Linear Boundary Value Problems for Systems of Ordinary Differential Equations on non Compact Intervals (*).

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Summary. – *Si stabiliscono teoremi di esistenza per problemi ai limiti lineari su intervalli aperti a destra in caso di risonanza.*

Introduction.

In this paper we shall prove some theorems which assure the existence of a bounded solution defined on the right open interval $[a, b)$, $(-\infty < a < b \leqslant +\infty)$, for the BVP, boundary value problem:

$$(*)\qquad \begin{cases} \dot{x}(t) - A(t)x(t) = f(t, x(t)) \\ Tx = r \end{cases}$$

where T is a linear operator.

A BVP on an infinite (right open) interval implies, beside some sort of initial conditions, a certain condition at infinity (at b). Such a condition may be a boundedness condition or the existence of the limit or some other kind of asymptotic behaviour. It is important to note that most BVPs on infinite intervals have been suggested by the study of problems in physics, as in the case of the famous Thomas-Fermi equation [23], or the Emden-Fowler equation [22].

The corresponding problem on a compact interval $[a, b]$ has been deeply studied. For a review of the methods of solution and for an ample bibliography see R. Conti [7]. Among the different methods used for solving this kind of BVPs we recall the alternative method of L. Cesari [6], which has been applied to a number of problems for differential equations, see the survey works of L. Cesari [4], [5]. Following J. Mawhin's method the problem $(*)$ is reduced to the abstract equation

$$Lx = Nx$$

which is solved, in the case that L is a Fredholm operator, by the local degree theory of Leray-Schauder. For the applications of this method and for an ample bibliography

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see J. MAWHIN - R. GAINES [13]. The problem (*) on a right open interval has been studied by many authors. We recall the admissibility theory introduced by J. L. MASSERA and J. J. SCHÄFFER [20]; W. A. COPPEL [8], [9], PH. HARTMAN [14], C. CORDUNEANU [10], [11] and C. AVRAMESCU [1], [2] use a similar approach to solve many BVPs. For a topological method see A. G. KARTSATOS [15], [16], [17], [18], who solves this problem under the hypothesis that the linear operator T restricted to the kernel of $d/dt - A(t)$ is invertible. In this paper we shall omit this last hypothesis and we shall not suppose that the linear operator associated with the system (*) is Fredholm, because this last hypothesis does not occur in the problems we are considering.

The method we are going to use is to reduce the problem (*) to the search for fixed points of an operator M that we shall construct using a theorem of P. L. ZEZZA [25]. We note that this operator may not be either completely continuous or defined in the whole space. It will be then useful to impose appropriate conditions to overcome this difficulties.

In the case that M is not completely continuous, using a method already employed by G. VILLARI [24], and A. G. KARTSATOS [15], we shall prove the existence of a fixed point by means of a diagonal process. The nonlinear boundary value problem is treated by the authors in [3].

§ 1. - Let $C = C[[a, b), R^n]$ be the locally convex space of continuous functions from $[a, b)$ into R^n, and let $BC = \{x(t) \in C \text{ such that } \sup_{t\in[a,b)} \|x(t)\| < +\infty\}$; BC is a Banach space with respect to the norm

$$\|x\|_{BC} = \sup_{t\in[a,b)} \|x(t)\| \qquad (-\infty < a < b \leqslant +\infty).$$

Let us consider the equation

$$\dot{x}(t) - A(t)x(t) = f(t, x(t)) \tag{1.1}$$

with the boundary condition

$$Tx = r\,, \quad r \in R^m\,, \quad m \leqslant n\,, \tag{1.2}$$

where $A(t)$ is a $n\times n$ matrix, continuous for $t \in [a, b)$ and such that the linear system associated to (1.1)

$$\dot{y}(t) - A(t)y(t) = 0 \tag{1.3}$$

is stable: i.e., the space D of all BC-solutions of (1.3) has dimension n.

Let

$$f\colon [a, b)\times R^n \to R^n$$

be a continuous function and

$$T : \operatorname{dom} T \subset BC \to R^m \qquad (m \leqslant n)$$

be a linear continuous operator such that $D \subset \operatorname{dom} T$ and its restriction to D is onto R^m, i.e. $T(D) = R^m$.

REMARK. – These conditions assure that the linear problem associated to (∗) for $f(t, x) \equiv 0$ has a solution for every $r \in R^m$.

Let L be the linear operator

$$L : \operatorname{dom} L \subset BC \to C \times R^m$$

defined by

$$x(t) \to \big(\dot{x}(t) - A(t)x(t), Tx\big)$$

where

$$\operatorname{dom} L = BC \cap C^1[[a, b), R^n] \cap \operatorname{dom} T ;$$

and let N be the operator

$$N : \operatorname{dom} N \subset BC \to C \times R^m$$

defined by

$$x(t) \to \big(f(\cdot, x(\cdot)), r\big) .$$

REMARK. – Because of the hypothesis of continuity on f dom $N = BC$.

The system (1.1)-(1.2) is equivalent to

$$Lx = Nx . \tag{1.4}$$

In general it is not possible to decompose the operator L as done in J. Mawhin's theory because it is not a Fredholm operator.

In fact:

a) $\operatorname{Im} L$ may not be a closed subspace of $C \times R^m$.

EXAMPLE. – Let $A(t) \equiv 0$, $Tx \equiv 0$, $m = 0$, $n = 1$, $a = 1$, $b = +\infty$, then the operator L becomes:

$$Lx = \frac{d}{dt} x(t) ;$$

if we choose $x_n(t) = -nt^{-1/n}$, then

$$Lx_n(t) = \frac{d}{dt} x_n(t) = t^{-(1+1/n)} = y_n(t) \in \operatorname{Im} L$$

but $\lim_{n\to+\infty} y_n(t) = y(t) = 1/t$, uniformly, and $1/t \notin \operatorname{Im} L$ because

$$\int_1^t y(s)\,ds = \int_1^t 1/s\,ds = \log t$$

in not bounded on $[1, +\infty)$.

b) The codimension of $\operatorname{Im} L$ may be infinity.

EXAMPLE. – Under the hypotheses of the preceding example, let

$$y_n(t) = t^{-1/n}\,, \qquad y_n \notin \operatorname{Im} L$$

because

$$x_n(t) = \int_1^t y_n(s)\,ds = \int_1^t s^{-1/n}\,ds = (t^{1-1/n})\,n/(n-1)$$

are not bounded and moreover $y_n(t)$ are linearly indipendent.

In the following we shall use this equivalence theorem for the equation (1.4) (See P. L. ZEZZA [25]).

THEOREM 1.1. – Let X, Y be linear spaces. Let

$$L\colon \operatorname{dom} L \subset X \to Y$$

be a linear operator and

$$N\colon \operatorname{dom} N \subset X \to Y$$

be an operator possibly non linear.

Then the equation (1.4) is equivalent to

$$\left\{\begin{array}{l} x = Mx \\ x \in \mathcal{A} \end{array}\right. \tag{1.5}$$

where

$$\mathcal{A} = \{x \in X\colon Nx \in \operatorname{Im} L\} = N^{-1}(\operatorname{Im} L)\,,$$

$$M\colon x \to Px + K_p Nx$$

and

$$P\colon X \to \operatorname{Ker} L$$

is a projection onto Ker L and

$$K_p = (L_{|\mathrm{dom} L \cap \mathrm{Im}(I-P)})^{-1}. \quad \square$$

We shall furthermore use the following fixed point theorem. (See P. L. ZEZZA [25]).

THEOREM 1.2. – Suppose that:

X is a Banach space,

dim Ker L is finite,

the operator M is completely continuous.

If Ω is an open, bounded neighbourhood of $0 \in X$, $\bar{\Omega} \subset \mathrm{dom}\, M$, such that

(1.6) $$x \in \partial\Omega, \quad \lambda \in (0.1) \Rightarrow Lx \neq \lambda Nx$$

or

$$x \in \partial\Omega, \quad \lambda \in (0.1) \Rightarrow x \neq \lambda K_p N x$$

then the operator M has at least one fixed point in $\bar{\Omega}$. $\square$

For the Theorem 1.1 this means that the equation (1.4) has at least one solution in $\bar{\Omega}$.

§ 2. – In this section we shall construct the operators we need for defining the operator M. Under our hypotheses $k = \dim \mathrm{Ker}\, L = n - m$ ($k \neq 0$ if $m < n$). Let $\varphi_1, \ldots, \varphi_k$ be a basis of Ker L; let us extend it to obtain a basis of D:

$$\varphi_1, \ldots, \varphi_k, \varphi_{k+1}, \ldots, \varphi_n; \quad \varphi_i \in BC.$$

Letting $X(t) = (\varphi_1, \ldots, \varphi_n)$ we get a fundamental matrix for equation (1.3). From our hypotheses we have:

$$\exists H > 0 \quad \text{such that } \|X(t)\| \leqslant H.$$

Let us consider the two operators:

(2.2) $$P_1: BC \to D, \quad P_1: x(t) \to X(t) X^{-1}(a) x(a)$$

(2.3) $$P_2: D \to \mathrm{Ker}\, L, \quad P_2: y(t) = \sum_{i=1}^{n} \lambda_i \varphi_i \to \sum_{i=1}^{k} \lambda_i \varphi_i;$$

the following lemmata hold:

LEMMA 2.1. – Under the hypotheses of section 1, P_1 and P_2 are topological projections (i.e. linear, continuous and idempotent).

PROOF. – Linearity follows immediately from definition.

For the idempotency:

$$P_1^2(x) = P_1(P_1(x)) = P_1[X(t)X^{-1}(a)x(a)] = X(t)X^{-1}(a)x(a) = P_1(x)\,.$$

$$P_2^2(x) = P_2(P_2(x)) = P_2\Big(P_2\Big(\sum_{i=1}^{n}\lambda_i\varphi_i\Big)\Big) = P_2\Big(\sum_{i=1}^{k}\lambda_i\varphi_i\Big) = \sum_{i=1}^{k}\lambda_i\varphi_i = P_2(x)\,, \qquad \text{for } x\in D.$$

For the continuity:

$$\|P_1(x)\| = \|X(t)X^{-1}(a)x(a)\|_{BC} = \sup_{t\in[a,b)}\|X(t)X^{-1}(a)x(a)\| \leqslant H\|X^{-1}(a)\|\,\|x\|$$

then P_1 is continuous. The continuity of P_2 follows immediately from its linearity because D is finite dimensional. □

LEMMA 2.2. – Let P, Q be two topological projections, $P\colon X\to X_1$, $Q\colon X_1\to X_2$, with X, X_1, X_2 linear topological spaces, $X\supset X_1\supset X_2$, then also the operator $(Q\circ P)(x) = Q(P(x))$ is a topological projection.

PROOF. – See [12], part I, pg. 481. □

Recalling that $\operatorname{Ker} L\subset D$, from the preceding lemmata we can immediately infer that:

$$P = P_2\circ P_1\colon BC\to \operatorname{Ker} L$$

is a topological projection.

Therefore if $x\in BC$, then Px is a solution of the system

$$\begin{cases} \dot y(t) - A(t)y(t) = 0 \\ Ty = 0 \end{cases} \tag{2.4}$$

and, moreover, if $w\in BC$ is a solution of (2.4) then $Pw = w$.

THEOREM 2.1. – *Under the hypotheses of section 1, if we fix $(b(t), r)\in \operatorname{Im} L$ then there exists one and only one solution $z(t)\in \operatorname{dom} L\subset BC$ of the system*

$$\begin{cases} a)\quad \dot z(t) - A(t)z(t) = b(t) \\ b)\quad Tz = r \end{cases} \tag{2.5}$$

such that $P(z) = 0$.

PROOF. – The existence of a solution for (2.5) follows from the choice of $(b(t), r)$. Let us now prove that there exists at least one solution $w = w(t)$ of (2.5) such that $P(w) = 0$. Let $z(t)$ be a solution of (2.5), we know that $P(z)$ is a solution of (2.4), hence also $w = z - P(z)$ is a solution of (2.5) and for this we have

$$P(w) = P(z - P(z)) = P(z) - P(z) = 0 . \tag{2.6}$$

This solution is unique. Let z_1 and z_2 be two solutions of (2.5) such that $P(z_1) = P(z_2) = 0$ and let $w_1 = z_1 - z_2$; w_1 is hence a solution of (2.4) and therefore $P(w_1) = w_1$, but

$$z_1 - z_2 = w_1 = P(w_1) = P(z_1 - z_2) = P(z_1) - P(z_2) = 0 . \quad \square$$

From now on we denote with $x(t, t_0, x_0)$, $y(t, t_0, y_0)$, $z(t, t_0, z_0)$, respectively, the solutions of (1.1), (1.3) and of (2.5) a), where x_0, y_0, z_0 are the values of the solutions for $t = t_0$.

Put $BC_{I-p} = \operatorname{Im}(I - P)$, and denote by K_p the linear operator

$$K_p\colon \operatorname{Im} L \to \operatorname{dom} L \subset BC_{I-p} \qquad K_p\colon (b(t), r) \to z(t)$$

where $z(t)$ is the unique solution of (2.5) such that $P(z) = 0$. We determine now the explicit form of K_p. Let $z(t) = K_p(b(t), r)$, by the variation of constants formula we have:

$$z(t) = X(t)X^{-1}(a)z(a) + \int_a^t X(t)X^{-1}(s)b(s)\,ds = X(t)X^{-1}(a)z(a) + z(t, a, 0) ; \tag{2.7}$$

applying the operator P we get

$$P(z) = P_2(P_1(z)) = P_2(X(t)X^{-1}(a)z(a)) = 0 .$$

Putting $c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = X^{-1}(a)z(a)$, the preceding formula can be written as

$$P_2(X(t)c) = 0 . \tag{2.8}$$

From the definition of P_2 and from (2.8) it follows

$$c_1 = c_2 = \ldots = c_k = 0$$

and so c is of the form $c = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ c_{k+1} \\ \vdots \\ c_n \end{pmatrix}$. The equation (2.7) hence becomes

$$z(t) = X(t)c + z(t, a, 0) . \tag{2.9}$$

REMARK. – In (2.9) $z(t) \in \operatorname{dom} T$ because it is a solution of (2.5), $X(t)c \in \operatorname{dom} T$ because we have supposed that $D \subset \operatorname{dom} T$ and then also $z(t, a, 0) \in \operatorname{dom} T$.

From the second equation in (2.5) we have

$$TX(t)c = r - Tz(t, a, 0) . \tag{2.10}$$

From our hypotheses and from the choice of φ_i it follows

$$TX(t) = (T\varphi_1, \dots, T\varphi_k, T\varphi_{k+1}, \dots, T\varphi_n) = (0, \dots, 0, T\varphi_{k+1}, \dots, T\varphi_n) \tag{2.11}$$

therefore if we call T_0 the $m \times m$ matrix $(T\varphi_{k+1}, \dots, T\varphi_n)$ recalling that $T(D) = R^m$ we infer

$$\det T_0 \neq 0 . \tag{2.12}$$

Calling $\bar{c} = \begin{pmatrix} c_{k+1} \\ \vdots \\ c_n \end{pmatrix}$, the linear system (2.10) is equivalent to

$$T_0 \bar{c} = r - Tz(t, a, 0) . \tag{2.13}$$

If we denote with J the immersion of R^m in R^n

$$J: R^m \to R^n , \qquad \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_m \end{pmatrix} \to \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \beta_{k+1} \\ \vdots \\ \beta_n \end{pmatrix}$$

where $\beta_{k+i} = \gamma_i$, $i = 1, 2, \dots, m$, we have

$$c = JT_0^{-1}(r - Tz(t, a, 0)) \tag{2.14}$$

and from (2.9) and (2.14) we conclude

$$\begin{aligned} z(t) = K_p(b(t), r) &= X(t)JT_0^{-1}(r - Tz(t, a, 0)) + z(t, a, 0) = \\ &= X(t)JT_0^{-1}\Big(r - T\int_a^t X(t)X^{-1}(s)b(s)\,ds\Big) + \int_a^t X(t)X^{-1}(s)b(s)\,ds . \end{aligned} \tag{2.15}$$

REMARK. – The operator K_p defined in (2.15) depends on P, because the choice of the fundamental matrix $X(t)$ is related to the form of P.

REMARK. – If $m = n$ this construction can be semplified: in fact in this case $P_2 = 0$ and the matrix $TX(t)$ is invertible; hence:

$$K_p(b(t), r) = X(t)(TX(t))^{-1}\Big(r - T\int_a^t X(t)X^{-1}(s)b(s)\,ds\Big) + \int_a^t X(t)X^{-1}(s)b(s)\,ds . \tag{2.16}$$

In this case A. G. KARTSATOS in [15], [16], [17], [18], has obtained some existence theorems that can be deduced from the results we shall state in section 4.

§ 3. – The equation (1.4), or the system (1.1)-(1.2), are equivalent, as it is stated in Theorem 1.1, to

$$\left\{\begin{array}{l} x = Mx \\ x \in \mathcal{A} \end{array}\right. \tag{3.1}$$

where

$$M: \operatorname{dom} M = \mathcal{A} \subset BC \to BC$$

$$M: x \to Px + K_p N x$$

$$\mathcal{A} = \{x \in BC: Nx \in \operatorname{Im} L\} = N^{-1}(\operatorname{Im} L)\,.$$

Let, in addition to the hypotheses of section 1, the following hold:

there are two functions $p(t)$, $q(t) \in C[[a, b), R]$, non-negative integrable on $[a, b)$ and such that

i) $\int_a^b p(t)\,dt = \Gamma < +\infty\,, \qquad \int_a^b q(t)\,dt = \Lambda < +\infty$

ii) $\|X^{-1}(t) f(t, u)\| \leq p(t)\|u\| + q(t)\,.$

REMARK. – From (2.15) we can easily see that the operator M is defined on

$$\mathcal{A} = \left\{g \in BC: \int_a^t X(t)\,X^{-1}(s)\,f\big(s, g(s)\big)\,ds = x(t, a, 0) \in \operatorname{dom} T\right\}.$$

Recalling that $\operatorname{dom} T \subset BC$, and that $T(D) = R^m$, the following lemmata hold:

LEMMA 3.1. – Under these hypotheses, if $\operatorname{dom} T = BC$, the operator M is defined on BC and it is continuous.

PROOF. – From the preceding remark if $g \in BC$ then

$$Ng = \big(f(\cdot, g(\cdot)), r\big) \in \operatorname{Im} L$$

if and only if

$$\int_a^t X^{-1}(s)\,f\big(s, g(s)\big)\,ds \in BC = \operatorname{dom} T\,.$$

From i) and ii) we have

$$\left\| \int_a^t X^{-1}(s) f(s, g(s))\, ds \right\| \leqslant \Gamma \|g\| + \Lambda$$

hence $Ng \in \operatorname{Im} L$.

Let us now prove the continuity of $M = P + K_p N$. We have already proved that the projection P is continuous.

Let $\{x_n\}$, $n \in N$, be a sequence in BC converging to $x \in BC$; we have to prove that $\{K_p N x_n\}$ converges to $K_p N x$; for (2.15) it is sufficient to show that the sequence

$$\int_a^t X^{-1}(s)\,[f(s, x_n(s)) - f(s, x(s))]\, ds \qquad n \in N \tag{3.2}$$

converges to zero.

Because of the continuity of f the sequence

$$X^{-1}(t)\,[f(t, x_n(t)) - f(t, x(t))] \qquad n \in N \tag{3.3}$$

converges to zero, and moreover

$$\begin{aligned} \|X^{-1}(t)\,[f(t, x_n(t)) - f(t, x(t))]\| &\leqslant \|X^{-1} f(t, x_n(t))\| + \|X^{-1}(t) f(t, x(t))\| \leqslant \\ &\leqslant \|x_n\| p(t) + \|x\| p(t) + 2q(t) \leqslant (2\|x\| + \varepsilon)\, p(t) + 2q(t) \end{aligned}$$

for $n \geqslant n_\varepsilon$. Hence the sequence (3.2) converges to zero for the Lebesgue dominated convergence theorem. □

LEMMA 3.2. – The operator M transforms bounded sets into sets of equibounded and equicontinuous functions.

PROOF. – Since P is a linear operator and its image is finite dimensional (hence compact) it is sufficient to prove the statement for the operator $K_p N$.

Let Ω be a bounded set, $\bar{\Omega} \subset \mathcal{A}$; then

$$x \in \Omega \Rightarrow \|x\| \leqslant \mu$$

then

$$\begin{aligned} \|K_p N x\| &\leqslant \|X(t) J T_0^{-1}(r - Tx(t, a, 0))\| + \|x(t, a, 0)\| \leqslant \\ &\leqslant \|X(t)\| \|J T_0^{-1}(r - Tx(t, a, 0))\| + \|X(t)\| \left\| \int_a^t X^{-1}(s) f(s, x(s))\, ds \right\| \leqslant \\ &\leqslant H \|J T_0^{-1}\| (\|r\| + \|T\| H(\Gamma\mu + \Lambda)) + H(\Gamma\mu + \Lambda) \end{aligned}$$

the equiboundedness is proved.

Let us now prove the equicontinuity. Let $t_1, t_2 \in [a, b)$ and if we put

$$\delta(t, x(t)) = \int_a^t X^{-1}(s) f(s, x(s))\, ds$$

and

$$V = JT_0^{-1}\left[r - TX(t)\,\delta(t, x(t))\right]$$

we have

$$\begin{aligned}
\|(K_p Nx)(t_2) - (K_p Nx)(t_1)\| &= \|X(t_2)V + X(t_2)\,\delta(t_2, x(t_2)) - X(t_1)V - X(t_1)\,\delta(t_1, x(t_1))\| \leqslant \\
&\leqslant \|X(t_2) - X(t_1)\|\,\|V\| + \|X(t_2)\,\delta(t_2, x(t_2)) - X(t_1)\,\delta(t_2, x(t_2)) + \\
&\qquad - X(t_1)\int_{t_1}^{t_2} X^{-1}(s) f(s, x(s))\, ds\| \leqslant \\
&\leqslant \|X(t_2) - X(t_1)\|\,\|V\| + \|X(t_2) - X(t_1)\|\,\Big\|\int_a^{t_2} X^{-1}(s) f(s, x(s))\, ds\Big\| + \\
&\qquad + \Big\|\int_{t_2}^{t_1} X(t_1) X^{-1}(s) f(s, x(s))\, ds\Big\| \leqslant \\
&\leqslant \|X(t_2) - X(t_1)\|\,\{\|JT_0^{-1}\|(\|r\| + \|T\| H(\Gamma\mu + \Lambda)) + \Gamma\mu + \Lambda\} + \\
&\qquad + H\Big(\mu\int_{t_1}^{t_2} p(s)\, ds + \int_{t_1}^{t_2} q(s)\, ds\Big)
\end{aligned}$$

from which the statement follows. □

§ **4.** – In this section we are going to state some existence theorems for the solutions of the system (1.1)-(1.2); this problem, as we have seen, is equivalent to the one of finding the solutions of the equation

$$\text{(4.1)}\qquad \begin{cases} x = Mx = Px + K_p Nx \\ x \in \mathcal{A}. \end{cases}$$

We start from a special case: the existence of solutions of (4.1) in the space $BC_l \subset BC$

$$BC_l = \{x \in BC : \lim_{t \to b} x(t) = l_x\}\ (\|l_x\| < +\infty)\,.$$

The following lemma holds:

LEMMA 4.1. – Suppose that, for the system (1.1)-(1.2) the following hypotheses hold:

(4.2) $A(t)$ is a real valued $n \times n$ matrix, defined and continuous on $[a, b)$ $(-\infty < a < b \leqslant +\infty)$ and such that if $X(t)$ is a fundamental matrix of (1.3), defined as in section two, we have $\|X(t)\| \leqslant H$.

(4.3) $$\lim_{t\to b} X(t) = W\,, \quad \text{i.e. } D \subset BC_l$$

(4.4) $f \in C[[a, b)\times R^n, R^n]$ and such that

$$\|X^{-1}(t)f(t, u)\| \leqslant p(t)\|u\| + q(t)$$

where $p(t)$, $q(t) \in C[[a, b), R]$ are non negative, integrable, functions such that

$$\int_a^b p(t)\,dt = \Gamma < +\infty\,, \qquad \int_a^b q(t)\,dt = \Lambda < +\infty$$

(4.5) T is a bounded, linear operator from dom $T = BC_l$ onto R^m and the matrix $TX(t)$ has rank m.

Then the operator M is defined on BC, its image is contained in BC_l and it is completely continuous.

Proof. – Let us observe that in Lemma 3.1. we have proved that

$$\int_a^b X^{-1}(s)f(s, x(s))\,ds \in BC$$

but, moreover, this integral for (4.4) is absolutely convergent and it is convergent on $[a, b)$, i.e.

$$\mathcal{A} = \text{dom}\, M = BC\,.$$

Moreover, recalling that $\text{Im}\, P = D \subset BC_l$, we have from (4.2) and (4.5)

$$\text{Im}\, M \subset BC_l\,.$$

Furthermore, still from Lemma 3.1, it follows that M is a continuous operator; hence it is sufficient to show that K_pN transforms bounded sets into relatively compact sets.

It is known that a subset Φ of BC_l is relatively compact if and only if it is ([1]):

1) equibounded;

2) equicontinuous;

3) uniformly convergent, in the following sense:

$$\forall\varepsilon > 0\ \exists\delta(\varepsilon) > 0 \quad \text{such that } \forall t > \delta(\varepsilon),\ \forall g \in \Phi \Rightarrow \|g(t) - l_g\| \leqslant \varepsilon\,.$$

The equicontinuity and the equiboundedness of $K_pN(\Omega)$ has been already proved (Lemma 3.2). Let us now prove the uniform convergence.

Let $\Omega \subset \operatorname{dom} M$ be bounded, i.e.

$$x \in \Omega \Rightarrow \|x\| \leq \mu ;$$

from (2.15) and from our hypoteses we have

$$\|(K_p Nx)(t) - \{\lim_{t\to b} (K_p Nx)(t)\}\| \leq \|W - X(t)\| \left\{ \|JT_0^{-1}\| \left[\|r\| + \|T\| H(\Gamma\mu + \Lambda)\right]\right\} + \\ + \left\| W\int_a^b X^{-1}(s) f(s, x(s))\, ds - X(t)\int_a^t X^{-1}(s) f(s, x(s))\, ds \right\|,$$

but

$$\left\| W\int_a^b X^{-1}(s) f(s, x(s))\, ds - X(t)\int_a^t X^{-1}(s) f(s, x(s))\, ds \right\| = \\ = \left\| W\int_a^b X^{-1}(s) f(s, x(s))\, ds - X(t)\int_a^b X^{-1}(s) f(s, x(s))\, ds + X(t)\int_t^b X^{-1}(s) f(s, x(s))\, ds \right\| \leq \\ \leq \|W - X(t)\|(\Gamma\mu + \Lambda) + H\left(\mu\int_t^b p(s)\, ds + \int_t^b q(s)\, ds\right)$$

and then

$$\|(K_p Nx)(t) - \lim_{t\to b} \{(K_p Nx)(t)\}\| \leq \\ \leq \|W - X(t)\| \left\{ \|JT_0^{-1}\| (\|r\| + \|T\| H(\Gamma\mu + \Lambda)) + \Gamma\mu + \Lambda \right\} + \\ + H\left(\mu\int_t^b p(s)\, ds + \int_t^b q(s)\, ds\right);$$

the uniform convergence is proved. □

To get theorems which assure the existence of solutions of (1.1)-(1.2), that is of fixed points for the operator M in BC_l, it is sufficient to add to the conditions of Lemma 4.1. an a-priori bound and to use some fixed point theorems.

For example the following theorems hold:

THEOREM 4.1. – If the conditions (4.2), (4.3), (4.4), (4.5) are verified and if

$$H^2 \|JT_0^{-1}\| \|T\| \Gamma \exp(H\Gamma) < 1 \tag{4.6}$$

then the operator M has at least one fixed point in BC_l.

PROOF. – The operator $M: BC_l \to BC_l$ is completely continuous because of Lemma 4.1. For Theorem 1.2 we have to show that there exists $\Omega \subset BC$, open, bounded neighbourhood of 0 such that

$$x \neq \lambda K_p Nx \qquad x \in \partial\Omega \qquad \lambda \in (0, 1) .$$

Let $\Omega = \{x \in BC_l : \|x\| \le \varrho\}$, if there exists $\bar{x} \in \partial\Omega$ such that

$$\bar{x} = \lambda K_p N \bar{x} \quad \text{for some } \lambda \in (0, 1)$$

then $\forall t \in [a, b)$ we have

$$\begin{aligned}
\|\bar{x}(t)\| &< \frac{1}{\lambda}\|\bar{x}(t)\| = \|(K_p N\bar{x})(t)\| \le \\
&\le \left\| X(t) J T_0^{-1}\left[r - T\int_a^t X(t) X^{-1}(s) f(s, \bar{x}(s))\, ds\right]\right\| + \left\|\int_a^t X(t) X^{-1}(s) f(s, \bar{x}(s)(ds\right\| \le \\
&\le H\|JT_0^{-1}\|[\|r\| + \|T\| H(\Gamma\|\bar{x}\| + \Lambda)] + H\Lambda + H\int_a^t p(s)\|\bar{x}(s)\| ds
\end{aligned}$$

and applying Gromwall's Lemma

$$\|\bar{x}(t)\| < \{H\|JT_0^{-1}\|[\|r\| + \|T\| H(\Gamma\|\bar{x}\| + \Lambda)] + H\Lambda\} \exp(H\Gamma)$$

then

$$\|\bar{x}(t)\| \le [H\|JT_0^{-1}\|\|r\| + H^2\|JT_0^{-1}\|\|T\|\Gamma\|\bar{x}\| + H^2\|JT_0^{-1}\|\|T\|\Lambda + H\Lambda] \exp(H\Gamma)$$

namely

$$\begin{aligned}
&(1 - H^2\|JT_0^{-1}\|\|T\|\Gamma \exp(H\Gamma))\|\bar{x}\| \le \\
&\qquad\qquad \le [H\|JT_0^{-1}\|\|r\| + H^2\|JT_0^{-1}\|\|T\|\Lambda + H\Lambda] \exp(H\Gamma)
\end{aligned}$$

but, recalling (4.6), this is a contraddiction for ϱ sufficiently large. The theorem is hence proved. □

THEOREM 4.2. – If the conditions (4.2), (4.3), (4.4), (4.5) are verified and if

$$H^2\|JT_0^{-1}\|\|T\|\Gamma + H\Gamma < 1 \tag{4.7}$$

then the operator M has at least one fixed point in BC_l.

PROOF. – As in Theorem 4.1. Let $\Omega = \{x \in BC_l : \|x\| \le \varrho\}$, and suppose that there exists $\bar{x} \in \partial\Omega$ such that

$$\bar{x} = \lambda K_p N\bar{x} \quad \text{for some } \lambda \in (0, 1);$$

we have

$$\|\bar{x}\| = \|\lambda K_p N\bar{x}\| < \|K_p N\bar{x}\| \le H\|JT_0^{-1}\|[\|r\| + \|T\| H(\Gamma\|\bar{x}\| + \Lambda)] + H\Lambda + H\Gamma\|\bar{x}\| .$$

From this

$$(1 - H^2\|JT_0^{-1}\|\,\|T\|\Gamma + H\Gamma)\|\bar{x}\| \leqslant H\|JT_0^{-1}\|\,\|r\| + H^2\|JT_0^{-1}\|\,\|T\|\Lambda + H\Lambda$$

but, recalling (4.7), this is a contraddiction for ϱ sufficiently large. The theorem is hence proved. □

We can now consider a more general case: the existence of solutions of equation (4.1) in BC (omitting the hypothesis (4.3)). Let us suppose that dom $T = BC$. From Lemma 4.1. we can affirm that $M: BC \to BC$, but it is not possible to repeat in BC the same reasoning used in BC_l, because the compactness theorem is not true anymore.

The existence of fixed points for the operator M shall be proved via Theorem 4.1 together with a diagonal process.

Let $\{a_i\}$, $i \in N$, an increasing sequence of real numbers such that

$$a_1 = a\,, \qquad \lim_{i \to +\infty} a_i = b\,;$$

let $I_i = [a, a_i]$, if $g(t) \in C[I_i, R^n]$ call E_i the set of every function $\bar{g}(t)$ defined in this way:

$$\bar{g}(t) = \begin{cases} g(t) & \text{if } t \in I_i \\ g(a_i) & \text{if } t \in [a_i, b)\,; \end{cases}$$

E_i is a Banach space with respect to the norm

$$\|\bar{g}\| = \sup_{t \in I_i} \|g(t)\|\,;$$

moreover E_i is isomorphic to $C[I_i, R^n]$:

The following lemma holds:

LEMMA 4.2. – Suppose that, for the system (1.1)-(1.2) the following hypotheses hold:

(4.8) $A(t)$ is a real valued $n \times n$ matrix, defined and continuous on $[a, b)$ $(-\infty < a < b \leqslant +\infty)$ and such that if $X(t)$ is a fundamental matrix of (1.3), defined as in the second section, we have $\|X(t)\| \leqslant H$.

(4.9) $f \in C[[a, b) \times R^n, R^n]$ and such that

$$\|X^{-1}(t) f(t, u)\| \leqslant p(t)\|u\| + q(t)$$

where $p(t)$ and $q(t)$ are non negative, integrable, real-valued functions such that

$$\int_a^b p(t)\,dt = \Gamma < +\infty \qquad \int_a^b q(t)\,dt = \Lambda < +\infty\,.$$

(4.10) T is a bounded, linear operator from dom $T = BC$ onto R^m and the matrix $TX(t)$ has characteristic m.

If, moreover, the condition (4.6) is satisfied, then the operator

$$M_i\colon \operatorname{dom} M_i \subseteq E_i \to E_i$$

defined by

$$M_i\colon \bar{g}(t) \to \bar{x}(t)$$

where

$$x(t) = (M\bar{g})(t) \qquad g(t) \in C[I_i, R^n]\,, \qquad t \in I_i$$

has at least one fixed point in E_i.

PROOF. – The complete continuity of the operator M_i can be proved via the Ascoli-Arzelà theorem whose hypotheses are easily verified from the Lemmata 3.1 and 3.2. The statement follows from (4.6) as in Theorem 4.1. □

We can now show that a solution of system (1.1)-(1.2) exists in BC.

THEOREM 4.3. – If the conditions (4.6), (4.8), (4.9), (4.10) are satisfied, then the system (1.1)-(1.2) has at least one solution in BC.

PROOF. – Because of Lemma 4.2. there exists a sequence $\{x_i\}$, $\bar{x}_i \in E_i$ such that

$$\bar{x}_i = M_i \bar{x}_i\,.$$

from the definition of M_i we have

$$x_i(t) = (M_i\bar{x}_i)(t) = M\bar{x}_i(t) \qquad t \in I_i\,. \tag{4.11}$$

The sequence $\{x_i\}$ is equibounded and equicontinuous in $C[I_1, R^n]$: the equiboundedness follows from the proof of Theorem 4.1 and the equicontinuity as in Lemma 3.2. Hence, for the Ascoli-Arzelà Theorem, there exists a subsequence $\{x_i^1(t)\}$ that converges uniformly to $z_1(t) \in C[I_1, R^n]$, i.e.

$$\lim_{i\to+\infty} x_i^1(t) = z_1(t) \qquad \text{uniformly } \forall t \in I_1\,.$$

Analogously there exists a subsequence $\{x_i^2(t)\}$ of $\{x_i^1(t)\}$ that converges uniformly to $z_2(t)$ on I_2 such that $z_2(t) = z_1(t)$ $\forall t \in I_1$ We can repeat this reasoning $\forall i \in N$. In this way we obtain a family of subsequences of x_i:

$$\begin{array}{ll} x_1^1,\ x_2^1,\ x_3^1,\ \dots & \text{unif. conv. on } I_1 \\ x_1^2,\ x_2^2,\ x_3^2,\ \dots & \text{unif. conv. on } I_2 \\ \dots\dots\dots & \\ \dots\dots\dots & \\ x_1^n,\ x_2^n,\ x_3^n,\ \dots & \text{unif. conv. on } I_n \\ \cdot & \\ \cdot & \\ \dots\dots\dots & \end{array}$$

Let $\{x_i^i(t)\}$ the subsequence of $\{x_i(t)\}$ obtained with a diagonal process: the sequence $\{\bar{x}_i^i\}$ converges uniformly in each compact of $[a, b)$; then there exists $z(t) \in C[[a, b), R^n]$ such that

$$\lim_{i\to+\infty} \|\bar{x}_i^i(t) - z(t)\| = 0 \tag{4.12}$$

uniformly on each compact of $[a, b)$.

Moreover $z(t)$ is bounded on $[a, b)$ because $\{\bar{x}_i(t)\}$ is equibounded. It remains to prove that $z(t)$ is a solution of our problem.

Let

$$y(t) = Mz(t) = Pz(t) + K_p Nz(t)\,.$$

For fixed $c \in [a, b)$, $\forall t \in [a, c]$ and for i sufficiently large from (4.11) we have

$$\begin{aligned} \|\bar{x}_i^i(t) - y(t)\| &= \|M\bar{x}_i(t) - y(t)\| \leqslant \\ &\leqslant \|P\|\,\|\bar{x}_i^i(t) - z(t)\| + \|K_p N\bar{x}_i^i(t) - K_p Nz(t)\| \leqslant \\ &\leqslant \|P\|\,\|\bar{x}_i^i(t) - z(t)\| + H(H\|JT_0^{-1}\|\,\|T\| + 1)\int_a^b \|X^{-1}(s)[f(s, \bar{x}_i^i(s)) - f(s, z(s))]\|\,ds\,. \end{aligned}$$

From (4.12) and applying the Lebesgue dominated convergence theorem, we can infer:

$$\lim_{i\to+\infty} \|\bar{x}_i^i(t) - y(t)\| = 0 \qquad t \in [a, c]. \tag{4.13}$$

Comparing (4.12) and (4.13) we can conclude

$$y(t) = z(t) = Mz(t) \qquad t \in [a, c]$$

Since c is arbitrary then

$$z(t) = Mz(t) \qquad t \in [a, b).$$

The theorem is hence proved. □

Likewise, we can state a theorem similar to Theorem 4.2.

THEOREM 4.4. – If the conditions (4.7), (4.8), (4.9), (4.10) are satisfied, then the system (1.1)-(1.2) has at least one solution in BC.

The proof is similar to the one of the preceding theorem.

REMARK. – In the Lemmata 3.1, 3.2, 4.1 the condition

$$\|X^{-1}(t) f(t, u)\| \leqslant p(t)\|u\| + q(t)\,,$$

with $p, q \in C[[a, b), R^+]$ such that

$$\int_a^b p(t)\,dt = \Gamma < +\infty \qquad \int_a^b q(t)\,dt = \Lambda < +\infty$$

can be replaced with the less restrictive one

$$\|X^{-1}(t) f(t, u)\| \leqslant g(t, \|u\|) + q(t)\,,$$

with $g \in C[[a, b) \times R^+, R^+]$, $q \in C[[a, b), R^+]$ such that

$$\int_a^b g(t, \|u\|)\,dt < +\infty\,, \qquad u \in BC$$

$$\int_a^b q(t)\,dt = \Lambda < +\infty\,.$$

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