Z. Wahrscheinlichkeitstheorie verw. Gebiete
61, 255–260 (1982)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Calculation of Some Conditional Excursion Formulae

Paul McGill

Department of Mathematics, The New University of Ulster, Coleraine BT52 1SA, Northern Ireland

Let $\{f_n\}$ be a sequence of bounded continuous real-valued functions defined on R. If B_t is a Brownian motion process whose excursion σ-field above zero is denoted by $\tilde{\mathscr{B}}_\infty$, we show here how to evaluate conditional expectations of the form

$$E\left[\int_0^{\tau_t} e^{-\lambda(n)t_n} f_n(B_{t_n})\,dt_n \int_0^{t_n}\ldots\int_0^{t_2} e^{-\lambda(1)t_1} f_1(B_{t_1})\,dt_1 \,|\, \tilde{\mathscr{B}}_\infty\right]$$

where $\{\lambda(n)\}_{n\geqq 1}$ is a sequence of positive terms. Williams, in [4], has considered related conditional excursion formulae but our method is simpler in that it involves little more than a judicious use of martingale calculus. In the interests of clarity we work only with Brownian motion but the method is quite general. It applies equally well to any recurrent process and, although we do not do this here, it can also be used in the transient case provided that we work on an enlarged filtration.

1. Preliminaries

Let B_t be a Brownian motion started at zero and having natural filtration $\mathscr{B}_t$. We write $A_t=\int_0^t 1_{\{B_s\geqq 0\}}\,ds$, $C_t=t-A_t$ and let τ_t denote the right continuous inverse of A_t. Then τ_t is a $\mathscr{B}_t$ stopping time. Next introduce the local time L_t^a of B_t which is always taken to be jointly continuous and normalised so that the occupation density formula becomes

$$\int_0^t f(B_s)\,ds=\int_R f(a)\,L_t^a\,da.$$

We also require the generalised Ito formula [3] for functions f which can be written as the difference of two convex functions. This states that

0044-3719/82/0061/0255/$01.20

$$f(B_t)=f(0)+\int_0^t f'(B_s)\,dB_s+\tfrac{1}{2}\int_R L_t^a\,\mu_f(da)$$

where μ_f is the second distributional derivative of f. Using this we get

$$B_t^+=\int_0^t 1_{\{B_s\geqq 0\}}\,dB_s+\tfrac{1}{2}L_t^0.$$

The process $\tilde{B}_t=B_{\tau_t}$ takes non-negative values only and time-changing the above equation gives

$$\tilde{B}_t=\tilde{\beta}_t+\tfrac{1}{2}L_{\tau_t}^0. \qquad (*)$$

Here $\tilde{\beta}_t=\int_0^{\tau_t} 1_{\{B_s\geqq 0\}}dB_s$ is, from the Levy martingale characterisation, a new Brownian motion. Now let us regard (*) as a stochastic differential equation with $\tilde{\beta}_t$ given.

Theorem 1.1 [1]. *The equation (*) has a unique $\tilde{\beta}_t$ adapted solution $\tilde{B}_t$ whose local time at zero is given by $\tilde{L}_t^0=\frac{1}{2}L_{\tau_t}^0$.*

Write $\tilde{\mathscr{B}}_t$ to denote the filtration of $\tilde{B}_t$ and note that $\tilde{\mathscr{B}}_t\subseteq\mathscr{B}_{\tau_t}$. Our key result on the connection between the two filtrations is the following.

Lemma 1.2. *Let M_t be a square integrable $\mathscr{B}_{\tau_t}$ martingale such that $\langle M,\tilde{\beta}\rangle=0$. Then $E[M_\infty-M_0|\tilde{\mathscr{B}}_\infty]=0$.*

Proof. Choose $F\in L_2(\tilde{\mathscr{B}}_\infty,P)$ and write $F_t=E[F|\tilde{\mathscr{B}}_t]$. By Theorem 1.1 $\tilde{\mathscr{B}}_t$ is the filtration of $\tilde{\beta}_t$ so by Ito's representation theorem there is a $\tilde{\mathscr{B}}_t$ predictable process ϕ_t with $E\left[\int_0^\infty \phi_s^2\,ds\right]<+\infty$ such that

$$F_t=F_0+\int_0^t \phi_s\,d\tilde{\beta}_s.$$

Since $\langle M,F\rangle=0$ we have

$$E[M_\infty F]=E[M_0F_0]=E[M_0F]$$

which completes the proof.

2. First Order Formulae

Let $T=\inf\{t\colon B_t\geqq 0\}$. For $\lambda>0$ and f a bounded continuous function defined on the real line write

$$R_\lambda f(x)=E_x\left[\int_0^T e^{-\lambda t}f(B_t)\,dt\right].$$

Thus $u(x)=R_\lambda f(x)$ satisfies

$$\tfrac{1}{2}u'' = \lambda u - f \qquad (x<0)$$
$$u(x) = 0 \qquad (x \geqq 0).$$

If $\mu > 0$ and $\eta = \sqrt{2\mu}$ write $g(x) = \exp\{\eta(x \wedge 0)\}$. We can use the generalised Ito formula to see that the following processes are local martingales orthogonal to $\int_0^t 1_{\{B_s \geqq 0\}} dB_s$.

$$N_t(\mu,\gamma;\lambda,f) = \exp\{-\mu C_t - \lambda t + \tfrac{1}{2}\gamma L_t^0\}(R_{\lambda+\mu} f)(B_t)$$
$$+ \tfrac{1}{2}(R_{\lambda+\mu} f)'(0-) \int_0^t \exp\{-\mu C_s - \lambda s + \tfrac{1}{2}\gamma L_s^0\}\, dL_s^0$$
$$+ \int_0^t \exp\{-\mu C_s - \lambda s + \tfrac{1}{2}\gamma L_s^0\} f(B_s)\, 1_{\{B_s<0\}}\, ds.$$
$$M_t(\mu) = g(B_t)\exp\{-\mu C_t + \tfrac{1}{2}\eta L_t^0\}.$$

Lemma 2.1 [5]. $E[\exp\{-\mu\tau_t\} | \tilde{\mathscr{B}}_\infty] = \exp\{-\mu t - \eta \tilde{L}_t^0\}$.

Proof. Introduce the $\tilde{\mathscr{B}}_t$ stopping times $T_n = \inf\{t: \tilde{L}_t^0 \geqq n\}$. We can apply Lemma 1.2 to $M_{\tau_t} 1_{\{t \leqq T_n\}}$ and use Theorem 1.1 to get the result when t is replaced by $t \wedge T_n$. But $\lim T_n = +\infty$ a.s. and the general result follows by taking limits.

Theorem 2.2. Suppose that $\zeta \geqq 0$ is a $\tilde{\mathscr{B}}_\infty$ measurable random variable. Then

$$E\left[\int_0^{\tau_\zeta} e^{-\lambda s} f(B_s)\, ds \,|\, \tilde{\mathscr{B}}_\infty\right]$$
$$= \int_0^\zeta \exp\{-\lambda s - \sqrt{2\lambda}\,\tilde{L}_s^0\} f(\tilde{B}_s)\, dx$$
$$- (R_\lambda f)'(0-) \int_0^\zeta \exp\{-\lambda s - \sqrt{2\lambda}\,\tilde{L}_s^0\}\, d\tilde{L}_s^0. \qquad (**)$$

Proof. We prove this first for $\zeta = t$. It is enough to consider two cases.

(a) If f is supported on $[0, +\infty)$ the integral can be time-changed to give $\int_0^t \exp\{-\lambda\tau_s\} f(\tilde{B}_s)\, ds$. Now use the previous lemma.

(b) If f is supported on $(-\infty, 0]$ note that $N_{\tau_t}(0,0;\lambda,f)$ is an $L_2\, \mathscr{B}_{\tau_t}$ martingale. The result follows from Lemma 1.2 and the previous lemma by stopping at t.

We have shown that (**) holds if ζ is a simple $\tilde{\mathscr{B}}_\infty$ measurable non-negative random variable so the general case now follows by taking limits.

Remarks. (a) On comparing the above with [2] 4.2 we find that $-(R_\lambda f)'(0-)$ is the Laplace transform of the excursion law of B_t from zero. This provides a probabilistic interpretation of the theorem (see [4]) and gives a method of calculating the excursion law.

(b) By integration, if f is supported on $(-\infty, 0]$, we get

$$E\left[\int_0^\infty e^{-\lambda s} f(B_s)\, ds\right] = -(R_\lambda f)'(0-)\, E\left[\int_0^\infty \exp\{-\lambda \tau_s\}\, d\tilde{L}_s^0\right].$$

3. Higher Order Formulae

In order to avoid undue complication we shall introduce the following notation.

$$\boldsymbol{\lambda} = (\lambda(1), \lambda(2), \ldots) \qquad \lambda(n) \geqq 0$$
$$\mathbf{f} = (f_1, f_2, \ldots)$$

f_n bounded continuous and supported on $(-\infty, 0]$.

$$\begin{aligned} K_t(n, \boldsymbol{\lambda}, \mathbf{f}) &= K_t(\lambda(1), \lambda(2), \ldots, \lambda(n); f_1, f_2, \ldots, f_n) \\ &= \int_0^t dt_n\, e^{-\lambda(n) t_n} f_n(B_{t_n}) \int_0^{t_n} dt_{n-1} \ldots \int_0^{t_2} dt_1\, e^{-\lambda(1) t_1} f_1(B_{t_1}). \end{aligned}$$

It is convenient to write $K_t(0, \lambda, \mathbf{f}) \equiv 1$. We also need the local martingales

$$N_t^{(n)}(\mu, \gamma; \boldsymbol{\lambda}, \mathbf{f}) = \int_0^t K_s(n-1, \boldsymbol{\lambda}, \mathbf{f})\, dN_s(\mu, \gamma; \lambda(n), f_n).$$

Finally, if $1 \leqq r \leqq n$, write

$$R_\lambda^\mu(n, r)\, \mathbf{f} = R_{\rho(n,r)} f_r [R_{\rho(n, r+1)} f_{r+1} \cdots [R_{\rho(n,n)} f_n] \ldots]$$

where $\rho(n, r) = \mu + \sum_{i=r}^{n} \lambda(i)$.

Lemma 3.1.

$$\begin{aligned} E&\left[\int_0^{\tau_t} \exp\{-\mu C_s - \lambda(n) s + \tfrac{1}{2}\gamma L_s^0\} f_n(B_s)\, K_s(n-1, \boldsymbol{\lambda}, \mathbf{f})\, ds \,|\, \tilde{\mathscr{B}}_\infty\right] \\ &= -\sum_{r=1}^{n} [R_\lambda^\mu(n, r)\, \mathbf{f}]'(0-) \int_0^t e^{\gamma \tilde{L}_s^0} \\ &\quad \cdot E\left[\exp\left\{-\mu C_{\tau_s} - \left(\sum_{i=r}^{n} \lambda(i)\right) \tau_s\right\} K_{\tau_s}(r-1, \boldsymbol{\lambda}, \mathbf{f}) \,|\, \tilde{\mathscr{B}}_\infty\right] d\tilde{L}_s^0 \end{aligned}$$

Proof. By stopping at T_n, as in Lemma 2.1, we can use Lemma 1.2 and the $\mathscr{B}_{\tau_t}$ local martingale $N_{\tau_t}^{(n)}(\mu, \gamma; \boldsymbol{\lambda}, \mathbf{f})$ to obtain

$$E\left[\int_0^{\tau_t} \exp\{-\mu C_s - \lambda(n) s + \tfrac{1}{2}\gamma L_s^0\} f_n(B_s) K_s(n-1, \boldsymbol{\lambda}, \mathbf{f})\, ds \,|\, \tilde{\mathscr{B}}_\infty\right] =$$

$$-[R_{\lambda(n)+\mu} f_n]'(0-) \int_0^t e^{\gamma \tilde{L}_s^0}$$

$$\cdot E[\exp\{-\mu C_{\tau_s} - \lambda(n)\tau_s\} K_{\tau_s}(n-1, \boldsymbol{\lambda}, \mathbf{f}) \,|\, \tilde{\mathscr{B}}_\infty]\, d\tilde{L}_s^0$$

$$+ E\left[\int_0^{\tau_t} \exp\{-\mu C_s - (\lambda(n) + \lambda(n-1)) s + \tfrac{1}{2}\gamma L_s^0\}\right.$$

$$\left.\cdot [f_{n-1} R_{\lambda(n)+\mu} f_n](B_s) K_s(n-2, \boldsymbol{\lambda}, \mathbf{f})\, ds \,|\, \tilde{\mathscr{B}}_\infty\right]$$

Now iterate to get the required formula.

If $\mu = \gamma = 0$ this lemma expresses $E[K_{\tau_t}(n, \boldsymbol{\lambda}.\mathbf{f}) \,|\, \tilde{\mathscr{B}}_\infty]$ in terms of conditional expectations of the type $E[e^{-\mu\tau_t} K_{\tau_t}(m, \boldsymbol{\lambda}, \mathbf{f}) \,|\, \tilde{\mathscr{B}}_\infty]$ where $m < n$.

Lemma 3.2.

$$E[e^{-\mu\tau_t} K_{\tau_t}(n, \boldsymbol{\lambda}, \mathbf{f}) \,|\, \tilde{\mathscr{B}}_\infty]$$

$$= \exp\{-\mu t - \eta \tilde{L}_t^0\}$$

$$\cdot E\left[\int_0^{\tau_t} \exp\{-\mu C_s - \lambda(n) s + \tfrac{1}{2}\eta L_s^0\} (g f_n)(B_s)\right.$$

$$\left.\cdot K_s(n-1, \boldsymbol{\lambda}, \mathbf{f})\, ds \,|\, \tilde{\mathscr{B}}_\infty\right].$$

Proof. Putting $\zeta = t \wedge T_m$ and using Ito's formula together with Lemma 1.2 we have

$$E[\exp\{-\mu C_{\tau_\zeta} + \eta \tilde{L}_\zeta^0\} K_{\tau_\zeta}(n, \boldsymbol{\lambda}, \mathbf{f}) \,|\, \tilde{\mathscr{B}}_\infty]$$

$$= E \cdot [M_{\tau_\zeta}(\mu) K_{\tau_\zeta}(n, \boldsymbol{\lambda}, \mathbf{f}) \,|\, \tilde{\mathscr{B}}_\infty]$$

$$= E\left[\int_0^{\tau_\zeta} M_s(\mu)\, dK_s(n, \boldsymbol{\lambda}, \mathbf{f}) \,|\, \tilde{\mathscr{B}}_\infty\right]$$

$$= E\left[\int_0^{\tau_\zeta} \exp\{-\mu C_s - \lambda(n) s + \tfrac{1}{2}\eta L_s^0\} (g f_n)(B_s) K_s(n-1, \boldsymbol{\lambda}, \mathbf{f})\, ds \,|\, \tilde{\mathscr{B}}_\infty\right].$$

Now take the limit in m and the result is clear.

This formulation of Lemma 3.2 was pointed out to us by T. Jeulin and is simpler than our previous version. Note that the formulae of this section are valid only if the functions $\{f_n\}$ are supported on $(-\infty, 0]$. If not, then the integrals involved must be decomposed as in the previous section. In any case it is clear that these two results enable us to reduce the evaluation of an nth order conditional excursion formula of the type described above to the evaluation of lower order formulae.

Remark. Suppose that the stochastic differential equation

$$X_t = X_0 + \int_0^t \sigma(X_s)\, dB_s + \int_0^t \chi(X_s)\, ds$$

has a unique strong recurrent solution. Then all of the above results remain valid if B_t is replaced by X_t, except that $g(x)$ must be replaced by the solution of

$$\tfrac{1}{2}\sigma^2 u'' + \chi u' = \mu u; \qquad u'(-\infty) = 0, \quad u(x) = 0 \quad \text{if } x \geqq 0$$

and η must be replaced by $u'(0-)$. Of course in this case $R_\lambda f$ will denote the resolvent of the process X_t killed at T.

References

1. El-Karoui, N., Chaleyat-Maurel, M.: Un problème de réflexion et ses applications au temps local et aux équations différentielles stochastiques sur R – cas continu. In: Temps Locaux, pp. 117–144. Astérisque No. 52-3 published by Société Mathématique de France. Paris 1978
2. Maisonneuve, B.: Exit systems. Ann. Probab. **3**, 399–411 (1975)
3. Meyer, P.A.: Un cours sur les intégrales stochastiques. In: Sém. de Probab. X pp. 245–400, Lecture Notes in Math. **511**. Berlin-Heidelberg-New York: Springer 1976
4. Williams, D.: Conditional excursion theory. In: Sém. de Probab. XIII pp. 490–494, Lecture Notes in Math. **721**. Berlin-Heidelberg-New York: Springer 1979
5. Williams, D.: Markov properties of Brownian local time. Bull. Amer. Math. Soc. **75**, 1035–1036 (1969)

Received December 15, 1981; in final form April 26, 1982