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Non-Stable Laws With All Projections Stable

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0. Introduction

We will show that for α small enough

$$\exp(i\alpha r\cos(3\theta)-r^{\gamma}), \qquad \text{for } 0<\gamma\leqq 1,$$

is the characteristic function of a probability law on $\mathbb{R}^2$. We then show that for $0<\gamma<1$ this law is not stable even though all of its projections are stable of index γ. This provides a counterexample to Theorem 4 of [4].

Definitions. Let (r,θ) be the polar coordinate system on $\mathbb{R}^2$. Fix $\gamma\in(0,1]$. Let n_0 be the smallest integer satisfying $n_0\gamma\geqq 4$. Define functions on $\mathbb{R}^2$ as follows:

$$P=P_\alpha=\exp(i\alpha r\cos 3\theta-r^{\gamma}),$$
$$Q=\exp(-r^{\gamma}),$$
$$G=G_\alpha=\sum_{m=1}^{4}\sum_{n=0}^{n_0}\left(\frac{(i\alpha r\cos 3\theta)^m}{m!}\frac{(-r^{\gamma})^n}{n!}\right),$$
$$H=H_\alpha=\sum_{m=1}^{4}\left(\frac{(i\alpha r\cos 3\theta)^m}{m!}\left(e^{-r^{\gamma}}-\sum_{n=0}^{n_0}\frac{(-r^{\gamma})^n}{n!}\right)\right),$$

and

$$F=F_\alpha=\left(e^{i\alpha r\cos 3\theta}-\sum_{m=0}^{4}\frac{(i\alpha r\cos 3\theta)^m}{m!}\right)e^{-r^{\gamma}}.$$

So $P=Q+G+H+F$. The plan is to Fourier transform Q, G, H, and F. We will show H and F are C^4 and use this to show their transforms are $O(\|\xi\|^{-4})$. The transform of each term of G is homogeneous of degree at most -3. The transform of Q is $O(\|\xi\|^{-2-\gamma})$ and no better. Then, by choosing α small, we can show the Fourier transform of P is positive near infinity. Since Q is known to be a characteristic function we will be able to choose α so that the transform of P is close enough to that of Q to be positive away from infinity. Since G, H, and F are not integrable we will take their Fourier transforms as tempered distributions. Section 1 presents the necessary information about Schwartz distributions and handles G. Section 2 handles Q, H, and F. Section 3 finishes the proof that P is a characteristic function for α small. Section 4 shows the projections are stable, but the law is not.

1. Schwartz Distributions

We present a crash course in distributions (generalized functions). Many of the assertions made below have nontrivial proofs. See any book on the subject, for example [13], [5], [15], or [3]. If we are in $\mathbb{R}^n$ then a *multi-index* β is an ordered n-tuple of nonnegative integers:

$$\beta=(\beta_1,\dots,\beta_n).$$

We write D_i for $\frac{\partial}{\partial x_i}$. Then the differential operator D^β is defined by

$$D^\beta=(D_1)^{\beta_1}\dots(D_n)^{\beta_n}.$$

The *order* of D^β is

$$|\beta|=\beta_1+\dots+\beta_n.$$

If $|\beta|=0$ then $D^\beta f=f$.

If Ω is an open subset of $\mathbb{R}^n$ we define

$$\begin{aligned}
C(\Omega)&=C^0(\Omega)=\{f\colon\ \Omega\to\mathbb{C}\,|\,f \text{ is continuous}\},\\
C^k(\Omega)&=\{f\,|\,D^\beta f\in C(\Omega) \text{ for all } \beta \text{ satisfying } |\beta|\leqq k\},\\
C^\infty(\Omega)&=\{f\,|\,D^\beta f\in C(\Omega) \text{ for all } \beta\},\\
C_0^k(\Omega)&=\{f\in C^k(\Omega)\,|\,\operatorname{supp}(f) \text{ is compact}\},
\end{aligned}$$

and

$$\mathscr{D}(\Omega)=C_0^\infty(\Omega).$$

If Ω is clear from the context (usually $\mathbb{R}^n$) we write simply $C^0, C^\infty, \mathscr{D}$, etc. Here $\operatorname{supp}(f)=$ *support* of $f=$ closure in Ω of $\{x\,|\,f(x)\neq 0\}$. We place a pseudo-topology on $\mathscr{D}(\Omega)$ by saying "$\phi_i\to 0$ in $\mathscr{D}(\Omega)$" if there is a compact subset K of Ω with $\operatorname{supp}(\phi_i)\subset K$ for all i and $\lim_{i\to\infty}\sup_K|D^\beta\phi_i|=0$ for all β. We now define the dual space $\mathscr{D}'(\Omega)=$ *distributions* in Ω by

$$\mathscr{D}'(\Omega)=\{T\colon\ \mathscr{D}(\Omega)\to\mathbb{C}\,|\,T \text{ is linear over } \mathbb{C} \text{ and if } \{\phi_i\} \text{ is any sequence in } \mathscr{D}(\Omega) \text{ and } \phi_i\to 0 \text{ in } \mathscr{D}(\Omega) \text{ then } T(\phi_i)\to 0\}.$$

The space of *rapidly decreasing functions* is

$$\mathscr{S}_n=\{f\in C^\infty(\mathbb{R}^n)\,|\,P_N(f)<\infty \text{ for } N=0,1,2,\dots\}$$

where

$$P_N(f) = \sup_{\substack{x\in\mathbb{R}^n \\ |\beta|\leqq N}} (1+\|x\|^2)^N |D^\beta f(x)|.$$

We give $\mathscr{S}_n$ the topology generated by the seminorms P_N for $N=0,1,2,\ldots$. The *tempered distributions*, $\mathscr{S}_n'$, are those distributions which may be extended to be continuous on $\mathscr{S}$, i.e.

$$\mathscr{S}_n' = \{T\colon \mathscr{S}_n \to \mathbb{C} \mid T \text{ is continuous and linear over } \mathbb{C}\}.$$

So $\mathscr{S}_n' \subset \mathscr{D}'(\mathbb{R}^n)$ by restriction. This inclusion is one-to-one. If f is a locally integrable function then we define $[f]\in\mathscr{D}'$ by

$$[f](\phi) = \int f\phi \qquad \text{for } \phi\in\mathscr{D},$$

where all integrals are with respect to Lebesgue measure. If f doesn't grow too fast near infinity then $[f]$ can be extended to be in $\mathscr{S}'$ by

$$[f](\phi) = \int f\phi \qquad \text{for } \phi\in\mathscr{S}.$$

For example, if f is bounded by some polynomial near infinity and is locally integrable then $[f]\in\mathscr{S}'$. If f and g are in L^1 then $[f]$ is in $\mathscr{S}'$ and

$$[f]=[g] \;\Rightarrow\; f=g \quad \text{a.e.}$$

We define the distribution $D^\beta T$ by $D^\beta T(\phi) = T((-1)^{|\beta|} D^\beta \phi)$.

For $f\in L^1(\mathbb{R}^n)$ we define the *Fourier transform*, $\hat{f} \equiv \mathscr{F}(f)$, by

$$\hat{f}(\xi) = \int f(x)\, e^{-ix\cdot\xi}\, dx.$$

(Many books use slightly different definitions.) $\mathscr{F}$ is a linear homeomorphism from $\mathscr{S}$ onto $\mathscr{S}$. For $T\in\mathscr{S}'$ we define the *Fourier transform*, $\hat{T} \equiv \mathscr{F}(T)$, by

$$\hat{T}(\phi) = T(\hat{\phi}) \qquad \text{for } \phi\in\mathscr{S}.$$

Notice, $\phi\in\mathscr{S}$ implies $\phi\in L^1$. $\mathscr{F}$ is a linear bijection of $\mathscr{S}_n'$ onto $\mathscr{S}_n'$. If $f\in L^1$ then

$$\widehat{[f]} = [\hat{f}].$$

If $T\in\mathscr{D}'(\Omega)$ and $\Omega'\subset\Omega$ is open then the *restriction* of T to Ω' is

$$T|_{\Omega'} = T|_{\mathscr{D}(\Omega')},$$

and $T|_{\Omega'}\in\mathscr{D}'(\Omega')$. If $T\in\mathscr{D}'(\Omega)$ and $\phi\in C^\infty(\Omega)$ then ϕT defined by

$$(\phi T)(\psi) = T(\phi\psi) \qquad \text{for } \psi\in\mathscr{D}(\Omega),$$

is in $\mathscr{D}'(\Omega)$. The *support* of T is the smallest closed (in Ω) subset X satisfying $T|_{\Omega\setminus X}=0$. We write $X=\operatorname{supp}(T)$. The *singular support* of T (sing supp(T)) is the smallest closed subset X such that there is an $f\in C^\infty(\Omega\setminus X)$ satisfying

$$T|_{\Omega\setminus X} = [f].$$

We say a function f is *homogeneous* of degree m if $f(tx)=t^m f(x)$ for $t>0$ and all x. If ϕ is a function then we denote by $\phi(\cdot/\lambda)$ the function defined by $\phi(\cdot/\lambda)(x)=\phi(x/\lambda)$, where $\lambda\in\mathbb{R}\setminus\{0\}$. If $T\in\mathscr{D}'(\mathbb{R}^n)$ then let $T_\lambda\in\mathscr{D}'(\mathbb{R}^n)$ be defined by $T_\lambda(\phi)=\lambda^{-n}T(\phi(\cdot/\lambda))$. We say T is *homogeneous* of degree m if

$$T_\lambda=\lambda^m T \quad \text{for } \lambda>0.$$

If f is homogeneous of degree m then so is $[f]$. We also have a converse.

Lemma 1.1. *Let T be in $\mathscr{D}'(\mathbb{R}^n)$ and homogeneous of degree m. Let $f\in C^0(\mathbb{R}^n\setminus\{0\})$ and suppose $T|_{\mathbb{R}^n\setminus\{0\}}=[f]$. Then f is homogeneous of degree m.*

Proof. Choose $\phi\in\mathscr{D}$, $\phi\geqq 0$, and $\int\phi=1$. Let $\phi_j(x)=j^n\phi(jx)$. The sequence $\{\phi_j\}$ is called an approximate identity. Let $y\in\mathbb{R}^n\setminus\{0\}$ and $\lambda>0$. We want to show $f(\lambda y)=\lambda^m f(y)$. Let $\psi_j(x)=\phi_j(x-y)$. Then for $g\in C^0(\mathbb{R}^n\setminus\{0\})$ we have

$$\lim_{j\to\infty}\int\psi_j g=g(y).$$

($\int\psi_j g$ may not make sense for small j, but for j large enough $\operatorname{supp}\psi_j\subset\mathbb{R}^n\setminus\{0\}$.)

$$\begin{aligned} f(\lambda y)&=\lim_{j\to\infty}\int f(\lambda z)\psi_j(z)\,dz\\ &=\lim_{j\to\infty}\lambda^{-n}\int f(x)\psi_j(x/\lambda)\,dx\\ &=\lim_{j\to\infty}\lambda^{-n}T(\psi_j(\cdot/\lambda))\\ &=\lim_{j\to\infty}T_\lambda(\psi_j)\\ &=\lim_{j\to\infty}\lambda^m T(\psi_j)\\ &=\lim_{j\to\infty}\lambda^m\int f\psi_j\\ &=\lambda^m f(y). \quad \dashv \end{aligned}$$

For $m\in\mathbb{R}$ we define $S^m(\mathbb{R}^n)$, the *symbols* of degree m, by

$$S^m(\mathbb{R}^n)=\{f\in C^\infty(\mathbb{R}^n)\,|\,\forall\beta\,\exists\, C(|D^\beta f(x)|\leqq C(1+\|x\|)^{m-|\beta|})\}.$$

And set $S^\infty(\mathbb{R}^n)=\bigcup_m S^m(\mathbb{R}^n)$. For a reference see any book on pseudo-differential operators, e.g. [3]. We now show that functions which are homogeneous near infinity are symbols.

Lemma 1.2. *Let $f\in C^\infty(\mathbb{R}^n)$. Let R and m satisfy*

$$f(tx)=t^m f(x) \quad \textit{for } t>0, \quad \|x\|\geqq R.$$

Then $f\in S^m(\mathbb{R}^n)$.

Proof. For $\|x\|\geqq R$ and $t>0$ we have

$$t(D_i f)(tx)=\frac{\partial}{\partial x_i}(f(tx))=\frac{\partial}{\partial x_i}(t^m f(x))=t^m D_i f(x).$$

So

$$D_i f(t x) = t^{m-1} D_i f(x).$$

Etc. ⊣

If f and g are in S^∞ then so is $f+g$. Also, $\mathscr{S}$ is contained in S^m for all m.

Lemma 1.3. *If f is a symbol then there is a $T \in \mathscr{S}'$ and a $g \in \mathscr{S}$ satisfying*

a) $\operatorname{supp}(T)$ *compact,*

b) $\operatorname{sing\,supp}(T) \subset \{0\}$, *and*

c) $\widehat{[f]} = T + [g]$.

Proof. First note that $x^\beta f$ is a symbol for all β. For $T \in \mathscr{S}'$ we have

$$\widehat{D^\beta T} = (i\,\xi)^\beta\,\hat{T}$$

and

$$\widehat{x^\beta T} = (i\,D)^\beta\,\hat{T}.$$

If g is a $C^{|\beta|}$ function then $D^\beta[g] = [D^\beta g]$. If $g \in C^\infty$ and $x^\beta g \in L^1$ for $|\beta| \leqq k$ then (by Lebesgue Dominated Convergence) $\hat{g} \in C^k$.

Fix k. Since f is a symbol we may choose $\beta = (\beta_1, 0, \ldots, 0)$ large enough so that

$$x^\delta D^\beta f \in L^1 \qquad \text{for } |\delta| \leqq k.$$

Then $\widehat{D^\beta f} \in C^k$ so there is a $g \in C^k$ satisfying $[g] = (i\,\xi)^\beta \widehat{[f]} = i^{\beta_1} \xi_1^{\beta_1} \widehat{[f]}$. Since k is arbitrary this implies $\operatorname{sing\,supp} \widehat{[f]} \subset \{\xi_1 = 0\}$. Similarly we get $\operatorname{sing\,supp} \widehat{[f]} \subset \{\xi_j = 0\}$ for $j = 1, \ldots, n$. This implies $\operatorname{sing\,supp} \widehat{[f]} \subset \{0\}$. Since $x^\beta f$ is a symbol, if we choose δ large enough we have

$$D^\delta(x^\beta f) \in L^1.$$

So,

$$(i\,\xi)^\delta (i\,D)^\beta \widehat{[f]} = (D^\delta(x^\beta f))\hat{}\in L^\infty.$$

Hence, if $\phi \in \mathscr{D}$ and equal to one on the unit ball we see there is a $g \in \mathscr{S}$ satisfying

$$(1-\phi)\,\widehat{[f]} = [g].$$

Hence, we may take $T = \phi\,\widehat{[f]}$. ⊣

Lemma 1.4. *Let $T \in \mathscr{D}'(\mathbb{R}^n)$ be homogeneous of degree m with* $\operatorname{sing\,supp}(T) \subset \{0\}$. *Then $T \in \mathscr{S}'$, $\hat{T}$ is homogeneous of degree $-m-n$, and there is an $f \in C^\infty(\mathbb{R}^n \setminus \{0\})$, homogeneous of degree $-m-n$, satisfying $\hat{T}|_{\mathbb{R}^n \setminus \{0\}} = [f]$.*

Remark. This lemma appears as an exercise in [3], Chap. 1, Sect. 10.2, page 64.

Proof. It is clear that T can be extended to be in $\mathscr{S}'$ since outside of the origin it is given by a function which is $O(\|x\|^m)$ as $\|x\| \to \infty$. Now for the homogeneity of $\hat{T}$. Let $\phi \in \mathscr{S}$. Then

$$(\phi(\cdot/\lambda))\hat{}(\xi) = \int \phi(x/\lambda)\, e^{-ix\cdot\xi}\, dx = \lambda^n \int \phi(y)\, e^{-i\lambda y\cdot\xi}\, dy = \lambda^n \hat{\phi}(\lambda\,\xi),$$

and so

$$\begin{aligned}(\hat{T})_\lambda(\phi) &= \lambda^{-n}\,\hat{T}(\phi(\cdot/\lambda)) = \lambda^{-n}\,T((\phi(\cdot/\lambda))\hat{\ })\\ &= \lambda^{-n}\,T(\lambda^n\,\hat{\phi}(\cdot/\lambda^{-1})) = \lambda^{-n}(\lambda^n\,T(\hat{\phi}(\cdot/\lambda^{-1})))\\ &= \lambda^{-n}\,T_{\lambda^{-1}}(\hat{\phi}) = \lambda^{-n-m}\,T(\hat{\phi}) = \lambda^{-n-m}\,\hat{T}(\phi).\end{aligned}$$

This shows $\hat{T}$ is homogeneous of degree $-m-n$. Now let $\phi\in\mathcal{D}(\mathbb{R}^n)$ and equal to one on a neighborhood of the origin. We have

$$\hat{T} = (\phi\,T)\hat{\ } + ((1-\phi)\,T)\hat{\ }.$$

The Paley-Wiener Theorem (see [13], Chap. 7, Thm. 7.23, p. 183) implies that the Fourier transform of a distribution with compact support has empty singular support. Hence

$$\operatorname{sing\,supp}(\widehat{\phi\,T}) = \emptyset.$$

Since $\operatorname{sing\,supp}(T)\subset\{0\}$ there is an $h\in C^\infty(\mathbb{R}^n\backslash\{0\})$ with $T|_{\mathbb{R}^n\backslash\{0\}} = [h]$. Lemma 1.1 implies h is homogeneous of degree m. So, $(1-\phi)\,T = [(1-\phi)\,h]$, and $(1-\phi)\,h$ is a symbol by Lemma 1.2. Now, Lemma 1.3 gives

$$\operatorname{sing\,supp}(((1-\phi)\,T)\hat{\ }) \subset \{0\}.$$

So $\operatorname{sing\,supp}(\hat{T})\subset\{0\}$. Hence, there is an $f\in C^\infty(\mathbb{R}^n\backslash\{0\})$ with $\hat{T}|_{\mathbb{R}^n\backslash\{0\}} = [f]$, and Lemma 1.1 gives the desired homogeneity of f. ⊣

Theorem 1.5. *For all $C>0$ there is an $\alpha_0>0$ such that $|\widehat{G_\alpha}(\xi)| \leqq C\,\|\xi\|^{-3}$ for $\|\xi\|\geqq 1$ and $0\leqq\alpha\leqq\alpha_0$.*

Proof. Recall

$$G_\alpha = \sum_{m=1}^{4}\sum_{n=0}^{n_0}\left(\frac{(i\,\alpha\,r\cos 3\,\theta)^m}{m\,!}\,\frac{(-r^\gamma)^n}{n\,!}\right).$$

Now, $\dfrac{(i\,r\cos 3\,\theta)^m(-r^\gamma)^n}{m\,!\,n\,!}$ is homogeneous of degree $m+\gamma\,n$ so

$$\left[\frac{(i\,r\cos 3\,\theta)^m(-r^\gamma)^n}{m\,!\,n\,!}\right] \quad \text{is also.}$$

Also, it is C^∞ except at the origin. Hence, by Lemma 1.4 there exists $f_{m,n}\in C^\infty(\mathbb{R}^2\backslash\{0\})$, homogeneous of degree $-2-m-\gamma\,n$, satisfying

$$\left[\frac{(i\,\alpha\,r\cos 3\,\theta)^m(-r^\gamma)^n}{m\,!\,n\,!}\right]^{\wedge}\Bigg|_{\mathbb{R}^2\backslash\{0\}} = \alpha^m\,[f_{m,n}].$$

For $\|\xi\|\geqq 1$ we have

$$\left|\sum_{m=1}^{4}\sum_{n=0}^{n_0}\alpha^m f_{m,n}(\xi)\right| \leqq \sum_{m=1}^{4}\sum_{n=0}^{n_0}\Big(\sup_{\|\xi\|=1}|f_{m,n}(\xi)|\Big)\,\alpha^m\,\|\xi\|^{-2-m-\gamma n}\Big).$$

Since $-2-m-\gamma n \leqq -3$ and $m \geqq 1$ (so we can make α^m small), the lemma is proved. $\dashv$

In the previous theorem we wrote "$\widehat{G_\alpha}(\xi)$". Now $G_\alpha \notin L^1$ so we haven't defined $\widehat{G_\alpha}$. We have, however, defined $\widehat{[G_\alpha]}$. But this is a tempered distribution so even if there is a g with $[g]=\widehat{[G_\alpha]}$, this g is only determined almost everywhere. However, we know that $\operatorname{sing\,supp}(\widehat{[G_\alpha]}) \subset \{0\}$ so there is a canonical C^∞ choice for g in $\mathbb{R}^2 \backslash \{0\}$. To clarify what is happening, here is a lemma.

Lemma 1.6. *Let $f \in L^1$. Let g and h satisfy $f=g+h$, $\operatorname{sing\,supp}\widehat{[g]} \subset \{0\}$, and $\operatorname{sing\,supp}\widehat{[h]} \subset \{0\}$. Let $\tilde{g}, \tilde{h} \in C^\infty(\mathbb{R}^n \backslash \{0\})$ satisfy*

$$\widehat{[g]}|_{\mathbb{R}^n \backslash \{0\}} = [\tilde{g}] \quad \textit{and} \quad \widehat{[h]}|_{\mathbb{R}^n \backslash \{0\}} = [\tilde{h}].$$

Then $\hat{f}|_{\mathbb{R}^n \backslash \{0\}} = \tilde{g} + \tilde{h}$.

Proof. $\widehat{[f]} = \widehat{[g]} + \widehat{[h]}$. Since $f \in L^1$ we also have $\widehat{[f]} = [\hat{f}]$. So

$$[\hat{f}]|_{\mathbb{R}^n \backslash \{0\}} = [\tilde{g}] + [\tilde{h}] = [\tilde{g} + \tilde{h}].$$

So $\hat{f}|_{\mathbb{R}^n \backslash \{0\}} = \tilde{g} + \tilde{h}$ almost everywhere. But $\hat{f}$, $\tilde{g}$, and $\tilde{h}$ are continuous so $\hat{f}|_{\mathbb{R}^n \backslash \{0\}} = \tilde{g} + \tilde{h}$. $\dashv$

With the preceding as justification we will allow some confusion of functions and distributions.

2. Bounds Near Infinity

The main lemma is the following (cf. [1], Chap. IX, § 44.4, pp. 244–245).

Lemma 2.1. *Let $f: \mathbb{R}^2 \to \mathbb{C}$. For $j=1,2$, $\beta=0,1,2,3,4$ suppose*

$$D_j^\beta f \in L^1 \cap C^0 \quad \textit{and} \quad \lim_{\|x\| \to \infty} |D_j^\beta f(x)| = 0.$$

Then $|\hat{f}(\xi)| \leqq 4 \max\{\|D_1^4 f\|_{L^1}, \|D_2^4 f\|_{L^1}\} \|\xi\|^{-4}$.

Proof. The hypotheses allow us to integrate by parts four times, giving

$$\hat{f}(\xi) = \int e^{-ix \cdot \xi} f(x)\, dx = \int (\xi_j)^{-4} e^{-ix \cdot \xi} D_j^4 f(x)\, dx.$$

So $|\hat{f}(\xi)| \leqq \xi_1^{-4} \|D_1^4 f\|_{L^1}$ and $|\hat{f}(\xi)| \leqq \xi_2^{-4} \|D_2^4 f\|_{L^1}$. If $|\xi_1| \leqq |\xi_2|$ then

$$\|\xi\|^{-4} = (\xi_1^2 + \xi_2^2)^{-2} \geqq (2\xi_2^2)^{-2} = (1/4)\, \xi_2^{-4}$$

and $\xi_2^{-4} \leqq \xi_1^{-4}$ so $\min\{\xi_2^{-4}, \xi_1^{-4}\} \leqq 4 \|\xi\|^{-4}$. If $|\xi_2| \leqq |\xi_1|$ we also get $\min\{\xi_1^{-4}, \xi_2^{-4}\} \leqq 4 \|\xi\|^{-4}$. Hence

$$\begin{aligned} |\hat{f}(\xi)| &\leqq \min\{\xi_1^{-4} \|D_1^4 f\|_{L^1}, \; \xi_2^{-4} \|D_2^4 f\|_{L^1}\} \\ &\leqq \max\{\|D_1^4 f\|_{L^1}, \; \|D_2^4 f\|_{L^1}\} \min\{\xi_1^{-4}, \xi_2^{-4}\} \\ &\leqq 4 \max\{\|D_1^4 f\|_{L^1}, \; \|D_2^4 f\|_{L^1}\} \|\xi\|^{-4}. \quad \dashv \end{aligned}$$

If $f \in C_0^4$ then f satisfies the hypotheses of the lemma. To show H and F are C^4 it seems best to use the following elementary calculus lemma.

Lemma 2.2. *Let B be open in $\mathbb{R}^2$. Let $f=\sum_{i=1}^{\infty} f_i$ where each $f_i \in C^4(B)$. Suppose $\sum_{i=1}^{\infty} \sup_B |D^\beta f_i| < \infty$ for $|\beta| \leqq 4$. Then $f \in C^4(B)$ and $D^\beta f = \sum_{i=1}^{\infty} D^\beta f_i$ for $|\beta| \leqq 4$.*

For the essence of the proof cf. [14], Chap. 23, Corollary to Thm. 3, p. 472.

If we are going to show things are C^4 then we had better take some derivatives. Here goes. First note that for $r>0$ we have $D_i r = x_i/r$, $D_1\theta = -x_2/r^2$, and $D_2\theta = x_1/r^2$.

Lemma 2.3. *Let $\beta=(\beta_1,\beta_2)$ be a multi-index. Let $y \geqq 0$. Then for $r>0$ we may write*

$$D^\beta(r^y) = \sum_{j=1}^{2|\beta|} f_j(y)\, r^{y-|\beta|} \frac{x^{\mu_j}}{r^{|\mu_j|}}$$

where μ_j is a multi-index, $|\mu_j| \leqq |\beta|$, and $f_j(y)$ is a polynomial in y (for a given β, of course).

Proof. By induction on $|\beta|$. $\beta=0$ is clear. To simplify notation we take a generic case:

$$D_1\left(f(y)\, r^{y-|\beta|} \frac{x^\mu}{r^{|\mu|}}\right) = f(y)(y-|\beta|-|\mu|)\, r^{y-|\beta|-1} \frac{x^\mu x_1}{r^{|\mu|+1}} + f(y)\,\mu_1\, r^{y-|\beta|-1} \frac{x_1^{\mu_1-1} x_2^{\mu_2}}{r^{|\mu|-1}}. \quad \dashv$$

Lemma 2.4. *For β a multi-index, $r>0$, we may write*

$$D^\beta e^{-r^\gamma} = \sum_{j=1}^{3|\beta|} h_j\, e^{-r^\gamma} r^{-y_j} \frac{x^{\mu_j}}{r^{|\mu_j|}}$$

where $0 \leqq y_j \leqq |\beta|$, $|\mu_j| \leqq |\beta|$, and h_j is a number (depending on β, of course).

Proof. Induction. $\beta=0$ is O.K. (Recall $\gamma \in (0,1]$ is fixed).

$$\begin{aligned} D_1\left(h e^{-r^\gamma} r^{-y} \frac{x^\mu}{r^{|\mu|}}\right) &= h e^{-r^\gamma}\left(-\gamma r^{\gamma-1}\frac{x_1}{r}\right) r^{-y} \frac{x^\mu}{r^{|\mu|}} \\ &\quad + h\, e^{-r^\gamma}(-y-|\mu|)\, r^{-y-|\mu|-1} \frac{x_1}{r} x^\mu \\ &\quad + h\, e^{-r^\gamma} r^{-y-|\mu|}\, \mu_1\, x_1^{\mu_1-1} x_2^{\mu_2} \\ &= -\gamma\, h\, e^{-r^\gamma} r^{-y-(1-\gamma)} \frac{x_1 x^\mu}{r^{|\mu|+1}} \\ &\quad + (-y-|\mu|)\, h\, e^{-r^\gamma} r^{-y-1} \frac{x_1 x^\mu}{r^{|\mu|+1}} \\ &\quad + \mu_1\, h\, e^{-r^\gamma} r^{-y-1} \frac{x_1^{\mu_1-1} x_2^{\mu_2}}{r^{|\mu|-1}}. \quad \dashv \end{aligned}$$

Lemma 2.5. *For k a positive integer, β a multi-index, $r>0$, we may write*

$$D^{\beta}(\cos^k 3\theta)=\sum_{j=1}^{4|\beta|} g_j(k)\sin^{n_j} 3\theta \cos^{m_j} 3\theta \frac{x^{\mu_j}}{r^{|\mu_j|}} r^{-|\beta|}$$

where $|\mu_j|\leqq|\beta|$, $n_j+m_j=k$, $n_j\geqq 0$, $m_j\geqq 0$, n_j *and* m_j *are integers, and* $|g_j(k)|\leqq(3k(|\beta|+1))^{|\beta|}$, *and we allow* $(n_j, m_j, \mu_j)=(n_i, m_i, \mu_i)$ *for* $i\neq j$.

Proof. $D_1\left(g\sin^n 3\theta\cos^m 3\theta\frac{x^{\mu}}{r^{|\mu|}}r^{-|\beta|}\right)$

$$\begin{aligned}
&=g\,n\sin^{n-1}3\theta\cos^{m+1}3\theta\frac{3(-x_2)}{r^2}\frac{x^{\mu}}{r^{|\mu|}}r^{-|\beta|}\\
&\quad -g\,m\sin^{n+1}3\theta\cos^{m-1}3\theta\frac{3(-x_2)}{r^2}\frac{x^{\mu}}{r^{|\mu|}}r^{-|\beta|}\\
&\quad +g\sin^n 3\theta\cos^{m-1}3\theta\,\mu_1 x_1^{\mu_1-1}x_2^{\mu_2}r^{-|\beta|-|\mu|}\\
&\quad +g\sin^n 3\theta\cos^m 3\theta\, x^{\mu}(-|\beta|-|\mu|)r^{-|\beta|-|\mu|-1}\frac{x_1}{r}\\
&=-3g\,n\sin^{n-1}3\theta\cos^{m+1}3\theta\frac{x_2 x^{\mu}}{r^{|\mu|+1}}r^{-|\beta|-1}\\
&\quad +3g\,m\sin^{n+1}3\theta\cos^{m-1}3\theta\frac{x_2 x^{\mu}}{r^{|\mu|+1}}r^{-|\beta|-1}\\
&\quad +g\,\mu_1\sin^n 3\theta\cos^m 3\theta\frac{x_1^{\mu_1-1}x_2^{\mu_2}}{r^{|\mu|-1}}r^{-|\beta|-1}\\
&\quad +g(-|\beta|-|\mu|)\sin^n 3\theta\cos^m 3\theta\frac{x_1 x^{\mu}}{r^{|\mu|+1}}r^{-|\beta|-1}.
\end{aligned}$$

Notice, we don't really have any negative powers of sin or cos since if $n=0$ (or $m=0$) the term with $n-1$ (or $m-1$) has a factor of n (or m). The bound on g is fairly obvious:

$$|-3g\,n|\leqq|-3g\,k|\leqq(3k(|\beta|+1))^{|\beta|}3k\leqq(3k(|\beta|+2))^{|\beta|+1},$$
$$|g\,\mu_1|\leqq|g|\,\mu||\leqq|g\,|\beta||\leqq(3k(|\beta|+1))^{|\beta|}|\beta|\leqq(3k(|\beta|+2))^{|\beta|+1},$$

and

$$|g(-|\beta|-|\nu|)|\leqq|g\,2|\beta||\leqq(3k(|\beta|+1))^{|\beta|}2|\beta|\leqq(3k(|\beta|+2))^{|\beta|+1}. \quad \dashv$$

We want to use Lemma 2.1 to bound $\hat{H}$. So we show H is C^4.

Lemma 2.6. $\dfrac{(i r\cos 3\theta)^m}{m!}\left(e^{-r^{\gamma}}-\sum_{n=0}^{n_0}\dfrac{(-r^{\gamma})^n}{n!}\right)\in C^4(\mathbb{R}^2)$ *for* $m=0,1,2,3,4$.

Proof. It is clearly in $C^{\infty}(\mathbb{R}^2\setminus\{0\})$. We want to use Lemma 2.2 so write

$$\frac{(i r\cos 3\theta)^m}{m!}\left(e^{-r^{\gamma}}-\sum_{n=0}^{n_0}\frac{(-r^{\gamma})^n}{n!}\right)=\sum_{n=n_0+1}^{\infty}\frac{(i r\cos 3\theta)^m(-r^{\gamma})^n}{m!\,n!}.$$

From Lemmas 2.3 and 2.5 we have (in $\mathbb{R}^2\setminus\{0\}$)

$$\begin{aligned}
&D^{\beta}((i\,r\cos 3\theta)^m(-r^{\gamma})^n)\\
&\quad=(-1)^n \sum_{\substack{\delta,\varepsilon\\ \delta+\varepsilon=\beta}} C_{\delta,\varepsilon}\, D^{\delta}((i\cos 3\theta)^m)\, D^{\varepsilon}(r^{\gamma n+m})\\
&\quad=(-1)^n i^m \sum_{\substack{\delta,\varepsilon\\ \delta+\varepsilon=\beta}} C_{\delta,\varepsilon}\left\{\sum_{j=1}^{4|\delta|} g_j(m)\sin^{n_j} 3\theta\cos^{m_j} 3\theta\,\frac{x^{\mu_j}}{r^{|\mu_j|}}\,r^{-|\delta|}\right\}\\
&\qquad\cdot\left\{\sum_{k=1}^{2|\varepsilon|} f_k(\gamma n+m)\,r^{\gamma n+m-|\varepsilon|}\,\frac{x^{\tilde{\mu}_k}}{r^{|\tilde{\mu}_k|}}\right\}.
\end{aligned}$$

In each bracket the g_j, n_j, m_j, μ, etc. depend on δ or ε. We want to check that the limit exists as we approach the origin. Consider a typical term after multiplying out. As $(x_1/r)=\cos\theta$ and $(x_2/r)=\sin\theta$, we see that

$$g_j(m)\sin^{n_j} 3\theta\cos^{m_j} 3\theta\,\frac{x^{\mu_j+\tilde{\mu}_k}}{r^{|\mu_j|+|\tilde{\mu}_k|}}\,f_k(\gamma n+m)$$

is bounded as $r\to 0$. The remaining factor is $r^{-|\delta|+\gamma n+m-|\varepsilon|}=r^{\gamma n+m-|\beta|}$. As $n>n_0$, $m\geqq 0$, and $|\beta|\leqq 4$ imply $\gamma n+m-|\beta|>0$, we have $r^{\gamma n+m-|\beta|}\to 0$ as $r\to 0$. Hence

$$\lim_{r\to 0} D^{\beta}(i\,r\cos 3\theta)^m(-r^{\gamma})^n=0.$$

Using that the derivative of a continuous function cannot have a removable discontinuity (by L'Hôpital's Rule), we may induct on $|\beta|$ to conclude

$$\frac{(i\,r\cos 3\theta)^m(-r^{\gamma})^n}{m!\,n!}\in C^4(\mathbb{R}^2)\quad\text{for } m=0,1,2,3,4,\quad n>n_0.$$

Now fix β with $|\beta|\leqq 4$. Then the above formula shows there is a constant C so that

$$\sup_{r<1}|D^{\beta}(i\,r\cos 3\theta)^m(-r^{\gamma})^n)|\leqq C\sup_k|f_k(\gamma n+m)|.$$

But each f_k is just a polynomial so

$$\sum_{n=n_0+1}^{\infty}\sup_{r<1}\left|\frac{D^{\beta}((i\,r\cos 3\theta)^m(-r^{\gamma})^n)}{m!\,n!}\right|<\infty\quad\text{for } m=0,1,2,3,4.$$

So Lemma 2.2 applies. ⊣

Theorem 2.7. *There exists a C satisfying*

$$|\widehat{H_{\alpha}}(\xi)|\leqq C\,\|\xi\|^{-4}\quad \textit{for}\ \|\xi\|\geqq 1\ \textit{and}\ 0\leqq\alpha\leqq 1.$$

There exists a C satisfying

$$\left|\left(e^{-r^{\gamma}}-\sum_{n=0}^{n_0}\frac{(-r^{\gamma})^n}{n!}\right)^{\wedge}(\xi)\right|\leqq C\,\|\xi\|^{-4}\quad \textit{for}\ \|\xi\|\geqq 1.$$

Remark. The second estimate will be used in Theorem 2.10.

Proof. Recall

$$H_\alpha = \sum_{m=1}^{4} \frac{(i\alpha r\cos 3\theta)^n}{m!}\left(e^{-r^\gamma} - \sum_{n=0}^{n_0} \frac{(-r^\gamma)^n}{n!}\right).$$

Let

$$f_m := \frac{(i r\cos 3\theta)^m}{m!}\left(e^{-r^\gamma} - \sum_{n=0}^{n_0} \frac{(-r^\gamma)^n}{m!}\right) \quad \text{for } m=0,1,2,3,4.$$

So $H_\alpha = \sum_{m=1}^{4} \alpha^m f_m$. Choose $\phi \in \mathscr{D}(\mathbb{R}^2)$ and equal to one on a neighborhood of zero. Then Lemma 2.6 implies $\phi f_m \in C_0^4(\mathbb{R}^2)$. Now

$$(1-\phi)e^{-r^\gamma} \in \mathscr{S} \quad \text{so} \quad (1-\phi)\frac{(i r\cos 3\theta)^m}{m!} e^{-r^\gamma} \in \mathscr{S}.$$

Also $(1-\phi)\dfrac{(i r\cos 3\theta)^m}{m!}\dfrac{(-r^\gamma)^n}{n!} \in S^{m+n\gamma}$ by Lemma 1.2. So, since the sum of symbols is a symbol, $(1-\phi)f_m$ is a symbol. Now, Lemma 1.3 gives a C_m such that

$$|((1-\phi)f_m)\hat{}(\xi)| \leqq C_m \|\xi\|^{-4} \quad \text{for } \|\xi\| \geqq 1.$$

Lemma 2.1 gives C'_m such that

$$|(\phi f_m)\hat{}(\xi)| \leqq C'_m \|\xi\|^{-4}.$$

So, for $\|\xi\| \geqq 1$,

$$\begin{aligned} |\widehat{H_\alpha}(\xi)| &= \left|\sum_{m=1}^{4} \alpha^m((\phi f_m)\hat{} + ((1-\phi)f_m)\hat{})\right| \\ &\leqq \sum_{m=1}^{4} \alpha^m (C_m + C'_m)\|\xi\|^{-4}. \end{aligned}$$

We may take $C = \sum_{m=1}^{4} C_m + C'_m$ for the first result and $C = C_0 + C'_0$ for the second. ⊣

We want to get a similar bound for $\hat{F}$. However we can't factor out α so the argument will be different.

Lemma 2.8. *$F_\alpha \in C^4(\mathbb{R}^2)$ and $\sup_{\substack{0 \leqq \alpha \leqq 1 \\ r<2}} |D^\beta F_\alpha| < \infty$ for $|\beta| = 4$.*

Proof. Recall

$$F_\alpha = \left(e^{i\alpha r\cos 3\theta} - \sum_{m=0}^{4} \frac{(i\alpha r\cos 3\theta)^m}{m!}\right)e^{-r^\gamma}.$$

So $F_\alpha \in C^\infty(\mathbb{R}^2 \setminus \{0\})$. We want to apply Lemma 2.2 (as usual). So we write

$$F_\alpha = \sum_{m=5}^{\infty} \frac{(i\alpha r\cos 3\theta)^m}{m!} e^{-r^\gamma}.$$

We only consider β satisfying $|\beta| \leqq 4$ and m satisfying $m \geqq 5$. Using Lemmas 2.3, 2.4, and 2.5 we see that $D^\beta \left(\frac{(i\alpha r \cos 3\theta)^m}{m!} e^{-r^\gamma}\right)$ is a finite sum of terms of the form

$$\frac{C(i\alpha)^m}{m!} g(m) \sin^{n'} 3\theta \cos^{m'} 3\theta \frac{x^{\mu'}}{r^{|\mu'|}} r^{-|\beta'|} f(m) r^{m-|\beta''|} \frac{x^{\mu''}}{r^{|\mu''|}} h e^{-r^\gamma} r^{-y} \frac{x^{\mu'''}}{r^{|\mu'''|}}$$

where $\beta' + \beta'' + \beta''' = \beta$ and $0 \leqq y \leqq |\beta'''|$. Factor out $r^{-|\beta'|+m-|\beta''|-y}$. What is left is a bounded function. For $m \geqq 5$ we have $-|\beta'|+m-|\beta''|-y \geqq m-|\beta| > 0$. This shows

$$\lim_{r\to 0} D^\beta \left(\frac{(i\alpha r \cos 3\theta)^m}{m!} e^{-r^\gamma}\right) = 0.$$

Hence (as in the proof of Lemma 2.6) we conclude $(i\alpha r \cos 3\theta)^m e^{-r^\gamma} \in C^4$. Now, since f is a polynomial and $|g(m)| \leqq (15m)^4$ we see there is a C_0 and a k s.t.

$$\sum_{m=5}^{\infty} \sup_{\substack{r<2 \\ 0\leqq\alpha\leqq 1}} \left| D^\beta \frac{((i\alpha r \cos 3\theta)^m e^{-r^\gamma})}{m!} \right| \leqq \sum_{m=5}^{\infty} \frac{C_0 m^k 2^m}{m!} < \infty.$$

So we may apply Lemma 2.2 to get $F_\alpha \in C^4$ and for $r<2$

$$D^\beta F_\alpha = \sum_{m=5}^{\infty} D^\beta \frac{((i\alpha r \cos 3\theta)^m e^{-r^\gamma})}{m!}.$$

Therefore

$$\sup_{\substack{0\leqq\alpha\leqq 1 \\ r<2}} |D^\beta F_\alpha| \leqq \sum_{m=5}^{\infty} \sup_{\substack{r<2 \\ 0\leqq\alpha\leqq 1}} \left| D^\beta \frac{((i\alpha r \cos 3\theta)^m e^{-r^\gamma})}{m!} \right| < \infty.$$

This proves the lemma. ⊣

Theorem 2.9. *There is a C satisfying*

$$|\widehat{F_\alpha}(\xi)| \leqq C\,\|\xi\|^{-4} \quad \textit{for } 0\leqq\alpha\leqq 1 \textit{ and all } \xi.$$

Proof. Choose $\phi \in \mathscr{D}$ with $\operatorname{supp}\phi \subset \{r \leqq 3/2\}$ and $\phi|_{\{r\leqq 5/4\}} = 1$. Then Lemma 2.8 and the product rule for differentiation give

$$\sup_{0\leqq\alpha\leqq 1} \|D^\beta(\phi F_\alpha)\|_{L^1} < \infty \quad \text{for } |\beta| = 4.$$

So Lemma 2.1 gives a C such that $\sup_{0\leqq\alpha\leqq 1} |\widehat{\phi F_\alpha}(\xi)| \leqq C\,\|\xi\|^{-4}$. For $m=0,1,2,3,4$

$$\frac{(i r \cos 3\theta)^m}{m!} e^{-r^\gamma}(1-\phi) \in \mathscr{S}$$

so its Fourier transform is in $\mathscr{S}$. Hence, there are constants C_m satisfying

$$\left| \left(\frac{(i r \cos 3\theta)^m}{m!} e^{-r^\gamma}(1-\phi)\right)^{\wedge} (\xi) \right| \leqq C_m \|\xi\|^{-4}.$$

Hence,

$$\begin{aligned}
&\sup_{0\leqq\alpha\leqq1}\left|\left(-\sum_{m=0}^{4}\frac{(i\alpha r\cos 3\theta)^m}{m!}e^{-r^\gamma}(1-\phi)\right)^{\wedge}(\xi)\right| \\
&\leqq \sup_{0\leqq\alpha\leqq1}\sum_{m=0}^{4}\alpha^m\left|\left(\frac{(ir\cos 3\theta)^m}{m!}e^{-r^\gamma}(1-\phi)\right)^{\wedge}(\xi)\right| \\
&\leqq \sup_{0\leqq\alpha\leqq1}\sum_{m=0}^{4}\alpha^m C_m\|\xi\|^{-4} \\
&\leqq \left(\sum_{m=0}^{4}C_m\right)\|\xi\|^{-4}.
\end{aligned}$$

The remaining term is $e^{i\alpha r\cos 3\theta}e^{-r^\gamma}(1-\phi)$. This is in $\mathscr{S}$. So, its transform is in $\mathscr{S}$. However we want a bound that is uniform in α. So suppose f is a smooth function. Then

$$D_1 e^f = e^f D_1 f,$$
$$D_1^2 e^f = e^f((D_1 f)^2 + D_1^2 f),$$
$$D_1^3 e^f = e^f((D_1 f)^3 + 3D_1 f D_1^2 f + D_1^3 f),$$

and

$$D_1^4 e^f = e^f((D_1 f)^4 + 6(D_1 f)^2 D_1^2 f + 4D_1 f D_1^3 f + 3(D_1^2 f)^2 + D_1^4 f).$$

And the same with "1" replaced by "2". Now let $g := ir\cos 3\theta$. Let $f := \alpha g$. Since g is homogeneous, for each β there is a constant C_β such that (cf. pf. of Lemma 1.2)

$$|D^\beta g| \leqq C_\beta r^{1-|\beta|} \quad \text{for } r \geqq 1.$$

So we have $\sup_{0\leqq\alpha\leqq1}|D^\beta f| \leqq C_\beta r^{1-|\beta|}$ for $r\geqq 1$. Since $|e^f|=1$, the above expressions imply there is a constant C such that

$$\sup_{\substack{0\leqq\alpha\leqq1\\ j=1,2\\ n=0,1,2,3,4}} |D_j^n e^f| \leqq C \quad \text{for } r\geqq 1.$$

The point is that C doesn't depend on α. Lemma 2.4 gives constants $\tilde{C}_\beta$ such that

$$|D^\beta e^{-r^\gamma}| \leqq \tilde{C}_\beta e^{-r^\gamma} \quad \text{for } r\geqq 1.$$

So, we see that for $n\leqq 4$, $j=1,2$ there is a constant K such that

$$\sup_{0\leqq\alpha\leqq1}|D_j^n(e^{i\alpha r\cos 3\theta}e^{-r^\gamma}(1-\phi)| \leqq K e^{-r^\gamma}.$$

Since $e^{-r^\gamma}\in L^1$ and goes to zero as $r\to\infty$, we may apply Lemma 2.1 to get a bound on $(e^{i\alpha r\cos 3\theta - r^\gamma}(1-\phi))^{\wedge}$ which is uniform in α. Combining our bounds on $\widehat{\phi F_\alpha}$,

$$\left(-\sum_{m=0}^{4} \frac{(i\alpha r\cos 3\theta)^m}{m!} e^{-r^\gamma}(1-\phi)\right)^\wedge, \quad \text{and} \quad (e^{i\alpha r\cos 3\theta - r^\gamma}(1-\phi))^\wedge$$

gives the theorem. ⊣

We want an estimate on $\hat{Q}$. Recall $Q=e^{-r^\gamma}$.

Theorem 2.10. *There exists a* $C>0$ *and an* R *satisfying*

$$|\hat{Q}(\xi)| \geqq C\,\|\xi\|^{-2-\gamma} \quad \textit{for} \quad \|\xi\| \geqq R.$$

Proof. Write

$$Q=\sum_{n=0}^{n_0} \frac{(-r^\gamma)^n}{n!} + \left\{e^{-r^\gamma} - \sum_{n=0}^{n_0} \frac{(-r^\gamma)^n}{n!}\right\}.$$

So

$$\widehat{[Q]} = \sum_{n=0}^{n_0} \frac{(-1)^n}{n!}[r^{\gamma n}]^\wedge + \left[e^{-r^\gamma} - \sum_{n=0}^{n_0} \frac{(-r^\gamma)^n}{n!}\right]^\wedge.$$

$r^{\gamma n}$ is rotationally invariant so $[r^{\gamma n}]^\wedge$ will be also (see [5], Chap. II, § 3.1, p. 191). Lemma 1.1 remains true if we replace "homogeneous of degree m" by "rotationally invariant" using the same method of proof. This and Lemma 1.4 imply there are constants K_n satisfying

$$[r^{\gamma n}]^\wedge|_{\mathbb{R}^2\setminus\{0\}} = K_n[\|\xi\|^{-2-\gamma n}].$$

When $n=0$ we have $\widehat{[1]}=(2\pi)^{-2}\delta$ so $K_0=0$. However, $K_1 \neq 0$ as we now show. Suppose $K_1=0$. Then $\widehat{[r^\gamma]}|_{\mathbb{R}^2\setminus\{0\}}=0$ so $\operatorname{supp}\widehat{[r^\gamma]}$ is compact. So by the Paley-Wiener Theorem $[r^\gamma]^{\wedge\wedge}$ is a C^∞ function (i.e. $\operatorname{sing\,supp}[r^\gamma]^{\wedge\wedge}=\emptyset$). But the Fourier Inversion Theorem gives $[r^\gamma]^{\wedge\wedge}=(2\pi)^{-2}[r^\gamma]$ which is definitely not C^∞. Therefore $K_1 \neq 0$.

For $n \geqq 2$ we have $-2-\gamma n < -2-\gamma < 0$, so there is an $R_0>0$ satisfying

$$|K_1/2|\,\|\xi\|^{-2-\gamma} \leqq \left|\sum_{n=0}^{n_0} K_n\,\|\xi\|^{-2-\gamma n}\right| \quad \text{for} \quad \|\xi\| \geqq R_0.$$

Theorem 2.7 gives a C_1 satisfying

$$\left|\left(e^{-r^\gamma} - \sum_{n=0}^{n_0} \frac{(-r^\gamma)^n}{n!}\right)^\wedge(\xi)\right| \leqq C_1\,\|\xi\|^{-4} \quad \text{for} \quad \|\xi\| \geqq 1.$$

Choose $R>R_0$ and large enough so that $C_1\,\|\xi\|^{-4} < |K_1/4|\,\|\xi\|^{-2-\gamma}$ for $\|\xi\| \geqq R$. Let $C=|K_1/4|$. The theorem is proved. ⊣

More explicit results are known: see [6], Chap. 34, § 4, p. 15 and [8], Lemma 1.1.

3. The Main Result

The proof of the next theorem is mostly from [9].

Theorem 3.1. $\hat{Q}$ *is strictly greater than zero.*

Proof. Let $p(\xi; N) := \int_{\mathbb{R}^N} e^{-i\xi\cdot x - \|x\|^\gamma} dx$. It is known that $e^{-\|x\|^\gamma}$ is the characteristic function of a stable probability law (see [10], Chap. VII, § 63, pp. 221–224; for a different proof see [7], pp. 222–224). The Fourier Inversion Theorem implies that the density of this law is $(2\pi)^{-N} p(\xi; N)$. Hence $p(\xi; N) \geqq 0$ for all ξ and all $N = 1, 2, 3, \ldots$. From [2], Chap. II, § 7. Thm. 40, p. 69 we get for $N = 2, 4, 6, \ldots$

$$p(\xi; N) = 2\pi(-4\pi)^{\frac{N}{2}-1} \left(\frac{d}{d(|\xi|^2)}\right)^{\frac{N}{2}-1} \int_0^\infty e^{-R^\gamma} R J_0(|\xi| R)\, dR,$$

where J_0 is the Bessel function, of the first kind, of order zero. Now, for $|\xi| > 0$

$$\begin{aligned}\frac{d}{d|\xi|} p(\xi; 2) &= 2|\xi| \frac{d}{d(|\xi|^2)} p(\xi; 2)\\ &= 2|\xi|\, 2\pi \frac{d}{d(|\xi|^2)} \int_0^\infty e^{-R^\gamma} R J_0(|\xi| R)\, dR\\ &= 2|\xi|(-4\pi)^{-1} p(\xi; 4)\\ &= -|\xi|(2\pi)^{-1} p(\xi; 4)\\ &\leqq 0.\end{aligned}$$

Since $e^{-\|x\|^\gamma} \in L^1$, we see that p is continuous (cf. Lemma 3.3). Let $f\colon [0, \infty) \to \mathbb{R}$ be defined by

$$f(|\xi|) = p(\xi; 2) := \hat{Q}(\xi).$$

Then f is continuous and $f' \leqq 0$ on $(0, \infty)$. Hence f is nonincreasing. But, Theorem 2.10 implies $f(y) \neq 0$ for y sufficiently large. We also know $f \geqq 0$. Therefore $f > 0$. ⊣

Minor modification of the above arguments gives

$$\frac{d}{d|\xi|} p(\xi; N) = -\frac{|\xi|}{2\pi} p(\xi; N+2) \quad \text{and} \quad p(\xi; N) > 0$$

for all ξ and N. Therefore

$$\frac{d}{d|\xi|} p(\xi; N) < 0 \quad \text{for } |\xi| > 0.$$

Hence, $(2\pi)^{-N} p(\xi; N)$ is a unimodal probability law for $N = 1, 2, 3, \ldots$.

Corollary 3.2. *There is a $C > 0$ and an R satisfying*

$$\hat{Q}(\xi) \geqq C\, \|\xi\|^{-2-\gamma} \quad \textit{for } \|\xi\| \geqq R.$$

Proof. Theorem 3.1 allows us to drop the absolute value bars from Theorem 2.10. ⊣

Lemma 3.3. *Let $S(\alpha, \xi) := \hat{P}_\alpha(\xi)$. Then S is jointly continuous on $\mathbb{R} \times \mathbb{R}^2$.*

Proof. Immediate by Lebesgue Dominated Convergence. ⊣

Theorem 3.4. *There is an $\alpha_1 > 0$ such that for $0 \leqq \alpha \leqq \alpha_1$ $e^{i\alpha r \cos 3\theta - r^\gamma}$ is the characteristic function of a probability law.*

Proof. Recall $P_\alpha = e^{i\alpha r \cos 3\theta} - r^\gamma$. Now, $P_\alpha \in L^1$ and $P_\alpha(x) = \overline{P_\alpha(-x)}$ so $\hat{P}_\alpha$ is real (the bar denotes complex conjugation). Recall $P = Q + G + H + F$. Let C_0, C_1, and C_2 be the C's that exist by Theorems 2.7, 2.9, and Corollary 3.2 respectively. Let R_0 be the R that exists by Corollary 3.2. In Theorem 1.5 take $C = C_2/3$. So, there is an $\alpha_0 > 0$ satisfying the conclusion of Theorem 1.5. We may suppose $\alpha_0 \leqq 1$. Now choose R so that $R > R_0$, $R > 1$, and

$$(C_0 + C_1) \|\xi\|^{-4} \leqq (C_2/3) \|\xi\|^{-2-\gamma} \quad \text{for } \|\xi\| \geqq R.$$

So,

$$\hat{P}_\alpha(\xi) \geqq (C_2/3) \|\xi\|^{-2-\gamma} > 0 \quad \text{for } \|\xi\| \geqq R \text{ and } 0 \leqq \alpha \leqq \alpha_0.$$

Since $\hat{P}_0 > 0$ (by Theorem 3.1) and $\hat{P}_\alpha(\xi)$ is jointly continuous (by Lemma 3.3), we may choose α_1 satisfying $0 < \alpha_1 < \alpha_0$ and

$$\hat{P}_\alpha(\xi) > 0 \quad \text{for } \|\xi\| \leqq R \text{ and } 0 \leqq \alpha \leqq \alpha_1.$$

Hence,

$$\hat{P}_\alpha > 0 \quad \text{for } 0 \leqq \alpha \leqq \alpha_1.$$

Since $P_\alpha \in L^1$, P_α is continuous at zero, and $\hat{P}_\alpha > 0$, we have (see [15], Chap. I, §1, Cor. 1.26, p. 15)

$$\int (2\pi)^{-2} \hat{P}_\alpha(\xi) e^{ix \cdot \xi} d\xi = P_\alpha(x) \quad \text{for } 0 \leqq \alpha \leqq \alpha_1,$$

and

$$\int (2\pi)^{-2} \hat{P}_\alpha(\xi) d\xi = P_\alpha(0) = 1 \quad \text{for } 0 \leqq \alpha \leqq \alpha_1.$$

Therefore, for $0 \leqq \alpha \leqq \alpha_1$ $(2\pi)^{-2} \hat{P}_\alpha$ is the density of a probability law which has characteristic function P_α. ⊣

4. Stability

For this section we require $0 < \gamma < 1$. By Theorem 3.4 choose $\alpha > 0$ so that

$$P = \exp(i \alpha r \cos 3\theta - r^\gamma)$$

is a characteristic function. Let μ be the probability measure on $\mathbb{R}^2$ with characteristic function P. We show μ is not stable even though all of its one-dimensional projections are stable.

Definitions. A probability measure, ν, on $\mathbb{R}^n$ is *stable* if for all A, $B > 0$ there is a $C > 0$ and an $s \in \mathbb{R}^n$ satisfying $\mathscr{L}(C(AX + BY) + s) = \nu$ where X and Y are independent random variables with $\mathscr{L}(X) = \mathscr{L}(Y) = \nu$. (Here $\mathscr{L}$ stands for "law of").

The *one-dimensional projections* of ν are the probability measures on $\mathbb{R}$ of the form $\nu\circ f^{-1}$ where f is a linear function from $\mathbb{R}^n\to\mathbb{R}$.

Theorem 4.1. *μ is not stable.*

Proof. Let X and Y be independent with law μ. Take $A=B=1$. Supposing μ is stable, there is a $C>0$ and an $s\in\mathbb{R}^2$ satisfying

$$\mu=\mathscr{L}(C(X+Y)+s).$$

Taking characteristic functions we have

$$\begin{aligned} e^{i\alpha r\cos 3\theta-r^\gamma}&=(e^{i\alpha Cr\cos 3\theta-(Cr)^\gamma})^2\, e^{ix\cdot s}\\ &=e^{i\alpha 2Cr\cos 3\theta-2(Cr)^\gamma+ix\cdot s}.\end{aligned}$$

So, we must have $C=2^{-1/\gamma}$. So

$$\alpha r\cos 3\theta-\alpha 2^{1-1/\gamma} r\cos 3\theta-x\cdot s\equiv 0 \bmod 2\pi.$$

Since it is continuous, there exists k satisfying

$$(1-2^{1-(1/\gamma)})\,\alpha r\cos 3\theta-x\cdot s=2\pi k.$$

Notice $1-2^{1-1/\gamma}\neq 0$ since $0<\gamma<1$. Choose θ so that $e^{i\theta}$ is orthogonal to s. Then for $x=re^{i\theta}$ we find $(1-2^{1-(1/\gamma)})\,\alpha r\cos 3\theta=2\pi k$ for all $r>0$. Therefore we must have $\cos 3\theta=0=k$. Now try $x=(0,1)$ so $x=e^{i\pi/2}$. This gives

$$s_2=x\cdot s=(1-2^{1-(1/\gamma)})\,\alpha\cos(3\pi/2)=0.$$

Now try $x=e^{i\pi/6}$ so

$$(s_1/\sqrt{3})=x\cdot s=(1-2^{1-(1/\gamma)})\,\alpha\cos(\pi/2)=0.$$

This leaves

$$(1-2^{1-(1/\gamma)})\,\alpha r\cos 3\theta=0$$

which isn't true, so μ is not stable. ⊣

Theorem 4.2. *If $f\colon\mathbb{R}^2\to\mathbb{R}$ is linear then $\mu\circ f^{-1}$ is stable.*

Proof. It is well known that functions of the form $e^{\phi(u)}$ where $\phi\colon\mathbb{R}\to\mathbb{C}$ is of the form

$$\begin{gathered}\phi(u)=iau-b\,|u|^\gamma\{1+ic(\operatorname{sgn}u)\tan(\pi\gamma/2)\},\\ a\in\mathbb{R},\quad b>0,\quad c\in[-1,1],\quad 0<\gamma<1\end{gathered}$$

are the characteristic functions of the non-degenerate stable laws of index γ (see [11], § 24.4, p. 339). Since f is linear, there is $y\in\mathbb{R}^2$ satisfying $f(x)=x\cdot y$. Then we have

$$\begin{aligned}(\text{ch. f.}\,(\mu\circ f^{-1}))(u)&=(\text{ch. f.}\,(\mu))(u\,y)\\ &=P(u\,y).\end{aligned}$$

If $y=0$ then $P(u\,y)=P(0)=1$ so $\mu\circ f^{-1}=\delta$ which is stable of index γ. Suppose $y \neq 0$. Let θ satisfy $e^{i\theta}=y/\|y\|$. Then

$$P(u\,y)=e^{i\alpha u\|y\|\cos 3\theta-\|uy\|^{\gamma}}.$$

This is in the above form if we set $a=\alpha\|y\|\cos 3\theta$, $b=\|y\|^{\gamma}$, and $c=0$. So $\mu\circ f^{-1}$ is stable of index γ. $\dashv$

This provides a counterexample to Theorem 4 (and Theorem 5) of "Zero-One Laws For Stable Measures" [4]. In the notation of that paper we take $S=\mathbb{R}^2$ and F=the vector space of linear forms on $\mathbb{R}^2$ (i.e. linear maps of $\mathbb{R}^2$ into $\mathbb{R}$). Then $\mathcal{S}(F)$=Borel sets. We use the above μ. It is elementary that $(\mathbb{R}^2, F)$ is a full pair. Our Theorem 4.2 now shows the hypotheses of Theorems 4 and 5 are fulfilled, but the conclusion is contradicted by our Theorem 4.1. However, Theorems 4 and 5 are true for $\gamma>1$, while the behavior for $\gamma=1$ is not known.

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