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# Asymptotic Expansions for Sums of Weakly Dependent Random Vectors

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**Summary.** It is shown that formal Edgeworth expansions are valid for sums of weakly dependent random vectors. The error of approximation has order $o(n^{-(s-2)/2})$ if

(i) the moments of order $s+1$ are uniformly bounded

(ii) a conditional Cramér-condition holds

(iii) the random vectors can be approximated by other random vectors which satisfy a strong mixing condition and a Markov type condition.

The strong mixing coefficients in (iii) are decreasing at an exponential rate. The above conditions can easily be checked and are often satisfied when the sequence of random vectors is a Gaussian, or a Markov, or an autoregressive process. Explicit formulas are given for the distribution of finite Fourier transforms of a strictly stationary time series.

## 1. Introduction and Summary

Consider a strictly stationary sequence $X_1, X_2, \ldots$ of $k$-variate random vectors with mean zero. If $X_1, X_2, \ldots$ are *independent* and if $X_1$ has a nonsingular covariance matrix, then the distribution of $n^{-1/2}(X_1+\ldots+X_n)$ is asymptotically normal. If third order moments exist, then the error of the normal approximation is of order $n^{-1/2}$. If higher order moments exist and the distribution of $X_1$ is smooth, then higher order approximations are valid. The smoothness condition usually imposed on $X_1$ is the following (Cramér):

For all $\varepsilon>0$ there exists a positive $\delta$ such that for $t\in\mathbb{R}^k$, $\|t\|\geqq\varepsilon$

$$(1.1)\qquad |E\exp(it^T X_1)|\leqq 1-\delta.$$

The higher order approximations are of the following kind:

$$(1.2)\qquad \Psi_{n,s}=\sum_{r=0}^{s-2} n^{-r/2} p_r.$$

Here $p_0$ is the normal distribution with mean zero and appropriate covariance matrix, and for $r=1,\dots,s-2$, $p_r$ is a finite signed measure with $p_0$-density $q_r$, say. The function $q_r$ is a polynomial with coefficients uniquely determined by the moments of $X_1$ up to order $r+2$. A detailed discussion of asymptotic expansions for independent random vectors is contained in Bhattacharya and Ranga Rao's monograph (1976).

It is well known that the central limit theorem remains true if $X_1, X_2, \dots$ are weakly dependent. Some concepts of weak dependence are $m$-dependence, uniform mixing, and strong mixing. The sequence $X_1, X_2, \dots$ is $m$-dependent if for all $p=1,2,\dots$ the sequences

$$(1.3) \qquad (X_1,\dots,X_p) \quad \text{and} \quad (X_{p+m+1}, X_{p+m+2}, \dots)$$

are stochastically independent. In the uniform mixing concept the dependence between the sequences (1.3) is measured by

$$\varphi(m) = \sup |P(A \cap B) - P(A)\,P(B)|/P(B)$$

where the sup is taken over all $p=1,2,\dots$ and all events $A, B, P(B)>0$, where $A$ is determined by $(X_1,\dots,X_p)$ and $B$ is determined by $(X_{p+m+1}, X_{p+m+2},\dots)$ (see Ibragimov (1962)). In the strong mixing concept the dependence in (1.3) is measured by

$$\alpha(m) = \sup |P(A \cap B) - P(A)\,P(B)|$$

where the sup is taken as above (see Rosenblatt (1956)). Another type of weak dependence occurs in models of the following kind. Given a sequence $Y_0, Y_{\pm 1}, Y_{\pm 2}, \dots$ of independent identically distributed random variables and a measurable function $f\colon \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^k$, define

$$X_j = f(Y_{j+p}, p \in \mathbb{Z}), \qquad j=1,2,\dots.$$

We call the sequence $X_1, X_2, \dots$ a weakly dependent shift if for all $j=1,2,\dots$, $E(X_j \mid Y_q : |j-q| \leqq m)$ converges to $X_j$ at a specified rate when $m$ tends to infinity. A weakly dependent shift will not necessarily satisfy a strong mixing condition (see Ibragimov (1962), pp. 374/5). It is unknown whether all $m$-dependent sequences are weakly dependent shifts.

Proofs of the central limit theorem usually require the additional assumption

$$(1.4) \qquad \Sigma = \lim_n \operatorname{Cov}(n^{-1/2}(X_1 + \dots + X_n)) \quad \text{exists and is nonsingular.}$$

When $X_1, X_2, \dots$ are *iid*, this assumption is satisfied whenever $X_1$ has a nonsingular covariance matrix. For weakly dependent random vectors $X_1, X_2, \dots$ condition (1.4) may fail even if $X_1$ has a nonsingular covariance matrix:

(1.5) *Example.* Let $Y_1, Y_2, \dots$ be a sequence of *iid* random variables, and for $j=1,2,\dots$ define

$$X_j = Y_{j+1} - Y_j.$$

The sequence $X_1, X_2, \dots$ is 1-dependent, and (1.4) never holds.

The following central limit theorems are known.

(1.6) **Theorem** (Ibragimov (1962), p. 360, Theorem 1.4). *Assume* (1.4) *and* $E\|X_1\|^{2+\delta}<\infty$ *for some positive* $\delta$. *If* $X_1, X_2, \ldots$ *is uniformly mixing with* $\lim \varphi(m)=0$, *then*

(1.7) $$\limsup_n \{|P\{n^{-1/2}(X_1+\ldots+X_n)\in C\}-N(C)|: C \text{ convex, measurable}\}=0$$

*where* $N$ *is the* $k$*-variate normal distribution with mean zero and covariance matrix* $\Sigma$.

(1.8) **Theorem** (Ibragimov (1962), p. 367, Theorem 1.7). *Assume* (1.4) *and* $E\|X_1\|^{2+\delta}<\infty$ *for some positive* $\delta$. *If* $X_1, X_2, \ldots$ *is strongly mixing with*

$$\Sigma\, \alpha(m)^{\delta/(2+\delta)}<\infty$$

*then* (1.7) *is true.*

The error of normal approximation has order $n^{-1/2}$ when third moments exist and $X_1, X_2, \ldots$ is $m$-dependent (Stein (1972), Tikhomirov (1980)) or Markov dependent (Bolthausen (1980), Statulevicius (1969, 1970)). For uniformly or strongly mixing sequences and for weakly dependent shifts the best available bounds are of order $n^{-1/2}(\log n)^{\beta}$, $\beta>0$ (Stein (1972), Tikhomirov (1980), Ibragimov (1967)). Higher order approximations of the form (1.2) were first derived by Statulevicius (1969, 1970) and Durbin (1980). The results of Statulevicius hold for finite order Markov chains, but not for weakly dependent shifts - these do not necessarily satisfy the RMT-condition of Statulevicius. Durbin's conditions are hard to check even in the simplest case when $X_1, X_2, \ldots$ is a weakly dependent shift which is $m$-dependent.

We shall prove the validity of higher order approximations for the distribution of $n^{-1/2}(X_1+\ldots+X_n)$ under (1.4), weak dependence assumptions, and a Cramér type condition. Our Cramér type condition is more restrictive then (1.1). In fact, assumption (1.1) is insufficient for our purpose even if $X_1, X_2, \ldots$ is an $m$-dependent shift. This is demonstrated by the following

(1.9) *Example.* Let $Y_1, Y_2, \ldots$ and $Z_1, Z_2, \ldots$ be two sequences of independent identically distributed random variables, the $Y$'s being independent of the $Z$'s. Assume that

$$P\{Y_1=-1/2\}=P\{Y_1=1/2\}=1/2$$

and that the distribution of $Z_1$ is normal with mean zero and unit variance. For $j=1,2,\ldots$ define

$$X_j=Y_j+Z_{j+1}-Z_j.$$

The sequence $X_1, X_2, \ldots$ is a 1-dependent shift, and (1.1) and (1.4) hold. However, the formal Edgeworth expansion

(1.10) $$P\{n^{-1/2}(X_1+\ldots+X_n)<t\}=\int_{-\infty}^{t}(2\pi)^{-1/2}\exp(-x^2/2)\,dx+o(n^{-1/2})$$

is *not* valid for $t \neq 0$. To see this we derive a *valid* higher order approximation for the distribution of

$$n^{-1/2}(X_1+\ldots+X_n)=n^{-1/2}(Y_1+\ldots+Y_n)+n^{-1/2}(Z_{n+1}-Z_1).$$

We have uniformly for $t \in \mathbb{R}$

$$P\{n^{-1/2}(Y_1+\ldots+Y_n)<t\}=\int_{-\infty}^{t}(2\pi)^{-1/2}\exp(-x^2/2)\,dx + n^{-1/2}(2\pi)^{-1/2}\exp(-t^2/2)\,S_1(n^{1/2}t) + o(n^{-1/2})$$

where $S_1$ is the 1st Bernoulli polynomial defined by

$$S_1(x)=x-1/2,\ 0\leqq x<1,\ S_1(x+1)=S_1(x), \qquad x\in\mathbb{R}.$$

Hence

$$\begin{aligned} &P\{n^{-1/2}(X_1+\ldots+X_n)<t\}\\ &=\int_{-\infty}^{t}\Big\{\int_{-\infty}^{t+n^{-1/2}2^{1/2}r}(2\pi)^{-1/2}\exp(-x^2/2)\,dx\\ &\quad+n^{-1/2}(2\pi)^{-1/2}\exp(-t^2/2)\,S_1(n^{1/2}t+2^{1/2}r)\Big\}(2\pi)^{-1/2}\exp(-r^2/2)\,dr\\ &\quad+o(n^{-1/2})\\ &=\int_{-\infty}^{t}(2\pi)^{-1/2}\exp(-x^2/2)\,dx\\ &\quad+n^{-1/2}(2\pi)^{-1/2}\exp(-t^2/2)\int_{-\infty}^{\infty}S_1(n^{1/2}t+2^{1/2}r)(2\pi)^{-1/2}\exp(-r^2/2)\,dr\\ &\quad+o(n^{-1/2}).\end{aligned}$$

For $t \neq 0$ the sequence

$$\int_{\infty}^{\infty}S_1(n^{1/2}t+2^{1/2}r)(2\pi)^{-1/2}\exp(-r^2/2)\,dr, \qquad n=1,2,\ldots$$

does not converge. Hence (1.10) is not true. Relation (1.10) holds for $t=n^{-1/2}m$, $m\in\mathbb{Z}$.

Cases as the one considered in (1.9) are excluded by our conditional Cramér condition which will be introduced and discussed in Sect. 2. Notice that, in (1.9), the conditional distributions of $X_1+X_2$, given $Y_p$, $Z_p$, $p\neq 2$, are lattice distributions.

In Theorem 2.8 we shall prove the validity of higher order approximations under general conditions (see (2.3)–(2.6)). These conditions are satisfied in the following examples (1.11)–(1.15). In other situations they can be checked easily.

(1.11) *Example.* Let $Y_1, Y_2, \ldots$ be a sequence of independent identically distributed random variables with Lebesgue density $g$, let $m\geqq 1$ and $h:\mathbb{R}^m\to\mathbb{R}$ be continuously differentiable, and define

$$X_j=h(Y_{j+1},\ldots,Y_{j+m}), \qquad j=1,2,\ldots.$$

The sequence $X_1, X_2, \ldots$ is an $m$-dependent shift.

We assume that there exist $y_1, \ldots, y_{2m-1} \in \mathbb{R}$ and an open subset $U \supset \{y_1, \ldots, y_{2m-1}\}$ such that $g > 0$ on $U$ and

$$\sum_{j=1}^{m} \frac{\partial}{\partial x_j} h(x_1, \ldots, x_m)|_{(x_1, \ldots, x_m) = (y_j, \ldots, y_{m+j-1})} \neq 0.$$

(1.12) *Example.* Let $Y_0, Y_{\pm 1}, Y_{\pm 2}, \ldots$ be a sequence of independent identically distributed random variables and $c_p$, $p \in \mathbb{Z}$ a sequence of real numbers satisfying

(i) $|c_p| \leqq \delta^{-1} \exp(-\delta |p|)$ for some positive $\delta$, and

(ii) $\sum_{p=-\infty}^{\infty} c_p \neq 0.$

Define

$$X_j = \sum_{p=-\infty}^{\infty} c_p Y_{j+p}, \quad j = 1, 2, \ldots$$

and assume that $Y_0$ satisfies Cramér's condition (1.1).

Notice that an arbitrary stationary autoregressive process

$$\beta_0 X_j + \beta_1 X_{j-1} + \ldots + \beta_r X_{j-r} = U_j, \quad j = 0, \pm 1, \pm 2, \ldots$$

can be written as

$$X_j = \sum_{p=0}^{\infty} c_p U_{j-p}$$

with $c_p$, $p \in \mathbb{Z}$, satisfying (i) and (ii) above whenever

$$\beta_0 z^r + \beta_1 z^{r-1} + \ldots + \beta_r = 0$$

has all its roots different and in $\{|z| < 1\}$ (see Anderson (1971), Sect. 5.2).

(1.13) *Example.* Let $\xi_0, \xi_1, \ldots$ be a homogeneous Markov chain, and $f$ a measurable function on its state space $I$. We define

$$X_j = f(\xi_j), \quad j = 1, 2, \ldots$$

and assume that $\xi_1, \xi_2, \ldots$ is strictly stationary, $X_1$ satisfies Cramér's condition (1.1), and the transition kernel $P(x, A)$ of the Markov chain satisfies

$$\sup |P(x, A) - P(x', A)| < 1$$

where the sup is taken over all $x, x'$, and $A$ (see Statulevicius (1969, 1970)). The last condition is satisfied whenever there exists a positive measure $\mu$ such that for all $x$ and $A$

$$P(x, A) \geqq \mu(A).$$

(1.14) *Example.* Let $Y_0, Y_{\pm 1}, Y_{\pm 2}, \ldots$ be a strictly stationary Gaussian process with positive analytic spectral density, and let $f$ be a nonconstant function which is continuously differentiable. Define

$$X_j = f(Y_j), \quad j = 1, 2, \ldots.$$

(1.15) *Example.* Let $X_1, X_2, \ldots$ be a strictly stationary Markov-dependent sequence of random variables satisfying the regularity conditions of Example (1.13). For fixed $p$ and $0 \leqq \lambda_1 < \ldots < \lambda_p \leqq \pi$, $\lambda_i + \lambda_j \neq 0 \bmod (2\pi)$, $1 \leqq i \neq j \leqq p$, and for $N = 1, 2, \ldots$ let

$$d^N(\lambda_1, \ldots, \lambda_p) = \left((2\pi N)^{-1/2} \sum_{n=1}^{N} X_n e^{-in\lambda_j}\right)_{j=1,\ldots,p}$$

be the finite Fourier transform of $X_1, \ldots, X_N$ at $\lambda_1, \ldots, \lambda_p$ (see Hannan (1970), Chap. IV, 3). If $E|X_1|^{s+1} < \infty$, then the distribution of $d^N(\lambda_1, \ldots, \lambda_p)$ admits a higher order approximation with an error term of order $O(N^{-(s-2)/2})$. The first term of the expansion is given in Anderson (1971), p. 482.

The paper is organized as follows. In Sect. 2 the regularity conditions and theorems are stated and applied to our Examples (1.11)–(1.15). At the end of Sect. 2 the technical lemmas of Sect. 3 are combined to yield proofs of our theorems. Formulas are deferred to Sect. 4. From now on we drop the assumption that the sequence under consideration is strictly stationary.

## 2. The Results

Let $X_1, X_2, \ldots$ be a sequence of $k$-variate random vectors on an abstract measure space $(\Omega, \mathscr{A}, P)$ with

(2.1) $$EX_j = 0, \quad j = 1, 2, \ldots$$

and

(2.2) $$E\|X_j\|^{s+1} \leqq \beta_{s+1} < \infty, \quad j = 1, 2, \ldots.$$

Define the integer $s_0 \leqq s$ by

$$s_0 = \begin{cases} s & \text{if } s \text{ is even} \\ s-1 & \text{if } s \text{ is odd.} \end{cases}$$

We assume that a sequence $\mathscr{D}_0, \mathscr{D}_{\pm 1}, \mathscr{D}_{\pm 2}, \ldots$ of sub-$\sigma$-fields of $\mathscr{A}$ are given and the following assumptions are satisfied.

(2.3) There exists a positive constant $d$ such that for $n, m = 1, 2, \ldots$ with $m > d^{-1}$ there exists a $\mathscr{D}_{n-m}^{n+m}$-measurable $k$-variate random vector $\bar{Y}_{n,m}$ for which

$$E\|X_n - \bar{Y}_{n,m}\| \leqq d^{-1} \exp(-dm).$$

Here $\mathscr{D}_p^q$ is the $\sigma$-field generated by $\mathscr{D}_j$, $p \leqq j \leqq q$.

(2.4) There exists $d > 0$ such that for all $m, n = 1, 2, \ldots$, $A \in \mathscr{D}_{-\infty}^n$, $B \in \mathscr{D}_{n+m}^{\infty}$,

$$|P(A \cap B) - P(A)P(B)| \leqq d^{-1} e^{-dm}.$$

(2.5) There exists $d > 0$ such that for all $m, n = 1, 2, \ldots$, $d^{-1} < m < n$, and all $t \in \mathbb{R}^k$ with $\|t\| \geqq d$

$$E|E(e^{it^T(X_{n-m} + X_{n-m+1} + \ldots + X_{n+m})} | \mathscr{D}_j : j \neq n)| \leqq e^{-d}.$$

(2.6) There exists $d>0$ such that for all $m,n,p=1,2,\ldots$ and $A\in\mathscr{D}_{n-p}^{n+p}$

$$E\,|P(A\,|\,\mathscr{D}_j: j\neq n)-P(A\,|\,\mathscr{D}_j: 0<|n-j|\leq m+p)|\leq d^{-1}\,e^{-dm}.$$

Write $S_n=n^{-1/2}(X_1+\ldots+X_n)$. For $r=0,\ldots,s$ let $\chi_{r,n}(t)$ be the cumulant of $t^T S_n$ of order $r$,

$$\chi_{r,n}(t)=\frac{d^r}{dx^r}\log E\,\exp(ixt^T S_n)|_{x=0}.$$

Here $t^T$ is the transpose of the $k$-vector $t$. Define the formal Edgeworth expansion $\Psi_{n,s}$ of $S_n$ by its characteristic function $\hat{\Psi}_{n,s}(t)=\exp(\chi_{2,n}(t))+\sum_{r=1}^{s-2} n^{-r/2}\,\tilde{P}_{j,n}(t)$, where the functions $\tilde{P}_{r,n}$, $r=1,2,\ldots$ are defined by the formal identity

$$\text{(2.7)}\qquad \exp\left(\chi_{2,n}(t)+\sum_{r=3}^{\infty}(r!)^{-1}\,\tau^{r-2}\,n^{(r-2)/2}\,\chi_{r,n}(t)\right)=\exp(\chi_{2,n}(t))+\sum_{r=1}^{\infty}\tau^r\,\tilde{P}_{r,n}(t).$$

(2.8) **Theorem.** *Let $f\colon \mathbb{R}^k\to\mathbb{R}$ denote a measurable function such that $|f(x)|\leq M(1+\|x\|^{s_0})$ for every $x\in\mathbb{R}^k$. Assume that* (2.1) (2.2), (1.4), (2.3)–(2.6) *hold. Then there exists a positive constant $\delta$ not depending on $f$ and $M$, and for arbitrary $\kappa>0$ there exists a positive constant $c$ depending on $M$ but not on $f$ such that*

$$|Ef(S_n)-\textstyle\int f\,d\Psi_{n,s}|\leq c\,\omega(f,n^{-\kappa})+o(n^{-(s-2+\delta)/2})$$

*where*

$$\omega(f,n^{-\kappa})=\textstyle\int\sup\{|f(x+y)-f(x)|:\ \|y\|\leq n^{-\kappa}\}\ \varphi_\Sigma(x)\,dx$$

*and $\varphi_\Sigma$ is the normal density with zero mean and covariance matrix $\Sigma$. The term $o(n^{-(s-2+\delta)/2})$ depends on $f$ through $M$ only.*

(2.9) **Corollary.** *Let the assumptions of Theorem* (2.8) *be satisfied. Then uniformly for convex measurable $C\subset\mathbb{R}^k$*

$$P\{S_n\in C\}=\Psi_{n,s}(C)+o(n^{-(s-2)/2}).$$

*As in* [6] *we can replace the smoothness conditions* (2.5), (2.6) *by smoothness of the function to be integrated. For nonnegative integral $k$-vectors $\alpha=(\alpha_1,\ldots,\alpha_k)$ define*

$$D^\alpha=\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\cdots\frac{\partial^{\alpha_k}}{\partial x_k^{\alpha_k}}.$$

(2.10) **Theorem.** *Let $f\colon\mathbb{R}^k\to\mathbb{R}$ denote an infinitely differentiable function such that $|f(x)|\leq M(1+\|x\|^{s_0})$ for every $x\in\mathbb{R}^k$ and $|D^\alpha f(x)|\leq M_\alpha(1+\|x\|^{p_\alpha})$ for every nonnegative integral $k$-vector $\alpha$ and positive constants $M_\alpha$, $p_\alpha$. Assume that* (2.1)–(2.4) *and* (1.4) *hold. Then*

$$Ef(S_n)-\textstyle\int f\,d\Psi_{n,s}=o(n^{-(s-2)/2}).$$

Actually, the expansion is valid for $f$ having a finite number $r(\varepsilon)>(s-2)/(2\varepsilon)$ of derivatives only. The number $\varepsilon$ depends on $\delta$, $s$, $\beta$, $d$, $k$, $\beta_{s+1}$, and $\Sigma$. We conjecture that $s-1$ derivatives are sufficient.

In our next theorem, smoothness conditions (2.5), (2.6) are not needed either.

(2.11) **Theorem.** *Under conditions* (2.1)–(2.4) *and* (1.4)

$$E(1+\|S_n\|^{s_0})\, 1_{\{\|S_n\|>((s-2)\log n)^{1/2}\}}=o(n^{-(s-2)/2}).$$

(2.12) *Remark.* Notice that the formal Edgeworth expansion $\Psi_{n,s}$ introduced here is not of the form (1.2): The functions $\tilde{P}_{r,n}$ depend on $n$. Hence the expansion $\Psi_{n,s}$ is not unique. However, under the assumptions of Theorem (2.8) an asymptotic expansion of the form (1.2) is valid whenever $X_1, X_2, \dots$ is *strictly stationary*. More precisely, if (2.1)–(2.4) and (1.4) are satisfied, then there exist polynomials $a_{r,j}(t)$, $r=2, \dots, s$, $j=r-2, \dots, s-2$, such that uniformly for $\|t\| \leqq 1$, $r=2, \dots, s, j=0, \dots, s-2$

$$\chi_{r,n}(t)=\sum_{j=r-2}^{s-2} n^{-j/2}\, a_{r,j+r}(t)+o(n^{-(s-2)/2}).$$

In particular, $a_{22}(t)=-t^T \Sigma t/2$. Define the formal expansion $\Psi_{n,s}^*$ by its characteristic function

$$\tilde{\Psi}_{n,s}^*(t)=\exp(-t^T \Sigma t/2)+\sum_{r=1}^{s-2} n^{-r/2}\, \tilde{P}_r^*(t)$$

where $\tilde{P}_r^*(t)$ is defined by the formal identity

$$\exp\left(-t^T \Sigma t/2+\sum_{j=1}^{s-2} \tau^j a_{2,j+2}(t)+\sum_{r=3}^{\infty} (r!)^{-1} \sum_{j=0}^{s-r} \tau^{r+j-2}\, a_{r,r+j-2}(t)\right)$$
$$=\exp(-t^T \Sigma t/2)+\sum_{r=1}^{\infty} \tau^r\, \tilde{P}_r^*(t).$$

Then

$$\int (1+\|x\|^s)\, d\,|\Psi_{n,s}-\Psi_{n,s}^*|=o(n^{-(s-2)/2})$$

where $|\ |$ denotes the variation measure.

Our conditions (2.3)–(2.6) will now be discussed in detail. Their wide applicability is due to the fact that we may choose the $\sigma$-fields $\mathscr{D}_j$. If $\mathscr{D}_j$ is the $\sigma$-field generated by $X_j$, the condition (2.3) is satisfied, but conditions (2.5) and (2.6) can hardly be checked, when $X_1, X_2, \dots$ is a weakly dependent shift. Here we show that conditions (2.3)–(2.6) are satisfied in Examples (1.11)–(1.15) when appropriate $\sigma$-fields are chosen.

(i) In Example (1.11) we choose $\mathscr{D}_j$ the $\sigma$-field generated by $Y_j$. Then (2.3), (2.4) and (2.6) hold. By assumption, there exists an open subset $W$ of $\mathbb{R}^{2m-1}$ with $P\{(Y_1, \dots, Y_{2m-1}) \in W\}>0$ such that the map $H$:

$$(x_1, \dots, x_{2m-1})$$
$$\to \left(x_1, \dots, x_{m-1}, x_{m+1}, \dots, x_{2m-1}, \sum_{j=1}^{m} h(x_j, x_{j+1}, \dots, x_{m+j-1})\right)$$

is a local isomorphism on $W$.

Write $P_W$ for the conditional distribution of $Y_1, Y_2, \ldots,$ given $(Y_1, \ldots, Y_{2m-1}) \in W$, and let $G$ be a Lebesgue-density of the distribution of $H$ under $P_W$. The Riemann-Lebesgue lemma and the dominated convergence theorem imply that

$$\lim_{|t|\to\infty} \int \left(\int \exp(it\, h_{2m-1})\, G(h_1, \ldots, h_{2m-1})\, dh_{2m-1}\right) dh_1 \ldots dh_{2m-2} = 0.$$

This implies that for $|t| \geqq d$

$$E\,|E(e^{it(X_1+\ldots+X_m)}|\, Y_j\colon j \neq m)| \leqq e^{-d}\, P(W) + 1 - P(W)$$

whence (2.5) follows.

(ii) In Example (1.12) let $\mathscr{D}_j$ be the $\sigma$-field generated by $Y_j$. Then (2.4) and (2.6) are obvious, (2.3) follows from (1.12(i)), and (2.5) follows from (1.12(ii)) and the relation

$$E\,|E(e^{it(X_{n-m}+\ldots+X_{n+m})}|\, \mathscr{D}_j\colon j \neq n)| = \left| E \exp\left(it \sum_{p=-m}^{m} c_p Y_n\right)\right|.$$

The condition (1.12(i)) and (1.12(ii)) are satisfied for an autoregressive process defined in (1.12). Condition (1.12(i)) follows from the representation of the coefficients $c_p$ as

$$c_p = \sum_{1}^{r} k_j x_j^p$$

with constants $k_1, \ldots, k_r$ and $x_1, \ldots, x_r$ the different zeroes of

$$\beta_0 z^r + \ldots + \beta_r = 0.$$

Condition (1.12(ii)) follows from

$$\sum_{p=0}^{\infty} c_p = (\beta_0 + \ldots + \beta_r)^{-1}$$

which cannot be zero.

(For details see Anderson (1971), Sect. 5.2.)

(iii) In Example (1.13) the $\sigma$-fields $\mathscr{D}_j$ are generated by $\xi_j$. Then (2.3) and (2.6) are satisfied. Condition (2.4) follows from

$$\sup |P(x, A) - P(x', A)| < 1$$

(see Statulevicius (1969) II, p. 644, Lemma 4).

Condition (2.5) follows from Cramér's condition for $X_1$ and Lemma 2 in Statulevicius (1969) II, p. 638.

(iv) In Example (1.14) the $\sigma$-fields $\mathscr{D}_j$ are generated by $Y_j$. Then (2.3) is obvious. Condition (2.4) is satisfied since the spectral density of $Y_0, Y_{\pm 1}, \ldots$ is analytic (see Ibragimov (1970), p. 35, Theorem 6). For arbitrary $n$ the conditional distribution of $Y_n$, given ($Y_j$, $j \in \mathbb{Z}$, $j \neq n$), is a *nondegenerate* normal distribution. Hence (2.5) holds. In order to prove (2.6) we notice that the conditional distribution of $Y_0$, given $(Y_j: j \neq 0)$, and given $(Y_j: 0<|j| \leqq m)$, are both normal. Hence it suffices to show that there exists a positive constant $d$ such that for $m=1,2,\ldots$

$$E(E(Y_0 | Y_j: j \neq 0) - E(Y_0 | Y_j: 0<|j| \leqq m))^2$$
$$= E E(Y_0 | Y_j: j \neq 0)^2 - E E(Y_0 | Y_j: 0<|j| \leqq m)^2 \leqq d^{-1} e^{-dm}.$$

This, however, follows from Grenander and Szegö (1958), p. 189, Theorem.

(v) In Example (1.15) we take the $\sigma$-fields $\mathscr{D}_j$ generated by $X_j$. Then condition (2.3) is true for the $2p$-vectors

$$Z_n = (X_n \cos(n\lambda_j), X_n \sin(n\lambda_j); j=1,\ldots,p), \qquad n=1,2,\ldots.$$

As in Example (1.13) we obtain that (2.4) and (2.6) hold. However, condition (2.5) will not be satisfied in general. Notice first that for $t=(t_1,t_2)\in\mathbb{R}^{2p}$

$$|E(\exp(it^T(Z_{n-m}+\ldots+Z_{n+m}))|X_j: j \neq n)|$$
$$= |E(\exp(it_1^T(\cos n\lambda_1,\ldots,\cos n\lambda_p) X_n$$
$$+ it_2^T(\sin n\lambda_1,\ldots,\sin n\lambda_p) X_n)| X_j: j=n-1, n+1)|.$$

If

$$(2.13) \qquad |t_1^T(\cos n\lambda_1,\ldots,\cos n\lambda_p) + t_2^T(\sin n\lambda_1,\ldots,\sin n\lambda_p)| \geqq d$$

then $E|E(\exp(it^T Z_n)|X_j: j \neq n)| \leqq e^{-d}$ follows from the fact that (2.5) holds for the sequence $X_1, X_2, \ldots$. However, given $d>0$ there will not exist $d'>0$ such that (2.13) holds whenever $\|t\| \geqq d'$. According to Remark (3.44) the weaker assumption (3.45) will be sufficient for the validity of higher order approximations. We have to show that (2.13) holds for sufficiently many $n$. Let $\Sigma$ be the asymptotic covariance matrix of $\sum_1^N Z_j$, i.e.

$$\Sigma = \lim n^{-1} \operatorname{Cov}\left(\sum_1^N Z_j\right).$$

The assumption $\lambda_i + \lambda_j \neq 0 \bmod (2\pi)$ implies that $\Sigma$ is nonsingular. Hence there exists $\varepsilon>0$ such that $N \geqq \varepsilon^{-1}$ implies

$$(2.14) \qquad E\left(t^T \sum_1^N Z_j\right)^2 \geqq \varepsilon N \|t\|^2.$$

Write $H_n = t_1^T(\cos n\lambda_1,\ldots,\cos n\lambda_p) + t_2^T(\sin n\lambda_1,\ldots,\sin n\lambda_p)$. Then

$$E\left(t^T \sum_1^N Z_j\right)^2 = \sum_{i,j=1}^N H_i H_j E X_i X_j$$
$$\leqq d \|t\| p \sum_{i,j=1}^N |E X_i X_j| + \|t\|^2 p^2 \sum' |E X_i X_j|$$

where the second sum extends over all $i,j \in \{1, \ldots, N\}$ for which (2.13) holds. We choose $d>0$ such that

$$d^{1/2} < \varepsilon p^{-1} \left(\sum_1^\infty |E X_1 X_j|\right)^{-1}$$

and

$$\|t\| \geqq d^{1/2}.$$

Then (2.14) implies that

$$N^{-1} \# \{j \in \{1, \ldots, N\} : (2.13) \text{ holds for } j\}$$

remains bounded away from zero.

*Outline of proofs for Theorems* (2.8)-(2.12). Theorem (2.8) follows from Lemmas (3.3) and (3.33). If we take $\kappa = (s-2+\delta)/2$ in (2.8), then (2.9) follows with Sazonov's lemma (see [2], p. 24, Corollary 3.2).

For Theorem (2.10) we sketch the proof. For details see [6]. Notice first that

$$E f(S_n) = E f(S_n^*) + o(n^{-(s-2)/2})$$

(see the proof of Lemma (3.3)). As in [6], for nonnegative $k$-vector $\alpha$ we can expand $E D^\alpha f(S_n^* + n^{-\varepsilon} U)$, where $U$ is a random vector with distribution $K$ from Lemma (3.3). Finally, $E f(S_n^* + n^{-\varepsilon} U - x U)$ is expanded in a Taylor series in $x$ at $x = n^{-\varepsilon}$.

*Proof of Theorem* (2.11). Lemma (3.3) and Lemma (3.33), applied for $\kappa = \varepsilon$, imply that

$$\begin{aligned} &E \|S_n\|^{s_0} 1_{\{\|S_n\| \geqq ((s-2)\log n)^{1/2}\}} \\ &\quad = \int \|x\|^{s_0} 1_{\{\|x\| \geqq ((s-2)\log n)^{1/2}\}} \Psi_{n,s}(dx) + \omega(g; n^{-\varepsilon}) + o(n^{-(s-2)/2}) \end{aligned}$$

where

$$g(x) = \|x\|^{s_0} (1 + \|x\|^{s_0})^{-1} 1_{\{\|x\| > ((s-2)\log n)^{1/2}\}}.$$

Since $g$ is constant on $\{\|x\| < ((s-2)\log n)^{1/2}\}$ and bounded on $\mathbb{R}^k$,

$$\omega(g; n^{-\varepsilon}) \leqq c \Phi_\Sigma \{\|x\| > ((s-2)\log n)^{1/2} - 1\} = o(n^{-(s-2)/2}).$$

Now the relation

$$\int \|x\|^{s_0} 1_{\{\|x\| > ((s-2)\log n)^{1/2}\}} \Psi_{n,s}(dx) = o(n^{-(s-2)/2})$$

implies the assertion.

## 3. Lemmas

To simplify our notations we use the following convention. Primary variables are the numbers $s$, $s_0$, $k$, $\beta_{s+1}$, the covariance matrix $\Sigma$ in (1.4), and the constant $d$ in (2.3)-(2.6). The symbols $\varepsilon$, $c$, $C$ will be used for finite positive generic constants which depend on the primary variables, but not on $n$. The symbol $\beta$ is used for a constant depending on our primary variables which has to be chosen appropriately. The variables $m$ and $K$ may depend on the primary variables and on $n$. We denote

$$S_n = n^{-1/2}(X_1 + \ldots + X_n)$$

and define

$$T(x) = \begin{cases} x & \text{if } \|x\| \leqq n^\beta \\ x n^\beta \, \psi(\|x\| \, n^{-\beta})/\|x\| & \text{otherwise} \end{cases} \tag{3.1}$$

where $\psi \in C^\infty(0, \infty)$ satisfies

$$\psi(r) = r \quad \text{if } r \leqq 1,$$
$$\psi \text{ is increasing,}$$
$$\psi(r) = 2 \quad \text{if } r \geqq 2.$$

For $j = 1, \ldots, n$ let

$$Y_j = T(X_j),$$

and

$$Z_j = Y_j - E\, Y_j.$$

We suppress the index $n$ at $T$, $Y_j$, and $Z_j$.

Define

$$S_n^* = n^{-1/2}(Z_1 + \ldots + Z_n)$$

and

$$H_n(t) = E \exp(i t^T S_n^*),$$

where $t^T a$ denotes the scalar product of $t$ and $a$. For nonnegative integral $k$-vector

$$\alpha = (\alpha_1, \ldots, \alpha_k)$$

write

$$|\alpha| = \alpha_1 + \ldots + \alpha_k$$

and

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial t_1^{\alpha_1} \ldots \partial t_k^{\alpha_k}}.$$

(3.2) **Lemma** (Petrov). *Let $X$ be a $k$-variate random vector on an abstract measure space $(\Omega, \mathscr{A}, P)$, let $\mathscr{B} \subset \mathscr{A}$ be a sub-$\sigma$-field, and $c$ be positive. Suppose that for $t \in \mathbb{R}^k$, $\|t\| \geqq c$*

$$E\,|E(\exp(i t^T X)\,|\,\mathscr{B})| \leqq 1 - c.$$

*Then there exists a positive $d$ such that for $\|t\| \leqq c$*

$$E\,|E(\exp(i t^T X)\,|\,\mathscr{B})| \leqq \exp(-d\,\|t\|^2).$$

*Proof.* Let $X_1, \mathscr{B}_1$ and $X_2, \mathscr{B}_2$ be two independent copies of $X, \mathscr{B}$ on some measure space $(\Omega', \mathscr{A}', P')$, and write $\mathscr{B}_3$ for the $\sigma$-field generated by $\mathscr{B}_1 \cup \mathscr{B}_2$.

Then for $t \in \mathbb{R}^k$

$$E|E(\exp(it^T X)|\mathscr{B})|^2 = E|E(\exp(it^T X_1)|\mathscr{B}_3)|^2$$

and

$$\begin{aligned}|E(\exp(it^T X_1)|\mathscr{B}_3)|^2 &= E(\exp(it^T(X_1 - X_2))|\mathscr{B}_3)\\ &= E(\cos t^T(X_1 - X_2)|\mathscr{B}_3) \quad \text{a.e.}\end{aligned}$$

The relation

$$1 - \cos(2x) \leqq 4(1 - \cos x), \qquad x \in \mathbb{R}$$

implies that for $m = 0, 1, 2, \ldots$

$$1 - \cos(2^m x) \leqq 4^m(1 - \cos x)$$

Let

$$g(t) = E|E(\exp(it^T X)|\mathscr{B})|^2.$$

Then

$$1 - g(2^m t) \leqq 4^m(1 - g(t)), \qquad m = 0, 1, 2, \ldots.$$

Fix $t \in \mathbb{R}^k$ with $\|t\| \leqq c$. Choose $m \geqq 0$ such that

$$2^{m+1}\|t\| > c \geqq 2^m \|t\|.$$

Then

$$g(t) \leqq 1 - c \qquad \text{for } \|t\| \geqq c$$

implies

$$\begin{aligned}1 - g(t) &\geqq 4^{-m-1}(1 - g(2^{m+1} t))\\ &\geqq 4^{-m-1} c \geqq c^{-1}\|t\|^2/4\end{aligned}$$

and hence

$$g(t) \leqq \exp(-c^{-1}\|t\|^2/4).$$

This proves the lemma.

(3.3) **Lemma** (Bhattacharya [2], Sweeting [14]). *Let $f\colon \mathbb{R}^k \to \mathbb{R}$ denote a function such that $|f(x)| \leqq M(1 + \|x\|^{s_0})$ for every $x \in \mathbb{R}^k$. Then for $\kappa > 0$*

$$\begin{aligned}&|Ef(S_n) - \int f\, d\Psi_{n,s}|\\ &\quad \leqq cM \sup_{|\alpha| \leqq k+1+s_0} \int |D^\alpha[(H_n(t) - \hat{\Psi}_{n,s}(t))\, \hat{K}(n^{-\kappa} t) \exp(it^T e_n)]|\, dt\\ &\qquad + c\,\omega(g\colon n^{-\kappa}) + o(n^{-(s-2+\delta)/2})\end{aligned}$$

where

$$\delta > 0,\ g(x) = f(x)/(1 + \|x\|^{s_0}),$$
$$\omega(g\colon n^{-\kappa}) = \int \sup\{|g(x+y) - g(x)|\colon \|y\| \leqq n^{-\kappa}\}\ \Phi_{\Sigma_n}(dx),$$

and

$$e_n = n^{-1/2} \sum_1^n E Y_j.$$

Here $\Sigma_n = \operatorname{Cov}(S_n)$ and $\hat{K}$ is a continuous function with compact support.

*Proof.* Let $S'_n = n^{-1/2} \sum_1^n Y_j$. Let $\varepsilon > 0$ to be determined later, and define

$$A = \{\|S_n\| \leqq n^\varepsilon\}, \quad B = \{\|S'_n\| \leqq n^\varepsilon\}.$$

Then

$$E\|S_n-S'_n\|^{s_0}=E\|S_n-S'_n\|^{s_0}(1_{AB}+1_{A\bar{B}}+1_{\bar{A}B}+1_{\bar{A}\bar{B}})$$
$$=I_1+I_2+I_3+I_4, \quad \text{say.}$$

Using $(x+y)^n\leqq 2^n(x^n+y^n)$, $n,x,y>0$, we obtain

$$I_1\leqq(2n^\varepsilon)^{s_0}P\{S_n\neq S'_n\},$$
$$I_2\leqq 2^{s_0}(E\|S_n\|^{s_0}1_{A\bar{B}}+E\|S'_n\|^{s_0}1_{\bar{B}})\leqq 2^{s_0}((n^\varepsilon)^{s_0}P\{S_n\neq S'_n\}+E\|S'_n\|^{s_0}1_{\bar{B}}),$$
$$I_3\leqq 2^{s_0}(E\|S_n\|^{s_0}1_{\bar{A}}+(n^\varepsilon)^{s_0}P\{S_n\neq S'_n\}),$$
$$I_4\leqq 2^{s_0}(E\|S_n\|^{s_0}1_{\bar{A}}+E\|S'_n\|^{s_0}1_{\bar{B}}).$$

Hence

$$E\|S_n-S'_n\|^{s_0}\leqq c(E\|S_n\|^{s_0}1_{\bar{A}}+E\|S'_n\|^{s_0}1_{\bar{B}}+n^{\varepsilon s_0}P\{S_n\neq S'_n\}).$$

Furthermore, Lemma (3.30) implies that for some positive $\delta$

$$E\|S_n\|^{s_0}1_{\bar{A}}=E\|S_n\|^{s_0}-E\|S_n\|^{s_0}1_A$$
$$\leqq|E\|S'_n\|^{s_0}-E\|S_n\|^{s_0}|+|E\|S'_n\|^{s_0}-E\|S'_n\|^{s_0}1_B|+|E\|S'_n\|^{s_0}1_B-E\|S_n\|^{s_0}1_A|$$
$$\leqq o(n^{-(s-2+\delta)/2})+2n^{\varepsilon s_0}P\{S_n\neq S'_n\}+E\|S_n\|^{s_0}1_{\bar{B}}.$$

This yields

$$E\|S_n-S'_n\|^{s_0}\leqq c(E\|S'_n\|^{s_0}1_{\bar{B}}+n^{\varepsilon s_0}P\{S_n\neq S'_n\})+o(n^{-(s-2+\delta)/2})$$

Lemma (3.33) implies that for arbitrary positive integer $r$

$$\sup_n E\|S'_n\|^r<\infty$$

and hence a choice

$$0<\varepsilon<(s+1)\beta-(s-2)/2$$

yields

$$E\|S_n-S'_n\|^{s_0}=o(n^{-(s-2)/2}).$$

This implies

$$|Ef(S_n)-Ef(S'_n)|=o(n^{-(s-2)/2}).$$

Finally,

$$e_n=O(n^{1/2}n^{-s\beta})=o(n^{-(s-2+\delta)/2}),$$

and

$$\int f(\cdot+e_n)\,d\Psi_{ns}=\int f\,d\Psi_{ns}+o(n^{-(s-2+\delta)/2}).$$

This together with $H_n(t)\exp[it^T e_n]=E\exp[it^T S'_n]$, Lemma (11.6), p. 98 of Bhattacharya and Rao (1976) (which estimates the $L^1(\mathbb{R}^k)$ norm of functions by the integral over derivatives of the Fourier-transform) and the smoothing inequality Lemma 5 of Sweeting (1977) proves the lemma.

Let $[a]$ denote the largest integer smaller or equal to $a$. For $x\in\mathbb{R}^k$ let

$$f_{j,n}(x)=\prod_{p=1}^{k_j}(a_{p,j}^T x)^{\alpha_{p,j}}\exp(in^{-1/2}t^T x), \quad j=1,\dots,n,$$

where $a_{p,j}$ are bounded vectors in $\mathbb{R}^k$, and $\alpha_{p,j}$ are nonnegative integers. Then

(3.4) **Lemma.** *Let* $U_1=\prod_{j=1}^{k} f_{j,n}(Z_j)$, $U_2=\prod_{j=k+l}^{n} f_{j,n}(Z_j)$ *and* $r=\sum_{p,j}\alpha_{p,j}$. *If condition* (2.3) *and* (2.4) *are fulfilled we have*

(i) $$|\operatorname{cov}(U_1, U_2)| \leqq c n^{\beta r} \exp(-d[l/3]).$$

*If in addition* $0 \leqq r \leqq s$ *we obtain*

(ii) $$|\operatorname{cov}(U_1, U_2)| \leqq c(1+\beta_{s+1}) \exp(-d[l/3]/(s+1)).$$

*The inequality* (ii) *still holds when we replace* $Z_j$ *by* $X_j$ *in the definition of* $U_1$ *and* $U_2$.

*Proof.* Let $g=[l/3]$. Using condition (2.3) there exists a $\mathscr{D}_{j-g}^{j+g}$ measurable random vector $\bar{X}_j$ such that $E\|X_j-\bar{X}_j\| \leqq \frac{1}{d}\exp(-dg)$. Let $\bar{U}_p$, $p=1,2$ be defined similarly as $U_p$, $p=1,2$ with $Z_j$ replaced by $\bar{X}_j - E Y_j$. Since $U_1$ is $\mathscr{D}_1^{k+g}$-measurable and $U_2$ is $\mathscr{D}_{k+l-g}^{n}$-measurable, condition (2.4) entails

(3.5) $$\sup_{A,B} |P(\bar{U}_1\in A, \bar{U}_2\in B) - P(\bar{U}_1\in A)\, P(\bar{U}_2\in B)| \leqq \frac{1}{d}\exp(-dg).$$

Using

(3.6) $$|U_1 U_2 - \bar{U}_1 \bar{U}_2| \leqq |U_1|\,|U_2-\bar{U}_2| + |\bar{U}_2|\,|U_1-\bar{U}_1|,$$

we derive (the truncation function $T$ is Lipschitz!)

(3.7) $$|\operatorname{cov}(U_1, U_2) - \operatorname{cov}(\bar{U}_1, \bar{U}_2)| \leqq c n^{r\beta} \exp(-dg).$$

By (3.5), $|\operatorname{cov}(\bar{U}_1, \bar{U}_2)| \leqq c n^{r\beta}\exp(-dg)$, thus proving (i).

(ii) Truncation of $X_j$ at $M$, i.e. replacing $X_j$ by $T_M(X_j)$, where

$$T_M(x) = \begin{cases} x & \text{if } \|x\| \leqq M \\ x M \psi(\|x\|\, M^{-1})/\|x\| & \text{otherwise,} \end{cases}$$

yields random variables

$$U_{1,M} = \prod_1^k f_{j,n}(T_M(Z_j)) \quad \text{and} \quad U_{2,M} = \prod_{k=l}^{n} f_{j,n}(T_M(Z_j)).$$

It is not hard to show that

(3.8) $$|E\, U_{1,M} U_{2,M} - E\, U_1 U_2| + |E\, U_{1,M}\, E\, U_{2,M} - E\, U_1\, E\, U_2| \leqq c\beta_{s+1} M^{-(s+1-r)}.$$

Let $\bar{U}_{1,M} = \prod_1^k f_{j,n}(T_M(T(\bar{X}_j) - E Y_j))$ and let $\bar{U}_{2,M}$ be defined similarly. Note again that $T_M$ and $T$ are Lipschitz functions. Using (3.6) and (3.8) and condition (2.3) we deduce

(3.9) $$|\operatorname{cov}(U_1, U_2) - \operatorname{cov}(\bar{U}_{1,M}, \bar{U}_{2,M})| \leqq c\beta_{s+1} M^{-(s+1-r)} + c M^{r-1} \exp(-dg).$$

Relation (3.5) implies $|\operatorname{cov}(\bar{U}_{1,M}, \bar{U}_{2,M})| \leqq M^r \exp(-dg)$. This together with (3.9) and the optimal choice of $M$ proves the assertion.

In order to expand the c.f. $H_n(t)$ and its derivatives in terms of $n^{-1/2}$ we have to estimate the cumulants of $S_n^*$ (which determine the coefficients of the expansion) as well as the derivatives of the remainder term of the expansion.

It is convenient to introduce some more notations.

Let $E_t U = EU \exp[it^T S_n^*]/H_n(t)$.

Define the semiinvariants of order $p$

$$\kappa_t(a_1^T S_n^*, \ldots, a_p^T S_n^*) = \frac{\partial}{\partial \varepsilon_1} \cdots \frac{\partial}{\partial \varepsilon_p}\bigg|_{\varepsilon_1 = \ldots = \varepsilon_p = 0} \ln H_n(t + \varepsilon_1 a_1 + \ldots + \varepsilon_p a_p),$$

where $a_1, \ldots, a_p \in \mathbb{R}$. Write

$$\kappa_t(\underbrace{a^T S_n^*, \ldots, a^T S_n^*}_{j\text{-times}}, \underbrace{b^T S_n^*, \ldots, b^T S_n^*}_{l\text{-times}}) = \kappa_t(a^T S_n^{*j}, b^T S_n^{*l}).$$

The Taylor expansion of $\ln H_n(t)$ can be written

$$\ln H_n(t) = \sum_{r=2}^{s} \kappa_0(it^T S_n^{*r})\, r!^{-1} + R_{s+1}(t),$$

where

(3.10) $$R_{s+1}(t) = s!^{-1} \int_0^1 (1-\eta)^s \kappa_{\eta t}(it^T S_n^{*(s+1)})\, d\eta.$$

Since we have to evaluate derivatives of expansions in dimension $k > 1$ note that

(3.11) $$\frac{\partial^l}{\partial \varepsilon^l}\bigg|_{\varepsilon=0} R_{s+1}(t + \varepsilon a) = s!^{-1} \int_0^1 d\eta (1-\eta)^s \sum{}^* c_{pq} \eta^{p+q-s-1} \kappa_{\eta t}(it^T S_n^{*p}, ia^T S_n^{*q})$$

where $c_{pq} \geqq 0$ are combinatorial coefficients and the summation $\sum^*$ extends over all $p, q \geqq 0$ such that $0 \leqq q \leqq l$, $0 \leqq p \leqq s+1$ and $p + q \geqq s+1$.

Note that the semiinvariants are multilinear forms in the random variables. Hence,

(3.12) $$\kappa_{\eta t}(t^T S_n^{*p}, a^T S_n^{*q}) = \sum \kappa_{\eta t}(t^T Z_{j_1}, \ldots, t^T Z_{j_p}, a^T Z_{l_1}, \ldots, a^T Z_{l_q})\, n^{-(p+q)/2}$$

where the summation extends over all indices $1 \leqq j_1, \ldots, j_p \leqq n$ and $1 \leqq l_1 \leqq \ldots \leqq l_q \leqq n$ and the semiinvariants on the r.h.s. are given by

$$\frac{\partial}{\partial \varepsilon_1} \cdots \frac{\partial}{\partial \varepsilon_{p+q}}\bigg|_{\varepsilon_1 = \ldots = \varepsilon_{p+q} = 0} \ln E \exp[i\eta t^T S_n^* + \varepsilon_1 t^T Z_{j_1} + \ldots + \varepsilon_p t^T Z_{j_p} + \varepsilon_{p+1} a^T Z_{l_1} + \ldots + \varepsilon_{p+q} a^T Z_{l_q}].$$

Similarly one defines a semiinvariant of order $r$ for random variables $V_{j_1}, \ldots, V_{j_r}$ (instead of $t^T Z_{j_r}$), where $V_{j_r}$ is $\sigma(Z_{j_r})$-measurable. It is well known that (see e.g. Zhurbenko (1972))

$$\kappa_t(V_{j_1}, \ldots, V_{j_r}) = \sum_{v=1}^{r} \sum{}^{*} c(I_1, \ldots, I_v)(-1)^v \prod_{j=1}^{v} E_t \prod_{l \in I_j} V_l \tag{3.13}$$

where $c(I_1, \ldots, I_r)$ denote combinatorial coefficients and the summation extends over all decompositions $\{1, 2, \ldots, r\} = I_1 \cup I_2 \cup \ldots \cup I_v$ into $v$ disjoint parts. Assume that $j_1 \leqq j_2 \leqq \ldots \leqq j_p < j_{p+1} \leqq j_{p+2} \leqq \ldots \leqq j_r$ and that $j_{p+1} - j_p$ is maximal. Let $V_j'$, $Z_j'$ denote independent copies of $V_j$, $Z_j$, $1 \leqq j \leqq n$ (having the same joint distribution).

The following identity easily follows from (3.12):

$$\begin{aligned} 0 = \frac{\partial}{\partial \varepsilon_1} \cdots \frac{\partial}{\partial \varepsilon_r}\bigg|_{\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_r = 0} & \ln E \exp[it^T(S_n^* + S_n^{*\prime}) \\ & + \varepsilon_1 V_{j_1} + \ldots + \varepsilon_p V_{j_p} + \varepsilon_{p+1} V'_{j_{p+1}} + \ldots + \varepsilon_r V'_{j_r}] \\ = \sum_{v=1}^{r} \sum{}^{*} c(I_1, \ldots, I_r)(-1)^{v-1} & \prod_{j=1}^{v} (E_t \prod_{l \in I_j \cap J_1} V_l) E_t \prod_{l \in I_j \cap J_2} V_l \end{aligned} \tag{3.14}$$

where $J_1 = \{1, 2, \ldots, p\}$ and $J_2 = \{p+1, \ldots, r\}$.

Let $S_I^{(r)} = in^{-1/2} t^T \sum{}^{*} Z_j$, where $\sum{}^{*}$ extends over all $1 \leqq j \leqq n$ such that $|j - j_1| > mr$ for every $j_1 \in I \subset \{1, 2, \ldots, n\}$. Furthermore let

$$Z_I = \prod_{j \in I} \prod_{p=1}^{r_j} (a_{jp}^T Z_j), \quad \text{where} \quad a_{jp} \in \mathbb{R}^k, \ \|a_{jp}\| \leqq 1. \tag{3.15}$$

(3.16) **Lemma.** *Under conditions* (2.1)–(2.4) *we have*

$$\begin{aligned} |E_t Z_I| \leqq c\{E|Z_I| [\sup\{|E \exp[S_I^{(v)}]| : 0 \leqq v \leqq K\} + \rho^K] \\ + c \exp(-dm/3) 2^K n^{|I|\beta}\} |H_n(t)|^{-1} \end{aligned}$$

*for every* $\|t\| \leqq cn^{1/2} m^{-1/2}$ *and some* $0 < \rho < 1$.

*Proof.* Using a method similar to that of A.N. Tikhomirov (1980) we arrive at the identity (write $\Delta_I^{(r)} = \exp[S_I^{(r-1)} - S_I^{(r)}] - 1$)

$$H_n(t) E_t Z_I = E Z_I \exp[S_I^{(1)}] + E Z_I \Delta_I^{(1)} \exp[S_I^{(1)}]$$

and repeating this step

$$= \sum_{v=1}^{K} E Z_I \left( \prod_{r=1}^{v-1} \Delta_I^{(r)} \right) \exp[S_I^{(v)}] + E Z_I \prod_{r=1}^{K} \Delta_I^{(r)} \exp[S_I^{(K)}].$$

Since $\exp[S_I^{(v)}]$ is weakly dependent on $\Delta_I^{(r)}$ and $Z_I$, $r \leqq v-1$, Lemma 3.3 shows

$$E Z_I \prod_{r=1}^{v-1} \Delta_I^{(r)} \exp[S_I^{(v)}] = \left( E Z_I \prod_{r=1}^{v-1} \Delta_I^{(r)} \right) E \exp[S_I^{(v)}] + 2^v O(\exp(-dm/3) n^{|I|\beta}.$$

Furthermore,

$$\left| E Z_I \prod_{r=1}^{v-1} \Delta_I^{(r)} \right| \leqq E |Z_I \prod{}' \Delta_I^{(r)}| 2^{v/2},$$

where $\prod'$ denotes the product over all even indices $r \leqq v-1$,

$$\leqq E|Z_I| \prod' E|\Delta_I^{(r)}| 2^{v/2} + v 2^v O(\exp(-dm/3) n^{|I|\beta}).$$

Since $E|\Delta_I^{(r)}| \leqq (E|S_I^{(r-1)} - S_I^{(r)}|^2)^{1/2} \leqq (|I| mcn^{-1} \|t\|^2)^{1/2} < \rho/2$, $\rho < 1$, we have

$$|H_n(t) E_t Z_I| \leqq cE|Z_I| [\sup\{|E \exp[S_I^{(v)}]| : v \leqq K-1\} + \rho^K] + O(2^K n^{|I|\beta} \exp(-md/3))$$

thus proving Lemma (3.16).

Let $Z_{I_1}$, $Z_{I_2}$ denote random variables defined as in (3.15), where $\min I_2 - \max I_1 \geqq m$.

Then the following estimate holds

(3.17) **Lemma.** *For every $t$ fulfilling $\|t\| \leqq \varepsilon n^{\frac{1}{2}-\beta} m^{-1}$, $0 < \varepsilon < 1$,*

$$|E_t Z_{I_1} Z_{I_2} - E_t Z_{I_1} E_t Z_{I_2}| \leqq c n^{\beta(|I_1|+|I_2|)} [(\exp(-dm^{1/2}) + (m^{1/2})!^{-1} \varepsilon^{m^{1/2}}] |H_n(t)|^{-2}.$$

*Proof.* Let $Z'_j$, $1 \leqq j \leqq n$ denote an independent copy of the series $Z_j$, $1 \leqq j \leqq n$, and let $Z'_{I_j}$ be defined in the same way as $Z_{I_j}$, $j=1,2$. Furthermore, introduce $\tilde{T}_j = \sum_p^{(j)} in^{-1/2} t^T (Z_p + Z'_p)$, $j=1,2$ and $\tilde{S}_m = in^{-1/2} \sum_p^{(3)} t^T (Z_p + Z'_p)$, where $\sum_p^{(1)}$ denotes summation over $p=1, \ldots, \max(I_1)$, $\sum_p^{(3)}$ denotes summation over $p = \max(I_1)+1, \ldots, \max(I_1)+m-1$ and $\sum_p^{(2)}$ denotes the summation over the remaining indices. Let $U_j = (Z_{I_j} - Z'_{I_j}) \exp[\tilde{T}_j]$.

Write

$$\begin{aligned}(3.18) \qquad \Delta &= (E_t Z_{I_1} Z_{I_2} - E_t Z_{I_1} E_t Z_{I_2}) H_n(t)^2 = EU_1 \exp[\tilde{S}_m] U_2 \\ &= \sum_{r=0}^{K-1} r!^{-1} EU_1 \tilde{S}_m^r U_2 + E|U_1 U_2| \theta(t) (mn^{\beta-\frac{1}{2}} \|t\|)^K K!^{-1}\end{aligned}$$

where $|\theta(t)| \leqq 1$. Let $T_j = in^{-1/2} t^T (Z_j + Z'_j)$.

Expanding $\tilde{S}_m^r$ we have

$$EU_1 \tilde{S}_m^r U_2 = \sum_{j_1, \ldots, j_r} EU_1 T_{j_1} \ldots T_{j_r} U_2.$$

Note that every sequence $U_1, T_{j_1}, \ldots, T_{j_r}, U_2$ of random variables contains a 'gap' of length at least $[3m/(r+1)]$, say between $T_{j_p}$ and $T_{j_{p+1}}$.

Hence,

$$\begin{aligned}(3.19) \qquad & E(U_1 T_{j_1} \ldots T_{j_p})(T_{j_{p+1}} \ldots T_{j_r} U_2) \\ &= EU_1 T_{j_1} \ldots T_{j_p} E T_{j_{p+1}} \ldots T_{j_r} U_2 + o(n^{\beta(|I_1|+|I_2|)} \exp(-md/(r+1)))\end{aligned}$$

using Lemma (3.4).

Note that by construction $EU_1 T_{j_1} \ldots T_{j_p} = 0$.

Relations (3.18) and (3.19) imply

$$|\Delta| \leqq n^{\beta(|I_1|+|I_2|)} \sum_{r=0}^{K-1} \exp(-md/(r+1)) r!^{-1} + E|U_1 U_2| \varepsilon^K K!^{-1}.$$

Choosing $K=[m^{1/2}]$ we obtain

$$|\Delta| \leqq c n^{\beta(|I_1|+|I_2|)}(\exp(-dm^{1/2})+(m^{1/2})!^{-1}\varepsilon^{m^{1/2}})$$

which proves the Lemma (3.17).

(3.20) **Lemma.** *Let $\theta_n(t)$ be as in* (3.27). *Then*

$$\left|\frac{\partial^l}{\partial\varepsilon^l}\right|_{\varepsilon=0} R_{s+1}(t+\varepsilon a)\left| \leqq c(1+\beta_{s+1})n^{-(s-2)/2-\varepsilon}(1+\theta_n(t)^{s+1+l})(1+\|t\|^{s+1}),\right.$$

$0<\varepsilon<1$, *for every t fulfilling*

$$\theta_n(t)<\infty, \qquad \|t\| \leqq \varepsilon n^{-\varepsilon-\beta+1/2}, \tag{3.21}$$

*where $a \in \mathbb{R}^k$ with $\|a\|<1$.*

*Proof.* Relations (3.11) and (3.12) show that it is sufficient to estimate

$$\sum_{g=0}^{n-r} c(r) \sum{}^{(g)} |\kappa_{t\eta}(V_{j_1},\ldots,V_{j_r})|, s+1 \leqq r \leqq s+1+l, \tag{3.22}$$

where $V_j=a_j^T Z_j$, $\|a_j\| \leqq 1$ and the summation $\sum^{(g)}$ extends over all indices $j_1 \leqq j_2 \leqq \ldots \leqq j_r$ such that $g=\sup\{|j_{p+1}-j_p|: 1 \leqq p \leqq r\}$. When the 'maximal gap' $g$ is smaller than $n^\varepsilon$, Lemma (3.16) with $K=cm$, $c>0$ and (3.13) together imply

$$|\kappa_{t\eta}(V_{j_1},\ldots,V_{j_r})| \leqq c(r)((1+\beta_{s+1})n^{\beta(r-s-1)_+})+\{1+[v_{cm}^{(n)}(t)+\rho^{Cm}]^r |H_n(t)|^{-r}\}, \tag{3.23}$$

where

$$v_l^{(n)}(t)=\sup\{|E\exp[S_I^{(p)}]|: p \leqq l, |I| \leqq r\}, \qquad x_+=\max(x,0), \tag{3.24}$$

choosing $m=O(n^\varepsilon)$, $o<\varepsilon<1$, small and $o<p<1$ sufficiently close to 1. On the other hand, when $g$ is larger or equal to $m=o(n^\varepsilon)$ relation (3.13) and identity (3.14) show that it is sufficient to estimate

$$\prod_{j=1}^{v} E_t \prod_{l \in I_j} V_l - \prod_{j=1}^{v} E_t \prod_{l \in I_j \cap J_1} V_l \, E_t \prod_{l \in I_j \cap J_2} V_l$$

which can be done by successive applications of Lemma (3.17).

Hence, we get

$$|\kappa_{t\eta}(V_{j_1},\ldots,V_{j_r})| \leqq c\exp(-cn^{\varepsilon/2})|H_n(t)|^{-r}, \qquad c>0. \tag{3.25}$$

Since there are at most $n(g+1)^{r-1}$ indices having a maximal gap smaller or equal to $g$, the sum in (3.22) can be bounded by

$$\begin{aligned} &cn(m+1)^r(1+\beta_{s+1})n^{\beta(r-s-1)}+\{[v_{cm}^{(n)}(t)+\rho^{Cm}]|H_n(t)|^{-1}\}^r \\ &+n\sum_{g \geqq m}^{n}(g+1)^{r-1}\exp(-cn^{\varepsilon/2})|H_n(t)|^{-r}. \end{aligned} \tag{3.26}$$

Hence relation (3.11) and (3.12) together with (3.26) prove

$$\left|\frac{\partial^l}{\partial \varepsilon^l}\right|_{\varepsilon=0} R_{s+1}(t+\varepsilon a)\Bigg| \leqq c(1+\beta_{s+1}) n^{-(s-2)/2} n^{-\frac{1}{2}+r\varepsilon}(\theta_n(t)^{s+1+l}+1)(1+\|t\|^{s+1}),$$

$$(3.27) \qquad \theta_n(t)=(v_m^{(n)}(t)+\exp(-cn^{\varepsilon/2}))/|H_n(t)|$$

thus proving Lemma (3.20).

(3.28) **Lemma.** *For* $2 \leqq r \leqq s$ *we have*

$$(3.29) \qquad |\kappa_0(a_1^T S_n^*, \ldots, a_r^T S_n^*)| \leqq c n^{-(r-2)/2} \beta_{s+1}^{r/s+1} \|a_1\| \ldots \|a_r\|.$$

*Proof.* Using Lemma (3.4) together with (3.12)-(3.14) (with $\eta=0$) and the decomposition (3.22) we can argue similarly as in the proof of Lemma (3.20). [When the maximal gap in the sequence of indices $j_1 \leqq \ldots \leqq j_r$ is $g$, we have

$$|\kappa_0(a_1^T Z_{j_1}, \ldots, a_r^T Z_{j_r})| \leqq c \exp(-dm/3(s+1)) |a_1\| \ldots \|a_r\| \beta_{s+1}^{r/s+1}.$$

This together with $\sum_{g=0}^{\infty} (g+1)^{r-1} \exp(-cg/3) \leqq c(r) < \infty$ proves Lemma (3.28)]. For a detailed proof see Bulinskii and Zhurbenko (1976).

(3.30) **Lemma.** *For any* $a_1, \ldots, a_r \in \mathbb{R}^k$ *we have*

$$|\kappa_0(a_1^T S_n^*, \ldots, a_r^T S_n^*) - \kappa_0(a_1^T S_n, \ldots, a_r^T S_n)|$$
$$\leqq c\|a_1\| \ldots \|a_r\| (1+\beta_{s+1}^2) n^{-(s-2)/2-\varepsilon}, \qquad 1 \leqq r \leqq s \quad \text{and} \quad 0<\varepsilon<1.$$

*Proof.* The proof is similar to the proof of Lemma (3.20). Let $V_j = a_j^T X_j$ and $V_j' = a_j^T Z_j = a_j^T(T(X_j) - ET(X_j))$. By Čebyšev's inequality $\|ET(X_j)\| \leqq n^{-s\beta} \beta_{s+1}$. Relation (3.12) and (3.13) entail that it is sufficient to bound

$$(3.31) \qquad \sum_{g=0}^{n-r} c(r) \sum{}^{(g)} |\kappa_0(V_{j_1}, \ldots, V_{j_r}) - \kappa_0(V_{j_1}', \ldots, V_{j_r}')| n^{-r/2}.$$

Here we used the same notation as in Lemma (3.20). When the maximal gap, say g, of the sequence $j_1 \leqq \ldots \leqq j_r$ is smaller than $n^\varepsilon$ the mixed cumulants may not be small and we estimate their difference $|\kappa_0(V_{j_1}, \ldots, V_{j_r}) - \kappa_0(V_{j_1}', \ldots, V_{j_r}')|$, using the inequality

$$|EV_{l_1} \ldots V_{l_p} - EV_{l_1}' \ldots V_{l_p}'|$$
$$(3.32) \qquad \leqq \sum_{q=0}^{p-1} |EV_{l_1}' \ldots V_{l_q}'(V_{l_{q+1}} - V_{l_{q+1}}') V_{l_{q+2}} \ldots V_{l_p}|$$
$$\leqq c(n^{-s\beta} \beta_{s+1} \beta_{s+1}^{(p-1)/(s+1)} + n^{-(s+1-p)\beta} \beta_{s+1}) \|a_{l_1}\| \ldots \|a_{l_p}\|$$

which follows from Čebyšev and Hölder-type inequalities. By similar arguments applied to products of moments we obtain

$$\sum_{g=0}^{[n^\varepsilon]} \sum{}^{(g)} |\kappa_0(V_{j_1}, \ldots, V_{j_r}) - \kappa_0(V_{j_1}', \ldots, V_{j_r}')| n^{-r/2}$$
$$(3.33) \qquad \leqq cn(1+\beta_{s+1}^2)(n^\varepsilon+1)^r (n^{-\beta s} + n^{-(s+1-r)_+ \beta}) \, n^{-r/2} \|a_{j_1}\| \ldots \|a_{j_r}\|.$$

When the maximal gap $g$ is larger than $n^\varepsilon$ we may apply Lemma (3.4) and Lemma (3.16) to derive (as in the proof of Lemma (3.20) and Lemma (3.30)) that $\kappa_0(V_{j_1}, \ldots, V_{j_r})$ and $\kappa_0(V'_{j_1}, \ldots, V'_{j_r})$ are neglegable. The corresponding part of the sum in (3.31) is of order

$$\sum_{g \geqq [n^\varepsilon]} n(g+1)^{r-1} \exp(-dm/3(s+1))(1+\beta_{s+1}^2).$$

Choosing $\beta$ sufficiently close to $\frac{1}{2}$ this together with (3.32) proves the assertion.

(3.33) **Lemma.** *For every $t$ fulfilling $\|t\| \leqq cn^\varepsilon$, we have*

$$|D^\alpha(H_n(t) - \hat{\Psi}_{n,s}(t))| \leqq c(1+\beta_{s+1})(1+\|t\|^{3(s-1)+|\alpha|}) \exp(-c\|t\|^2)\, n^{-(s-2)/2-\varepsilon}.$$

*Proof.* By Lemma (3.28) we obtain for $\|t\| \leqq cn^\varepsilon$

$$(3.34) \qquad |D^\alpha \sum_{r=3}^{s} \kappa_0(it^T S_n^{*r}) r!^{-1}| = 0 \begin{cases} 1 & s \leqq |\alpha| \\ n^{-(|\alpha|-2)/2} & 3 < |\alpha| < s \\ n^{-1/2}\|t\|^{3-|\alpha|} & |\alpha| < 2. \end{cases}$$

Using (3.10) we have

$$(3.35) \qquad \begin{aligned} \Delta &= D^\alpha \left(H_n(t) - \exp\left[\sum_{r=2}^{s} \kappa_0(it^T S_n^{*r}) r!^{-1}\right]\right) \\ &= \sum_{\alpha_1+\alpha_2=\alpha} c_{\alpha_1\alpha_2} D^{\alpha_1} \exp\left[\sum_{r=2}^{s} \kappa_0(it^T S_n^{*r}) r!^{-1}\right] D^{\alpha_2}(\exp[R_{s+1}(t)]-1). \end{aligned}$$

Relation (3.34) together with Lemma (3.20) entail

$$(3.36) \qquad \begin{aligned} |\Delta| &\leqq c(1+\|t\|^{|\alpha|})\|t\|^{s+1}(1+\beta_{s+1}) \\ &\quad \cdot n^{-(s-2)/2-\varepsilon} \exp(-c\|t\|^2)(1+\theta_n(t)^{|\alpha|}). \end{aligned}$$

Note that for complex $a \in \mathbb{C}^k$ with $\|Im(a)\| \leqq \eta$ Lemma (3.30) still holds (with the Euclidean norm replaced by the Euclidean norm $\|a\|$ in $\mathbb{C}^k$). By Lemma (3.30) and Lemma (9.7), p. 73 in Bhattacharya and Rao (1976) we have for complex vectors $t$ described above

$$(3.37) \qquad \begin{aligned} &\exp\left(\sum_{r=2}^{s} \kappa_0(it^T S_n^{*r}) r!^{-1}\right) \\ &= \exp[o(n^{-(s-1)/2+\varepsilon}\|t\|^s)](\hat{\Psi}_{n,s}(t) + \bar{R}_s(t)) \\ &= \hat{\Psi}_{n,s}(t) + \bar{\bar{R}}_s(t), \text{ say} \end{aligned}$$

where

$$|\bar{R}_s(t)| \leqq cn^{-(s-1)/2}(1+\beta_{s+1})(\|t\|^{s+1} + \|t\|^{3(s-1)}) \exp(-c_2\|\mathrm{Re}\, t\|^2 + c_3\eta^2).$$

Since the r.h.s. of (3.37) is an analytic function of $t$, Cauchy's inequalities for derivatives of analytic functions can be used to estimate the derivatives $D^\alpha \bar{\bar{R}}_s(t)$

by means of

$$\max\{|\bar{\bar{R}}_s(z)|:\|z_j-t_j\|=\eta,\ z\in\mathbb{C}^k, j=1,\dots,k\}\,\eta^{-|\alpha|}.$$

Using $|\hat{\Psi}_{n,s}(t)|\leq c\exp(-c|\mathrm{Re}\,t\|^2)$, $c>0$ for $\|\mathrm{Im}\,t\|\leq\eta$ and $\|\mathrm{Re}\,t\|\leq cn^{\varepsilon}$ we obtain

$$|D^{\alpha}\bar{\bar{R}}_s(t)|\leq cn^{-(s-2)/2-\varepsilon}(1+\|t\|^{3(s-1)})\exp(-c\|t\|^2) \tag{3.38}$$

for every $t\in\mathbb{R}^k$, $\|t\|\leq cn^{\varepsilon}$.

Hence,

$$\begin{aligned}&|D^{\alpha}(H_n(t)-\hat{\Psi}_{n,s}(t))|\\&\leq c(1+\beta_{s+1})(1+\|t\|^{3(s-1)+|\alpha|})(1+\theta_n(t)^{|\alpha|+1})\exp(-c\|t\|^2)n^{-(s-2)/2-\varepsilon}.\end{aligned} \tag{3.39}$$

It remains to estimate the function $\theta_n(t)$ defined in (3.27). Let $T_l=n^{-1/2}\sum_{p=1}^{l} Z_{j_p}$, $1\leq l\leq n$ for any sequence $j_1<j_2<\dots<j_l$ and define $H(T_l,t)=E\exp[it^T T_l]$. Relation (3.10), Lemma (3.20) and Lemma (3.28) together can be used to prove for $1\leq l\leq n$ and $\alpha=0$:

$$H(T_l,t)=\exp[-\tfrac{1}{2}\kappa_0(t^T T_l^2)+cn^{-3/2-\varepsilon}\|t\|^3 l\theta_{l,3}(t)],$$

where

$$|\theta_{l,3}(t)|\leq\sup\{|(\exp(-cn^{\varepsilon})+H(T_{I,l}^{(p)},t))/H(T_l,t)|^3:\ |I|\leq 3, 0\leq p\leq m, T_l\} \tag{3.40}$$

and $T_{I,l}^{(p)}$ is related to $T_l$ as $S_I^{(p)}$ is related to $S_n^*$ in Lemma (3.16). Here, $\varepsilon$, $c$ and $m$ are independent of $l$, $1\leq l\leq n$. We claim that

$$\sup\{|\theta_{l,3}(t)|:\|t\|\leq n^{\varepsilon}\}\leq 2 \qquad \text{for } l=1,2,\dots,n \tag{3.41}$$

provided $n$ is sufficiently large (depending on $s$, $k$, $d$, $\beta_{s+1}$).

Inequality (3.41) trivially holds for $l=1$. Suppose it holds for $1\leq l\leq r$. Assuming that it does not hold for $l=r+1$, i.e. there exists a $t_0$, such that $\|t_0\|<n^{\varepsilon}$ and $|\theta_{r+1,3}(t_0)|=2$, we have

$$2\leq\sup_{I,p}|\exp(-cn^{2\varepsilon})/H(T_{r+1},t_0)+H(T_{I,r+1}^{(p)},t_0)/H(T_{r+1},t_0)|^3.$$

These assumptions together with (3.40) imply

$$\begin{aligned}2\leq\sup_{I,r}|&\exp(-cn^{2\varepsilon})+\exp[\tfrac{1}{2}|\kappa_0(t^T T_{I,r+1}^{(p)2})-\kappa_0(t^T T_{r+1}^2)|\\&-4cn^{-3/2-\varepsilon}(r+1)\|t\|^3]|^3\end{aligned} \tag{3.42}$$

provided $n$ is sufficiently large.

Since

$$\begin{aligned}&|\kappa_0(t^T T_{I,r+1}^{(p)2})-\kappa_0(t^T T_{r+1}^2)|\\&\quad\leq c_2\|t\|^2 E\|T_{r+1}-T_{I,r+1}^{(p)}\|\,\|T_{r+1}+T_{I,r+1}^{(p)}\|\\&\quad\leq c\|t\|^2 mn^{-1/2}\\&\quad\leq n^{\varepsilon-\frac{1}{2}},\qquad \varepsilon<\tfrac{1}{2},\end{aligned}$$

relation (3.42) yields a contradiction when $n$ is sufficiently large, thus proving (3.41) for $l=r+1$.

This completes the proof of (3.41).

Hence,

$$|\theta_n(t)| \leqq 2^{(s+|\alpha|)/3} \text{ for every } \|t\| \leqq n^{\varepsilon}$$

which completes the proof of Lemma (3.33).

(3.43) **Lemma.** *For every $c$, $\varepsilon$, and $E>0$ there exists a positive $\delta$ such that*

$$c n^{\varepsilon} \leqq \|t\| \leqq n^{E}$$

*implies*

$$|D^{\alpha} H_n(t)| \leqq \delta^{-1} \exp(-n^{\delta}), \qquad |\alpha| \leqq k+1.$$

*Proof.* It suffices to show that there exists a positive $\delta$ such that for $r=0, \ldots, k+1$ and $a \in \mathbb{R}^k$, $\|a\|=1$

$$\left| \frac{d^r}{d\lambda^r} H_n(t+\lambda a) \bigg|_{\lambda=0} \right| \leqq \delta^{-1} \exp(-n^{\delta}).$$

The left hand side equals

$$|E(a^T S_n^*)^r \exp(i t^T S_n^*)|$$

which is bounded by

$$\sum |E(a^T Z_{j_1}) \ldots (a^T Z_{j_r}) \exp(i t^T S_n^*)|$$

where the sum extends over $j_1, \ldots j_r \in \{1, \ldots, n\}$. These are $n^r$ terms. We shall give upper bounds for each term. Fix $j_1^0, \ldots, j_r^0 \in \{1, \ldots, n\}$, and for $m \in \mathbb{N}$ to be determined later let

$$I = \{j \in \{1, \ldots, n\} : |j - j_p^0| \geqq 3m, p=1, \ldots, r\}.$$

Divide $I$ into blocks $A_1, B_1, \ldots, A_l, B_l$ as follows. Define $j_1, \ldots, j_l$ by

$$\begin{aligned} j_1 &= \inf I, \\ j_{p+1} &= \inf\{j \geqq j_p + 7m : j \in I\} \end{aligned}$$

and let $l$ be the smallest integer for which the inf is undefined. Write

$$\begin{aligned} A_p &= \prod \{e^{itn^{-1/2} Z_j} : |j - j_p| \leqq m\}, \qquad p=1, \ldots, l, \\ B_p &= \prod \{e^{itn^{-1/2} Z_j} : j_p + m + 1 \leqq j \leqq j_{p+1} - m - 1\}, \qquad p=1, \ldots, l-1, \\ B_l &= \prod \{e^{itn^{-1/2} Z_j} : j > j_l + m + 1\} \end{aligned}$$

and

$$R = (a^T Z_{j_1^0}) \ldots (a^T Z_{j_r^0}) \prod \{e^{itZ_j} : j \notin I\}.$$

Then

$$(a^T Z_{j_1^0}) \ldots (a^T Z_{j_r^0}) e^{itS_n^*} = \prod_1^l A_p B_p R.$$

We have

$$\left|ER\prod_1^l A_pB_p - ER\prod_1^l B_pE(A_p|\mathcal{D}_j:j\neq j_p)\right|$$
$$\leq \sum_{q=1}^{l}\left|ER\prod_1^{q-1} A_pB_p(A_q - E(A_q|\mathcal{D}_j:j\neq j_q))\prod_{q+1}^{l} B_pE(A_p|\mathcal{D}_j:j\neq j_p)\right|.$$

For $p=1,\ldots,l$ we shall approximate $A_p$, $B_p$, $R$ by random variables $A'_p$, $B'_p$, $R'$ such that $A_p$ is bounded by 1 and $\mathcal{D}_{j_p-2m}^{j_p+2m}$-measurable, $B'_p$ is bounded by 1 and $\mathcal{D}_{j_p+1}^{j_{p+1}-1}$-measurable, and $R'$ is bounded by $2n^\beta$ and measurable with respect to the $\sigma$-field generated by $\{\mathcal{D}_l$: there exists $j\notin I$ with $|l-j|\leq m\}$.

Using condition (2.3) we see that this can be done with the following accuracy:

$$E|A_p - A'_p| \leq n\,n^E d^{-1}\exp(-dm),$$
$$E|B_p - B'_p| \leq n\,n^E d^{-1}\exp(-dm),$$
$$E|R-R'| \leq (n+r)(2n^\beta)^r n^E d^{-1}\exp(-dm).$$

Condition (2.6) yields that for $p=1,\ldots,l$

$$E|E(A'_p|\mathcal{D}_j\colon j\neq j_p) - E(A'_p|\mathcal{D}_j\colon 0<|j-j_p|\leq 3m)| \leq 2d^{-1}e^{-dm}.$$

Furthermore,

$$E|E(A'_p|\mathcal{D}_j\colon j\neq j_p) - E(A_p|\mathcal{D}_j\colon j\neq j_p)| \leq E|A_p - A'_p| \leq n^{E+1}d^{-1}e^{-dm}.$$

Hence

$$\left|ER\prod_1^l A_pB_p - ER'\prod_1^l B'_pE(A'_p|\mathcal{D}_j\colon 0<|j-j_p|\leq 3m)\right|$$
$$\leq \left|ER\prod_1^l A_pB_p - ER'\prod_1^l A'_pB'_p\right| + \left|ER'\prod_1^l A'_pB'_p - ER'\prod_1^l B'_pE(A'_p|\mathcal{D}_j\colon 0<|j-j_p|\leq 3m)\right|$$
$$= I_1 + I_2, \text{ say.}$$

We have

$$I_1 \leq 2(2l+1)(n+r)(2n^\beta)^r n^E d^{-1}e^{-dm}$$

and

$$I_2 \leq \sum_{q=1}^{l}\left|ER'\prod_1^{q-1} A'_pB'_p(A_q - E(A_q|\mathcal{D}_j\colon j\neq j_q))\right.$$
$$\left.\cdot\prod_{q+1}^{l} B'_pE(A'_p|\mathcal{D}_j\colon 0<|j-j_p|\leq 3m)\right| + \sum_{q=1}^{l}(2n^\beta)^r\cdot 4\cdot d^{-1}e^{-dm}.$$

The first sum vanishes since

$$R'\prod_1^{q-1} A'_pB'_p \quad\text{and}\quad \prod_{q+1}^{l} B'_pE(A'_p|\mathcal{D}_j\colon 0<|j-j_p|\leq 3m)$$

are both measurable with respect to the $\sigma$-field generated by $\mathcal{D}_j\colon j\neq j_q$.

Recall that the functions

$$E(A'_p|\mathscr{D}_j:0<|j-j_p|\leqq 3m),\, p=1,\ldots,l$$

are weakly dependent since $j_{p+1}-j_p\geqq 7m$, $p=1,\ldots,l-1$. Using condition (2.4) we obtain

$$\left|ER'\prod_1^l B'_p E(A'_p|\mathscr{D}_j:0<|j-j_p|\leqq 3m)\right|\leqq(2n^\beta)^r E\left|\prod_1^l E(A'_p|\mathscr{D}_j:0<|j-j_p|\leqq 3m)\right|$$

$$\leqq(2n^\beta)^r\prod_1^l E|E(A'_p|\mathscr{D}_j:0<|j-j_p|\leqq 3m)|+(2n^\beta)^r l\cdot 4d^{-1}e^{-dm}.$$

With condition (2.5) and Lemma (3.2) we find an upper bound for

$$E|E(A'_p|\mathscr{D}_j:0<|j-j_p|\leqq 3m)|.$$

We have for $\|t\|\geqq d$ the relation $E|E(A_p|\mathscr{D}_j:j\neq j_p)|\leqq e^{-d}$, and hence by Lemma (3.2) for all $t\in\mathbb{R}^k$ $\|t\|\leqq d$

$$E|E(A_p|\mathscr{D}_j:\, j\neq j_p)|\leqq\exp(-d\,\|t\|^2/n).$$

We have for all $t\in\mathbb{R}^k$

$$\begin{aligned}&E|E(A'_p|\mathscr{D}_j:0<|j-j_p|\leqq 3m)|\\&\quad\leqq n^{E+1}d^{-1}e^{-dm}+E|E(A_p|\mathscr{D}_j:0<|j-j_p|\leqq 3m)|\\&\quad\leqq n^{E+1}d^{-1}e^{-dm}+E|E(A_p|\mathscr{D}_j:j\neq j_p)|\\&\quad\leqq n^{E-1}d^{-1}e^{-dm}+\max(\exp(-d\,\|t\|^2/n),e^{-d}).\end{aligned}$$

If we choose $K$ appropriately and let $m$ be the integral part of $K\log n$, then the assertion of the lemma follows from

$$\begin{aligned}&\exp(-d\,\|t\|^2/n)^{\frac{n}{m}}\leqq\exp(-d\,\|t\|^2/K\log n))\\&\quad\leqq\exp(-d' n^{\varepsilon/2})\quad\text{for}\quad\|t\|\geqq cn^\varepsilon\\&\quad\text{and some } d'>0.\end{aligned}$$

(3.44) *Remark.* For some applications, e.g. for finite Fourier transforms, assumption (2.5) is too restrictive. From the above proof we observe that

$$|E(\exp(it(X_{j-m}+\ldots+X_{j+m})|\mathscr{D}_l:l\neq j)|\leqq 1-\rho$$

must hold for a sufficiently large number of $j'_p$'s. Hence the proof works with the following weaker assumption:

(3.45) There exists a positive constant $\rho$ such that for every $n\geqq\rho^{-1}$

$$\begin{aligned}&\#\{j\in\{1,\ldots,n\}:\text{for all }\rho^{-1}<m<n,\\&\quad|E(\exp(it(X_{j-m}+\ldots+X_{j+m})|\mathscr{D}_l:\,l\neq j)|\leqq 1-\rho\}\geqq\rho n.\end{aligned}$$

## 4. Formulas

Explicite formulas for the functions $\tilde{P}_j^*$ of Remark (2.12) are given for $s=4$ (including terms up to order $n^{-1}$) and strictly stationary sequences $X_1, X_2, \ldots$ of random vectors. We first consider the case $k=1$ (univariate random variables). Write $\chi_{n,r}$ for the cumulant of $S_n = n^{-\frac{1}{2}} \sum_1^n X_j$ of order $r$. Then, when $EX_1 = 0$,

$$\chi_{n,1} = 0,$$

$$\chi_{n,2} = EX_1^2 + 2 \sum_{j=1}^{\infty} EX_1 X_{j+1} - n^{-1} 2 \sum_{j=1}^{\infty} jEX_1 X_{j+1} + o(n^{-1}),$$

$$n^{1/2} \chi_{n,3} = EX_1^3 + 3 \sum_{j=1}^{\infty} (EX_1^2 X_{j+1} + EX_1 X_{j+1}^2) + 6 \sum_{i,j=1}^{\infty} EX_1 X_{1+i} X_{1+i+j} + o(n^{-1/2}),$$

$$\begin{aligned} n\chi_{n,4} = EX_1^4 - 3(EX_1^2)^2 &+ 4 \sum_{j=1}^{\infty} (EX_1 X_{j+1}^3 + EX_1^3 X_{j+1}) \\ &+ 6 \sum_{j=1}^{\infty} E(X_1^2 - EX_1^2) X_{j+1}^2 \\ &+ 12 \sum_{i,j=1}^{\infty} EX_1 X_{i+1} (X_{j+i+1}^2 - EX_1^2) \\ &+ 4 \sum_{h,i,j=1}^{\infty} E(X_1 X_{h+1} - EX_1 X_{h+1}) X_{i+h+1} X_{j+i+h+1} \\ &+ o(n^0). \end{aligned}$$

Let $f_2, f_3, f_4$ be cumulant spectral densities of $X_1, X_2, \ldots$ of order 2, 3, 4, i.e. for $j_1, j_2, j_3 = 0, 1, 2, \ldots$

$$EX_0 X_{j_1} = \int_{-\pi}^{\pi} e^{-ij_1 x} f_2(x)\, dx,$$

$$EX_0 X_{j_1} X_{j_2} = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{-ij_1 x} e^{-ij_2 y} f_3(x, y)\, dx\, dy$$

and

$$\operatorname{cum}^{(4)}(X_0, X_{j_1}, X_{j_2}, X_{j_3}) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{-ij_1 x} e^{-ij_2 y} e^{-ij_3 z} f_4(x, y, z)\, dx\, dy\, dz$$

where

$$\operatorname{cum}^4(X, Y, U, V) = E\bar{X}\bar{Y}\bar{U}\bar{V} - E\bar{X}\bar{Y}E\bar{U}\bar{V} - E\bar{X}\bar{U}E\bar{Y}\bar{V} - E\bar{X}\bar{V}E\bar{Y}\bar{U}$$

and $\bar{X} = X - EX$, $\bar{Y}, \bar{U}, \bar{V}$ are defined analoguously. Whenever $f_2, f_3, f_4$ are smooth,

$$\chi_{n,2}=2\pi f_2(0)+n^{-1}\int_{-\pi}^{\pi}\frac{\sin x}{\sin^2 x/2}f_2'(x)\,dx+o(n^{-1}),$$
$$\chi_{n,3}=n^{-1/2}(2\pi)^2 f_3(0,0)+o(n^{-1}),$$
$$\chi_{n,4}=n^{-1}(2\pi)^3 f_4(0,0,0)+o(n^{-1}).$$

The functions $\tilde{P}_j^*$ of Remark (2.12) in this case read

$$\tilde{P}_0^*(t)=\exp(-\tfrac{1}{2}t^2(2\pi)^1 f_2(0)),$$
$$\tilde{P}_1^*(t)=\tfrac{1}{6}(it)^3(2\pi)^2 f_3(0,0)\,\tilde{P}_0^*(t),$$
$$\tilde{P}_2^*(t)=\Big(-\int_{-\pi}^{\pi} f_2'(x)\frac{\sin x}{\sin^2 x/2}\,dx\,t^2 +(24)^{-1}(2\pi)^3 f_4(0,0,0)\,t^4 -(72)^{-1}(2\pi)^4 f_3^2(0,0)\,t^6\Big)\tilde{P}_0^*(t).$$

We now derive the corresponding formulas for the finite Fourier transform $d^N(\lambda_1,\ldots,\lambda_p)$ (see Example (1.15)). The random vector $d^N$ has complex components. In order to apply our theory we split $d^N$ into real and imaginary parts and deal with a $2p$-variate random vector with real components. Write $d^N=(d_1^N,\ldots,d_{2p}^N)$ for this vector. For our formulas we need approximations for $\chi_{r,N}(t)$, $t\in\mathbb{R}^{2p}$, the $r$-th order cumulant of $t^T d^N$.

Since
$$\chi_{r,N}(t)=\sum_{j_1,\ldots,j_r=1}^{2p} t_{j_1}\ldots t_{j_r}\,\kappa_0(d_{j_1}^N,\ldots,d_{j_r}^N)$$

it suffices to derive approximations for $\kappa_0(d_{j_1}^N,\ldots,d_{j_r}^N)$, $j_1,\ldots,j_r=1,\ldots,2p$. For $\lambda\in[-\pi,\pi]$ write
$$D^N(\lambda)=\sum_{0}^{N-1} X_j e^{ij\lambda},$$
$$\Delta^N(\lambda)=\sum_{0}^{N-1} e^{ij\lambda}.$$

Then for $\lambda$, $\mu$, $\nu\in[-\pi,\pi]$

$$E D^N(\lambda)\,D^N(\mu)\,D^N(\nu)=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Delta^N(\nu-x)\,\Delta^N(\mu-y)\,\Delta^N(\lambda+x+y)\,f_3(x,y)\,dx\,dy$$
$$=\begin{cases}O(N^0) & \text{if}\quad \lambda+\mu+\nu\not\equiv 0\ (2\pi)\\ (2\pi)^2 N f_3(\mu,\nu)+O(N^0) & \text{if}\quad \lambda+\mu+\nu\equiv 0\ (2\pi)\end{cases}$$

and for $\lambda$, $\mu$, $\nu$, $\xi\in[-\pi,\pi]$

$$\operatorname{cum}^{(4)}(D^N(\lambda), D^N(\mu), D^N(\nu), D^N(\xi))$$
$$=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Delta^N(\mu-x)\,\Delta^N(\nu-y)\,\Delta^N(\xi-z)\,\Delta^N(\lambda+x+y+z)\,f_4(x,y,z)\,dx\,dy\,dz$$
$$=\begin{cases}O(N^0) & \text{if}\quad \lambda+\mu+\nu+\xi\not\equiv 0\ (2\pi)\\ (2\pi)^3 N f_4(\mu,\nu,\xi)+O(N^0) & \text{if}\quad \lambda+\mu+\nu+\xi\equiv 0\ (2\pi).\end{cases}$$

Finally, if $\lambda+\mu\equiv 0\ (2\pi)$, then

$$\begin{aligned} E D^N(\lambda)\, D^N(\mu) = & -2\pi f_2(\lambda) \frac{e^{iN(\lambda+\mu)}}{1-e^{iN(\lambda+\mu)}} \\ & +2\pi f_2(-\mu)/(1-e^{iN(\lambda+\mu)}) \\ & + \int_{-\pi}^{\pi} \frac{h(x)-h(-\mu)}{1-e^{i(x+\mu)}}\, dx + o(N^0) \end{aligned}$$

where $h(x)=(f_2(x)-f_2(\lambda))/(1-e^{i(\lambda-x)})$.

If $\lambda+\mu\equiv 0\ (2\pi)$, then

$$\begin{aligned} E|D^N(\lambda)|^2 &= E D^N(\lambda)\, D^N(-\lambda) \\ &= 2\pi f_2(0)\, N - \int_{-\pi}^{\pi} \frac{\sin(\lambda-x)}{\sin^2(\lambda-x)/2} (f_2'(x)-f_2'(\lambda))\, dx \\ &\quad + o(N^0). \end{aligned}$$

Approximations for $\kappa_0(d_{j_1}^N, \ldots, d_{j_r}^N)$ are obtained easily: e.g. for $r=2$:

$$\begin{aligned} &\kappa_0 \left( (2\pi N)^{-1/2} \sum_0^{N-1} X_j \cos \lambda j, (2\pi N)^{-1/2} \sum_0^{N-1} X_j \cos \mu j \right) \\ &\quad = (2\pi N)^{-1} \cdot \tfrac{1}{4} (E D^N(\lambda)\, D^N(\mu) + E D^N(\lambda)\, D^N(-\mu) + E D^N(-\lambda)\, D^N(\mu) \\ &\qquad + E D^N(-\lambda)\, D^N(-\mu)) \end{aligned}$$

$$\begin{aligned} &\kappa_0 \left( (2\pi N)^{-1/2} \sum_0^{N-1} X_j \cos \lambda j, (2\pi N)^{-1/2} \sum_0^{N-1} X_j \sin \mu j \right) \\ &\quad = (2\pi N)^{-1} \cdot \tfrac{1}{4} (E D^N(\lambda)\, D^N(\mu) - E D^N(\lambda)\, D^N(-\mu) - E D^N(-\lambda)\, D^N(\mu) \\ &\qquad + E D^N(-\lambda)\, D^N(-\mu)). \end{aligned}$$

Notice that in the non-stationary case considered here, the $N^{-1}$-term of $\chi_{2,N}(t)$ does not converge in general.

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