Multi-Dimensional Volumes and Moduli of Convexity in Banach Spaces (*) (**).

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Summary. – *Let E be a Banach space. Using the definition for the k-dimensional volume enclosed by $k+1$ vectors due to Silverman* [16], *one can define the modulus of k-rotundity of E. In* [22] *it was shown that k-uniformly rotund Banach spaces are isomorphic to uniformly rotund spaces and, indeed, have some of the same isometric properties with respect to non-expansive and nearest-point maps. The present paper examines the modulus of k-rotundity more thoroughly. Included are a result on the asymptotic behavior of the moduli for l^2; a generalization of Dixmier's Theorem on higher-duals of non-reflexive spaces; and an inequality relating the moduli of E^{**}/E and those of E. The modulus of 2-rotundity is shown to be equivalent to one of the moduli defined by V. D. Milman* [13] *and a necessary and sufficient condition for an l^p-product of spaces to be 2-uniformly rotund is given.*

1. – Introduction.

A definition of the k-dimensional voume enclosed by $k+1$ vectors in a Banach space, E, has been given by E. SILVERMAN [16]. Using this definition and extending the usual notion of the modulus of rotundity, one can define the modulus of k-rotundity of E. In [22] it was shown that k-uniformly rotund Banach spaces are isomorphic to uniformly rotund spaces and, indeed, have some of the same isometric properties with respect to non-expansive and nearest point maps. Our principal aim in the present paper is to examine the basic properties of the modulus of k-rotundity more thoroughly.

The motivation for this work comes from two sources: the results of J. J. SCHÄFFER concerning girth of spheres [18], [19] and the research of V. D. MILMAN on multi-dimensional moduli. Central to Schäffer's work is the definition of arc length in Banach spaces. The obvious generalizations are to surface area, volume and higher dimensional « hyper-volumes ». The idea of working with uniformities defined over all sub-spaces of some fixed finite dimension, k, comes from the point of view taken by V. D. MILMAN in his study of higher dimensional moduli of smoothness and convexity.

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In section 2 we give a geometrical result which is useful in studying k-dimensional moduli. We also apply a method of J. BERNAL to obtain information on the behavior of the k-dimensional moduli of Hilbert space for k increasing.

Section 3 is most closely related to the results of [22]. We show, first, the existence of non-trivial hypervolumes in the unit sphere of higher duals of non-reflexive spaces. This generalizes the result of Dixmier on the existence of line segments in the unit sphere of a fourth dual space [8]. A corollary of our Theorem is that a Banach space E is reflexive whenever the second dual is locally k-UR. The proof is based on the same sort of « local reflexivity » arguments as were used in [21]. Similar methods yield a connection between the moduli of E^{**}/E and those of E. The idea that k-dimensional subspaces of E^{**}/E determine properties of $2k$-dimensional subspaces of E comes from the work in [3] on the « $l^1_{(n)}$ problem ». (The deeper connections between existence of maximal hypervolumes and existence of subspaces almost isometric to $l^1_{(n)}$ remain to be examined.) Finally, results on duality for the two dimensional convexity moduli and relations with the two dimensional moduli defined by Milman are given.

Section 4 contains necessary and sufficient conditions for l^p products of Banach spaces to be 2-UR. These give a method for constructing a large class of non-trivial examples of spaces which are 2-UR but not 1-UR.

The remainder of this section consists of definitions and notation which will be used in the sequel.

A Banach space, E, is said to be 1-uniformly rotund (1-UR) if, for each $\varepsilon > 0$, there is a $\delta_E^{(1)}(\varepsilon) > 0$ such that if $\|x\|, \|y\| \leqslant 1$ and

$$\left\|\frac{x+y}{2}\right\| \geqslant 1 - \delta_E^{(1)}(\varepsilon)$$

then

$$\sup\left\{\begin{vmatrix} 1 & 1 \\ \langle g, x\rangle & \langle g, y\rangle \end{vmatrix} : g \in E^*,\ \|g\| \leqslant 1\right\} < \varepsilon .$$

Here, and throughout the sequel, the symbol $|\cdot|$ denotes the determinant. Hence, 1-UR is just the usual notion of uniform rotundity. To generalize this we define the 2-dimensional « area » enclosed by vectors (x, y, z) as

$$A(x, y, z) \equiv \sup\left\{\begin{vmatrix} 1 & 1 & 1 \\ \langle f, x\rangle & \langle f, y\rangle & \langle f, z\rangle \\ \langle g, x\rangle & \langle g, y\rangle & \langle g, z\rangle \end{vmatrix} : \|f\|,\ \|g\| \leqslant 1\right\} .$$

This idea is taken from the work of E. SILVERMAN [16], [17]. The k-dimensional volume enclosed by vectors $(x_1, x_2, \ldots, x_{k+1})$ is defined in the obvious way, and the modulus of k-rotundity is given by

$$\delta_E^{(k)}(\varepsilon) = \inf\left\{1 - \left\|\frac{x_1 + x_2 + \ldots + x_{k+1}}{k+1}\right\| : \|x_1\|, \|x_2\|, \ldots, \|x_{k+1}\| \leqslant 1, \right. \\ \left. A(x_1, x_2, \ldots, x_{k+1}) \geqslant \varepsilon\right\}.$$

When no confusion is possible we shall omit the subscript « E ». The space is said to k-UR if for all $\varepsilon > 0$, $\delta^{(k)}(\varepsilon) > 0$.

2. - Geometry of multi-dimensional volumes.

Our first result shows that the definition of enclosed volume makes sense geometrically. It says that the volume is greater than or equal to the « height » times the area of the « base ». For vectors $z, y_1, y_2, .., y_k$ in a Banach space E, dist $(z, [y_1, y_2, ..., y_k])$ denotes the distance from z to the affine span of the y_i.

LEMMA 1. - For all vectors $x_1, x_2, \ldots, x_k$ in E

$$A(x_1, x_2, \ldots, x_k) \geqslant \operatorname{dist}(x_1, [x_2, \ldots, x_k])\, A(x_2, \ldots, x_k)\,.$$

PROOF. - From the definition we have that

$$A(x_1, x_2, \ldots, x_k) =$$

$$= \sup\left\{\begin{vmatrix} 1 & 1 & \ldots & 1 \\ \langle f_1, x_1\rangle & \langle f_1, x_2\rangle & \ldots & \langle f_1, x_k\rangle \\ \vdots & & & \\ \langle f_{k-1}, x_1\rangle & \langle f_{k-1}, x_2\rangle & \ldots & \langle f_{k-1}, x_k\rangle \end{vmatrix} : \quad \|f_1\|, \|f_2\|, \ldots, \|f_{k-1}\| = 1\right\}$$

$$= \sup\{M_1\langle f_{k-1}, x_1\rangle + M_2\langle f_{k-1}, x_2\rangle + \ldots + M_k\langle f_{k-1}, x_k\rangle\}$$

Here $M_1, M_2, \ldots, M_k$ are the minors obtained by expanding along the last row of the determinant.

Since the f_i's can be chosen independently of each other, we have that

$$A(x_1, x_2, \ldots, x_k) = \sup\{\|M_1 x_1 + M_2 x_2 + \ldots + M_k x_k\| : \|f_1\|, \|f_2\|, \ldots, \|f_{k-2}\| = 1\}\,.$$

If the last row is replaced by 1's the determinant is zero; so that we must have $M_1 + M_2 + \ldots + M_k = 0$. Also, $f_1, f_2, \ldots, f_{k-2}$ can be chosen so that M_1 is close to $A(x_2, x_3, \ldots, x_k)$. Hence, $A(x_1, x_2, \ldots, x_k) \geqslant \|x_1 + \alpha_2 x_2 + \ldots + \alpha_k x_k\| A(x_2, \ldots, x_k)$ where $\alpha_2 + \ldots + \alpha_k = -1$. Therefore,

$$A(x_1, x_2, \ldots, x_k) \geqslant \operatorname{dist}(x_1, [x_2, \ldots, x_k])\, A(x_2, \ldots, x_k)$$

as required. Q.E.D.

It is not hard to see that this inequality is an equality in case E is a Hilbert space. However, a simple reverse inequality of this form is, in a certain sense, impossible for general spaces. The inequality of HADAMARD [10] says that if $r_1, r_2, \ldots, r_n$ are the rows of an $n\times n$ determinant with Euclidean norms $\|r_1\|_2$, $\|r_2\|_2, \ldots, \|r_n\|_2$ then

$$\det(r_1, r_2, \ldots, r_n) \leqslant \|r_1\|_2 \cdot \|r_2\|_2 \cdot \ldots \|r_n\|_2 .$$

It is known that equality can actually be attained for certain values of n by taking pairwise orthogonal rows of ± 1's. It is easy to see that this is the volume of the convex hull of the unit vectors in $l^1_{(n)}$ and is equal to $(n)^{n/2}$. Thus, one can't have an inequality of the form:

$$A(x_1, x_2, \ldots, x_n) \leqslant M \operatorname{dist}(x_1, [x_2, \ldots, x_n]) A(x_2, \ldots, x_n)$$

with M independent of n because it would say that $(n)^{n/2} \leqslant 2^n M^n$.

However, in the case $n = 3$, we have the following easy Lemma which will be needed later.

LEMMA 2. – For all $x, y, z \in E$

$$A(x, y, z) \leqslant 2\|x-y\| \operatorname{dist}(z, [x, y]) .$$

PROOF. – Using the same idea as in the previous lemma we can write, for all a

$$\begin{aligned} A(x, y, z) &= \sup\{\|\langle f, z-y\rangle x + \langle f, x-z\rangle y + \langle f, y-x\rangle z\| : \|f\| = 1\} \\ &\leqslant \sup\{\|\langle f, z-y\rangle x + \langle f, x-z\rangle y + \langle f, y-x\rangle (ax + (1-a)y)\|\} \\ &\quad + \sup\{\|\langle f, y-x\rangle (z - (ax + (1-a)y))\|\} \\ &= \sup\{\|\langle f, z - ax - (1-a)y\rangle (x-y)\|\} \\ &\quad + \sup\{\|\langle f, y-x\rangle (z - (ax + (1-a)y))\|\} . \end{aligned}$$

Hence, $A(x, y, z) \leqslant 2\|x-y\| \operatorname{dist}(z, [x, y])$. Q.E.D.

As has been mentioned, if E is Hilbert space one actually has the *equality*

$$A(x_1, x_2, \ldots, x_k) = \operatorname{dist}(x_1, [x_2, \ldots, x_k]) A(x_2, \ldots, x_k) .$$

Using this fact, Javier Bernal has been able to compute the exact values of the moduli $\delta_E^{(k)}(\varepsilon)$ for Hilbert space. The following result illustrates Bernal's technique and is reported here with his kind permission.

THEOREM 3. – If E is a Hilbert space then for fixed $\varepsilon > 0$

$$\lim_{k\to\infty} \delta_E^{(k)}(\varepsilon) = 1 .$$

PROOF. – We shall need an equality which is a consequence of the fact that the norm in E comes from an inner product.

It is immediate that for all $x, y \in E$ and positive integers k

$$\|x/k + (k-1)y/k\|^2 = \|x\|^2/k + (k-1)\|y\|^2/k - (k-1)\|x-y\|^2/k^2 .$$

From this an induction argument shows that for any $x_1, x_2, \dots, x_k \in E$

$$\left\|\frac{x_1 + x_2 + \dots + x_k}{k}\right\|^2 = 1 - \frac{1}{k}\left(\frac{1}{2}\|x_{k-1} - x_k\|^2 + \frac{2}{3}\left\|x_{k-2} - \frac{1}{2}(x_{k-1} + x_k)\right\|^2 + \frac{3}{4}\left\|x_{k-3} - \frac{1}{3}(x_{k-2} + x_{k-1} + x_k)\right\|^2 + \dots + \left(\frac{k-1}{k}\right)\left\|x_1 - \frac{1}{(k-1)}(x_2 + x_3 + \dots + x_k)\right\|^2\right).$$

Assume, now, that $x_1, x_2, \dots, x_k$ are norm-1 vectors and for each i, $1 \leqslant i \leqslant k$

$$d_i \equiv \operatorname{dist}(x_i, [x_{i+1}, \dots, x_k]) .$$

From the above we have that

$$\left\|\frac{x_1 + x_2 + \dots + x_k}{k}\right\|^2 \leqslant f(d_{k-1}, d_{k-2}, \dots, d_1) \equiv 1 - \frac{1}{k}\left(\frac{1}{2}d_{k-1}^2 + \frac{2}{3}d_{k-2}^2 + \frac{3}{4}d_{k-3}^2 + \dots + \left(\frac{k-1}{k}\right)d_1^2\right) \dots (*) .$$

Using again the fact that E is a Hilbert space, we have $A(x_1, x_2, \dots, x_k) = d_{k-1}\cdot d_{k-2} \dots d_2 \cdot d_1$. What remains is to show that the maximum of the concave function $f(d_{k-1}, d_{k-2}, \dots, d_1)$, subject to the constraint $d_{k-1}\cdot d_{k-2} \dots d_1 = \varepsilon$, converges to zero as k increases. This is done using the method of Lagrange multipliers.

Let

$$G(d_{k-1}, d_{k-1}, \dots, d_1, \lambda) \equiv f(d_{k-1}, d_{k-2}, \dots, d_1) + \lambda(d_{k-1} \dots d_1 - \varepsilon) .$$

Setting the partial derivatives of G to zero gives

$$-\frac{2}{k}\left(\frac{1}{2}d_{k-1}\right) + \lambda d_{k-2}\cdot d_{k-3} \dots d_1 = 0 ,$$
$$-\frac{2}{k}\left(\frac{2}{3}d_{k-2}\right) + \lambda d_{k-1}\cdot d_{k-3} \dots d_1 = 0 ,$$
$$\vdots$$
$$-\frac{2}{k}\left(\frac{k-1}{k}d_1\right) + \lambda d_{k-1}\cdot d_{k-2} \dots d_2 = 0 .$$

Simplifying we have

$$\frac{2}{k}\left(\frac{1}{2}d_{k-1}^2\right)=\frac{2}{k}\left(\frac{2}{3}d_{k-2}^2\right)=\ldots=\frac{2}{k}\left(\frac{k-1}{k}d_1^2\right)=\lambda\varepsilon\,,$$

so that, at the maximum, $f(d_{k-1}, d_{k-2}, \ldots, d_1) = 1-\left(\frac{k-1}{2}\right)\lambda\varepsilon$.

Multiplying together the above equalities results in

$$\lambda\varepsilon = \frac{2}{k^{k/(k-1)}}\,\varepsilon^{2/(k-1)}$$

and hence

$$f(d_{k-1}, d_{k-2}, \ldots, d_1) = 1-\frac{k-1}{k^{k/(k-1)}}\,\varepsilon^{2/(k-1)}$$

which converges to zero as k increases. Q.E.D.

An interesting way to think about Theorem 1 is this: If (x_n) is a norm-1 sequence in Hilbert space then, by passing to a sub-sequence, we may assume that either $\lim_k A(x_1, x_2, \ldots, x_k) = 0$ or

$$\lim_k \left\|\frac{x_1+x_2+\ldots+x_k}{k}\right\| = 0\,.$$

Using this point of view one can obtain an unusual and simple proof of the Banach-Saks property [13]. It is also possible to extract more information from the proof of the Theorem. In particular, as has been mentioned, the exact values of the moduli $\delta_E^{(k)}(\varepsilon)$ can be computed [1].

3. – Multi-dimensional moduli.

Let (e_n) denote the usual unit vector basis for l^1. It is clear that for each k

$$\left\|\frac{e_1+e_2+\ldots+e_k}{k}\right\| = 1$$

while $A(e_1, e_2, \ldots, e_k) > 1$. In fact, for some values of k one can attain the maximum

$$A(e_1, e_2, \ldots, e_k) = k^{k/2}\,.$$

A consequence is that, for each k, $\delta_{l^1}^{(k)}(1) = 0$. In [22] it was proved that if E is any Banach space which is not super-reflexive then for all k, $\delta_E^{(k)}(1) = 0$. However, this does *not* mean that there exist norm-1 vectors $x_1, x_2, \ldots, x_k$ such that

$$\left\| \frac{x_1 + x_2 + \ldots + x_k}{k} \right\| = 1$$

while $A(x_1, x_2, \ldots, x_k) > 0$. The space E might be strictly convex, for example.

We shall show that if E is any non-reflexive space then for each positive integer k there are norm-1 vectors $x_1, x_2, \ldots, x_k$ in the $2k$'th dual of E such that

$$\left\| \frac{x_1 + x_2 + \ldots + x_k}{k} \right\| = 1$$

and $A(x_1, x_2, \ldots, x_k) > 0$.

Before stating the Theorem we will need some notation. For a (non-reflexive) Banach space E, let $E^{(n)}$ denote the n'th dual of E. Elements of $E^{(n)}$ will be written $x^{(n)}$. There is a sequence of natural imbeddings

$$E \underset{Q_0}{\rightarrow} E^{(2)} \underset{Q_2}{\rightarrow} E^{(4)} \underset{Q_4}{\rightarrow} E^{(6)} \underset{Q_6}{\rightarrow} E^{(8)} \ldots$$

where $\langle Q_{2m} x^{(2m)}, x^{(2m+1)} \rangle = \langle x^{(2m+1)}, x^{(2m)} \rangle$. The imbeddings for the odd numbered duals are defined in the same way. Hence, for each i, we have

$$Q_i^* Q_{i+1} = id|_{E^{(i+1)}}$$

and $Q_{i+1} Q_i^*$ is a norm-1 projection on $E^{(i+3)}$ with range $Q_{i+1}[E^{(i+1)}]$.

Suppose, now, that Y is any non-reflexive Banach space and $y^{(2)}$ is a norm-1 vector in $Y^{(2)} \setminus Y$. Then, according to the result of DIXMIER [8],

$$Q_2 y^{(2)}|_{Y^*} = Q_0^{**} y^{(2)}|_{Y^*} \quad \text{so that} \quad Q_2 y^{(2)} - Q_0^{**} y^{(2)} \in Y^{*\perp} \subset Y^{(4)}$$

while $\|Q_2 y^{(2)} - Q_0^{**} y^{(2)}\| \geqslant \operatorname{dist}(y^{(2)}, Y)$. The idea is that $Q_0^{**} y^{(2)} \in Y^{\perp\perp}$ but $\operatorname{dist}(y^{(2)}, Y)$ is the supremum of terms $\langle Q_2 y^{(2)}, y^{\perp} \rangle$.

It is immediate that $Y^{(4)}$ cannot be strictly convex and, in fact, Y^* cannot be very smooth [21]. The following can be viewed as a generalization of Dixmier's Theorem:

THEOREM 1. – If E is not reflexive then there is a sequence of norm-1 vector $x^{(2)} \in E^{(2)}, x^{(4)} \in E^{(4)}, \ldots, x^{(2m)} \in E^{(2m)}, \ldots$ such that for all m

$$\left\| \frac{x^{(2)} + x^{(4)} + \ldots + x^{(2m)}}{m} \right\| = 1$$

and $A(x^{(2)}, x^{(4)}, \ldots, x^{(2m)}) > 0$.

PROOF. – Choose $x^{(2)} \in E^{(2)} \setminus E$ and let

$$x^{(4)} = Q_0^{**} x^{(2)},\ x^{(6)} = Q_2^{**} x^{(4)},\ \ldots,\ x^{(2m+2)} = Q_{2m-2}^{**} x^{(2m)},\ \ldots .$$

The fact that the averages have norm-1 is clear from the previous discussion. To show that the areas are non-zero note first that $x^{(2)} \in E^{(2)} \setminus E$ and $x^{(4)} \in E^{\perp\perp}$ while

$$x^{(4)}|_{E^*} = Q_2 x^{(2)}|_{E^*} .$$

This implies that, in fact, $x^{(4)} \in E^{(4)} \setminus E^{(2)}$ because $E^{\perp\perp} \cap E^{(2)} = E$. At the next level the same reasoning gives $x^{(6)} \in E^{(6)} \setminus E^{(4)}$ and, continuing, we get $x^{(2m+2)} \in E^{(2m+2)} \setminus E^{(2m)}$.

Hence, for each m

$$\begin{aligned} A(x^{(2)}, x^{(4)}, \ldots, x^{(2m)}, x^{(2m+2)}) \\ &\geqslant A(x^{(2)}, x^{(4)}, \ldots, x^{(2m)}) \operatorname{dist}\left(x^{(2m+2)}, [x^{(2)}, x^{(4)}, \ldots, x^{(2m)}]\right) \\ &\geqslant A(x^{(2)}, x^{(4)}, \ldots, x^{(2m)}) \operatorname{dist}\left(x^{2m+2}, E^{(2m)}\right) . \end{aligned}$$

A simple induction now completes the proof. Q.E.D.

A similar idea has been used by PERROTT [14] to study the relationship between super-reflexivity and ergodic properties.

Recall that a Banach space, E, is locally uniformly rotund if for all $\|x\| = 1$ and all norm-1 sequences (x_n), $\|x + x_n\| \to 2$ implies that $\|x - x_n\| \to 0$. Generalising this we say that E is locally k-UR if for each $\|x\| = 1$ and $\varepsilon > 0$ there is a $\delta = \delta(x; \varepsilon) > 0$ such that for all norm-1 $(x_1, x_2, \ldots, x_k)$, if

$$\left\| \frac{x + x_1 + x_2 + \ldots + x_k}{k+1} \right\| > 1 - \delta$$

then $A(x, x_1, x_2, \ldots, x_k) < \varepsilon$. It is an immediate consequence of Goldstine's Theorem that if E^{**} is locally UR then E is reflexive. In [22] we showed that a locally 2-UR second dual is reflexive. Using Theorem 1 and Goldstine's Theorem we get the following:

COROLLARY 2. – If, for any positive integer k, E^{**} is locally k-UR then E is reflexive.

We shall need the combination of Goldstine's Theorem and Helly's Theorem which LINDENSTRAUSS and ROSENTHAL called « local reflexivity » [23]. The form we shall use is due to DEAN [7]: If $A \subset E^{**}$ and $F \subset E^*$ are finite dimensional subspaces and $0 < \delta < 1$ is arbitrary, then there is a linear map $T: A \to E$ such that

(1) $T(a) = a$ for all $a \in A \cap E$;

(2) $\langle f, T(a) \rangle = \langle a, f \rangle$ for all $a \in A$ and $f \in F$;

(3) $(1 - \delta)\|a\| \leqslant \|T(a)\| \leqslant (1 + \delta)\|a\|$ for all $a \in A$.

Local reflexivity has been used by DAVIS, JOHNSON and LINDENSTRAUSS [3], [4] to obtain information on the relation between geometrical properties of the quotient $R(E) \equiv E^{**}/E$ and those of E. Following their lead we have:

THEOREM 3. – For all positive integers k and for all $\varepsilon > 0$, $\delta_E^{(2k+1)}(\varepsilon^2) \leqslant \delta_{R(E)}^{(k)}(\varepsilon)$.

PROOF. – Suppose that k and ε are given and that $a > \delta_{R(E)}^{(k)}(\varepsilon)$. We shall show that $a > \delta_E^{(2k+1)}(\varepsilon^2)$. From the definition of the modulus, there are norm-1 cosets $x_1^{**} + E, \ldots, x_{k+1}^{**} + E \in E^{**}/E$ such that

$$\left\| \frac{x_1^{**} + x_2^{**} + \ldots + x_{k+1}^{**}}{k+1} + E \right\| > 1 - a$$

while $A(x_1^{**} + E, \ldots, x_k^{**} + E) > \varepsilon$.

Without loss of generality, we may assume that the vectors (x_i^{**}) have norm arbitrarily close to 1. Recalling that $(E^{**}/E)^*$ is linearly isometric to $E^\perp$, there are norm-1 vectors $x_1^\perp, \ldots, x_k^\perp \in E^\perp$ such that

$$|D| = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ \langle x_1^\perp, x_1^{**}\rangle & \langle x_1^\perp, x_2^{**}\rangle & & \langle x_k^\perp, x_{k+1}^{**}\rangle \\ \vdots & & & \\ \langle x_k^\perp, x_1^{**}\rangle & \langle x_k^\perp, x_2^{**}\rangle & \ldots & \langle x_k^\perp, x_{k+1}^{**}\rangle \end{vmatrix} > \varepsilon .$$

Applying Diximier's Theorem, we have that

$$\left\| \frac{x_1^{**} + x_2^{**} + \ldots + x_{k+1}^{**} + Q_0^{**} x_1^{**} + \ldots + Q_0^{**} x_{k+1}^{**}}{2k+2} \right\| > 1 - a .$$

We need only show that $A(x_1^{**}, x_2^{**}, \ldots, Q_0^{**} x_{k+1}^{**}) > \varepsilon$ and an application of local reflexivity will then give the result. To obtain the last inequality, notice that there is a norm-1 $y^\perp \in E^\perp$ so that all $\langle y^\perp, x_i^{**}\rangle$ are close to 1 and norm-1 vectors $\{x_1^*, x_2^*, \ldots, x_k^*\} \in E^*$ so that for each i and j $\langle x_i^{**}, x_j^*\rangle$ is close to $\langle x_j^\perp, x_i^{**}\rangle$. Finally, recall that $Q_0^{**}[E^{**}] = E^{\perp\perp}$. Now, estimate $A(x_1^{**}, x_2^{**}, \ldots, Q_0^{**} x_{k+1}^{**})$ by evaluating the following determinant:

$$\begin{vmatrix} 1 & \ldots & 1 & 1 & \ldots & 1 \\ \langle x_1^\perp, x_1^{**}\rangle & & \langle x^\perp, x_{k+1}^{**}\rangle & \langle x^\perp, Q_0^{**} x_k^{**}\rangle & & \langle x_1^\perp, Q_0^{**} x_{k+1}^{**}\rangle \\ \vdots & & & & & \\ \langle x_k^\perp, x_1^{**}\rangle & \ldots & \langle x_k^\perp, x_{k+1}^{**}\rangle & \langle x_k^\perp, Q_0^{**} x_1^{**}\rangle & \ldots & \langle x_k^\perp, Q_0^{**} x_{k+1}^{**}\rangle \\ \langle y^\perp, x_1^{**}\rangle & \ldots & \langle y^\perp, x_{k+1}^{**}\rangle & \langle y^\perp, Q_0^{**} x_1^{**}\rangle & \ldots & \langle y^\perp, Q_3^{**} x_{k+1}^{**}\rangle \\ \langle x_1^{**}, x_1^*\rangle & \ldots & \langle x_{k+1}^{**}, x_1^*\rangle & \langle Q_0^{**} x_1^{**}, x_1^*\rangle & \ldots & \langle Q_0^{**} x_{k+1}^{**}, x_1^{**}\rangle \\ \vdots & & & & & \\ \langle x_k^{**}, x_k^*\rangle & \ldots & \langle x_{k+1}^{**}, x_k^*\rangle & \langle Q_0^{**} x_1^{**}, x_k^*\rangle & \ldots & \langle Q_0^{**} x_{k+1}^{**}, x_k^*\rangle \end{vmatrix}$$

It is not hard to check that after evaluating and interchanging rows (and possibly changing the sign) this has the form:

$$\begin{vmatrix} D & 0 \\ D & D \end{vmatrix}$$

Hence, $A(x_1^{**}, x_2^{**}, \ldots, Q_0^{**}x_{k+1}^{**}) \geqslant |D|^2 > \varepsilon^2$. Q.E.D.

An interesting special case of this Theorem is when $\delta^{(k)}_{R(E)}(\varepsilon) = 0$ for some $\varepsilon > 0$. For example, if E^{**}/E is not 1-UR then there are norm-1 vectors x_1, x_2, x_3, x_4 in E with $A(x_1, x_2, x_3, x_4) > 0$ while

$$\left\| \frac{x_1 + x_2 + x_3 + x_4}{4} \right\|$$

is arbitrarily close to 1. Stated very informally this says that if the sphere of E^{**}/E almost contains a line segment, then the sphere of E almost contains a tetrahedron. Another easy consequence along the same line is that if every tetrahedron on the unit sphere of E has volume less than 4 (i.e. $\delta^{(3)}_E(4) > 0$) then $\delta^{(1)}_{R(E)}(2) > 0$ and so E^{**}/E is super-reflexive.

Let P_1 denote the cononical projection of E^{***} onto $Q_1[E^*]$ and P_2 the projection of $E^{(4)}$ onto $Q_2[E^{**}]$. In [2] A. L. BROWN proved that $\|I - P_1\| = 1$ iff for all $x^{**} \in E^{**}$, $\|Q_2 x_0^{**} - Q^{**}x^{**}\| = \text{dist}\,(x^{**}, E)$. If $\|I - P_2\| = 1$ then $E^{*\perp}$ is isometric to $E^{(4)}/E^{**}$ and also $\|I - P_1\| = 1$ [20]. Combining these ideas with the Theorem of KADEC [11] we get the following:

COROLLARY 4. – Let E be a Banach space such that $\|I - P_2\| = 1$. If (x_n^{**}) is a sequence in E^{**} such that $\sum x_n^{**}$ converges unconditionally then

$$\sum \delta^{(3)}_{E^{**}}(\text{dist}\,(x_n^{**}, E)^2) < \infty\,.$$

PROOF. – Since $\sum x_n^{**}$ converges unconditionally $\sum(Q_2 x_n^{**} - Q_0^{**} x_n^{**})$ does also Using the fact that $\|I - P_2\| = 1$ we have, for each n, $\text{dist}\,(x_n^{**}, E) = \|(Q_2 x_n^{**} - Q_0^{**} x_n^{**})\| = \|(Q_2 x_n^{**} - Q_0^{**} x_n^{**}) + E^{**}\|$. Applying Kadec's theorem in $E^{(4)}/E^{**}$ gives

$$\sum \delta^{(1)}_{R(E^{**})}(\text{dist}\,(x_n^{**}, E)) < \infty$$

and the result follows from Theorem 3. Q.E.D.

Notice that if for some $a < 16$, $\delta^{(3)}_{E^{**}}(a) > 0$ then $\text{dist}\,(x_n^{**}, E) \to 0$ for any unconditionally summable sequence (x_n).

Duality for the multi-dimensional moduli appears to be a complicated and technical question. Intuitively, the dual of a triangle on the unit sphere of E should be a three dimensional corner on the sphere of E^*; just as the dual of a line segment is a two dimensional corner. This is the idea behind M. M. Day's definition of a *uniformly flattened* Banach space. The space E is *uniformly flattened* (UF) [6] iff for all pairs of sequences $(x_n)(y_n)$ if $\|x_n - y_n\| \to 0$ then

$$\frac{\|x_n\| + \|y_n\| - \|x_n + y_n\|}{\|x_n - y_n\|} \to 0 .$$

Day proved that E is UR iff E^* is UF. Generalizing Day's definition we say that E is 2-UF if for all sequences (x_n), (y_n), (z_n) if $\|x_n - y_n\|$ and $\|x_n - z_n\|$ converge to zero then

$$Q(x_n, y_n, z_n) \equiv \frac{(\|x_n\| + \|y_n\| + \|z_n\| - \|x_n + y_n + z_n\|)^2}{A(x_n, y_n, z_n)} \to 0 .$$

THEOREM 5. – If E^* is 2-UF then E is 2-UR.

PROOF. – If E is not 2-UR then for some $\varepsilon > 0$ there are norm-1 sequences (x_n), (y_n), (z_n) such that

$$\|x_n + y_n + z_n\| \geqslant 3 - \frac{\varepsilon}{4n}$$

while $A(x_n, y_n, z_n) \geqslant \varepsilon$. Using Lemma 1.2 we get that for all n $\|y_n - x_n\| \geqslant \varepsilon/4$, $\|z_n - x_n\| \geqslant \varepsilon/4$ and $\|y_n - z_n\| \geqslant \varepsilon/4$. Hence, there exist norm-1 sequences (f_n), (g_n), (h_n) such that for all n,

$$\langle f_n, x_n + y_n + z_n \rangle \geqslant 3 - \frac{\varepsilon}{4n}$$

and $\langle g_n, y_n - x_n \rangle$, $\langle h_n, z_n - x_n \rangle \geqslant \varepsilon/4$.

Consider now the sequences $(f_n + (1/n)g_n)$, $(f_n + (1/n)h_n)$ and $(f_n - (1/n)g_n - (1/n)h_n)$. All three differences converge to zero but

$$\left\|f_n + \frac{1}{n}g_n\right\| + \left\|f_n + \frac{1}{n}h_n\right\| + \left\|f_n - \frac{1}{n}g_n - \frac{1}{n}h_n\right\| - 3\|f_n\|$$
$$\geqslant \langle f_n, x_n + y_n + z_n \rangle - 3 + \frac{1}{n}\langle g_n, y_n - x_n \rangle + \frac{1}{n}\langle h_n, z_n - x_n \rangle \geqslant \frac{\varepsilon}{4n} .$$

On the other hand, for all n

$$A\left(f_n + \frac{1}{n}g_n, f_n + \frac{1}{u}h_n, f_n - \frac{1}{u}g_n - \frac{1}{n}h_n\right)$$
$$\leqslant 2 \cdot \frac{1}{n}\|g_n - h_n\| \operatorname{dist}\left(f_n - \frac{1}{n}g_n - \frac{1}{n}h_n, \left[f_n + \frac{1}{n}g_n, f_n + \frac{1}{n}h_n\right]\right)$$
$$\leqslant 2 \cdot \frac{1}{n}\|g_n - h_n\| \cdot \frac{1}{2n}\|g_n - h_n\| \leqslant \frac{4}{n^2} .$$

Thus,

$$Q\left(f_n + \frac{1}{n} g_n, f_n + \frac{1}{n} h_n, f_n - \frac{1}{n} g_n - \frac{1}{n} h_n\right) \geqslant \varepsilon^2/64$$

and E^* is not 2-UF. Q.E.D.

A duality theory for multi-dimensional moduli of rotundity has been developed by MILMAN [12], [13]. These moduli are defined using subspaces Y as follows:

$$\Delta^{(k)}(\varepsilon) = \inf_{\|x\|=1} \inf_{\substack{Y\subset E\\ \dim Y = k}} \sup_{\substack{\|y\|=1\\ y\in Y}} \{\|x + \varepsilon y\| - 1\}.$$

The space, E, is said to be k-uniformly convex if for all $\varepsilon > 0$, $\Delta^{(k)}(\varepsilon) > 0$. A Banach space is 1-uniformly convex iff it is 1-UR. A proof of this can be found in the paper of FIGIEL [9]. We extend Figiel's technique to the case $k = 2$. Again, the general k-dimensional result appears to be complicated and technical.

We shall need several preliminary results. In order to simplify the proofs we assume that $\dim(E) < \infty$. This is possible because for each ε

$$\delta_E^{(2)}(\varepsilon) = \inf\{\delta_{E^1}^{(2)}(\varepsilon): E^1 \subset E,\ \dim E^1 < \infty\}$$

and

$$\Delta_E^{(2)}(\varepsilon) = \inf\{\Delta_{E^1}^{(2)}(\varepsilon): E^1 \subset E,\ \dim E^1 < \infty\}.$$

LEMMA 6. – If $\dim(E) \geqslant 2$ then for each $\varepsilon > 0$ there are vectors v_1, v_2 and v_3 such that $\|v_1\| = \|v_2\| = \|v_3\| = \varepsilon$ and $v_1 + v_2 + v_3 = 0$.

PROOF. – Choose $\|v_1\| = \varepsilon$ arbitrarily and consider the continuous function on the ε-sphere given by

$$F(x) = \frac{\|v_1 + x\|}{3}.$$

Obviously $F(v_1) = 2\varepsilon/3$ and $F(-v_1) = 0$. The ε-sphere of E is connected so that for some $\|v_2\| = \varepsilon$

$$\varepsilon/3 = F(v_2) = \frac{\|v_1 + v_2\|}{3}.$$

To complete the proof simply let $v_3 = -v_1 - v_2$. Q.E.D.

LEMMA 7. – Let v_1, v_2, v_3 be as in the previous Lemma. Then $\operatorname{dist}(v_1, [v_2]) \geqslant \varepsilon/3$.

PROOF. – Notice first that

$$\|v_1 + v_2\| = \|-v_3\| = \varepsilon \quad \text{and} \quad \|v_1 - v_2\| = \|v_1 - (-v_1 - v_3)\| \geqslant 2\|v_1\| - \|v_3\| = \varepsilon.$$

We need only show that there is an $f \in E^* \|f\| \leqslant 1$ such that $\langle f, v_1 \rangle \geqslant \varepsilon/3$ and $\langle f, v_2 \rangle = 0$ for then we have $\|v_1 - av_2\| \geqslant \langle f, v_1 \rangle \geqslant \varepsilon/3$ for all real a. If for all $\|f\| \leqslant 1$ $\langle f, v_2 \rangle = 0$ implies $\langle f, v_1 \rangle < \varepsilon/3$ then from the Lemma of PHELPS [15] either $\|v_1 + v_2\| \leqslant 2\varepsilon/3$ or $\|v_1 - v_2\| \leqslant 2\varepsilon/3$. Q.E.D.

LEMMA 8. – For all $\varepsilon > 0$

$$\delta^{(2)}(s) \leqslant \frac{\Delta^{(2)}(\varepsilon)}{1 + \Delta^{(2)}(\varepsilon)}$$

where

$$s = \frac{\varepsilon^2}{(1 + \Delta^{(2)}(\varepsilon))^2}.$$

PROOF. – Recall first that

$$\Delta^{(2)}(\varepsilon) \equiv \inf_{\|x\|=1} \inf_{\dim Y=2} \sup_{\|y\|=1} \{\|x + \varepsilon y\| - 1\}.$$

By the remarks made earlier we may assume that there is a norm-1 vector u, and a two dimensional subspace $Y \subseteq E$ such that

$$\Delta^{(2)}(\varepsilon) = \sup_{\|y\|=1} \|u + \varepsilon y\| - 1.$$

Choose $v_1 \in Y$, $\|v_1\| = \varepsilon$ so that $1/a \equiv 1 + \Delta^{(2)}(\varepsilon) = \|u + v_1\|$, and select $\|v_2\| = \|v_3\| = \varepsilon$ with $v_1 + v_2 + v_3 = 0$. Let $x_1 = a(u + v_1)$, $x_2 = a(u + v_2)$ and $x_3 = a(u + v_3)$ and note that $\|x_1\|, \|x_2\|, \|x_3\| \leqslant 1$.

We have dist $(v_1, [v_2]) \geqslant \varepsilon/3$ by the previous lemma; so there is an $f_0 \in E^*$ such that $\|f_0\| = 1$, $f_0(v_2) = 0$ and $f_0(v_1) \geqslant \varepsilon/3$.

Now consider

$$\begin{aligned} A(x_1, x_2, x_3) = a^2 A(u + v_1, u + v_2, u + v_3) &= a^2 A(v_1, v_2, v_3) \\ &= a^2 \sup_f \|f(v_3 - v_2) v_1 + f(v_1 - v_3) v_2 + f(v_2 - v_1) v_3\| \\ &= a^2 \sup_f \|f(-3v_2) v_1 + f(3v_1) v_2\| \geqslant 3a^2 \|f_0(v_1) v_2 - f_0(v_2) v_1\| \\ &= 3a^2 |f_0(v_1)| \cdot \|v_2\| \geqslant \frac{\varepsilon^2}{(1 + \Delta^{(2)}(\varepsilon))^2}. \end{aligned}$$

Hence,

$$\begin{aligned} \delta^{(2)}(s) \leqslant 1 - \frac{1}{3} \|x_1 + x_2 + x_3\| &= 1 - \frac{1}{3} \|3au\| \\ &= 1 - \frac{1}{1 + \Delta^{(2)}(\varepsilon)} = \frac{\Delta^{(2)}(\varepsilon)}{1 + \Delta^{(2)}(\varepsilon)}. \qquad \text{Q.E.D.} \end{aligned}$$

LEMMA 9. – For all $\varepsilon > 0$

$$\Delta^{(2)}(s) \leqslant \frac{\delta^{(2)}(\varepsilon)}{1-\delta^{(2)}(\varepsilon)}$$

where

$$s = \frac{\varepsilon}{12(1-\delta^{(2)}(\varepsilon))}\,.$$

PROOF. – Given $\varepsilon > 0$, choose norm-1 vectors $x, y, z \in E$ such that

$$\|x+y+z\| = 3(1-\delta^{(2)}(\varepsilon))$$

and f, g norm-1 in E^* such that

$$\varepsilon = A(x, y, z) = \begin{vmatrix} 1 & 1 & 1 \\ \langle f, x\rangle & \langle f, y\rangle & \langle f, z\rangle \\ \langle g, x\rangle & \langle g, y\rangle & \langle g, z\rangle \end{vmatrix}$$

Let $u = \dfrac{x+y+z}{\|x+y+z\|}$ and evaluate the determinant to obtain vectors

$$v_1 = \frac{\langle f, z-y\rangle x + \langle f, x-z\rangle y + \langle f, y-x\rangle z}{4\|x+y+z\|}$$

and

$$v_2 = \frac{\langle g, y-z\rangle x + \langle g, z-x\rangle y + \langle g, x-y\rangle z}{4\|x+y+z\|}\,.$$

Notice that $s = \|v_1\| = \langle g, v_1\rangle = \|v_2\| = \langle f, v_2\rangle$ and $\langle f, v_1\rangle = 0 = \langle g, v_2\rangle$ so that v_1 and v_2 are linearly independent. We shall show that if $\|av_1 + bv_2\| = s$ then

$$\|u + av_1 + bv_2\| \leqslant \frac{1}{1-\delta^{(2)}(\varepsilon)}\,.$$

Hence,

$$\Delta^{(2)}(s) \leqslant \frac{1}{1-\delta^{(2)}(\varepsilon)} - 1 = \frac{\delta^{(2)}(\varepsilon)}{1-\delta^{(2)}(\varepsilon)}$$

as required.

If $s = \|av_1 + bv_2\|$ then $s \geqslant |a\langle g, v_1\rangle|$ and $s \geqslant |b\langle f, v_2\rangle|$ and $|a|, |b| \leqslant 1$. Define now

$$c_1 = 1 + \tfrac{1}{4}\langle af - bg, z-y\rangle$$

$$c_2 = 1 + \tfrac{1}{4}\langle af - bg, x-z\rangle$$

$$c_3 = 1 + \tfrac{1}{4}\langle af - bg, y-x\rangle$$

and notice that $c_1, c_2, c_3 \geqslant 0$ while $c_1 + c_2 + c_3 = 3$. Hence

$$\|u + av_1 + bv_2\| = \frac{\|c_1 x + c_2 y + c_3 z\|}{\|x + y + z\|} \leqslant \frac{3}{\|x + y + z\|} = \frac{1}{1 - \delta^{(2)}(\varepsilon)}. \qquad \text{Q.E.D.}$$

THEOREM 10. – A Banach space, E, is 2-UR iff it is 2-uniformly convex.

PROOF. – Combine Lemmas 8 and 9. Q.E.D.

4. – Products of uniformly rotund spaces.

In this section we give a necessary and sufficient condition for the l^p product of spaces to be 2-UR. Recall that for a sequence of Banach spaces (E_n), and $1 \leqslant p < \infty$ the l^p product, $(\Sigma \oplus E_n)_p$, is the space of all sequences (x_n), where for each n, $x_n \in E_n$ and $\Sigma \|x_n\|^p < \infty$. The norm is given by

$$\|(x_1, x_2, \ldots, x_n, \ldots)\| = (\Sigma \|x_n\|^p)^{1/p}.$$

For, $\varepsilon > 0$, let $\delta_n^{(1)}(\varepsilon)$ denote the 1-modulus of the space E_n. The sequence of spaces (E_n) is said to have a common modulus of convexity if for each $\varepsilon > 0$,

$$\inf_n \delta_n^{(1)}(\varepsilon) > 0.$$

The following is due to M. M. DAY [6].

THEOREM 1. – If (E_n) is a sequence of Banach spaces, then $(\Sigma \oplus E_n)_p$, $1 < p < \infty$, is uniformly rotund if and only if the sequence (E_n) has a common modulus of convexity.

Our result for 2-UR spaces is based on Day's; viz:

THEOREM 2. – If (E_n) is a sequence of Banach spaces, then $(\Sigma \oplus E_n)_p$, $1 < p < \infty$ is 2-UR if and only if all but one of the E_n are 1-UR with a common modulus of convexity and the remaining space is (2-UR).

The proof requires several preliminary lemmas.

LEMMA 3. – If E and F are Banach spaces such that $(E \oplus F)_p$, $1 < p < \infty$ is 2-UR then at least one of E or F is 1-UR.

PROOF. – If neither E nor F is 1-UR then there are norm-1 sequences $(x_n^{(1)})$, $(x_n^{(2)}) \subseteq E$, $(y_n^{(1)})$, $(y_n^{(2)}) \subseteq F$ such that $\|x_n^{(1)} + x_n^{(2)}\| \to 2$ while for all n,

$$\|x_n^{(1)} - x_n^{(2)}\| > \varepsilon_1 > 0$$

and $\|y_n^{(1)} + y_n^{(2)}\| \to 2$ while for all n, $\|y_n^{(1)} - y_n^{(1)}\| > \varepsilon_2 > 0$.

Let $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$. By passing to a subsequence we may assume that

$$\left\| \frac{1}{3} x_n^{(1)} + \frac{2}{3} x_n^{(2)} \right\| \geq 1 - \frac{1}{n}$$

and

$$\left\| \frac{1}{2} y_n^{(1)} + \frac{1}{2} y_n^{(2)} \right\| \geq 1 - \frac{1}{n} .$$

Let

$$x_n^{(3)} = \frac{\frac{1}{3} x_n^{(1)} + \frac{2}{3} x_n^{(2)}}{\|\frac{1}{3} x_n^{(1)} + \frac{2}{3} x_n^{(2)}\|},$$

$$y_n^{(3)} = \frac{\frac{1}{2} y_n^{(1)} + \frac{1}{2} y_n^{(2)}}{\|\frac{1}{2} y_n^{(1)} + \frac{1}{2} y_n^{(2)}\|}$$

and define sequences in $(E \oplus F)_p$ by $u_n = (x_n^{(1)}, y_n^{(1)})$, $v_n = (x_n^{(2)}, y_n^{(2)})$ and $w_n = (x_n^{(3)}, y_n^{(3)})$.

Clearly $\|u_n\| = \|v_n\| = \|w_n\| = 2^{1/p}$. Using the fact that $\|x_n^{(1)} + x_n^{(1)} + x_n^{(3)}\| \to 3$ and $\|y_n^{(1)} + y_n^{(2)} + y_n^{(3)}\| \to 3$ we have that $\|u_n + v_n + w_n\| \to 3 \cdot 2^{1/p}$. To complete the proof we need only show that $A(u_n, v_n, w_n)$ remains bounded away from zero.

To show this we use the fact that for each n,

$$A(u_n, v_n, w_n) \geq \|u_n - v_n\| \operatorname{dist}(w_n, [u_n, v_n]) .$$

Clearly, $\|u_n - v_n\| \geq \varepsilon \cdot 2^{1/p}$; and from the triangle inequality

$$\begin{aligned} &\operatorname{dist}(w_n, [u_n, v_n]) \\ &\quad \geq \inf \{\|x_n^{(3)} - (a x_n^{(1)} + (1-a) x_n^{(2)})\|^p + \|y_n^{(3)} - (a y_n^{(1)} + (1-a) y_n^{(2)})\|^p\}^{1/p} \\ &\quad \geq \inf_a \left\{ \left(\left| \frac{1}{3} - a \right| \varepsilon - \frac{1}{n} \right)^p + \left(\left| \frac{1}{2} - a \right| \varepsilon - \frac{1}{n} \right)^p \right\}^{1/p} . \end{aligned}$$

which is bounded away from zero. Q.E.D.

LEMMA 4. – If E is a Banach space and $x, y, z \in E$, then at least one of the altitudes of the triangle formed by the three points lies inside the triangle.

PROOF. – By translating and re-labeling the points we may assume that $x = 0$ and $\|y\| \geq \|y - z\| \geq \|z\|$. Choose $f \in E^*$ with $\|f\| = 1$ and $f(z) = \|z\|$.

If $f(y) \geq 0$, then for any $t < 0$ we have

$$\begin{aligned} \|z - ty\| &\geq f(z - ty) \\ &= \|z\| - t f(y) \\ &\geq \|z\| \end{aligned}$$

and for $t > 1$ we have $\|z - ty\| \geqslant \|z\|$ since $\|y - z\| \geqslant \|z\|$. Hence the best approximation to z on $[0, y]$ must be in the form ty where $0 \leqslant t \leqslant 1$.

If $f(y) < 0$, then for $t > 1$ we have

$$\begin{aligned}\|tz + (1-t)y\| &\geqslant f(tz + (1-t)y)\\ &= t\|z\| + (1-t)f(y)\\ &> \|z\|\end{aligned}$$

and for $t < 0$ we must have $\|tz + (1-t)y\| \geqslant \|z\|$ since $\|y\| \geqslant \|z\|$. Hence the best approximation to 0 an $[y, z]$ is a convex combination of y and z. Q.E.D.

LEMMA 5. – If E and F are Banach spaces such that E is 2-UR and F is 1-UR, then $(E \oplus F)_p$, $1 < p < \infty$, is 2-UR.

PROOF. – Set $\varepsilon > 0$ and let $x = (x_1, x_2)$, $y = (y_1, y_2)$, and $z = (z_1, z_2)$ be norme-1 elements of $(E \oplus F)_p$ with $A(x, y, z) \geqslant \varepsilon$. We first consider the proof for the case when $\|x_i\| = \|y_i\| = \|z_i\|$ for $i = 1, 2$.

By Lemma II.2 we have

$$\|x - y\| \operatorname{dist}(z, [x, y]) \geqslant \tfrac{1}{2}A(x, y, z) \geqslant \varepsilon/2 \equiv \varepsilon^1 ,$$

and since $\|x - y\| \leqslant 2$ and $\operatorname{dist}(z, [x, y]) \leqslant \|x - z\| \leqslant 2$ we have $\|x - y\| \geqslant \varepsilon^1/2$ and $\operatorname{dist}(z, [x, y]) \geqslant \varepsilon^1/2$. Similar reasoning shows $\|x - z\|$, $\|y - z\|$, $\operatorname{dist}(y, [x, z])$, and $\operatorname{dist}(x, [y, z]) \geqslant \varepsilon^1/2$.

We consider the following two cases:

Case (i):

$$\max\{\|x_2 - y_2\|, \|x_2 - z_2\|, \|y_2 - z_2\|\} \geqslant \varepsilon^1/4 .$$

We may assume that $\|x_2 - y_2\| \geqslant \varepsilon^1/4$. Set $\beta = \|x_2\| = \|y_2\|$ and note that $\beta \leqslant 1$ since $\|x\| = 1$ and $\beta \geqslant \varepsilon^1/8$, since $\|x_2 - y_2\| \geqslant \varepsilon^1/4$. Since F is 1-UR and

$$\left\| \frac{x_2}{\beta} - \frac{y_2}{\beta} \right\| \geqslant \varepsilon^1/4$$

there is a $\delta^{(1)}(\varepsilon^1/4) > 0$ such that $\|x_2 + y_2\| \leqslant 2\beta(1 - \delta^{(1)})$. Hence we have

$$\begin{aligned}\|x + y\| &= (\|x_1 + y_1\|^p + \|x_2 + y_2\|^p)^{1/p}\\ &\leqslant (2^p\|x_1\|^p + 2^p\beta^p(1 - \delta^{(1)})^p)^{1/p}\\ &= (2^p(1 - \beta^p) + 2^p\beta^p(1 - \delta^{(1)})^p)^{1/p}\\ &\leqslant \left(2^p - 2^p\frac{\varepsilon^{1p}}{8^p}\left(1 - (1 - \delta^{(1)})^p\right)\right)^{1/p}\\ &\equiv 2 - \delta^*(\varepsilon) .\end{aligned}$$

Thus $\|x + y + z\| \leqslant 2 - \delta^*(\varepsilon)$.

Case (ii):

$$\|x_2 - y_2\|,\ \|x_2 - z_2\|,\ \|y_2 - z_2\| < \varepsilon^1/4\,.$$

By Lemma 4 we may assume that

$$\operatorname{dist}(z_1, [x_1, y_1]) = \|z_1 - (a_1 x_1 + (1-a) y_1)\|$$

for some a_1 with $0 < a_1 \leqslant 1$. From above we have

$$\begin{aligned}\varepsilon^1/2 &\leqslant \operatorname{dist}(z, [x, y])\\ &\leqslant \Big(\|z_1 - (a_1x_1 + (1-a_1)y_1)\|^p + \|z_2 - (a_1x_2 + (1-a_1)y_2)\|^p\Big)^{1/p}\\ &\leqslant \operatorname{dist}(z_1, [x_1, y_1]) + a_1\|z_2 - x_2\| + (1-a_1)\|z_2 - y_2\|\\ &\leqslant \operatorname{dist}(z_1, [x_1, y_1]) + \varepsilon^1/4\end{aligned}$$

so that $\operatorname{dist}(z_1, [x_1, y_1]) \geqslant \varepsilon^1/4$. Since we also have

$$\varepsilon^1/2 \leqslant \|x - y\| \leqslant \|x_1 - y_1\| + \|x_2 - y_2\| \leqslant \|x_1 - y_1\| + \varepsilon^1/4\,,$$

then by setting $\beta = \|x_1\| = \|y_1\| = \|z_1\| \leqslant 1$ and applying Lemma II.1 we get

$$\begin{aligned}A\left(\frac{x_1}{\beta}, \frac{y_1}{\beta}, \frac{z_1}{\beta}\right) &= \frac{1}{\beta^2} A(x_1, y_1, z_1)\\ &\geqslant \frac{1}{\beta^2}\|x_1 - y_1\| \operatorname{dist}(z_1, [x_1, y_1])\\ &\geqslant (\varepsilon^1)^2/16\,.\end{aligned}$$

Since E is 2-UR there is a $\delta^{(2)}((\varepsilon^{1'})^3/16) > 0$ such that $\|x_1 + y_1 + z_1\| \leqslant 3\beta(1 - \delta^{(2)})$. Using the fact that $\beta \geqslant \varepsilon^1/8$ we have

$$\begin{aligned}\|x + y + z\| &= (\|x_1 + y_1 + z_1\|^p + \|x_2 + y_2 + z_2\|^p)^{1/p}\\ &\leqslant (3^p\beta^p(1 - \delta^{(2)})^p + 3^p\|x_2\|^p)^{1/p}\\ &= (3^p\beta^p(1 - \delta^{(2)})^p + 3^p(1 - \beta^p))^{1/p}\\ &\leqslant \left(3^p - 3^p\frac{\varepsilon^{1p}}{8^p}(1 - (1 - \delta^{(2)})^p)\right)^{1/p}\\ &\equiv 3 - \delta^1(\varepsilon)\,.\end{aligned}$$

Set $\delta_0(\varepsilon) = \min\{\frac{1}{3}\delta^1(\varepsilon), \frac{1}{3}\delta^*(\varepsilon)\}$ so that if $A(x, y, z) \geqslant \varepsilon$ and $\|x_i\| = \|y_i\| = \|z_i\|$ for $i = 1, 2$, then $\|x + y + z\| \leqslant 3(1 - \delta_0(\varepsilon))$.

For the general case, let $\varepsilon > 0$ and choose $\alpha < \varepsilon/16$ and $k < \frac{2}{3}$ such that

$$k\delta_{l_p}^{(1)}(\alpha) + \tfrac{2}{3}\alpha < \delta_0(\varepsilon/4)$$

where $\delta_{l_p}^{(1)}$ is the modulus of uniform convexity for l_p and δ_0 is as above.

If $x, y, z \in (E \oplus F)_p$ are norm-1 vectors with $\|x + y + z\| > 3(1 - k\delta_{l_p}^{(1)}(\alpha))$, then

$$\begin{aligned} 3(1 - k\delta_{l_p}^{(1)}(\alpha)) &< (\|x_1 + y_1 + z_1\|^p + \|x_2 + y_2 + z_2\|)^{p1/p} \\ &\leqslant \big((\|x_1\| + \|y_1\|)^p + (\|x_2\| + \|y_2\|)^p\big)^{1/p} + 1 \end{aligned}$$

which implies that

$$\big((\|x_1\| - \|y_1\|)^p + (\|x_2\| - \|y_2\|)^p\big)^{1/p} \leqslant \alpha\,.$$

Similarly we can show

$$\Big(\sum_{i=1}^{2}(\|x_i\| - \|z_i\|)^p\Big)^{1/p}, \qquad \Big(\sum_{i=1}^{2}(\|y_i\| - \|z_i\|)^p\Big)^{1/p} \leqslant \alpha\,.$$

Define $u, v \in (E \oplus F)_p$ by

$$u_i = \begin{cases} \dfrac{y_i\|x_i\|}{\|y_i\|} & \text{if } y_i \neq 0 \\ x_i & \text{if } y_i = 0 \end{cases}$$

for $i = 1, 2$ and with a similar definition for v with z_i replacing y_i. Then $\|u_i\| = \|v_i\| = \|x_i\|$ for $i = 1, 2$, and

$$\|u - y\| = \Big(\sum_{i=1}^{2}(\|x_i\| - \|y_i\|)^p\Big)^{1/p} \leqslant \alpha\,.$$

Similarly $\|v - z\| \leqslant \alpha$. Now we consider

$$\begin{aligned} \|x + u + v\| &\geqslant \|x + y + z\| - \|u - y\| - \|v - z\| \\ &\geqslant 3(1 - k\delta_{l_p}^{(1)}(2) - \tfrac{2}{3}\alpha) \\ &\geqslant 3(1 - \delta_0(\varepsilon/4)) \end{aligned}$$

which implies from above that $A(x, u, v) < \varepsilon/4$.

Finally, we note that

$$\begin{aligned} A(x, y, z) &= \sup_{\|f\|=1} \|f(z - y)x + f(x - z)y + f(y - x)z\| \\ &\leqslant \sup_f \big(\|f(v - u)x + f(x - v)y + f(u - x)z\| + \|f(z - v)x\| + \|f(u - y)x\| \\ &\qquad + \|f(v - z)y\| + \|f(y - u)z\|\big) \end{aligned}$$

$$
\begin{aligned}
&\leqslant \sup_f (\|f(v-u)x + f(x-v)u + f(u-x)v\| + \|f(x-v)(y-u)\| + \\
&\qquad\qquad + \|f(u-x)(z-v)\| + 2\|z-v\| + 2\|u-y\|) \\
&\leqslant A(x, u, v) + 4\|u-y\| + 4\|z-v\| \\
&\leqslant \varepsilon/4 + 8\alpha \\
&< \varepsilon
\end{aligned}
$$

which completes the proof. Q.E.D.

We now have all the results necessary for the proof of Theorem 2.

PROOF. – Suppose that all but one of the E_n's are (1-UR) with a common modulus of convexity and the remaining E_n is (2-UR). We may assume that E_1 is 2-UR and that $E_2, E_3, \dots$ are 1-UR and have a common modulus. By Theorem 1,

$$\Big(\sum_{n=2}^{\infty} \oplus E_n\Big)_p$$

is 1-UR and by Lemma 5

$$\Big(E_1 \oplus \sum_{n=2}^{\infty} \oplus E_n\Big)_p = \Big(\sum_{n=1}^{\infty} \oplus E_n\Big)_p$$

is 2-UR.

Conversely, suppose that

$$\Big(\sum_{n=1}^{\infty} \oplus E_n\Big)_p$$

is 2-UR. By Lemma 3 at most one of the E_n's can fail to be 1-UR so we may assume that $E_2, E_3, \dots$ are all 1-UR.

If $E_2, E_3, \dots$ do not have a common modulus of convexity, then there is an $\varepsilon > 0$ and norm-1 sequences (x_m) and (y_m) with $x_m, y_m \in E_{k_m}$ such that $\|x_m + y_m\| \to 2$ while $\|x_m - y_m\| > \varepsilon$, $\forall m$. If we define the sequences (u_m), (v_m), and (w_m) in

$$\Big(\sum_{n=1}^{\infty} \oplus E_n\Big)_p$$

by

$$
\begin{aligned}
u_m &= (0, \dots, 0, x_{2m-1}, 0, \dots, 0, x_{2m}, 0, \dots) \\
v_m &= (0, \dots, 0, y_{2m-1}, 0, \dots, 0, y_{2m}, 0, \dots) \\
w_m &= \left(0, \dots, 0, \frac{x_{2m-1} + 2y_{2m-1}}{\|x_{2m-1} + 2y_{2m-1}\|}, 0, \dots, 0, \frac{x_{2m} + y_{2m}}{\|x_{2m} + y_{2m}\|}, 0, \dots\right)
\end{aligned}
$$

and proceed as in the proof of Lemma 3, then we obtain a contradiction to

$$\Big(\sum_{n=1}^{\infty} \oplus E_n\Big)_p$$

being 2-UR. Hence $E_2, E_3, \dots$ must have a common modulus of convexity. Q.E.D.

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