_{\pi_\alpha}(f_\alpha)\, d\mu \leqq \int \partial(\mathscr{E}_i)\, d\mu = 0,$$

and the desired result follows from Theorem 9.

(ii) Sufficiency follows by (i) and necessity is established rather easily by consideration of point masses. □

The proof could equally well have been based on Theorem 10.

We have been unable to decide if the result holds without the restriction to τ-smooth measures. For necessity in (ii) no such restriction is needed. As is well known, this restriction is also superfluous if I consists of one element and $\mathscr{E}_i$ of one function (Theorem 8.1, (vii) of [19] – a proof based on the implication (b) ⇒ (a) of Theorem 5 can also be carried out).

In the present generality the result is new. If I contains just one element, it is well known under various restrictions, cf. [11, 2, 6].

Theorem 11 can be applied to a study of joint continuity of the map $(\mu, f) \to \int f\, d\mu$ and also to a study of preservation of weak convergence under mappings. Since one only obtains slight refinements and variants of previous results ([19], sections 16 and 17), we shall not embark on a discussion of this.

Another possible application of Theorem 11 is to establish the joint cintinuity of the formation of product measures $(\mu, \eta) \to \mu \otimes \eta$. Based on Theorem 11, this can be carried out for completely regular spaces and τ-smooth measures. This result, for Radon measures, has been announced previously by the author on various occasions. The reason why we do not give the details here is that quite recently, P. Ressel has proved a more general result by a nice direct argument, cf. [12].

We shall derive some uniformity results for classes of sets. We mainly work in an euclidean space.

Lemma 5. *Let $\mathscr{D}$ be one of the following classes of closed subsets of an euclidean space:* 1°. *The class of all hyperplanes.* 2°. *The class of all boundaries of convex sets.* 3°. *The class consisting of all hyperplanes and all euclidean spheres.*

Then, if Δ is the topological limit of any sequence (or net) of sets from $\mathscr{D}$, Δ is a subset of a set in $\mathscr{D}$.

Actually, Δ is even a member of $\mathscr{D}$, so that $\mathscr{D}$ is closed in the notion of topological limit, but this is of no significance for our applications.

Since the proof is straight forward, and since most of it is contained in [17] and [18], we leave the details to the reader. However, we do want to point out that with the general results of section 4 at hand, all one has to do in order to prove concrete uniformity results for classes of sets is to establish closure properties of the type appearing in the lemma. Therefore, the lemma is very essential. Observe that it is a pleasant, and somewhat surprising feature of our theory, that closure properties suffice – compactness is not needed.

Theorem 12. *Let $\mathscr{E}$ be one of the following classes of subsets of an euclidean space:* 1°. *The class of closed halfspaces;* 2°. *The class of measurable convex sets;* 3°. *The class of closed halfspaces and closed euclidean balls. Then $\mathscr{E}$ is an ideal uniformity class for weak convergence.*

Proof. Apply the corollary to Theorem 6 or the corollary to Theorem 7 in connection with Lemma 5. □

Combining with the corollary to Theorem 5, we see that for each m, the class of sets expressible as an intersection of at most m closed halfspaces is an ideal uniformity class for weak convergence.

Theorem 12 is not new. For instance, case 2° is due to Ranga Rao [11] and to Ahmad [1] ([1] is only an announcement and does not seem to have been followed up by a real publication). We also mention [2] and [18] and refer to the references and further results given there.

Theorem 13. *Let $\mathscr{E}$ consist of all closed balls in a compact metric space. Then $\mathscr{E}$ is a μ-uniformity class for weak convergence for every μ which vanishes on every sphere.*

Proof. Let $(B_n)_{n\geqq 1}$ be a sequence of closed balls in the compact metric space (X, d) and assume that the limit boundary F of (B_n) exists. Let x_n be the center and r_n the radius of B_n. By compactness, we may assume that (x_n) converges, say $x_n \to x$. By compactness, we may assume that the r_n's are bounded and then, we may assume that (r_n) converges, say $r_n \to r$. It is easy to check that $d(y, x) = r$ for every $y \in F$, i.e. F is contained in the sphere with center x and radius r. An application of Theorem 6 finishes the proof. □

It is not true in general that $\mathscr{E}$ in Theorem 13 is an ideal uniformity class for weak convergence – for instance, this does not hold in the unit interval with the usual metric. Of course, the result implies that if the compact metric space satisfies the additional requirement that every sphere is contained in the boundary of a ball, then the class of closed balls is an ideal uniformity class for weak convergence.

A slightly more general result is obtained if we instead of compactness assume that closed balls are compact and if we put an upper bound on the radia of the balls in $\mathscr{E}$.

We shall now prove some results for setwise convergence.

Let $(X, \mathscr{B})$ be a measurable space and let $\mathscr{A}$ be a subalgebra of $\mathscr{B}$. $\mathscr{A}$-convergence is then equivalent to setwise convergence on $\mathscr{A}$. Assume that $\mathscr{E}$ is a μ-uniformity class for $\mathscr{A}$-convergence. Then, for every $Y \in \mathscr{A}$, $\mathscr{E}|Y$ is a $\mu|Y$-uniformity class in the space Y for $\mathscr{A}|Y$-convergence. Here $\mathscr{E}|Y$ is the class of restrictions of the functions in $\mathscr{E}$ to Y, $\mu|Y$ is the restriction of μ to Y and $\mathscr{A}|Y$ is the restriction of $\mathscr{A}$ to Y. The stated result follows immediately from Theorem 5. The following lemma is a kind of converse saying that if uniformity holds for many restrictions, then uniformity holds in the original space.

Lemma 6. *Let $(X, \mathscr{B}, \mu)$ be a measure space, $\mathscr{A}$ a subalgebra of $\mathscr{B}$ and $\mathscr{E}$ a uniformly bounded class of measurable functions. Assume that for each ordinal number α less than a certain countable ordinal number γ, there is given a measurable set X_α. For $\alpha < \gamma$, denote by μ_α the restriction of μ to $X_\alpha \setminus \bigcup_{i<\alpha} X_i$, considered as a measure on X_α. If the following conditions are satisfied:*

(a) $X_\alpha \in \mathscr{A}$ for all $\alpha < \gamma$,

(b) $\bigcup_{\alpha<\gamma} X_\alpha = X$,

(c) *for all $\alpha < \gamma$, $\mathscr{E}|X_\alpha$ is a μ_α-uniformity class in the space X_α for $\mathscr{A}|X_\alpha$-convergence,*

then $\mathscr{E}$ is a μ-uniformity class (in X) for $\mathscr{A}$-convergence.

Proof. For $\alpha < \gamma$ put $Y_\alpha = X_\alpha \setminus \bigcup_{i<\alpha} X_i$. Given $\varepsilon > 0$, choose Λ, a finite subset of $[0, \gamma[$ such that

$$\mu(\bigcup \{Y_\alpha : \alpha \in [0, \gamma[\setminus \Lambda\}) < \varepsilon.$$

Denote by n the number of elements in Λ. Let $\alpha \in \Lambda$ and choose, according to Theorem 5, π_α, a finite subalgebra of the algebra $\mathscr{A}|X_\alpha$ in X_α, such that

$$\int_{X_\alpha} \partial_{\pi_\alpha}(f)\, d\mu_\alpha < \varepsilon/n \qquad \text{for all } f \in \mathscr{E}|X_\alpha.$$

Denote by π the finite subalgebra of $\mathscr{A}$ generated by all the sets in the π_α's; $\alpha \in \Lambda$.

If $|f(x)| \leqq K$ for all $f \in \mathscr{E}$, $x \in X$, it is seen that for $f \in \mathscr{E}$,

$$\begin{aligned}\int_X \partial_\pi(f)\,d\mu &= \sum_{\alpha<\gamma} \int_{Y_\alpha} \partial_\pi(f)\,d\mu \\ &\leqq 2K\varepsilon + \sum_{\alpha\in A} \int_{X_\alpha} \partial_{\pi_\alpha}(f)\,d\mu_\alpha \\ &\leqq 2K\varepsilon + \varepsilon.\end{aligned}$$

By Theorem 5, this shows that $\mathscr{E}$ is a μ-uniformity class for $\mathscr{A}$-convergence. □

Theorem 14. *Let $\mathscr{E}$ be one of the following two classes of subsets of an euclidean space:* 1°. *The class of all closed halfspaces,* 2°. *The class of all closed euclidean balls. Then $\mathscr{E}$ is an ideal uniformity class for setwise convergence.*

Proof. Let us limit the discussion to the space R^3. Consider case 1°. Let μ be any measure on R^3. We shall construct a certain scheme of subsets of R^3:

$$\begin{aligned}&x_0, x_1, \ldots\\ &\ell_0, \ell_1, \ldots\\ &\sigma_0, \sigma_1, \ldots\\ &R^3.\end{aligned}$$

Here $x_0, x_1, \ldots$ is chosen as a sequence of points in R^3 containing all atoms of μ. Then μ_c, the non-atomic part of μ, is given by

$$\mu_c = \mu | R^3 \setminus \{x_0, x_1, \ldots\}.$$

$\ell_0, \ell_1, \ldots$ is chosen as a sequence of lines containing all lines ℓ with $\mu_c(\ell) > 0$. Put

$$\mu_1 = \mu_c | R^3 \setminus \bigcup_0^\infty \ell_n.$$

Then $\mu_1(\ell) = 0$ for any line in R^3.

$\sigma_0, \sigma_1, \ldots$ is chosen as a sequence of planes containing all planes σ with $\mu_1(\sigma) > 0$. Put

$$\mu_2 = \mu_1 | R^3 \setminus \bigcup_0^\infty \sigma_n.$$

Then $\mu_2(\sigma) = 0$ for any plane σ in R^3.

Denoting by ω the first infinite ordinal, we define, for each $\alpha \leqq 3\omega$ a subspace X_α of R^3 in the following way:

$$\begin{aligned} X_n &= \{x_n\} && \text{for } n = 0, 1, \ldots\\ X_{\omega+n} &= \ell_n && \text{for } n = 0, 1, \ldots\\ X_{2\omega+n} &= \sigma_n && \text{for } n = 0, 1, \ldots\\ X_{3\omega} &= R^3. \end{aligned}$$

We shall apply Lemma 6 with $\mathscr{A}=\mathscr{B}$, $\gamma=3\omega+1$ and X_α's as defined above. Consider an $\alpha\leqq 3\omega$. Then X_α can be considered as an euclidean space (for $\alpha<\omega$ we get a one-point space) and $\mathscr{E}|X_\alpha$ coincides with the class of halfspaces in X_α (including $\emptyset$ and X_α). Since, in the notation of Lemma 6, μ_α is $\mathscr{E}|X_\alpha$-continuous in the space X_α, we conclude by Theorem 12, 1°, that $\mathscr{E}|X_\alpha$ is a μ_α-uniformity class for weak convergence (in X_α). A fortiori, $\mathscr{E}|X_\alpha$ is a μ_α-uniformity class for setwise convergence (in X_α).

By Lemma 6 it now follows that $\mathscr{E}$ is a μ-uniformity class for setwise convergence (in R^3). This is the desired result.

Now consider case 2°. Since the proof in this case is very similar, we only give an outline.

Again we consider a fixed measure μ. This time the X_α's are constructed according to the scheme

$x_0, x_1, \ldots$	(X_α for $0\leqq\alpha<\omega$)
$\ell_0, \ell_1, \ldots$	(X_α for $\omega\leqq\alpha<2\omega$)
$c_0, c_1, \ldots$	(X_α for $2\omega\leqq\alpha<3\omega$)
$\sigma_0, \sigma_1, \ldots$	(X_α for $3\omega\leqq\alpha<4\omega$)
$s_0, s_1, \ldots$	(X_α for $4\omega\leqq\alpha<5\omega$)
R^3	($X_{5\omega}$).

The x's and the ℓ's are constructed as before, i.e. – intuitively speaking – by removing all mass concentrated on points and then by removing all mass concentrated on lines. The c's refer to circles and are constructed by removing all mass concentrated on circles. σ's are constructed as before by removing all mass concentrated on planes. s's denote spheres and are constructed by removing all mass concentrated on spheres.

The essential fact to establish, is then, that for each $0\leqq\alpha\leqq 5\omega$, $\mathscr{E}|X_\alpha$ is a μ_α-uniformity class for setwise convergence (in X_α). We even have uniformity for weak convergence (in X_α). For $\alpha<2\omega$ this is easy. For $2\omega\leqq\alpha<3\omega$ and for $4\omega\leqq\alpha<5\omega$, this follows from Theorem 13. For $3\omega\leqq\alpha<4\omega$ and for $\alpha=5\omega$, this follows from Theorem 12, 3°. □

Case 2° was first proved by Elker [4].

We turn to a closer study of uniformity over convex sets in R^N. Some preparations are needed. As a general reference to convex analysis, we mention [13].

Let $A\subseteq R^N$. co(A) denotes the convex hull and aff(A) the affine hull of A. If A is finite, co(A) is a polytope. For a convex set C, dim(C) denotes the dimension of C (= the dimension of aff(C)). By $ri(C)$ we denote the relative interior of C, and by $\partial_{\text{rel}}(C)$ the relative boundary of C; these concepts are defined by considering C as a subset of aff(C). For an affine subspace H of R^N, $\mathscr{C}(H)$ denotes the class of measurable convex subsets of H. We write $\mathscr{C}$ instead of $\mathscr{C}(R^N)$. The following result is intuitive and easy to prove by standard techniques as developed in Chapters 1–2, [13].

Lemma 7. *Let C be a k-dimensional subset, and Δ a finite subset of R^N. Then the set*

$$E = \operatorname{co}(\bar{C} \cap \Delta) \cap \partial_{\mathrm{rel}}(C)$$

is a finite union of polytopes of dimension at most $(k-1)$.

Our next lemma is an "affine" decomposition property for measures which was really allready employed in the proof of the first part of Theorem 14. It is the same property as that derived in Section 7 of [11]. Recall, that a measure is supported by a set A if the complement of A has measure 0.

Lemma 8. *Let μ be a measure on R^N.*

(i) *For each $1 \leqq k \leqq N$ there exists uniquely determined measures $\mu'_{[k]}$ and $\mu''_{[k]}$ such that $\mu = \mu'_{[k]} + \mu''_{[k]}$, such that $\mu'_{[k]}$ is supported by a countable union of $(k-1)$-dimensional affine subspaces and such that $\mu''_{[k]}$ vanishes on every $(k-1)$-dimensional affine subspace.*

(ii) *If ξ is a measure vanishing on every $(k-1)$-dimensional affine subspace and if $\xi \leqq \mu$, then $\xi \leqq \mu''_{[k]}$.*

(iii) *There exist uniquely determined measures $\mu_{[0]}, \mu_{[1]}, \ldots, \mu_{[N]}$ such that $\mu = \sum_0^N \mu_{[k]}$ and such that each $\mu_{[k]}$ is supported by a countable union of k-dimensional affine subspaces and vanishes on every affine subspace of dimension less than k.*

As to the proof we just mention, that one may start proving existence in (i), then (ii), then uniqueness in (i) and lastly (iii).

Clearly, $\mu'_{[k]} = \sum_{i<k} \mu_{[i]}, \quad \mu''_{[k]} = \sum_{i \geqq k} \mu_{[i]}$.

We call $\mu_{[k]}$ the *k-dimensional part* of μ. $\mu_{[0]} = \mu'_{[1]}$ is the atomic part of μ and $\mu''_{[1]}$ the non-atomic part.

Theorem 15. *Let μ be a probability measure on R^N. The following conditions are equivalent:*

(a) *For every net (μ_α) of measures which converges to μ on $\mathscr{C}$, the convergence is uniform, i.e.*

$$\lim_\alpha \sup_{C \in \mathscr{C}} |\mu_\alpha(C) - \mu(C)| = 0;$$

(b) *For every random net $(\mu_{\alpha\omega})$ of measures such that, for each $C \in \mathscr{C}$, $\mu_{\alpha\omega}(C)$ converges to $\mu(C)$, almost surely $[\omega]$, we have*

$$\lim_\alpha \sup_{C \in \mathscr{C}} |\mu_{\alpha\omega}(C) - \mu(C)| = 0 \quad \text{a.s.} \quad [\omega];$$

(c) *Denoting by $(\mu_{n\omega})$ the empirical measures pertaining to the theoretical distribution μ, we have*

$$\lim_{n \to \infty} \sup_{C \in \mathscr{C}} |\mu_{n\omega}(C) - \mu(C)| = 0 \quad \text{a.s.} \quad [\omega];$$

(d) *For every $C \in \mathscr{C}$, we have*

$$\mu_{[k]}(\partial_{\mathrm{rel}} C) = 0 \quad \textit{where} \quad k = \dim(C).$$

Before the proof, let us comment on the significance of condition (d). In a different wording, (d) says that for every affine subspace A of R^N, the measure obtained by restricting μ to A and removing all "affine singularities", that is all concentration of mass on affine subspaces of A of lower dimension than that of A, should be well enough "smeared" out over A so that it vanishes on the boundary – calculated in A – of every convex subset of A.

Naturally, in (d), we need only consider sets C with $ri(C) \neq \emptyset$. Also notice that we may replace $\mu_{[k]}$ by $\mu''_{[k]}$ in (d). Since the requirement in (d) is automatically fulfilled for sets of dimension at most 1, it follows by the inclusion $\partial_{\text{rel}}(C) \subseteq \partial(C)$ and the inequalities

$$\mu \geqq \mu''_{[1]} \geqq \mu''_{[2]} \geqq \cdots \geqq \mu''_{[N]}$$

that the condition

$$\mu''_{[2]}(\partial C) = 0 \qquad \text{for all } C \in \mathscr{C}$$

is sufficient for (a), (b) and (c) to hold.

(c) was first proved by Ranga Rao (Theorem 7.1 of [11]) under the condition $\mu''_{[1]}(\partial C) = 0$ for $C \in \mathscr{C}$. For a stronger result derived under this condition, see [18]. Fabian [6] was the first to establish (a); he assumed that $\mu(\partial C) = 0$ for $C \in \mathscr{C}$. Stute [14] considers (a) and (c); he imposed the condition that μ be absolutely continuous with respect to some product measure (that this condition implies (d) may be proved directly, cf. [8], but is not entirely trivial). In [15] Stute proves a result which essentially amounts to the equivalence of (c) and (d) when $N = 2$.

It has been pointed out to me that Elker obtained essentially the same result as our Theorem 15 (Theorem 3.3 of [4]). Elker was then the first to obtain the result. The main difference between the proofs is that Elker only has available the "η-criterion", whereas we make use of the "ζ-criterion".

The reader who does not want to return to the proof below, should watch the effect a replacement of $\mathscr{C}$ by the class of closed convex sets will have.

Proof of Theorem 15. For an affine subspace A, let $\mathscr{A}(A)$ denote the algebra of subsets of A spanned by $\mathscr{C}(A)$. Put $\mathscr{A} = \mathscr{A}(R^N)$. It is easy to see, for instance by P 11 of [19], that if μ_α converges to μ on $\mathscr{C}$, then μ_α converges to μ on $\mathscr{A}$. Therefore, (a) asserts that $\mathscr{C}$ is a μ-uniformity class for $\mathscr{A}$-convergence. Similarly, if $\mu_{\alpha\omega}(C) \to \mu(C)$ a.s. $[\omega]$ for $C \in \mathscr{C}$, then $\mu_{\alpha\omega}(E) \to \mu(E)$ a.s. $[\omega]$ for $E \in \mathscr{A}$. The implication (a) $\Rightarrow$ (b) follows from these remarks in connection with the discussion preceding Theorem 2.

The implication (b) $\Rightarrow$ (c) follows by the strong law of large numbers.

(c) $\Rightarrow$ (d): Let $C \in \mathscr{C}$ and put $k = \dim(C)$. Let $X_1, X_2, \ldots$ be a sequence of independant random vectors, all with distribution μ. For each $n \geqq 1$ and each ω in the background probability space, put

$$C_{n\omega} = \text{co}(\bar{C} \cap \{X_1(\omega), \ldots, X_n(\omega)\}).$$

Then $C_{n\omega} \in \mathscr{C}$. Clearly,

$$C_{n\omega} \cap \{X_1(\omega), \ldots, X_n(\omega)\} = \bar{C} \cap \{X_1(\omega), \ldots, X_n(\omega)\},$$

hence $\mu_{n\omega}(C_{n\omega})=\mu_{n\omega}(\bar{C})$, and it follows that

$$\mu_{n\omega}(C_{n\omega})\to\mu(\bar{C}) \quad \text{a.s.} \quad [\omega]. \tag{15}$$

We have that

$$\begin{aligned}\mu(C_{n\omega})&=\mu(C_{n\omega}\cap ri(C))+\mu(C_{n\omega}\cap\partial_{\text{rel}}(C))\\&\leqq\mu(ri(C))+\mu(C_{n\omega}\cap\partial_{\text{rel}}(C))\end{aligned}$$

and by an application of Lemma 7, that

$$\mu(C_{n\omega})\leqq\mu(ri(C))+\mu'_{[k]}(\partial_{\text{rel}}(C)). \tag{16}$$

(15) and (16) implies that, almost surely,

$$\begin{aligned}&\limsup_{n\to\infty}\sup_{C\in\mathscr{C}}|\mu_{n\omega}(C)-\mu(C)|\\&\quad\geqq\limsup_{n\to\infty}|\mu_{n\omega}(C_{n\omega})-\mu(C_{n\omega})|\\&\quad\geqq\mu(\bar{C})-\mu(ri(C))-\mu'_{[k]}(\partial_{\text{rel}}(C))\\&\quad=\mu''_{[k]}(\partial_{\text{rel}}(C))\\&\quad=\mu_{[k]}(\partial_{\text{rel}}(C)).\end{aligned}$$

Therefore, by (c), $\mu_{[k]}(\partial_{\text{rel}}C)=0$.

(d) $\Rightarrow$ (a): For simplicity, assume that $N=3$. Construct x's, ℓ's and σ's and corresponding sets X_α and associated measures μ_α ($\alpha\leqq 3\omega$) precisely as in the first part of the proof of Theorem 14. Denote by $\mathscr{C}_\alpha$ the restriction of $\mathscr{C}$ and by $\mathscr{A}_\alpha$ the restriction of $\mathscr{A}$ to X_α. Then $\mathscr{C}_\alpha=\mathscr{C}(X_\alpha)$, $\mathscr{A}_\alpha=\mathscr{A}(X_\alpha)$.

Fix $\alpha\leqq 3\omega$, and let $k=\dim(X_\alpha)$. By Lemma 8, (ii), $\mu_\alpha\leqq\mu''_{[k]}$. Denoting by ∂_α the boundary operation in the space X_α, it follows from (d) that

$$\mu_\alpha(\partial_\alpha C)=0 \qquad \text{for all } C\in\mathscr{C}(X_\alpha) \tag{17}$$

(you first get this with ∂_{rel} in place of ∂_α, but the transition to ∂_α is clear).

Now remark, that there exists a countable subclass of $\mathscr{A}_\alpha$ which generates the usual topology on X_α. Combining this fact with Theorem 6, (i), with Lemma 5, 2° and with (17), we conclude that $\mathscr{C}_\alpha$ is a μ_α-uniformity class (in X_α) for $\mathscr{A}_\alpha$-convergence. This being so for all $\alpha\leqq 3\omega$, it follows by Lemma 6, that $\mathscr{C}$ is a μ-uniformity class for $\mathscr{A}$-convergence. □

Let $\mathscr{C}^*$ denote the class of all closed convex subsets of R^N and let (a*), (b*) and (c*) be the statements obtained from (a), (b) and (c), respectively, by replacing $\mathscr{C}$ with $\mathscr{C}^*$. Clearly, such a replacement will not affect (d). With (a*), (b*) and (c*) instead of (a), (b) and (c), and $\mathscr{C}^*$ instead of $\mathscr{C}$, all steps in the above proof go through. Accordingly, we have the following result:

The conditions (a), (b), (c), (d), (a*), (b*) *and* (c*) *are equivalent.*

From the outset, it is clear that (c) $\Rightarrow$ (c*). As to (a) and (a*) no implication seems obvious beforehand. In this connection we notice that convergence on $\mathscr{C}^*$ does not imply convergence on $\mathscr{C}$: Consider μ_r = uniform distribution in R^2 on the sphere with center 0 and radius r. By an argument, which we shall leave to the reader, it can be seen that, as $r \uparrow 1$, $\mu_r(C) \to \mu_1(C)$ for every $C \in \mathscr{C}^*$. Clearly, for the open unit ball B, $\mu_r(B) \to 1$ but $\mu B = 0$ so that the convergence cannot be extended to $\mathscr{C}$.

We suspect that if μ satisfies (d), and if μ_α converges to μ on $\mathscr{C}^*$, then μ_α converges to μ on $\mathscr{C}$. So far, we have only been able to show that $\mu_\alpha(C) \to \mu(C)$ for every relatively open convex set.

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References

1. Ahmad, S.: Sur le Théorème de Glivenko-Cantelli. C.R. Acad. Sci. Paris Sér. A–B **252**, 1413–1417 (1961)
2. Billingsley, P., Topsøe, F.: Uniformity in weak convergence. Z. Wahrscheinlichkeitstheorie verw. Gebiete **7**, 1–16 (1967)
3. Dunford, N., Schwartz, J.T.: Linear operators: Part I, general theory. New York: Interscience publishers 1963
4. Elker, J.: Unpublished "Diplomarbeit" from the Ruhr university, Bochum 1975
5. Engelking, R.: Outline of general topology. Amsterdam: North-Holland 1968
6. Fabian, V.: On uniform convergence of measures. Z. Wahrscheinlichkeitstheorie verw. Gebiete **15**, 139–143 (1970)
7. Gaenssler, P.: On convergence of sample distributions. Bull. Inst. Internat. Statist., Proc. 39th Session **45**, 427–432 (1973)
8. Gaenssler, P., Stute, W.: On uniform convergence of measures with applications to uniform convergence of empirical distributions. In Empirical Distributions and Processes. Lecture notes in mathematics, 566. Berlin-Heidelberg-New York: Springer 1976
9. Halmos, P.R.: Mesure theory. Princeton: D. Van Nostrand 1950
10. Bhaskara Rao, M., Bhaskara Rao, K.P.S.: Borel σ-algebra on $[0, \Omega]$. Manuscripta math. **5**, 195–198 (1971)
11. Ranga Rao, R.: Relations between weak and uniform convergence of measures with applications. Ann. math. Statistics **33**, 659–680 (1962)
12. Ressel, P.: Some continuity and measurability results on spaces of measures. Copenhagen University Preprint Series No 12, 1976
13. Rockafellar, R.T.: Convex analysis. Princeton: Princeton University Press 1970
14. Stute, W.: On a generalization of the Glivenko-Cantelli Theorem. Z. Wahrscheinlichkeitstheorie verw. Gebiete **35**, 167–175 (1976)
15. Stute, W.: A necessary condition for the convergence of the isotrope discrepancy. In Empirical Distributions and Processes. Lecture notes in mathematics, 566. Berlin-Heidelberg-New York: Springer 1976
16. Stute, W.: On uniformity classes of functions with an application to the speed of Mean Glivenko-Cantelli convergence. Ruhr-University Bochum, Preprint Series No. 5, 1974

17. Topsøe, F.: On the connection between P-continuity and P-uniformity in weak convergence. Theor. Probability Appl. **12**, 281–290 (1967)
18. Topsøe, F.: On the Glivenko-Cantelli theorem. Z. Wahrscheinlichkeitstheorie verw. Gebiete **14**, 239–250 (1970)
19. Topsøe, F.: Topology and measure. Lecture Notes in Math. 133. Berlin-Heidelberg-New York: Springer 1970

Added Reference

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